Computability on subsets of Euclidean space I: closed and compact subsets

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Abstract

In this paper we introduce and compare computability concepts on the set of closed subsets of Euclidean space. We use the language and framework of Type 2 Theory of Effectivity (TTE) which supplies a concise language for distinguishing a variety of effectivity properties and which admits highly effective versions of classical theorems. In particular, Type 2 Theory of Effectivity allows to separate topological from computational aspects of effectivity. We consider three different computability concepts on the set of closed subsets, each of which is characterized by several representations which are proved to be equivalent. The three induced types of computable closed sets have already been considered by many authors, however, under different and partly inconsistent names. Our characterizations show that they can be regarded as straightforward generalizations of the r.e., co-r.e., and recursive subsets of natural numbers. Therefore, we suggest to call them the recursively enumerable, the co-recursively enumerable, and the recursive closed subsets of Euclidean space. Open subsets obtain the dual names. We extend the investigation by introducing several natural representations of the compact subsets of Euclidean space and proving equivalences. The paper extends and generalizes earlier definitions, adds new ones and compares them in a single framework. The resultant canonical computability concepts induce computability of objects as well as computability of operators on the space of closed and compact subsets. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Classical recursion theory studies computational properties of subsets of the natural numbers \( \mathbb{N} \). A subset \( A \subseteq \mathbb{N} \) is called recursive, if there is an algorithm which decides, whether a given number \( k \in \mathbb{N} \) is in \( A \) or not, and \( A \) is called recursively enumerable (r.e.), if there is an algorithm that lists all numbers \( k \in A \), see e.g., [29, 33, 36]. These concepts can be extended easily from the natural numbers to other countable sets. For subsets of the real numbers, however, the situation is more complicated. In the past,
several computability definitions have been proposed for subsets of the Euclidean space $\mathbb{R}^n$. They are based on various definitions of computable real functions and on topological and on measure theoretical concepts. Presently, the terminology is confusing, since some of the concepts have got different names and, what is worse, names like "recursive" have been used for different concepts. In this contribution, we use exclusively the notion of computable real functions of the "Polish recursive analysis" introduced by Grzegorczyk and Lacombe [15, 24] and further generalized by Hauk, Kreitz and Weihrauch, Pour-El and Richards, Ko, and others [7, 16, 17, 21, 31, 40]. In this notion a real function is called computable if each approximation of the output can be computed from an approximation of the input. Probably, the first definitions of effective subsets of $\mathbb{R}^n$ based on this concept of computability have been proposed by Kreisel and Lacombe in 1957 [20, 25]. Later on these investigations have been continued by Metakides, Nerode, Huang, Kreitz, Weihrauch, Ko, Friedman, Ge, Zhou, Zhong, Brattka and others [8, 10, 12, 13, 17, 18, 23, 26, 42, 43, 45].

In this paper we present a number of computability definitions, the old and some new ones, for subsets of the Euclidean space and investigate their relations. For resolving the present confusion in terminology, we suggest the names recursively enumerable, co-recursively enumerable, and recursive for the three most important types of computable closed subsets of $\mathbb{R}^n$ (and dual terms for the open subsets). We embed our studies in a more comprehensive theory, "Type 2 Theory of Effectivity" (TTE), where not only computable objects but complete computability theories are defined on the full sets under consideration, e.g., on the set of real numbers, the set of continuous real functions, the set of open subsets or the set of compact subsets of the Euclidean space $\mathbb{R}^n$ [22, 36, 38]. In many situations this more comprehensive view gives much deeper insight, admits to prove more general and powerful theorems and has the additional advantage that topological aspects ("approximation") can be separated clearly from computational ones.

As an example, consider the Mandelbrot set $M \subseteq \mathbb{R}^2$ with its fascinating microscopic fine structures (cf. Fig. 1). In his popular book "The Emperor's New Mind" [30] Roger Penrose raised the question: when should a set like this be called "computable", "recursive" or "recursively enumerable"?

A reasonable recursiveness definition should extend or generalize one of the standard definitions of the recursive subsets of $\mathbb{N}^n$. If we start from the definition of recursive sets by computable characteristic functions, $A \subseteq \mathbb{N}^n$ is recursive, if and only if its characteristic function $\text{cf}_A : \mathbb{N}^n \to \mathbb{N}, \text{cf}_A(x) = (0$ if $x \in A, 1$ otherwise), is computable", then (R1) as well as (R2) seem to be suitable generalizations:

(R1) $A \subseteq \mathbb{R}^n$ is recursive, if and only if the function $\text{cf}_A : \mathbb{R}^n \to \mathbb{N}, \text{cf}_A(x) = (0$ if $x \in A, 1$ otherwise), is computable.

(R2) $A \subseteq \mathbb{R}^n$ is recursive, if and only if the metric distance function $d_A : \mathbb{R}^n \to \mathbb{R}, d_A(x) := \inf_{a \in A} d(a, x)$, is computable.

Notice that $\text{cf}_A : \mathbb{N}^n \to \mathbb{N}$ is the metric distance function, if we consider the discrete metric on $\mathbb{N}$. Fig. 2 shows the two generalizations $\text{cf}_A$ and $d_A$ for the set $A = [0; 1] \subseteq \mathbb{R}$. 
As a basic fact, every computable real function is continuous. Since a characteristic function \( \text{cf}_A : \mathbb{R}^n \to \mathbb{N} \) is continuous only if \( A = \emptyset \) or \( A = \mathbb{R}^n \), no non-trivial property of the real numbers is recursive according to Definition (R1). Therefore, this generalization which seems to be the most natural one at first glance is useless. In fact, (R2) will be our notion of recursive (closed) subsets of \( \mathbb{R}^n \).

A graphic explanation of this kind of recursiveness leads to an important application in computer science: in the case \( n = 2 \) a recursive closed subset \( A \subseteq [0; 1]^2 \), that is a subset with a computable distance function \( d_A \), can be plotted with arbitrary precision. Suppose, we have a screen representing the square \( [0, 1]^2 \) which is divided into \( k \times k \) pixels. From a program computing the function \( d_A \) an algorithm can be constructed, which for \( i, j = 1, \ldots, k \) determines whether pixel \( p_{i,j}^k \) will become white or black: if \( x \in [0; 1]^2 \) is the center of pixel \( p_{i,j}^k \), “the first” rational numbers \( a, b \) are computed such that \( d_A(x) \in [a; b] \) and \( b - a \leq 1/(2k) \). Then pixel \( p_{i,j}^k \) is set to black if \( a < 3/(2k) \) and
to white otherwise. Consequently, a pixel becomes black if $A$ intersects it and a pixel becomes white if neither the pixel itself nor any of its immediate neighbors intersect $A$. Hence, the $n$th approximation $A_n := \bigcup \{ p_{i,j}^{*n} \mid p_{i,j}^{*n} \text{ is black} \}$ of $A$ covers $A$, i.e., $A \subseteq A_n$ but it covers $A$ even very narrowly (in fact we have $d_H(A, A_n) < 2^{-n}$ for the Hausdorff distance $d_H$).

