



# Forked Temperley–Lieb algebras and intermediate subfactors

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## Abstract

We consider noncommuting pairs  $P, Q$  of intermediate subfactors of an irreducible, finite-index inclusion  $N \subset M$  of  $\text{II}_1$  factors such that  $P$  and  $Q$  are supertransitive with Jones index less than 4 over  $N$ . We show that up to isomorphism of the standard invariant, there is a unique such pair corresponding to each even value  $[P : N] = 4 \cos^2 \frac{\pi}{2n}$  but none for the odd values  $[P : N] = 4 \cos^2 \frac{\pi}{2n+1}$ . We also classify the angle values which occur between pairs of intermediate subfactors with small index over their intersection: if  $[P : N], [Q : N] < 4$ , then the unique nontrivial angle value is always  $\cos^{-1} \frac{1}{[P : N] - 1}$ .

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## 1. Introduction

A fundamental example of a subfactor is the fixed-point algebra of an outer action of a finite group on a von Neumann algebra with trivial center. In this case the structure of the subfactor is determined by the structure of the group. One then thinks of a general subfactor as a “quantum” version of a finite group, and subfactor theory as a “non-commutative Galois theory.” In this spirit it is natural to consider an intermediate subfactor  $N \subset P \subset M$  as an analogue of a subgroup, and indeed, the intermediate subfactors of the fixed-point subfactor of an action of a finite group are precisely the fixed-point subfactors of its subgroups.

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The main classification theory for subfactors involves a standard invariant, which was axiomatized by Popa [20] and then described by Jones using a diagrammatic apparatus called planar algebras [13]. A major structural feature of subfactors is duality: there are two algebraic structures applicable to the standard invariant (analogous to Hopf-algebra duality for groups) which are reflected in the planar algebra picture by a choice of shading. Bisch discovered the remarkable fact that intermediate subfactors are characterized by “biprojections”—elements of the standard invariant that are projections in both algebra structures [1]. Bisch’s theorem thus reduces the problem of intermediate subfactors to a problem of planar algebras.

Bisch and Jones then studied the planar algebra generated by a single biprojection, which yields a generic construction of an intermediate subfactor, or more generally, a chain of intermediate subfactors [2]. This led naturally to the approach of trying to construct more general families of intermediate subfactors by considering planar algebras generated by multiple biprojections.

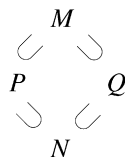
There is a notion of commutativity for pairs of intermediate subfactors (defined simply by commutativity of the corresponding projections in  $L^2(M)$ ). Sano and Watatani considered the set of angles between two subfactors, a numerical invariant which measures the degree of non-commutativity of a pair of intermediate subfactors [21]. It turns out that pairs of commuting subfactors can be constructed simply via a tensor product, but the construction of intermediate subfactors with nontrivial angles has proven much more difficult.

In [9], Jones and the present author set out to construct generic pairs of noncommuting intermediate subfactors by assuming *no extra structure*, i.e. that the standard invariants of the elementary subfactors involved were just Temperley–Lieb algebras, a situation also referred to as supertransitivity. It was hoped that since the bimodule components of the intermediate subfactors obey the Temperley–Lieb fusion rules, that information together with invariants such as angles and indices would provide sufficient rigidity to suggest appropriate planar relations among biprojections and yield generic constructions of planar algebras generated by two biprojections.

Instead we found the surprising result that there are essentially only two quadrilaterals with no extra structure. One is the fixed point algebra of an outer action of  $S_3$  on a factor (where the intermediate subfactors are fixed point algebras of distinct order 2 subgroups) and the other is a new example with irrational angles and indices coming from the GHJ family of subfactors.

The dearth of examples with no extra structure does not however mean that the original project of constructing planar algebras generated by multiple biprojections is unsound. Rather, because of the rigidity of intermediate subfactors, we must adjust our hypotheses if we wish to obtain more examples.

In the present work we continue this approach by considering a (noncommuting, irreducible, finite index) quadrilateral



in which only the lower two elementary subfactors  $N \subset P$  and  $N \subset Q$  are assumed to be supertransitive. We also assume that the lower indices  $[P : N]$  and  $[Q : N]$  are less than 4 (and therefore equal to  $4 \cos^2 \frac{\pi}{k}$  for some  $k \geq 3$  by Jones’ theorem in [12]).

We first show that for any quadrilateral with lower indices  $[P : N]$  and  $[Q : N]$  less than 4 there is a unique nontrivial angle value, equal to  $\cos^{-1} \frac{1}{[P:N]-1}$ . We then describe a series of examples in the GHJ family—first discovered in [9]—of quadrilaterals whose lower subfactors are supertransitive with indices  $[P : N] = [Q : N] = 4 \cos^2 \frac{\pi}{2k}$ ,  $k \geq 3$ .

Ultimately we prove the following result.

**Theorem 1.0.1.** *The series of GHJ subfactors for  $D_n$  at the trivalent vertex gives a complete classification of noncommuting irreducible hyperfinite quadrilaterals such that  $N \subset P$  and  $N \subset Q$  are supertransitive subfactors with index less than 4.*

In particular it turns out there is a unique such quadrilateral for each even value  $[P : N] = 4 \cos^2 \frac{\pi}{2k}$  but none for the odd values  $[P : N] = 4 \cos^2 \frac{\pi}{2k-1}$ .

The methods of proof involve both the planar apparatus for intermediate subfactors developed in [9] and a notion of forked Temperley–Lieb algebras, in which a sequence of Temperley–Lieb projections  $e_1, e_2, e_3, \dots$  in a von Neumann algebra can be initially extended by either of two orthogonal projections  $p$  and  $q$ . This turns out to be a special case of  $AF$  algebras associated to  $T$ -shaped graphs, which were studied by Evans and Gould in [5], and precisely captures the structure of noncommuting supertransitive intermediate subfactors with small index.

