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L_p-Norms of Polynomials with Positive Real Part

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We derive an estimate for

$$\Delta_{n,1} = \sup\left\{(2\pi)^{-1}\int_0^{2\pi} |p(e^{it})|\,dt : p(z) = 1 + a_1 z + \cdots + a_n z^n,\ \operatorname{Re}(p(z)) > 0 \text{ for } |z| < 1\right\}.$$

In particular it is shown that

$$\Delta_{n,1} \leqslant 1 + \log(C_1(n+1)+1),$$

where $C_1 = 0.686981293\ldots$. It is also shown that $2/\pi \leqslant \liminf_{n\to\infty} \Delta_{n,1}/\log n$. Finally, upper bounds are found for the L_p-norms of polynomials with positive real part on the unit disk. © 1991 Academic Press, Inc.

Let $\mathscr{A}_n$ denote the collection of polynomials in z which have positive real part on the open unit disk centered at 0 and which take the value 1 at 0. Let $\|f\|_p = ((2\pi)^{-1}\int_0^{2\pi} |f(e^{it})|^p\,dt)^{1/p}$ for $p \geqslant 1$ and let $\|f\|_\infty$ denote the corresponding L_∞-norm of f. In this paper we study the following extremal problems:

$$\Delta_{n,p} = \sup\{\|f\|_p : f \in \mathscr{A}_p\}, \qquad 1 \leqslant p \leqslant \infty, \quad n \geqslant 1. \tag{1}$$

Our work has its origins in the problem posed by Holland in [1], namely, to solve (1) in the case where $p = 2$. Of course, by "solving" (1) we mean

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calculating $\Delta_{n,p}$ and characterizing the extremal functions. It is only in the case $p=\infty$ that (1) has been completely solved. (See [5].)

In the case $p=2$ the problem (1) is related to the following:

$$\Lambda_n = \sup\{\|r\|_2^2 : r \in \mathscr{T}_n\}, \tag{2}$$

where $\mathscr{T}_n$ denotes the set of non-negative trigonometric polynomials having degree $\leqslant n$ and constant term equal to 1. Indeed, it is easy to see that $\Lambda_n = 2\Delta_{n,2}^2 - 1$. The following continuous analogue of (2) has a strong bearing on our present investigations:

$$C_1 = \sup\{\textstyle\int_{-1}^{1} |f(x)|^2\, dx : f \text{ is positive definite and continuous on } (-\infty, \infty), \text{ vanishes outside of } [-1, 1], \text{ and } f(0)=1\}. \tag{3}$$

Equation (3) has been studied extensively by Garsia, Rodemich, and Rumsey in [4]. They have calculated C_1 to many decimal places via a certain iterative scheme. ($C_1 = 0.686981293....$) In [2] we were able to show that

$$C_1(n+1) \leqslant \Lambda_n \leqslant 1 + C_1(n+1). \tag{4}$$

We will use (4) together with Theorem 2 below to obtain an estimate for $\Delta_{n,1}$. Once we have established our estimate for $\Delta_{n,1}$, we will combine it with results from [2, 5] to obtain upper bounds for $\Delta_{n,p}$.

We begin with a result which is in the spirit of discussions found in Zygmund [7, p. 261–262].

THEOREM 1. *Suppose that $f = u + iv$ is analytic and has positive real part in some neighborhood of the closed unit disk centered at* 0. *Suppose also that $f(0)=1$. Then*

$$(2\pi)^{-1} \int_0^{2\pi} |f(e^{it})|\, dt \leqslant 1 + (2\pi)^{-1} \int_0^{2\pi} u(e^{it}) \log u(e^{it})\, dt.$$