Since the distance function $d_A$ of a set $A \subseteq \mathbb{R}^n$ coincides with the distance function of the closure $\bar{A}$ of $A$, we restrict the investigation in this paper to closed subsets of Euclidean space. All our results on closed subsets have a natural dualization to open subsets which will be left to the reader. In the sense of Borel hierarchy the closed and the open subsets are the most simple sets to start with. In contrast to the set $2^\mathbb{R}$ of all subsets of real numbers, the hyperspace of closed subsets has the cardinality of the continuum; thus, the tools of TTE apply to it.

We will transfer the definitions of the recursively enumerable and the co-recursively enumerable subsets of $\mathbb{N}^n$ to closed subsets of $\mathbb{R}^n$ as follows: a closed subset $A \subseteq \mathbb{R}^n$ is recursively enumerable (co-recursively enumerable), if and only if its distance function $d_A$ is upper (lower) semi-computable, which means that we can compute a list of all upper (lower) rational bounds of the distance $d(A,x)$ from approximations of the input $x \in \mathbb{R}^n$. Since the complement of a closed set is not closed in general, these concepts are not symmetrical w.r.t. the complement. Nevertheless, the theorem "$A$ is recursive, if and only if it is recursively enumerable and co-recursively enumerable" holds also for closed subsets of $\mathbb{R}^n$.

"Well-behaved" closed sets like $\{x \in \mathbb{R} \mid x \geq 0\}$, $\{(x,y) \in \mathbb{R}^2 \mid x \geq y\}$ or the epigraphs of computable functions $f : \mathbb{R} \to \mathbb{R}$, c.g. the set $\{(x,y) \in \mathbb{R}^2 \mid \exp(x) \leq y\}$, are recursive. While it is easy to see that the Mandelbrot set is co-recursively enumerable, it is still an open and challenging question whether it is recursive.

Among the results of this paper there are several characterizations of recursive, recursively enumerable and co-recursively enumerable subsets, some of which are
generalizations of classical characterizations. For instance, a closed set $A \subseteq \mathbb{R}^n$ is recursively enumerable, if and only if there is a computable sequence of points which is dense in $A$ and it is co-recursively enumerable if it is the set of zeros of a computable function $f : \mathbb{R}^n \to \mathbb{R}$. Moreover, a compact set $A \subseteq \mathbb{R}^n$ is co-recursively enumerable, if and only if there is a function (which we will call "Heine–Borel function") which determines a finite subcover of each open covering of $A$, and $A$ is recursive, if and only if it can be approximated effectively by finite rational subsets w.r.t. the Hausdorff metric.

TTE allows to express these characterizations in a very uniform way. For each of these characterizations we define a representation of the hyperspace, i.e., of the set $\mathcal{A} := \{ A \subseteq \mathbb{R}^n | A \text{ closed} \}$ of all closed subsets of Euclidean space. Since a representation induces a full computability theory on the represented set, we can investigate operators like the union $\cup : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(A, B) \mapsto A \cup B$ or the boundary operator $\partial : \mathcal{A} \to \mathcal{A}$, $A \mapsto \partial A$ and their effectivity properties. Furthermore, we can express highly effective versions of classical theorems. In this sense our approach is more uniform and general than former ones.

We close this section with a short survey on the organization of this paper: in the preliminary section we will sketch some basic concepts of Type 2 Theory of Effectivity. Representations of the set of closed subsets will be introduced and characterized in the succeeding section. The special situation of the set of compact subsets will be discussed in a further section. A proof of an effective version of the Heine–Borel Theorem and of a Hausdorff approximation property will be included. We close the paper with some final remarks.

2. Preliminaries

We assume that the reader is familiar with the basic concepts of ordinary (Type 1) computability (or recursion) theory, see e.g. [29, 33, 36].

By $f : \subseteq A \to B$ we denote a partial function from $A$ to $B$ with domain $\text{dom}(f) \subseteq A$. If $A = \text{dom}(f)$, we write $f : A \to B$ as usual. Let $\mathbb{N}$ be the set $\{0, 1, \ldots \}$ of natural numbers. For any finite alphabet $\Sigma$, $\Sigma^*$ is the set of all finite words over $\Sigma$, and $\Sigma^\omega$ is the set of all infinite sequences over $\Sigma$. In the following let $\Sigma$ be a finite alphabet which contains all symbols we will use later. We call an infinite sequence $p \in \Sigma^\omega$ computable, if and only if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that $f(0^i) = p(i)$ for all $i \in \mathbb{N}$.

We will use the language of Type 2 Theory of Effectivity (TTE). In the following we summarize some concepts and facts. No proofs will be included. They are trivial or can be found in [22, 36, 38–41].

- We introduce our standard computability theory on $\Sigma^*$ and $\Sigma^\omega$.
- We introduce naming systems, notations and representations, and define the computability concepts induced by them.
- We introduce computation spaces and the very natural representations associated with them.
We introduce and discuss three standard computability concepts on the set of real numbers.

Computability for functions $f: \subseteq Y_1 \times \cdots \times Y_n \rightarrow Y_0$ with $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ is defined by Type-2 Turing machines (TT-machines for short) which are Turing machines with finite or one-way infinite input and output tapes. In particular, $x \in \Sigma^*$ is the result of a computation, if and only if the machine halts with $x$ written on the output tape; and $p \in \Sigma^\omega$ is the result of a computation, if and only if the machine computes forever writing $p$ on the output tape. Note that the type (finite or infinite) for each input and output tape must be defined in advance for each TT-machine $M$. The function computed by a TT-machine $M$ is denoted by $f_M$. TT-machines can be considered as oracle machines [17,33].

Type 2 Theory of Effectivity uses some basic notations and facts from topology (see e.g., [11] or any other textbook on topology). We will consider the discrete topology $\tau_d := \{ A | A \subseteq \Sigma^* \}$ on $\Sigma^*$ and the Cantor topology $\tau_c := \{ A \Sigma^\omega | A \subseteq \Sigma^* \}$ on $\Sigma^\omega$. The set $\{ x \Sigma^\omega | x \in \Sigma^* \}$ is a base of $\tau_c$. As a fundamental property, every function computed by a TT-machine is continuous. This is the mathematical way of expressing that for any TT-machine any finite portion of the output depends only on finite portions of the inputs.

From classical computability theory we know that for the set of computable functions $f: \subseteq \mathbb{N} \rightarrow \mathbb{N}$ there is an “admissible Gödel numbering” $\varphi$ satisfying the utm-theorem and the smn-theorem [33,36]. For continuous functions from $\Sigma^a$ to $\Sigma^b$, $a,b \in \{*,\omega\}$, there are representations with similar properties [35,36].

Let $F^{***}$ be the set of all (continuous) functions $f: \subseteq \Sigma^* \rightarrow \Sigma^*$, let $F^{*\omega}$ be the set of all (continuous) functions $f: \subseteq \Sigma^* \rightarrow \Sigma^\omega$, let $F^{\omega*}$ be the set of all continuous functions $f: \subseteq \Sigma^\omega \rightarrow \Sigma^*$ with open domain, and let $F^{\omega\omega}$ be the set of all continuous functions $f: \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with $G_\delta$-domain. For all $a,b \in \{*,\omega\}$, every continuous function $f: \subseteq \Sigma^a \rightarrow \Sigma^b$ has an extension in $g \in F^{ab}$, i.e. $f(x) = g(x)$ for all $x \in \text{dom}(f)$.