The paper is organized as follows. In Section 2 we review some basic facts about subfactors and planar algebras, intermediate subfactors, and supertransitive subfactors. In Section 3 we describe a construction of pairs of supertransitive intermediate subfactors in the GHJ family (there is overlap here as well with [9], but the presentation is slightly different). In Section 4 we prove that if  $[P : N]$  and  $[Q : N]$  are less than 4 then the angle is uniquely determined: it is always  $\cos^{-1} \frac{1}{[P:N]-1}$ . In Section 5 we prove the main result, 1.0.1. The result of Section 4 on angles was substantially generalized by M. Izumi using results of [11]: it holds whenever  $N \subset P$  and  $N \subset Q$  are 3-supertransitive. Izumi also computed the opposite angles of the forked Temperley–Lieb quadrilaterals. These results appear in [10].

## 2. Background

### 2.1. Subfactors and planar algebras

Let  $N \subset M$  be an inclusion of  $\text{II}_1$  factors. Let  $L^2(M)$  be the Hilbert space completion of  $M$  with respect to the unique normalized trace, and let  $e_1$  be the projection of  $L^2(M)$  onto  $L^2(N)$ . Let  $M_1$  be the von Neumann algebra on  $L^2(M)$  generated by  $M$  and  $e_1$ . This is called the “basic construction.”

If  $M$  is finitely generated as a module over  $N$  then  $M_1$  is a  $\text{II}_1$  factor. In this case the index  $[M : N]$  is defined to be  $\tau^{-1}$ , where  $\tau = \text{tr}(e_1)$ ; otherwise the index is defined to be infinity. Some properties of the index are:  $[M : N] \geq 1$ ,  $[M_1 : M] = [M : N]$ , and if  $N \subseteq P \subseteq M$ , then  $[M : N] = [M : P][P : N]$ . It turns out that not all numbers greater than 1 can occur as index values: either  $[M : N] \geq 4$  or  $[M : N] = 4 \cos^2 \frac{\pi}{k}$  for some integer  $k$  (Jones [12]).

If  $[M : N] < \infty$ , one can iterate the basic construction to obtain a sequence of projections  $e_1, e_2, \dots$ , and a tower of  $\text{II}_1$  factors  $M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \dots$ , where  $M_{-1} = N$ ,  $M_0 = M$ ,

$e_k$  is the projection onto  $L^2(M_{k-2})$  in  $\mathbf{B}(L^2(M_{k-1}))$ , and  $M_k$  is the von Neumann algebra generated by  $M_{k-1}$  and  $e_k$ , for  $k \geq 1$ . Restricting the tower to those elements which commute with  $N$ , one obtains a smaller tower of algebras—which turn out to be finite-dimensional—called the tower of relative commutants  $N' \cap M_k$ .

The tower of relative commutants can also be thought of as algebras of bimodule intertwiners of tensor powers of the  $N$ – $N$  bimodule  $L^2(M)$ : as  $N$ – $N$  bimodules,  $L^2(M_k) \cong L^2(M) \otimes_N \cdots \otimes_N L^2(M)$ , ( $k + 1$  factors). Moreover,  $\text{Hom}_{N-N} L^2(M) \cong N' \cap M_{2k+1}$ . So an  $N$ – $N$  bimodule decomposition of  $\otimes_N^{k+1} L^2(M)$  corresponds to a decomposition of the identity in  $N' \cap M_{2k+1}$ . Under this correspondence projections in  $N' \cap M_{2k+1}$  correspond to submodules of  $\otimes_N^{k+1} L^2(M)$ , minimal projections correspond to irreducible submodules (those which have no proper nonzero closed submodules), and minimal central projections to equivalence classes of irreducible submodules.

The lattice of finite-dimensional algebras

$$\begin{array}{ccccccc} N' \cap N \subseteq N' \cap M \subseteq N' \cap M_1 \subseteq N' \cap M_2 \subseteq \cdots & & & & & & \\ & \cup & & \cup & & \cup & \\ & M' \cap M \subseteq M' \cap M_1 \subseteq M' \cap M_2 \subseteq \cdots & & & & & \end{array}$$

is called the *standard invariant*.

Note that the standard invariant always contains the Jones projections  $e_1, e_2, e_3, \dots$ , but in general will contain more structure as well. The Jones projections satisfy the following relations: (i)  $e_i e_{i \pm 1} e_i = \tau e_i$ ; and (ii)  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$ , where  $\tau = [M : N]^{-1}$ . The algebras they generate are in a sense “minimal” standard invariants, and are known as the Temperley–Lieb algebras [22].

The standard invariant was axiomatized by Popa [20], who showed that it is a complete invariant under certain conditions, and also introduced a construction of subfactors having a given standard invariant. It was noticed by Jones and others that many seemingly complicated linear relations in the standard invariants of subfactors had simple pictorial representations. To capture this phenomenon, Jones introduced the notion of a planar algebra, which turned out to be equivalent to Popa’s axiomatization of the standard invariant.

A *planar tangle* consists of a disc in the plane, together with some disjoint internal discs and nonintersecting strings connecting certain boundary points of the discs, all defined up to planar isotopy. There is a natural notion of composition of tangles (by isotoping one tangle to fit in an internal disc of another tangle and removing the boundary). One then thinks of the internal discs as “inputs” and the external boundary as the “output,” and the collection of planar tangles forms an operad.

If  $N \subset M$  is a subfactor, its planar algebra is essentially an action of the operad of planar tangles on the central vectors of the tensor powers of the  $N$ – $N$  bimodule  $M$ , i.e. the standard invariant. In this picture, the trivial tangles—those with no internal discs—correspond to the Temperley–Lieb algebras, which are present for any subfactor, and more complicated tangles may reflect additional structure.

For more precise details on definitions and examples of planar algebras and the meaning of various pictures we direct the reader to [13]. Note, however, that we adopt the drawing conventions of [9].