Proof. By straightforward calculations involving the use of the Cauchy–Riemann equations we have

$$\nabla^2 |f(z)| = |f'(z)|^2 / |f(z)|$$

$$\nabla^2 u(z) \log u(z) = |f'(z)|^2 / u(z),$$

where ∇^2 denotes the Laplacian. Since $u < |f|$, it follows that the function $|f| - u \log u$ is subharmonic. Hence, we have

$$(2\pi)^{-1} \int_0^{2\pi} (|f(e^{it})| - u(e^{it}) \log u(e^{it}))\, dt$$

$$\leqslant |f(0)| - u(0) \log u(0) \leqslant 1.$$

THEOREM 2. *Suppose that f belongs to the usual Hardy space H_2 of the unit disk. Suppose also that f has positive real part on the interior of the unit disk and that $f(0)=1$. Then*

$$(2\pi)^{-1}\int_0^{2\pi}|f(e^{it})|\,dt\leqslant 1+\log\left((2\pi)^{-1}\int_0^{2\pi}(\operatorname{Re}f(e^{it}))^2\,dt\right).$$

Proof. We prove the inequality first in the case where $f=u+iv$ is analytic and has positive real part in a neighborhood of the closed unit disk. Let the measure v be defined by $dv=u(e^{it})\,dt/2\pi$. We note that v is a positive measure with total mass equal to 1. By Theorem 1 we have

$$(2\pi)^{-1}\int_0^{2\pi}|f(e^{it})|\,dt\leqslant 1+\int_0^{2\pi}\log u(e^{it})\,dv.$$

It follows from the concavity of the log function that

$$\begin{aligned}(2\pi)^{-1}\int_0^{2\pi}|f(e^{it})|\,dt&\leqslant 1+\log\int_0^{2\pi}u(e^{it})\,dv\\&\leqslant 1+\log\left((2\pi)^{-1}\int_0^{2\pi}(\operatorname{Re}f(e^{it}))^2\,dt\right).\end{aligned}$$

To handle the general case we simply replace f by the function f_r defined by $f_r(z)=f(rz)$, where $r<1$, and then use a limiting argument.

THEOREM 3. $\Delta_{n,1}\leqslant 1+\log(C_1(n+1)+1)$.

Proof. Consider a polynomial p in $\mathscr{A}_n$. By Theorem 2 it follows that

$$(2\pi)^{-1}\int_0^{2\pi}|p(e^{it})|\,dt\leqslant 1+\log\left((2\pi)^{-1}\int_0^{2\pi}(\operatorname{Re}p(e^{it}))^2\,dt\right).$$

Since $\operatorname{Re}p$ belongs to $\mathscr{T}_n$, it follows from (4) that

$$(2\pi)^{-1}\int_0^{2\pi}|p(e^{it})|\,dt\leqslant 1+\log(C_1(n+1)+1).$$

THEOREM 4.

$$\Delta_{n,p}\leqslant\begin{cases}(1+\log(C_1(n+1)+1))^{2/p-1}(C_1(n+1)/2+1)^{1-1/p}, & 1\leqslant p\leqslant 2.\\(C_1(n+1)/2+1)^{1/p}\,(n+1)^{1-2/p}, & 2\leqslant p\leqslant\infty.\end{cases}$$

Proof. The theorem follows from Theorem 3, the inequality (4), the result of Holland [5, Th. 2], and the Riesz–Thorin interpolation theorem (see [6, p. 96]).

By Theorem 3 the sequence $\{\Delta_{n,1}\}$ is bounded above, term by term, by a sequence which behaves asymptotically like $\log n$. We will show that there is a sequence $\{t_n\}$ which behaves asymptotically like $(2/\pi)\log n$ and satisfies $t_n \leqslant \Delta_{n,1}$ for large n. The idea is to use the polynomial

$$f(z) = 1 + 2\sum_{k=1}^{n} (1 - k/(n+1))z^k.$$

Holland has shown that f is extremal for the problem (1) in the case where $p = \infty$. It follows from Hardy's inequality for H_1-functions that

$$(2\pi)^{-1}\int_0^{2\pi} |f(e^{it})|\, dt \geqslant \pi^{-1}\left(1 + 2\sum_{k=1}^{n} (1 - k/(n+1))/(k+1))\right).$$

Thus, we have

$$\Delta_{n,1} \geqslant t_n = \pi^{-1}\left(1 + 2\sum_{k=1}^{n} (1 - k/(n+1))/(k+1)\right).$$

Since t_n behaves asymptotically like $\log n$, we have proved the following:

THEOREM 5. $\liminf_{n\to\infty} \Delta_{n,1}/\log n \geqslant 2/\pi$.

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