**Theorem 2.1** (Representation of continuous functions). For any $a,b \in \{*,\omega\}$ there is a total function $\eta^{ab}$ from $\Sigma^a$ onto $F^{ab}$ such that:

1. (utm-Theorem) The function $u: \subseteq \Sigma^\omega \times \Sigma^a \rightarrow \Sigma^b$ defined by $u(p,q) := \eta^{ab}_p(q)$ is computable,
2. (smn-Theorem) For any computable function $f: \subseteq \Sigma^\omega \times \Sigma^a \rightarrow \Sigma^b$ there is a computable function $s: \Sigma^\omega \rightarrow \Sigma^\omega$ such that $f(p,q) = s_\delta(p)(q)$.

In Type 2 Theory of Effectivity, machines transform “names” of “abstract” objects, where names are words $x \in \Sigma^*$ or infinite sequences $p \in \Sigma^\omega$. A naming system of a set $M$ is either a notation, i.e., a surjective function $v: \subseteq \Sigma^* \rightarrow M$, or a representation, i.e., a surjective function $\delta: \subseteq \Sigma^\omega \rightarrow M$. With this terminology $\eta^{ab}$ is a representation of $F^{ab}$ ($a,b \in \{*,\omega\}$).

Naming systems can be compared by reducibilities, as follows:

**Definition 2.2** (Reducibility and equivalence). Consider naming systems $\gamma: \subseteq \Sigma^a \rightarrow M$ and $\gamma': \subseteq \Sigma^b \rightarrow M'$. Then define:
(1) \( y \leq_t y' \) (\( y \leq y' \)), if and only if there is some continuous (computable) function 
\( h : \subseteq \Sigma^a \rightarrow \Sigma^b \), such that \( \gamma(p) = \gamma'h(p) \) for all \( p \in \text{dom}(\gamma) \) (we will say "\( \gamma \) is \( t \)-reducible (reducible) to \( \gamma' \)).

(2) \( y \equiv_t y' \) if and only if \( y \leq_t y' \) and \( y' \leq_t y \); \( \gamma \equiv \gamma' \) if and only if \( \gamma \leq \gamma' \) and \( \gamma' \leq \gamma \) (we will say "\( \gamma \) is \( t \)-equivalent (equivalent) to \( \gamma' \)).

Computability and continuity are transferred from \( \Sigma^* \) and \( \Sigma^w \) by means of naming systems as follows:

**Definition 2.3 (Computability and continuity).** Consider naming systems \( \gamma : \subseteq \Sigma^a \rightarrow M \) and \( \gamma' : \subseteq \Sigma^b \rightarrow M' \). Then define:

1. \( x \in M \) is called \( \gamma \)-computable, if and only if \( \gamma(p) = x \) for some computable \( p \in \Sigma^a \).
2. \( f : \subseteq M \rightarrow M' \) is called \( (\gamma, \gamma') \)-continuous ((\( \gamma, \gamma' \))-computable), if and only if there is some continuous (computable) function \( h : \subseteq \Sigma^a \rightarrow \Sigma^b \), such that \( f\gamma(p) = \gamma'h(p) \) for all \( p \in \text{dom}(f \gamma) \) (we will say "\( h \) is a \( (\gamma, \gamma') \)-realization of \( f \)").
3. \( f : \subseteq M \rightarrow M' \) is called strongly \( (\gamma, \gamma') \)-continuous (strongly \( (\gamma, \gamma') \)-computable), if and only if (2) holds and additionally \( p \not\in \text{dom}(h) \) for \( p \in \text{dom}(\gamma) \setminus \text{dom}(f \gamma) \) (we will say "\( h \) is a strong \( (\gamma, \gamma') \)-realization of \( f \)").
4. \( Q \subseteq M \times M' \) is called \( (\gamma, \gamma') \)-continuous ((\( \gamma, \gamma' \))-computable), if and only if there is some continuous (computable) function \( h : \subseteq \Sigma^a \rightarrow \Sigma^b \) respectively, such that \( (\gamma(p), \gamma'h(p)) \in Q \) for all \( p \) such that \( \gamma(p) \) is in the first projection of \( Q \) (we will say that "\( h \) is a \( (\gamma, \gamma') \)-realization of \( Q \)").
5. \( f \) is called a choice function for the relation \( Q \), if and only if \( (x, f(x)) \in Q \) for all \( x \) in the first projection of \( Q \).

The definitions of computability and continuity can be extended easily from \( M \) to \( M_1 \times \cdots \times M_k \).

Two naming systems induce the same computability (continuity) theory on a set, if and only if they are equivalent (\( t \)-equivalent).

Two useful operations on naming systems are given by the following definition. Here, \( \langle \rangle : \Sigma^a \times \Sigma^b \rightarrow \Sigma^c \) denotes a suitable injective and computable standard pairing function, for instance, let \( \langle \rangle : \Sigma^a \times \Sigma^b \rightarrow \Sigma^c \) be defined by \( \langle p, q \rangle := p(0)q(0)p(1)q(1)\ldots \in \Sigma^w \) for all \( p, q \in \Sigma^w \). The other cases are defined similarly.

**Definition 2.4 (Operations).** Consider naming systems \( \gamma : \subseteq \Sigma^a \rightarrow M \) and \( \gamma' : \subseteq \Sigma^b \rightarrow M' \).

1. The conjunction \( \gamma \cap \gamma' : \subseteq \langle \Sigma^a, \Sigma^b \rangle \rightarrow M \cap M' \) is defined by \( (\gamma \cap \gamma')(p, q) = x \iff \gamma(p) = x \) and \( \gamma'(q) = x \) for all \( p \in \Sigma^a, q \in \Sigma^b \) and \( x \in M \cap M' \).
2. The product \( [\gamma, \gamma'] : \subseteq \langle \Sigma^a, \Sigma^b \rangle \rightarrow M \times M' \) is defined by \( [\gamma, \gamma'](p, q) := (\gamma(p), \gamma'(q)) \) for all \( p \in \Sigma^a, q \in \Sigma^b \).

The conjunction is, except for equivalence, the greatest lower bound w.r.t. \( \leq \) as well as w.r.t. \( \leq_t \). If \( \gamma : \subseteq \Sigma^a \rightarrow M \) is a naming system, then we define the naming system \( \gamma^n : \subseteq \Sigma^a \rightarrow M^n \) by \( \gamma^1 := \gamma \) and \( \gamma^{n+1} := [\gamma^n, \gamma] \) for all \( n \geq 1 \).
Only very few of the numerous (equivalence classes of) naming systems of a set \( M \) are of practical interest, in particular those which are compatible with some algebraic or topological structure on \( M \). For countable sets we will usually consider "standard" notations compatible with some characteristic algebraic structure which are often called "effective" (for a discussion see [32,36,39]). In particular, let \( v_N : \subseteq \Sigma^* \rightarrow \mathbb{N} \) be the binary notation of the natural numbers, and let \( v_Q : \subseteq \Sigma^* \rightarrow \mathbb{Q} \) be a notation of the rational numbers by pairs of integers in binary notation. If no confusion is expected the abbreviation \( \bar{u} \) will be used for \( v_Q(u) \). (We assume that \( \text{dom}(v_Q) \subseteq (\Sigma \setminus \{(,),\#\})^* \).) By \( \bar{v}_Q \) we denote the induced notation of \( \mathbb{Q}^n \). For sets with at most continuum cardinality, representations which are compatible with some natural topology are of particular interest. We introduce a very natural class of representations via computation spaces.