### 2.2. Intermediate subfactors

We now recall some facts about intermediate subfactors; for more details and proofs we refer the reader to [9].

Let  $N \subset M$  be an irreducible inclusion of  $\text{II}_1$  factors with finite index. (Irreducible here means that  $N' \cap M = \mathbb{C}Id$ .) Consider also the dual inclusion from the basic construction  $M \subset M_1$ . The first relative commutants  $N' \cap M_1$  and  $M' \cap M_2$  have the same vector space dimension, and the map

$$\phi: \begin{array}{|c|} \hline a \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \circlearrowleft \\ \hline \end{array}$$

is a linear isomorphism. Pulling back the multiplication from  $M' \cap M_2$  via  $\phi$  gives a second multiplicative structure on  $N' \cap M_1$ , called “comultiplication” and denoted by the symbol  $\circ$ : if  $a$  and  $b$  are elements of  $N' \cap M_1$ , then

$$a \circ b = \begin{array}{|c|} \hline a \quad b \\ \hline \end{array}$$

Bisch characterized intermediate subfactors in terms of the standard invariant: if  $p$  is an arbitrary projection in  $N' \cap M_1$ , then  $p$  projects onto an intermediate subfactor  $N \subseteq P \subseteq M$  iff  $p$  is also (a scalar multiple of) a projection in the dual algebra structure. Therefore an intermediate subfactor projection is called a biprojection. Landau discovered the following important relation among biprojections.

2.2.1. (Landau) Let  $P$  and  $Q$  be intermediate subfactors with biprojections  $e_P$  and  $e_Q$ . Then

$$e_P \circ e_Q = \begin{array}{|c|} \hline P \quad Q \\ \hline \end{array} = \delta \operatorname{tr}(e_P e_Q) e_{PQ},$$

where  $e_{PQ}$  is the projection onto the vector space  $PQ \subseteq M$ .

The following trace formulas follow.

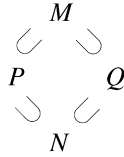
2.2.2. We have  $\operatorname{tr}(e_{PQ}) \operatorname{tr}(e_{PEQ}) = \operatorname{tr}(e_P) \operatorname{tr}(e_Q)$ . Also,

$$\operatorname{tr}(e_{PEQ}) = \frac{1}{\dim_M L^2(\bar{P}\bar{Q})},$$

where  $\bar{P}$  and  $\bar{Q}$  are the dual factors from the basic construction for  $P \subset M$  and  $Q \subset M$ , respectively.

A pair of intermediate subfactors of  $M$  is said to commute if the corresponding biprojections commute in the planar algebra. The subfactors are said to cocommute if the biprojections

cocommute. A quadrilateral of  $\text{II}_1$  factors is a diagram



such that  $P \vee Q = M$  and  $P \wedge Q = N$ .

2.3. Supertransitive subfactors

We recall some facts about supertransitive subfactors, which were defined by Jones in [14]. Let  $N \subset M$  be an inclusion of  $\text{II}_1$  factors with associated tower  $M_{-1} = N \subset M_0 \subset M_1 \subset \dots$  and Jones projections  $e_1, e_2, \dots$ . Each  $e_k$  commutes with  $N$ , so  $\{1, e_1, \dots, e_k\}$  generates a  $*$ -subalgebra, called  $TL_{k+1}$ , of the  $k$ th relative commutant  $N' \cap M_k$ .

A finite-index subfactor  $N \subset M$  is called  $k$ -supertransitive (for  $k > 1$ ) if  $N' \cap M_{k-1} = TL_k$ . We will say  $N \subset M$  is supertransitive if it is  $k$ -supertransitive for all  $k$ . Note that  $N \subset M$  is 1-supertransitive iff it is irreducible, i.e.  $N' \cap M \cong \mathbb{C}$  and is 2-supertransitive iff the  $N$ - $N$  bimodule  $L^2(M)$  has two irreducible components. Supertransitivity of  $N \subset M$  is the same as having principal graph  $A_n$  for some  $n = 2, 3, 4, \dots, \infty$ .

2.3.1. Suppose  $N \subset M$  is supertransitive. If  $[M : N] \geq 4$  then there is a sequence of irreducible  $N$ - $N$  bimodules  $V_0, V_1, V_2, \dots$  such that  $L^2(N) \cong V_0, L^2(M) \cong V_0 \oplus V_1$ , and  $V_i \otimes V_j \cong \bigoplus_{k=|i-j|}^{i+j} V_k$ . If  $[M : N] = 4 \cos^2(\frac{\pi}{n})$  then the sequence terminates at  $V_l$ , where  $l = [\frac{n-2}{2}]$ , and the fusion rule is:

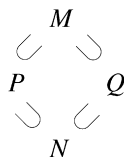
$$V_i \otimes V_j \cong \bigoplus_{k=|i-j|}^{(\frac{n-2}{2}) - (|\frac{n-2}{2} - (i+j)|)} V_k$$

(see [3]). In either case, we have

$$\dim_N V_k = [M : N]^k T_{2k+1} \left( \frac{1}{[M : N]} \right),$$

where  $\{T_k(x)\}$  is the sequence of polynomials defined recursively by  $T_0(x) = 0, T_1(x) = 1$ , and  $T_{k+2}(x) = T_{k+1}(x) - xT_k(x)$ . In particular,  $\dim_N V_1 = [M : N] - 1$  and  $\dim_N V_2 = [M : N]^2 - 3[M : N] + 1$ . If  $N \subset M$  is  $2k$ -supertransitive, then there is a sequence of irreducible bimodules  $V_0, \dots, V_k$  for which the above fusion rules and dimension formula hold as long as  $i + j \leq k$ .

If



is a quadrilateral of subfactors, the four subfactors  $N \subset P$ ,  $N \subset Q$ ,  $P \subset M$ , and  $Q \subset M$  are called the elementary subfactors. Quadrilaterals of supertransitive subfactors were studied by Jones and this author in [9], and the following result obtained.