**Definition 2.5 (Computation space).** A pair \((M,v)\) is called computation space, if and only if \( M \) is a set and \( v : \subseteq \Sigma^* \rightarrow 2^M \) is a function. We will say that \( v \) identifies points, if and only if \( \{ P \in \text{range}(v) \mid x \in P \} = \{ P \in \text{range}(v) \mid y \in P \} \Rightarrow x = y \) holds for all \( x, y \in M \).

Any subset \( P \subseteq M \) can be regarded as a property on \( M \). We will call the elements \( P \in \text{range}(v) \) the atomic properties of the computation space \((M,v)\). The topology and the standard representation induced by a computation space are defined as follows.

**Definition 2.6 (Topology and standard representation).** Let \((M,v)\) be a computation space. The topology \( \tau_v \) induced by \((M,v)\) is by definition the smallest topology on \( M \) containing \( \text{range}(v) \). Assume that \( v \) identifies points. The standard representation \( \delta_v : \subseteq \Sigma^\omega \rightarrow M \) of \( M \) induced by \( v \) is defined by

\[
\delta_v(p) = x \iff \{ w \mid x \in v(w) \} = \{ w \mid \text{"(w)" is a subword of } p \}
\]

for all \( p \in \Sigma^\omega \), and \( x \in M \). (Here and in all future applications we assume tacitly that \( \text{dom}(v) \subseteq (\Sigma \setminus \{(,),\#\})^* \).)

Obviously, \( \text{range}(v) \) is a subbase of the topology \( \tau_v \). On the other hand, by each notation \( v \) of a subbase of a topology \( \tau \) on \( M \) one obtains a computation space \((M,v)\). In particular, \( \tau \) is a \( T_0 \)-topology if and only if \( v \) identifies points. Informally speaking, a standard name \( p \in \Sigma^\omega \) of an element \( x \in M \) (i.e., \( \delta(p) - x \)) is a list (in any order) of all names of all atomic properties \( P \in \text{range}(v) \) which hold for \( x \). An element \( x \in M \) is \( \delta_v \)-computable, if and only if the set \( \{ w \mid x \in v(w) \} \) is r.e. In almost all applications we obtain an equivalent representation by listing "sufficiently many" (instead of all) atomic properties.

Standard representations have some remarkable properties (see [22, 36] where representations t-equivalent to them are called admissible):

**Theorem 2.7 (Properties of standard representations).** Let \( \delta \) and \( \delta' \) be the standard representations and let \( \tau \) and \( \tau' \) be the topologies induced by computation spaces
(\(M, \nu\)) and (\(M', \nu'\)), respectively. Then

1. \(\delta\) is continuous and open (in particular, \(\tau\) is the final topology of \(\delta\));
2. \(\gamma : \subseteq \Sigma^\omega \rightarrow M\) is (\(\tau_C, \tau\))-continuous, if and only if \(\gamma \leq \delta\);
3. \(f : \subseteq M \rightarrow M'\) is (\(\delta, \delta'\))-continuous, if and only if it is (\(\tau, \tau'\))-continuous.

The last property holds accordingly for the general case \(f : \subseteq M_1 \times \cdots \times M_k \rightarrow M'\).

If \((M, \nu)\) is a computation space, then the set \(\text{range}(\nu)\) of atomic properties introduces a concept of approximation (a topology) and the notation \(\nu\) introduces a concept of computability on the set \(M\). On the other hand, both types of information seem to be necessary for defining a computability theory on \(M\). It depends on the application which finite portions of information about the (usually infinite) objects \(x \in M\) are considered as atomic, i.e., which finite portions of information are available as input for a computation and which will be produced, and which notations are used for them.

We will need three representations of the real numbers [36, 42]:

**Definition 2.8 (Standard representations of the real numbers).** Define computation spaces \((\mathbb{R}, \nu^<_R), (\mathbb{R}, \nu^>_R),\) and \((\mathbb{R}, \nu_R)\) by

1. \(x \in \nu^<_R(w) : \iff \overline{w} < x\),
2. \(x \in \nu^>_R(w) : \iff x < \overline{w}\),
3. \(x \in \nu_R(0\#w) : \iff \overline{w} < x\) and \(x \in \nu_R(1\#w) : \iff x < \overline{w}\),

for all \(x \in \mathbb{R}\) and \(w \in \text{dom}(\nu_Q)\). Let \(\rho^<, \rho^>, \rho : \subseteq \Sigma^\omega \rightarrow \mathbb{R}\) denote the corresponding standard representations.

The final topologies on \(\mathbb{R}\) are the lower topology \(\tau^<_R = \{(a; \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\}\), the upper topology \(\tau^>_R = \{(-\infty; a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\}\) and the real line topology \(\tau_\mathbb{R}\), respectively.

The above representations are invariant under “unimportant” modifications. If, for instance, the notation of the rational numbers is replaced by an equivalent one or by a standard notation of the dyadic rational numbers, the induced representations are equivalent to the corresponding given ones. The three representations are related as follows:

**Theorem 2.9 (Representations of the real numbers).**

1. \(\rho^< \not\leq \rho^>\) and \(\rho^> \not\leq \rho^<\),
2. \(\rho \equiv \rho^< \cap \rho^>\).

Occasionally, we will need representations \(\overline{\rho}^<, \overline{\rho}^>, \overline{\rho} : \subseteq \Sigma^\omega \rightarrow \overline{\mathbb{R}}\) of the extended real numbers \(\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}\). We obtain them from Definition 2.8 if we substitute “\(x \in \overline{\mathbb{R}}\)” for “\(x \in \mathbb{R}\)”. With this definition, a \(\overline{\rho}^<\)-name of \(\infty\) is just a list of all rational numbers and a \(\overline{\rho}^>\)-name of \(\infty\) is an empty list. Obviously, \(\overline{\rho} \mid \mathbb{R} \equiv \rho\) and corresponding properties hold for \(\rho^<, \rho^>\).

Let \(I^n\) be some standard notation of the set \(\text{Int}^n\) of all open \(n\)-dimensional cubes from \(\mathbb{R}^n\) with edges parallel to the coordinate axes and with rational vertices. For
short we often write $I_u$ for $I^u(u)$ and Int instead of Int$^u$. Obviously, Int is a base of the Euclidean topology on $\mathbb{R}^n$. If we consider the maximum metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $d(x,y) := \max_{i=1,\ldots,n} d(x_i,y_i)$ for all $x,y \in \mathbb{R}^n$, the elements of Int are the open balls $B(c,r) := \{ x \in \mathbb{R}^n \mid d(x,c) < r \}$ with rational center $c$ and rational radius $r$. Moreover, $\overline{B}(c,r)$ denotes the corresponding closed ball.

By $\rho^n : \subseteq \Sigma^0 \to \mathbb{R}^n$ we denote the representation which is induced by the computation space $(\mathbb{R}^n,I^n)$. Obviously, $\rho^1 \equiv \rho, \rho^2 \equiv [\rho, \rho]$ and so on.

Again, the representation is stable against various modification: if, for instance, the maximum metric is replaced by the usual Euclidean metric, an equivalent representation is obtained.

For the effective points w.r.t. to the three introduced representations of the real numbers we will use the following terminology.

**Definition 2.10 (Computable points).** We will call
1. $x \in \mathbb{R}^n$ computable, if and only if $x$ is $\rho^n$-computable,
2. $x \in \mathbb{R}$ lower semi-computable, if and only if $x$ is $\rho^\leq$-computable,
3. $x \in \mathbb{R}$ upper semi-computable, if and only if $x$ is $\rho^\geq$-computable.

For the sets $\mathbb{N}, \mathbb{Q}^n$, and $\mathbb{R}^n$ we will use $v_\mathbb{N}, v_\mathbb{Q}^n$, and $\rho^n$ respectively as fixed standard naming systems. For instance, for a function $f : \mathbb{R}^n \to \mathbb{Q}$ we will say for short that it is computable, if and only if it is $(\rho^n,v_\mathbb{Q})$-computable.

The maximum metric or the Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are examples of computable functions.

For the standard representations on the real numbers we will introduce some further notions of computability (which specialize continuous and semi-continuous real-valued functions).

**Definition 2.11 (Computable real-valued functions).** Let $f : \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function. Then
1. $f$ is called computable, if and only if $f$ is $(\rho^n,\overline{\rho})$-computable,
2. $f$ is called lower semi-computable, if and only if $f$ is $(\rho^n,\rho^\leq)$-computable,
3. $f$ is called upper semi-computable, if and only if $f$ is $(\rho^n,\rho^\geq)$-computable.

Obviously, a function is computable if and only if it is lower and upper semi-computable.

3. Topological representations of closed sets

In this section we introduce computability concepts on the set $\mathcal{A}(\mathbb{R}^n)$ of closed subsets of the Euclidean space $\mathbb{R}^n$. In the following $n \geq 1$ is a fixed natural number and hence we will write for short $\mathcal{A}$ instead of $\mathcal{A}(\mathbb{R}^n)$. According to the principles of Type 2 Theory of Effectivity, we introduce representations $\delta : \subseteq \Sigma^0 \to \mathcal{A}$ of $\mathcal{A}$ for this purpose. For any such representation there is a dual representation $\co-\delta : \subseteq \Sigma^0 \to \mathcal{C}$ of
the set $\mathcal{O}$ of open subsets of $\mathbb{R}^n$, defined by $\text{co-}\delta(p) := \mathbb{R}^n \setminus \delta(p)$. Therefore, we obtain computability theories on the open sets simultaneously. Some results presented in this section can be found in [23, 38, 42].

3.1. The definitions

In this subsection we introduce three different computability concepts on the set $\mathcal{A}$. First we define three standard representations derived from computation spaces on the set of closed sets, then we define several other representations each equivalent to one of the standard representations.

Every closed subset $A \in \mathcal{A}$ can be characterized by the set $\{J \in \text{Int} \mid A \cap J \neq \emptyset\}$ of all rational open balls intersecting it. Therefore, with any $J \in \text{Int}$ we can associate a property $\{A \in \mathcal{A} \mid A \cap J \neq \emptyset\}$ on $\mathcal{A}$. Together with the standard notation of $\text{Int}$ we obtain a computation space on $\mathcal{A}$, a derived standard representation and a derived topology on $\mathcal{A}$.

Correspondingly, every closed subset $A \in \mathcal{A}$ can be characterized by the set $\{J \in \text{Int} \mid A \cap J = \emptyset\}$ of closed rational balls disjoint from $A$. Here, for each $B \subseteq \mathbb{R}^n$ we denote by $\overline{B}$ its closure w.r.t. the Euclidean topology. This gives rise to another computation space. Finally, a combination of both types of information generates a third computation space. Fig. 4 shows the two types of information about a closed set.

**Definition 3.1 (Standard representations of the closed sets).** Define computation spaces $(\mathcal{A}, v^<)$, $(\mathcal{A}, v^>)$, and $(\mathcal{A}, v^=)$ on the set $\mathcal{A}$ of the closed subsets of $\mathbb{R}^n$ as follows:

1. $A \in v^<(w) :\iff A \cap \overline{I_w} \neq \emptyset$,
2. $A \in v^>(w) :\iff A \cap I_w = \emptyset$,
3. $A \in v^=(0#w) :\iff A \cap I_w \neq \emptyset, A \in v^=(1#w) :\iff A \cap \overline{I_w} = \emptyset$

for all $w \in \text{dom}(I)$ and $A \in \mathcal{A}$. Let $\delta^<, \delta^>, \delta^= : \Sigma^\omega \to \mathcal{A}$ be the corresponding standard representations and let $\tau^<, \tau^>, \tau^= \in \mathcal{T}$ be the induced topologies. (We tacitly assume that $\text{dom}(I) \subseteq (\Sigma \setminus \{(, \#\})^\ast$.)

Roughly speaking, $\delta^<(p) = A$, if and only if $p$ is a list of all rational open balls which intersect the closed set $A$, $\delta^>(p) = A$, if and only if $p$ is a list of all rational
closed balls disjoint from $A$, and $\delta^-(p) = A$, if and only if $p$ is a list of all rational open balls which intersect $A$ and of all rational closed balls disjoint from $A$.

The final topologies $\tau^<, \tau^>, \tau^=$ of our representations $\delta^<, \delta^>, \delta^=$ are well-known hyperspace topologies (cf. [1]): $\tau^=$ is the extended Fell topology which is generated from the subbase consisting of all the sets $\{A \in \mathcal{A} | A \cap U \neq \emptyset \}$ and $\{A \in \mathcal{A} | A \cap K = \emptyset \}$ with open subsets $U$ and compact subsets $K$. Usually, the Fell topology is considered on the set of non-empty closed subsets. In this case the topology $\tau^= (\tau^<, \tau^>)$ is equivalent to the Wijsman topology (lower, upper Wijsman topology) and to the Attouch–Wets topology. It is worth noticing that the extension of the Fell topology to the closed subsets including the empty subset corresponds to the one-point compactification: $(\mathcal{A}, \tau^=)$ is a compact metrizable space.

The choice of the two properties $A \cap I_w \neq \emptyset$ and $A \cap \overline{I}_w = \emptyset$ in the above definition deserves some explanation. First, we observe that the above definition is "stable". If, for instance, the rational numbers are replaced by some other dense subset like the dyadic rational numbers or the maximum metric is replaced by the Euclidean metric, the induced representations are equivalent to the given ones and hence induce the same computability concepts on $\mathcal{A}$. Consider a modification of Definition 3.1, where $I_w$ and $\overline{I}_w$ are exchanged. These definitions are no longer stable. Every change of the dense subset now changes the topologies and hence the computability concepts. Probably only very few users will need one of these sensitive modified definitions. Therefore, in this paper we will consider the important stable variant, Definition 3.1, exclusively.