2.3.2. Suppose  $N' \cap M = \mathbb{C}$ ,  $[M : N] < \infty$ , and all the elementary inclusions are supertransitive. Then either  $[M : N] = 6$  and  $N$  is the fixed point algebra for an outer action of  $S_3$  on  $M$  with  $P$  and  $Q$  being the fixed point algebras for two transpositions in  $S_3$  (in this case the angle between  $P$  and  $Q$  is  $\pi/3$ ), or  $[M : N] = (2 + \sqrt{2})^2$  and the planar algebra of  $N \subseteq M$  is the same as that coming from the GHJ subfactor (see [7]) constructed from the Coxeter graph  $D_5$  with the distinguished vertex being the trivalent one (in this case the angle between  $P$  and  $Q$  is  $\theta = \cos^{-1}(\sqrt{2} - 1)$ ).

### 3. Supertransitive intermediate subfactors in the GHJ family

#### 3.1. GHJ subfactors

Goodman, de la Harpe and Jones introduced a construction of subfactors from Coxeter graphs [7]. Let  $G$  be a Coxeter–Dynkin diagram of type  $A$ ,  $D$ , or  $E$ , with a distinguished vertex  $*$  and a bipartite structure. Let  $A_0 \subset A_1$  be an inclusion of finite-dimensional von Neumann algebras the underlying graph of whose Bratteli diagram is  $G$ , with  $*$  corresponding to a particular simple direct summand of  $A_0$ .

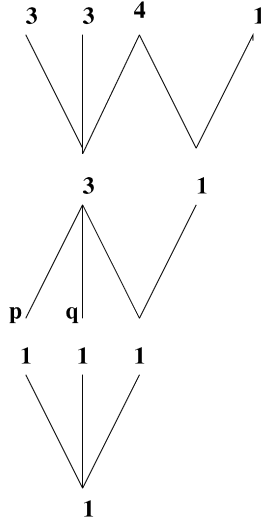
Perform the basic construction on the inclusion  $A_0 \subset A_1$  with respect to the Markov trace, and iterate to obtain a tower  $A_0 \subset A_1 \subset A_2 \subset A_3 \subset \dots$ . Inside this tower are the Jones projections  $e_1, e_2, e_3, \dots$ . Let  $B_i$  be the subalgebra of  $A_i$  generated by  $e_1, \dots, e_{i-1}$ , for each  $i = 2, 3, \dots$ . Then the towers  $B_i \subset A_i$  form commuting squares, and the von Neumann algebra  $B$  generated by  $\bigcup_{i=2}^\infty B_i$  is a subfactor of the von Neumann algebra  $A$  generated by  $\bigcup_{i=2}^\infty A_i$  on  $L^2(\bigcup_{i=2}^\infty A_i)$ . This subfactor has finite index but is not in general irreducible. Let  $r$  be the projection in  $A_0$  corresponding to  $*$ . Then  $r \in B'$ , and  $rB \subset rAr$  is an irreducible subfactor with finite index, called the GHJ subfactor for  $G, *$ .

3.1.1. The Jones projections  $e_1, e_2, e_3, \dots$  satisfy the Temperley–Lieb relations: (i)  $e_i e_{i\pm 1} e_i = \tau e_i$  and (ii)  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$ . Here  $\tau = \frac{1}{4 \cos^2 \frac{\pi}{k}}$ , where  $k$  is the Coxeter number of  $G$  ( $n + 1$  for  $A_n$  and  $2n - 2$  for  $D_n$ ). The cutdown Jones projections  $re_1, re_2, re_3, \dots$  satisfy the same relations, and we will suppress the “ $r$ ” when dealing with them. Because of the Temperley–Lieb relations, there is a unitary representation of the braid group inside the algebras  $B_i$  (or  $rB_i$ ), sending the usual braid group generators  $\sigma_i$  to  $g_i = (t + 1)e_i - 1$ , where  $t = e^{2\pi i/k}$ ,  $k$  again being the Coxeter number of  $G$ .

The principal graphs of the GHJ subfactors were computed by Okamoto in [17].

#### 3.2. The GHJ subfactor for $D_n$ at the trivalent vertex, and a pair of intermediate subfactors

The Bratteli diagram [4] for the tower  $rA_0r \subset rA_1r \subset rA_2r \subset \dots$  is determined by the string algebra for  $G$  based at  $*$  (see [6]). Thus for example the initial steps  $rA_0r \subset rA_1r \subset rA_2r \subset rA_3r$  for  $D_n$  at the trivalent vertex have the following diagram:



Let  $N \subset M$  be the GHJ subfactor for  $D_n$  at the trivalent vertex. Let  $p$  and  $q$  be the projections corresponding to the two terminal vertices of the fork, as in the above figure. Then evidently  $p$  and  $q$  are orthogonal, and commute with  $e_2, e_3, \dots$ . From the string algebra construction it follows that  $p$  and  $q$  have the same trace as the  $e_i$ , and therefore  $pe_1p = \tau p$  and  $e_1pe_1 = \tau e_1$ , and similarly for  $q$ . Thus  $p$  and  $q$  can each be used independently to initially extend the sequence of Temperley–Lieb projections which generate  $N$ . We recall the following result from [12].

**Theorem 3.2.1 (Jones).** *Let  $M$  be a von Neumann algebra generated by a sequence of projections  $e_1, e_2, e_3, \dots$  satisfying the Temperley–Lieb relations with  $\tau = \frac{1}{4\cos^2 \frac{\pi}{k}}$ , and let  $N$  be the subalgebra generated by  $e_2, e_3, \dots$ . Then  $N$  and  $M$  are  $\text{II}_1$  factors and the principal graph for  $N \subset M$  is  $A_{k-1}$ .*

3.2.2. The converse is also true in the hyperfinite case: by results of Ocneanu and Popa [16,19] any finite-depth hyperfinite  $\text{II}_1$  subfactor is determined by the standard invariant; in particular a hyperfinite subfactor with principal graph  $A_n$  can be expressed as the subalgebra obtained by dropping the first of a sequence of Temperley–Lieb projections.