The atomic properties of closed sets are independent in a strong way: no atomic property $P(A)$: $\forall A \in \mathcal{A}$ such that $A \cap J \neq \emptyset$ can be concluded from finitely many properties $Q_i(A)$: $\forall A \in \mathcal{A}$ and vice versa. Proposition 3.2(1) below is an immediate consequence of this observation.

**Proposition 3.2** (Standard representations of the closed sets).

1. $\delta^< \not\subseteq \delta^>$ and $\delta^> \not\subseteq \delta^<$.
2. $\delta^= \equiv \delta^< \cap \delta^>$. 

**Proof.** (1) Obviously, $\emptyset \cap \overline{I}_w = \emptyset$ and $\mathbb{R}^n \cap I_w \neq \emptyset$ for all $w \in \text{dom}(I)$ with $I_w \neq \emptyset$. Thus $\tau^< \not\subseteq \tau^>$ and $\tau^> \not\subseteq \tau^<$ and consequently $\delta^< \not\subseteq \delta^>$ and $\delta^> \not\subseteq \delta^<$. 

(2) This is an easy consequence of the definition. \(\square\)

By Property (2), $\delta^=$ is, except for equivalence, the greatest lower bound of $\delta^<$ and $\delta^>$. If $\delta : \Sigma^+ \rightarrow M$ is a standard representation for a computation space $(M, v)$ identifying points such that $\{(u,v) \mid v(u) = v(v)\}$ is r.e., then $x \in M$ is $\delta$-computable if and only if $\{w \in \Sigma^+ \mid x \in v(w)\}$ is a r.e. set of words. For the $\delta^<=, \delta^>=, \text{and } \delta^=$-computable closed sets and their complements we suggest the following standard names (cf. [12,45]).

**Definition 3.3** (*Recursive, r.e., and co-r.e. sets*). We call a closed set $A \subseteq \mathbb{R}^n$

1. *recursive*, if and only if $A$ is $\delta^=$-computable,
(2) recursively enumerable (r.e. for short), if and only if $A$ is $\delta^\leq$-computable,
(3) co-recursively enumerable (co-r.e. for short), if and only if $A$ is $\delta^\geq$-computable.
We call an open set $B \subseteq \mathbb{R}^n$ r.e., co-r.e. or recursive, if and only if its complement is co-r.e., r.e. or recursive, respectively.

Corollary 3.13 below gives several characterizations of these types of "computable" closed sets, which show in particular that Definition 3.3 generalizes the classical definitions for r.e., co-r.e., and recursive subsets of $\mathbb{N}^k$ straightforwardly. The above three types of computable closed or open sets have been considered by many authors under different (e.g. "recursively co-semi-located") and partly inconsistent names. To end the confusion we suggest to use the above names in future.

By a basic theorem of recursion theory, a set $B \subseteq \mathbb{N}$ is recursive, if and only if it is r.e. and co-r.e. As an easy consequence of Proposition 3.2(2), a closed set is recursive, if and only if it is r.e. and co-r.e. Notice however, that in contrast to the case of natural numbers this is no theorem but an almost trivial consequence of the definitions: $\delta^r \equiv \delta^\leq \cap \delta^\geq$ implies that $A$ is $\delta^r$-computable, if and only if it is $\delta^\leq$- and $\delta^\geq$-computable.

We give some examples of recursive, recursively enumerable and co-recursively enumerable sets:

Example 3.4 (Recursive and r.e. sets). (1) The sets $\mathbb{R}^n$ and $\emptyset$ are both, recursive open and recursive closed.

(2) $\{x\}$ is r.e. $\iff$ $\{x\}$ is co-r.e. $\iff$ $\{x\}$ is recursive $\iff$ $x$ is computable, for any $x \in \mathbb{R}^n$.

(3) The closed sets $\{(x, x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and $\{(x, y) \mid x \leq y\} \subseteq \mathbb{R}^2$ are recursive.

(4) The open set $\{(x, y) \mid x < y\} \subseteq \mathbb{R}^2$ is recursive.

(5) Let $x \in \mathbb{R}^n$ be a computable point and $r > 0$. The open ball $B(x, r)$ is r.e., co-r.e., or recursive, if and only if $r$ is lower semi-computable, upper semi-computable, or computable, respectively. The same holds for the closed balls $\overline{B}(x, r)$.

(6) An open interval $(a; b)$ is r.e., if and only if $a$ is upper and $b$ is lower semi-computable, it is co-r.e., if and only if $a$ is lower and $b$ is upper semi-computable, it is recursive, if and only if $a$ and $b$ are computable.

(7) The Mandelbrot set $M \subseteq \mathbb{R}^2$ is a co-r.e. closed set. It is a challenging open problem whether it is recursive or not.

In TTE for a representation $\delta : \subseteq \Sigma^o \rightarrow M$ a subset $X \subseteq M^n$ is called $\delta^n$-r.e., if and only if $(\delta^n)^{-1}(X) = A\Sigma^o \cap \text{dom}(\delta^n)$ for some r.e. subset $A \subseteq \Sigma^*$, and it is called $\delta^n$-decidable if and only if it is $\delta^n$-r.e. and also its complement is $\delta^n$-r.e. It turns out that an open subset $U \subseteq \mathbb{R}^n$ is r.e. if and only if it is $\rho^n$-r.e. This is essentially 2(a) $\iff$ 2(e) in Corollary 3.13. In particular, the set $L := \{(x, y) \in \mathbb{R}^2 \mid x < y\}$ from (4) is $\rho^2$-r.e. However for no representation $\delta : \subseteq \Sigma^o \rightarrow \mathbb{R}$ the set $L$ is $\delta^2$-decidable [39, 40].

The next proposition shows that the introduced notions are suitable generalizations of the classical ones. Here we assume that $\mathbb{N}$ is embedded in $\mathbb{R}$.
**Proposition 3.5** (Discrete subsets of the real numbers). A set $A \subseteq \mathbb{N}^n$ is r.e., co-r.e. or recursive in the classical sense, if and only if $A$ is r.e., co-r.e. or recursive, respectively, as a closed subset of Euclidean space $\mathbb{R}^n$.

We omit the easy proof. The next proposition considers isolated points.

**Proposition 3.6** (Isolated points of closed sets). Let $A \subseteq \mathbb{R}^n$ be a closed set which is r.e. or co-r.e. and let $x \in A$ be an isolated point of $A$. Then $x$ is computable.

**Proof.** First, assume $A$ is r.e., i.e., $\delta^<$-computable. Then the set $C := \{w | A \cap I_w \neq \emptyset\}$ is r.e. Since $x$ is isolated, $A \cap I_{w_0} = \{x\}$ for some $w_0 \in \Sigma^*$. We obtain

$$x \in I_w \iff (\exists v \in C)(I_v \subseteq I_{w_0} \cap I_w)$$

for all $w \in \text{dom}(I)$. Hence $\{w | x \in I_w\}$ is r.e., thus $x$ is $\rho^\rho$-computable.