Let  $P$  be the von Neumann algebra generated by  $N$  and  $p$ , and let  $Q$  be the von Neumann algebra generated by  $N$  and  $q$ . Because  $p$  extends initially the sequence of Temperley–Lieb projections which generate  $N$ , the above theorem applies with  $\tau = \frac{1}{4\cos^2 \frac{\pi}{2n-2}}$ , since the Coxeter number for  $D_n$  is  $2n - 2$ . Similarly for  $N \subset Q$ . Therefore the principal graphs for the subfactors  $N \subset P$  and  $N \subset Q$  are both  $A_{2n-3}$ .

**Remark 3.2.3.** In a similar way, the braid group representation mentioned above can be initially extended in two different ways, depending on whether one chooses  $p$  or  $q$  as  $e_0$ .

3.2.4. Example: the  $D_5$  case, and intermediate subfactors with no extra structure

For  $D_5$ , the GHJ subfactor at the trivalent vertex has index  $6 + 4\sqrt{2} = (2 + \sqrt{2})^2$ . The Coxeter number for  $D_5$  is 8, and  $4\cos^2(\frac{\pi}{8}) = 2 + \sqrt{2}$ . So we have  $[M : P] = [M : Q] = [P : N] = [Q : N] = 2 + \sqrt{2}$ . This was one construction of the quadrilateral with no extra structure in [9].



The other construction involved the “GHJ pair”  $\tilde{P}$  and  $\tilde{Q}$  coming from the two braid group representations of 3.2.3. The GHJ pair also has index  $2 + \sqrt{2}$  in  $M$ . One might have expected these two pairs of intermediate subfactors to coincide but in fact they give two distinct (but isomorphic) quadrilaterals with no extra structure.

However, for  $n > 5$ , the upper subfactors  $P \subset M$  and  $Q \subset M$  have extra structure, and the total index of the GHJ subfactor  $N \subset M$  tends toward infinity as  $n$  gets large.

#### 4. Angles between intermediate subfactors with small index

##### 4.1. Determination of possible angle values

In [21], Sano and Watatani studied the notion of the set of angles between subfactors: if  $P$  and  $Q$  are subfactors of  $M$ , then  $\text{Ang}(P, Q) = \text{spectrum}(\cos^{-1} \sqrt{e_P e_Q e_P \wedge e_Q})$ .

It turns out that for pairs of intermediate subfactors with index less than 4 over the intersection, the angles are completely determined by the index.

We first recall the following result from [18].

**Lemma 4.1.1** (Pimsner–Popa). *If the  $N$ – $N$ -bimodule decomposition of  $L^2(M)$  contains  $k$  copies of the  $N$ – $N$ -bimodule  $R$ , then  $k \leq \dim_N R$ . In particular,  $L^2(M)$  contains only one copy of  $L^2(N)$ .*

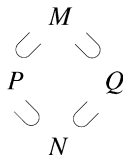
We will also need some results from [9], which we combine in the following lemma.

**Lemma 4.1.2.** *If  $N \subset P$  and  $N \subset Q$  are 2-transitive and the quadrilateral does not commute then  $L^2(P) \cong L^2(Q)$  as  $N$ – $N$ -bimodules, and therefore  $[P : N] = [Q : N]$ . Also  $e_P e_Q e_P = e_N + \lambda(e_P - e_N)$ , where*

$$\lambda = \frac{\text{tr}(e_{\tilde{P}\tilde{Q}})^{-1} - 1}{[P : N] - 1}.$$

*If  $N \subset P$  and  $N \subset Q$  are 4-supertransitive and the quadrilateral does not commute then the  $N$ – $N$ -bimodule  $L^2(PQ)$  isomorphic to one of the following:  $V_0 \oplus 2V_1 \oplus V_2$ ,  $V_0 \oplus 3V_1 \oplus V_2$ , or  $V_0 \oplus 3V_1$ , where the  $V_i$  are as in 2.3.1 (for the 4-supertransitive inclusion  $N \subset P$ ).*

**Theorem 4.1.3.** *Let*



*be a noncommuting irreducible quadrilateral with finite index, and suppose that  $[P : N]$  and  $[Q : N]$  are less than 4. Then  $\text{Ang}(P, Q)$  contains a unique nontrivial value, equal to  $\cos^{-1} \frac{1}{[P:N]-1}$ .*

**Proof.** Since  $[P : N]$  and  $[Q : N]$  are less than 4, the inclusions  $N \subset P$  and  $N \subset Q$  have principal graphs equal to  $A_n$  or  $D_{2n}$  for some  $n$ , or  $E_6$  or  $E_8$ . The only candidate which lacks

2-supertransitivity is  $D_4$ - however this is impossible by the noncommutativity hypothesis, since the  $N$ - $N$  bimodules of a  $D_4$  subfactor have dimension 1, and the corresponding projections would necessarily be central by 4.1.1. So we may assume 2-supertransitivity of  $N \subset P$  and  $N \subset Q$ . Then by 4.1.2  $L^2(P) \cong L^2(Q)$  as  $N$ - $N$  bimodules.

By 2-supertransitivity and 4.1.2 we have  $e_P e_Q e_P = e_N + \lambda(e_P - e_N)$ , where

$$\lambda = \frac{\text{tr}(e_{\bar{P}\bar{Q}})^{-1} - 1}{[P : N] - 1},$$

so there is a unique nontrivial angle value, and it is equal to  $\cos^{-1} \sqrt{\frac{\text{tr}(e_{\bar{P}\bar{Q}})^{-1} - 1}{[P : N] - 1}}$ , which by 2.2.2 can be rewritten as

$$\begin{aligned} \cos^{-1} \sqrt{\frac{\frac{[M:N]}{\dim_N L^2(PQ)} - 1}{[P : N] - 1}} &= \cos^{-1} \sqrt{\frac{\text{tr}(e_P e_Q)[M : N] - 1}{[P : N] - 1}} \\ &= \cos^{-1} \sqrt{\frac{\frac{\text{tr}(e_P) \text{tr}(e_Q)[M:N]}{\text{tr}(e_P e_Q)} - 1}{[P : N] - 1}} = \cos^{-1} \sqrt{\frac{\frac{[M:P]^{-2}[M:N]^2}{[M:N] \text{tr}(e_P e_Q)} - 1}{[P : N] - 1}} \\ &= \cos^{-1} \sqrt{\frac{\frac{[P:N]^2}{\dim_N L^2(PQ)} - 1}{[P : N] - 1}}. \end{aligned}$$

Suppose that  $N \subset P$  is also 4-supertransitive. Then by 4.1.2 we have that the  $N$ - $N$ -bimodule  $L^2(PQ)$  isomorphic to one of the following:

$$V_0 \oplus 2V_1 \oplus V_2, \quad V_0 \oplus 3V_1 \oplus V_2, \quad \text{or} \quad V_0 \oplus 3V_1,$$

where

$$\dim_N V_0 = 1, \quad \dim_N V_1 = [P : N] - 1, \quad \text{and} \quad \dim_N V_2 = [P : N]^2 - 3[P : N] + 1.$$

However, since  $[P : N] < 4$ , by 4.1.1  $L^2(M)$  can contain at most 2 copies of  $V_1$ . So in fact  $L^2(PQ) \cong V_0 \oplus 2V_1 \oplus V_2$ , and  $\dim_N L^2(PQ) = 1 + 2([P : N] - 1) + [P : N]^2 - 3[P : N] + 1 = [P : N]^2 - [P : N] = [P : N]([P : N] - 1)$ .

Plugging this into the angle formula above, we find that the angle is

$$\cos^{-1} \sqrt{\frac{\frac{[P:N]^2}{[P:N]([P:N]-1)} - 1}{[P : N] - 1}} = \cos^{-1} \sqrt{\frac{1}{[P:N]-1}} = \cos^{-1} \frac{1}{[P : N] - 1}.$$

All this assumed 4-supertransitivity. However, the only admissible principal graphs lacking 4-supertransitivity are  $D_6$  and  $E_6$ . But a noncommuting quadrilateral whose lower subfactors have principal graph  $E_6$  cannot exist (see [8]). For any noncommuting quadrilateral whose lower subfactors have principal graph  $D_6$ , by the  $D_6$  fusion rules we have  $L^2(PQ) = L^2(M) \cong L^2(N) \oplus 2V \oplus T_1 \oplus T_2$ , where  $\dim_N V = [P : N] - 1$  and  $\dim_N(T_1 \oplus T_2) = [P : N]^2 - 3[P : N] + 1$ .

(For more details see [8].) So in this case even without 4-supertransitivity the above argument holds.  $\square$

4.1.4. Example: GHJ subfactors for  $D_n$  at the trivalent vertex

We have already seen in 3.2 an example of a noncommuting irreducible quadrilateral whose lower two subfactors have principal graph  $A_n$ , for each odd value of  $n$ . The above theorem applies, but the angles can also be computed directly, as in [9]. We find that for the GHJ subfactor for  $D_n$  at the trivalent vertex, and the pair of intermediate subfactors constructed in 3.2, the angle between  $P$  and  $Q$  is  $\cos^{-1} \frac{1}{4 \cos^2 \frac{\pi}{2n-2} - 1}$ .

5. Subfactors coming from forked Temperley–Lieb algebras

5.1. Definition

We note some properties of the example in the preceding section.

**Lemma 5.1.1.** *Let  $N \subset M$  be the GHJ subfactor for  $D_n$  at the trivalent vertex. Let  $p$  and  $q$  be as in 3.2. Then we have:*

- (i)  $p$  is orthogonal to  $q$ .
- (ii) The sequence of projections  $p, e_1, e_2, e_3, \dots$  satisfy the Temperley–Lieb relations, with  $p$  playing the role of “ $e_0$ ” and  $\tau = \frac{1}{4 \cos^2 \frac{\pi}{2n-2}}$ . Similarly for the sequence  $q, e_1, e_2, \dots$
- (iii)  $N$  is generated by  $e_1, e_2, \dots$ , and  $M$  is generated by  $p, q, e_1, e_2, \dots$

**Proof.** These are all pretty trivial. (i) follows from the fact that  $p$  and  $q$  represent distinct vertices at the same level of the Bratteli diagram. (ii) follows from the discussion in 3.2. (iii) follows since the edges emanating from  $p$  and  $q$  generate the entire Bratteli diagram.  $\square$

This motivates the following definition.

**Definition 5.1.2.** We call a subfactor  $N \subset M$  a forked Temperley–Lieb subfactor with parameter  $\tau$  if there are projections  $p, q, e_1, e_2, e_3, \dots$  such that:

- (i)  $p$  is orthogonal to  $q$ .
- (ii) The sequence of projections  $p, e_1, e_2, e_3, \dots$  satisfy the Temperley–Lieb relations with parameter  $\tau$ , and with  $p$  playing the role of “ $e_0$ .” Similarly for the sequence  $q, e_1, e_2, \dots$
- (iii)  $N$  is generated by  $e_1, e_2, \dots$ , and  $M$  is generated by  $p, q, e_1, e_2, \dots$

Thus the GHJ subfactors for  $D_n$  at the trivalent vertex are forked Temperley–Lieb subfactors with parameter  $\tau = \frac{1}{4 \cos^2 \frac{\pi}{2n-2}}$ . The forked Temperley–Lieb algebras in the above definition are a special case of algebras associated to  $T$ -shaped graphs which were studied by Evans and Gould in [5]. We recall the following result.