Assume $A$ is co-r.e., i.e., $\delta^>$-computable. Then the set $D := \{w | A \cap \overline{I_w} = \emptyset\}$ is r.e. Since $x$ is isolated, $A \cap \overline{I_{w_0}} = \{x\}$ for some $w_0 \in \Sigma^*$. Since $\overline{I_{w_0}}$ is compact, we obtain

$$x \in I_w \iff (\exists v_1, \ldots, v_k \in D)(\overline{I_{w_0}} \subseteq I_w \cup \bigcup_{i=1}^k I_{v_i})$$

for all $w \in \text{dom}(I)$. Hence $\{w | x \in I_w\}$ is r.e., thus $x$ is $\rho^\rho$-computable. $\Box$

The result that isolated points of co-r.e. closed sets $A \subseteq \mathbb{R}$ are computable is already due to Lacombe [25].

### 3.2. Characterizations

In this subsection we introduce some further representations of the set $\mathcal{A}$ of closed subsets of $\mathbb{R}^n$. Each of them is equivalent to one of the three representations $\delta^<$, $\delta^>$, and $\delta^\rho$. Notice that equivalent representations induce the same computability concept for points. In classical recursion theory, a set $A \subseteq \mathbb{N}$ is recursive if and only if $A = f^{-1}\{0\}$ for some total recursive function $f : \mathbb{N} \to \mathbb{N}$. In the introduction we have seen that computability of functions $f : \mathbb{R}^n \to \mathbb{N}$ does not yield useful effective subsets $A = f^{-1}\{0\}$, since those sets have to be simultaneously open and closed. Nevertheless, we can generalize the classical notion in a different way: the characteristic functions in the discrete case can be seen as distance functions w.r.t. the discrete metric. In the case of the Euclidean space closed sets $A \in \mathcal{A}$ can also be characterized by their distance functions.

For each non-empty closed set $A \subseteq \mathbb{R}^n$ its distance function $d_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by $d_A(x) := d(x,A) := \min\{d(x,a) | a \in A\}$ (where $d$ is the maximum metric on $\mathbb{R}^n$). For technical reasons let $d_\phi : \mathbb{R}^n \to \overline{\mathbb{R}}, x \mapsto \infty$. Obviously, we have $d_\phi^{-1}\{0\} = A$. Since $d_A$ is continuous w.r.t. the Euclidean topology, we can realize $d_A$ by some function $\phi^p_\rho$. 
Definition 3.7 (Representations of closed sets by distance functions). Define representations $\delta_{\text{dist}}^\leq, \delta_{\text{dist}}^\geq, \delta_{\text{dist}}^= : \Sigma^\omega \rightarrow \mathcal{A}$ by

1. $\delta_{\text{dist}}^\leq(p) = A : \Leftrightarrow \eta^{\text{oo}}_p$ is a $((\rho^n, \bar{\rho})^\leq)$-realization of $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$,
2. $\delta_{\text{dist}}^\geq(p) = A : \Leftrightarrow \eta^{\text{oo}}_p$ is a $((\rho^n, \bar{\rho})^\geq)$-realization of $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$,
3. $\delta_{\text{dist}}^=(p) = A : \Leftrightarrow \eta^{\text{oo}}_p$ is a $(\rho^n, \bar{\rho})$-realization of $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$

for all $p \in \Sigma^\omega$ and $A \in \mathcal{A}$.

Note that the direction of “lower” and “upper” is related crosswise to the distance functions.

Proposition 3.8. $\delta_{\text{dist}}^\leq \equiv \delta_{\text{dist}}^= \cap \delta_{\text{dist}}^\geq$.

Proof. By Theorem 2.9 (generalized to $\mathbb{R}$) $\bar{\rho}$ is reducible to $\bar{\rho}^\leq$, i.e. there is a computable function $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with $\bar{\rho}(p) = \bar{\rho}^\leq f(p)$ for all $p \in \text{dom}(\bar{\rho})$. We apply Theorem 2.1. By the utm-theorem the function $F : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega, F(p, q) := f \eta^{\text{oo}}_q(q)$, is computable. By the smn-theorem there is a computable function $g : \Sigma^\omega \rightarrow \Sigma^\omega$ with $F(p, q) = \eta^{\text{oo}}_g(p, q)$. We show that $g$ translates $\delta_{\text{dist}}^\leq$ to $\delta_{\text{dist}}^\geq$. Let $d_A$ be $(\rho^n, \bar{\rho})$-realized by $\eta^{\text{oo}}_p$. Then for all $q \in \text{dom}(\rho^n)$ we obtain

$$d_A^\rho_p(q) = \bar{\rho} \eta^{\text{oo}}_g(q) = \bar{\rho}^\leq f \eta^{\text{oo}}_p(q) = \bar{\rho}^\leq \eta^{\text{oo}}_g(q),$$

hence $d_A$ is $(\rho^n, \bar{\rho}^\leq)$-realized by $\eta^{\text{oo}}_g$. We conclude $\delta_{\text{dist}}^\leq \leq \delta_{\text{dist}}^\geq$. The second reduction $\delta_{\text{dist}}^= \leq \delta_{\text{dist}}^\leq$ can be proved accordingly.

By Theorem 2.9 (generalized to $\mathbb{R}$) there is a computable function $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with $(\rho^\geq \cap \rho^\leq)(r) = \bar{\rho} f(r)$ for all $r \in \text{dom}(\rho^\geq \cap \rho^\leq)$. By the utm-theorem the function $F : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega, F(p, q, r) := f(\eta^{\text{oo}}_p(q), \eta^{\text{oo}}_q(r))$, is computable. By the smn-theorem $F(p, q, r) = \eta^{\text{oo}}_{g(p, q)}(r)$ for some computable function $g : \Sigma^\omega \rightarrow \Sigma^\omega$. Assume $A = \delta_{\text{dist}}^\leq \cap \delta_{\text{dist}}^\geq(p, q)$. Then $A = \delta_{\text{dist}}^\leq(p) = \delta_{\text{dist}}^\geq(q)$, hence $d_A^\rho_p(r) = \rho^\geq \eta^{\text{oo}}_p(r) = \rho^\leq \eta^{\text{oo}}_p(q, r)$. We obtain

$$d_A^\rho_p(r) = (\rho^\geq \cap \rho^\leq)(\eta^{\text{oo}}_p(r), \eta^{\text{oo}}_q(r)) = \bar{\rho} f(\eta^{\text{oo}}_p(r), \eta^{\text{oo}}_q(r)) = \bar{\rho} \eta^{\text{oo}}_{g(p, q)}(r).$$

Therefore, $g$ translates $\delta_{\text{dist}}^\leq \cap \delta_{\text{dist}}^\geq$ to $\delta_{\text{dist}}^\leq$. □

In classical recursion theory, a non-empty subset $A \subseteq \mathbb{N}$ is recursively enumerable if and only if $A = \text{range}(f)$ for some computable total function $f : \mathbb{N} \rightarrow \mathbb{N}$. Furthermore, a subset $A \subseteq \mathbb{N}$ is recursively enumerable, if and only if $A = \text{dom}(f)$ for some computable function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$. These characterizations give rise to further representations of $\mathcal{A}$.