**Theorem 5.1.3 (Evans–Gould).** *Let  $k \geq 2$ , and let  $e_1, e_2, \dots$  be a sequence of distinct projections satisfying the Temperley–Lieb relations with parameter  $\tau > 0$ , and let  $e_{\bar{k}}$  be another projection. Suppose that we have the following additional relations:*

- (i)  $e_n e_{\bar{k}} = e_{\bar{k}} e_n$  if  $n \neq k$ .
- (ii)  $e_{\bar{k}} e_k e_{\bar{k}} = \tau e_{\bar{k}}$  and  $e_k e_{\bar{k}} e_k = \tau(1 - e_1 \vee \dots \vee e_{k-2}) e_k$

and in the case  $k = 2$

- (iii)  $e_{\bar{k}} e_1 = 0$ .

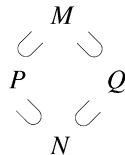
Let  $T_{k,n}$  ( $n \in \mathbb{N} \cup \{\infty\}$ ) be the graph which is constructed by adding one vertex to a Coxeter graph of type  $A_n$  and one edge connecting the  $k$ th vertex of  $A_n$  to the new vertex. Then if  $\tau > \frac{1}{\|T_{k,\infty}\|^2}$  then  $\tau = \frac{1}{\|T_{k,n}\|^2}$  for some  $n$ . Moreover, in this case the  $C^*$ -algebra generated by  $1, e_{\bar{k}}, e_1, e_2, \dots$  is uniquely determined.

The case  $k = 2$  corresponds to the forked Temperley–Lieb algebras (shifting the indices and letting  $p = e_1$  and  $q = e_2$ ). Then  $T_{2,\infty} = D_\infty$  has norm 2 and  $T_{k,n} = D_{n+1}$  have norm  $2 \cos \frac{\pi}{2n}$ . Thus if a forked Temperley–Lieb subfactor has parameter  $\tau > \frac{1}{4}$ , then we must have  $\tau = \frac{1}{4 \cos^2 \frac{\pi}{2n}}$  for some  $n$ . Also, it follows from the theorem that a forked Temperley–Lieb subfactor is determined by the parameter  $\tau$ .

### 5.2. Classification

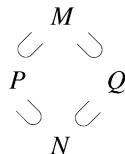
In fact the GHJ subfactors for  $D_n$  at the trivalent vertex completely classify noncommuting Type  $A_n$  (lower) intermediate subfactors.

**Theorem 5.2.1.** *Let*



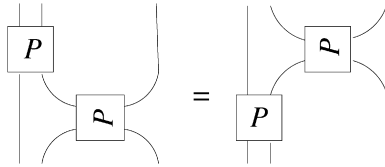
be a noncommuting, irreducible, hyperfinite quadrilateral with finite index such that  $N \subset P$  and  $N \subset Q$  are supertransitive with index less than 4. Then  $N \subset M$  is a forked Temperley–Lieb subfactor.

**Lemma 5.2.2.** *Let*

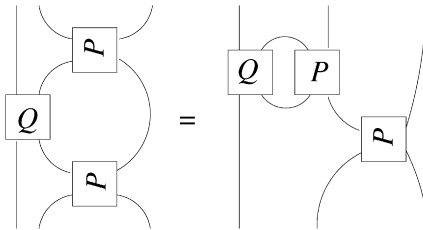


be a quadrilateral, and let  $P \subset M \subset \bar{P}$  be the basic construction. Then  $E_{\bar{P}}(e_Q)$  is a scalar multiple of  $e_{QP}$ .

**Proof.** There is an “exchange relation” for biprojections [1] (see also [15]):



which we can apply to obtain:



Recall that  $e_{\bar{P}}$  is a scalar multiple of



So the left-hand side is, up to a scalar,  $e_{\bar{P}}e_Qe_{\bar{P}} = E_{\bar{P}}(e_Q)e_{\bar{P}}$ , and by 2.2.1 the right-hand side is, again up to a scalar,  $e_QPe_{\bar{P}}$ . Since multiplication by  $e_{\bar{P}}$  is injective on  $\bar{P}$ ,  $E_{\bar{P}}(e_Q) = ce_{QP}$ .  $\square$

We can easily compute the scalar: since the conditional expectation preserves the trace,  $E_{\bar{P}}(e_Q) = ce_{QP}$  implies that  $c = \frac{\text{tr}(e_Q)}{\text{tr}(e_{QP})} = \frac{\text{tr}(e_Pe_Q)}{\text{tr}(e_P)}$  by 2.2.2. If  $[M : P] = \tau^{-1} < 4$ , then

$$\text{tr}(e_Pe_Q) = \frac{1}{\dim_N(L^2(\bar{P}\bar{Q}))} = \frac{1}{[M : P]([M : P] - 1)},$$

so  $c = \frac{\text{tr}(e_Pe_Q)}{\text{tr}(e_P)} = \frac{[M : P]}{[M : P]([M : P] - 1)} = \frac{\tau}{1 - \tau}$ . So  $E_{\bar{P}}(e_Q) = \frac{\tau}{1 - \tau}e_{QP}$ .

**Proof of 5.2.1.** Choose a downward basic construction  $N_{-1} \subset N \subset M$ , with corresponding  $P_{-1} \subset N \subset P$  and  $Q_{-1} \subset N \subset Q$ .

Since  $N \subset P$ —and therefore also  $P_{-1} \subset N$ —is supertransitive with index less than 4, it has principal graph  $A_n$  for some  $n = 3, 4, \dots$ . Then by the uniqueness of the hyperfinite  $A_n$  subfactor there exist projections  $e_1, e_2, \dots$  satisfying the Temperley–Lieb relations (with  $\tau = \frac{1}{4 \cos^2 \frac{\pi}{n+1}}$ ) such that  $N = \{e_1, e_2, e_3, \dots\}'$  and  $P_{-1} = \{e_2, e_3, e_4, \dots\}''$ . Let  $p = e_{P_{-1}} \in P$  be the Jones projection for the inclusion  $P_{-1} \subset N$ . Then  $p$ , considered as  $e_0$ , satisfies the Temperley–Lieb relations along with  $e_1, e_2, e_3, \dots$

Consider the  $N$ – $N$  bimodule map  $\phi = \tau^{-1}(E_N + (\tau - 1)E_Q) : P \rightarrow Q$ . By the preceding lemma  $\phi(p) = \tau^{-1}E_N(p) + \frac{\tau-1}{\tau}E_Q(p) = \tau^{-1}\tau + \frac{\tau-1}{\tau}\frac{\tau}{1-\tau}e_{P_{-1}Q_{-1}} = 1 - e_{P_{-1}Q_{-1}}$ .

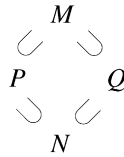
So  $q = 1 - e_{P_{-1}Q_{-1}}$  is a projection. Since  $P_{-1} \subset P_{-1}Q_{-1}$ ,  $q$  is orthogonal to  $p = e_{P_{-1}}$ . Because  $\phi$  is a bimodule intertwiner  $e_i q = e_i \phi(p) = \phi(e_i p) = e_i \phi(p) = e_i q$  if  $i > 1$  (since  $e_i \in N$ ), and similarly  $e_1 q e_1 = \tau e_1$ . Also,  $(q e_1 q)^2 = q(e_1 q e_1)q = \tau q e_1 q$ . Then  $\tau^{-1} q e_1 q$  is a subprojection of  $q$ , but  $\text{tr}(\tau^{-1} q e_1 q) = \tau^{-1} \text{tr}(e_1 q e_1) \tau^{-1} \tau \text{tr}(e_1) = \tau = \text{tr}(q)$ . So in fact  $q = \tau^{-1} q e_1 q$ , or  $q e_1 q = \tau q$ . Therefore  $q$  also satisfies the Temperley–Lieb relations (as  $e_0$ ) along with  $e_1, e_2, \dots$ . Finally, because  $P \vee Q = M$ ,  $p, q, e_1, e_2, \dots$  generate  $M$ , and we see that  $N \subset M$  is a forked Temperley–Lieb subfactor.  $\square$

**Corollary 5.2.3.** *The series of GHJ subfactors for  $D_n$  at the trivalent vertex gives a complete list (up to isomorphism) of noncommuting irreducible quadrilaterals such that  $N \subset P$  and  $N \subset Q$  are hyperfinite  $A_n$  subfactors.*

**Proof.** By 5.2.1 any such quadrilateral must be a forked TL subfactor, and by 5.1.3 the parameter must be  $4 \cos^2 \frac{\pi}{2k}$  for some  $k$ . Since the GHJ subfactors for  $D_n$  at the trivalent vertex give examples for each  $k$ , and a forked TL subfactor is determined by the parameter, it must be isomorphic to one of those GHJ subfactors.  $\square$

**Corollary 5.2.4.** *Up to isomorphism of the standard invariant, there is a unique noncommuting irreducible finite-index quadrilateral such that the lower elementary subfactors are supertransitive for each even index value  $[P : N] = 4 \cos^2 \frac{\pi}{2n}$ , but none for the odd values  $4 \cos^2 \frac{\pi}{2n+1}$ .*

**Proof.** Suppose



is a noncommuting irreducible finite-index quadrilateral such that the lower elementary subfactors are supertransitive with index less than 4. Since  $[M : N] < \infty$ , we must have  $M = P Q P Q P Q \dots$  for sufficiently many letters. Therefore as an  $N$ – $N$  bimodule,  $L^2(M)$  is a quotient of  $\otimes_N^k (L^2(P) \otimes_N L^2(Q)) \cong \otimes_N^{2k} L^2(P)$  (by 4.1.2) for some  $k$ . Since  $L^2(M)$  is composed of bimodules which obey the truncated fusion rules of 2.3.1,  $N \subset M$  must be finite depth. Then there is a hyperfinite subfactor with the same standard invariant as  $N \subset M$ , which necessarily has the same intermediate subfactor structure. But by 5.2.3 such subfactors are classified by the GHJ subfactors of  $D_n$  at the trivalent vertex, of which there is one for each even value but none for the odd values.  $\square$

**Corollary 5.2.5.** *The set of numbers which occur as angles between intermediate subfactors of an irreducible finite-index inclusion whose indices over their intersection are less than 4 is exactly  $\{\cos^{-1} \frac{1}{4 \cos^2 \frac{\pi}{2k} - 1} \mid k = 3, 4, 5, \dots\}$ .*

**Proof.** By 4.1.3 if  $[P : N], [Q : N] < 4$  then the angle is equal to  $\cos^{-1} \frac{1}{[P:N]-1}$ . On the other hand, we know from [12] that the only admissible index values less than 4 are  $4 \cos^2 \frac{\pi}{k}$ ,  $k = 3, 4, 5, \dots$ . So the only possible angle values for quadrilaterals with small lower indices are

given by the series  $\cos^{-1} \frac{1}{4 \cos^2 \frac{\pi}{k} - 1}$ . The GHJ subfactors for  $D_n$  at the trivalent vertex give examples of all these values for even  $k$ . To achieve the values for odd  $k$ ,  $N \subset P$  and  $N \subset Q$  would have to have as principal graphs Dynkin diagrams with odd Coxeter numbers, of which the only candidates are  $A_{2k}$ . But by 5.2.1 this would mean that the quadrilateral is a forked TL subfactor, and by 5.1.3 this is impossible for an odd Coxeter number.  $\square$

It is still an open question whether there exist noncommuting intermediate subfactors which are supertransitive with index greater than 4 over their intersection.

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