Definition 3.9 (Representation of closed sets by domains and ranges). Define representations $\delta_{\text{range}}, \delta_{\text{dom}} : \subseteq \Sigma^\omega \rightarrow \mathcal{A}$ by

1. $\delta_{\text{range}}(p) = A : \Leftrightarrow \eta^{\text{oo}}_p$ is a $(\rho^n, \rho^n)$-realization of a total function $f : \mathbb{N} \rightarrow \mathbb{R}^n$

such that $\text{range}(f)$ is dense in $A$ or

$(A = \emptyset$ and $\eta^{\text{oo}}_p(w) = \#^\omega$ for all $w \in \Sigma^\omega$),
\[ \delta_{\text{dom}}(p) = A : \iff \eta_{p}^{\omega} \text{ is a strong } (\rho^{\omega}, \nu_{\mathbb{N}}) \text{-realization of} \]
\[ \text{a function } f : \subseteq \mathbb{R}^{n} \to \mathbb{N} \text{ such that } \text{dom}(f) = A^{c}, \]
for all \( p \in \Sigma^{\omega} \) and \( A \in \mathcal{A} \).

In the first case \( p \) is a name of a non-empty closed set \( A \) if and only if \( \eta_{p}^{\omega} \) is a realization of a function \( f : \mathbb{N} \to \mathbb{R}^{n} \) which enumerates a dense subset of \( A \). In the second case \( p \) is a name of \( A \) if and only if \( \eta_{p}^{\omega} \) is a strong realization of a function \( f : \subseteq \mathbb{R}^{n} \to \mathbb{N} \) such that \( A \) is the complement of the domain of \( f \). Thus, given a name \( p \) of \( A \) as an "oracle" we have a procedure which for any \( x \in \mathbb{R}^{n} \) verifies the statement "\( x \not\in A \)" if and only if \( x \not\in A \).

As we have seen, the classical characterization of recursive sets as fibers \( f^{-1}\{0\} \) of computable functions \( f \) can be generalized in a certain sense by distance functions. If we allow arbitrary continuous functions \( f : \mathbb{R}^{n} \to \mathbb{R} \), we obtain another characterization of closed sets by fibers \( f^{-1}\{0\} \).

**Definition 3.10 (Representation of closed sets by fibers of functions).** Define a representation \( \delta_{\text{fiber}} : \subseteq \Sigma^{\omega} \to \mathcal{A} \) by
\[ \delta_{\text{fiber}}(p) = A : \iff \eta_{p}^{\omega} \text{ is a } (\rho^{\omega}, \rho) \text{-realization of} \]
\[ \text{a function } f : \mathbb{R}^{n} \to \mathbb{R} \text{ with } f^{-1}\{0\} = A \]
for all \( p \in \Sigma^{\omega} \) and \( A \in \mathcal{A} \).

In this case \( p \) is a name of a closed set \( A \) if and only if \( A \) is the set of zero-positions of a total (!) function \( f : \mathbb{R}^{n} \to \mathbb{R} \) which is realized by \( \eta_{p}^{\omega} \).

Our last representation of closed sets \( A \) is defined by enumerating open balls, whose union is the complement of \( A \). This is another way of generalizing the classical definition of co-r.e. sets.

**Definition 3.11 (Representation of closed sets by union of balls).** Define a representation \( \delta_{\text{union}} : \subseteq \Sigma^{\omega} \to \mathcal{A} \) by
\[ \delta_{\text{union}}(p) = A : \iff A^{c} = \bigcup \{ I_{w} \mid \text{"(w)" is a subword of } p \} \]
for all \( p \in \Sigma^{\omega} \) and \( A \in \mathcal{A} \).

Note that we do not require that \( p \) lists all words \( w \) with \( I_{w} \subseteq A^{c} \). This stronger requirement leads to a "sensitive" definition (cf. the remarks after Definitions 2.8 and 2.6). The language of TTE admits to formulate the main result of this section in a very condensed form:

**Theorem 3.12 (Equivalent representations of closed sets).**

1. \( \delta^{<} \equiv \delta_{\text{dist}}^{<} \equiv \delta_{\text{range}} \),
2. \( \delta^{>} \equiv \delta_{\text{dist}}^{>} \equiv \delta_{\text{dom}} \equiv \delta_{\text{fiber}} \equiv \delta_{\text{union}} \),
3. \( \delta^{=} \equiv \delta_{\text{dist}}^{=} \).
Since equivalent representations induce the same computability theory, the above ten representations of the set \( \mathcal{A} \) of closed subsets of \( \mathbb{R}^n \) induce only three computability concepts on \( \mathcal{A} \). By Proposition 3.2 these three computability concepts are different. In the following we outline the proof. Its various parts are almost straightforward, but of course, require a clear understanding of the underlying definitions. The utm-theorem and the smn-theorem (Theorem 2.1) are used, whenever indices of continuous functions are considered.

**Proof.** (1) We prove \( \delta^c \leq \delta_{\text{range}} \leq \delta_{\text{dist}} \leq \delta^c \).

"\( \delta^c \leq \delta_{\text{range}} \)". Let \( A \in \mathcal{A} \) and \( \delta^c(p) = A \). From \( p \) we can compute a list of all words \( w_i, i \in \mathbb{N} \) such that \( A \cap I_{w_i} \neq \emptyset \). If \( A \neq \emptyset \) then we can effectively find indices \( i_0, i_1, i_2, \ldots \) for each \( m \in \mathbb{N} \) such that

\[
I_{w_{i_0}} \supseteq I_{w_{i_1}} \supseteq I_{w_{i_2}} \supseteq \cdots \text{ and diam}(I_{w_m}) < 2^{-k}.
\]

By Cantor's Theorem, the completeness of \( \mathbb{R}^n \) yields an \( x_m \in \mathbb{R}^n \) with

\[
\{x_m\} = \bigcap_{k=0}^{\infty} I_{w_k} \subseteq I_{w_m}.
\]

Then \( x_m \subseteq A \). Let \( f_p : \mathbb{N} \to \mathbb{R}^n \) be defined by \( f_p(m) := x_m \). Our construction yields an algorithm for \( f_p \) such that range(\( f_p \)) is dense in \( A \). More precisely, we have a computable function \( H : \subseteq \Sigma^* \times \Sigma^* \to \Sigma^* \) such that \( \rho^n H(p, v) = f_p(m) \) (where \( m = \nu_n(v) \)) and \( H(p, v) = \#^o \) if \( p \) has no subword "(w)" with \( w \in \text{dom}(I) \). By the smn-Theorem there is a computable function \( F : \Sigma^* \to \Sigma^* \) such that \( H(p, v) = \eta^o_F(p)(v) \). We obtain \( \rho^n \eta^o_F(p)(v) - f_p \nu_n(v) \) for all \( v \in \Sigma^* \). Therefore, \( f_p \) is \((\nu_n, \rho^n)\)-realized by \( \eta^o_F \) if \( A \neq \emptyset \).

We obtain \( \delta_{\text{range}} F(p) = \delta^c(p) \) for each \( p \in \text{dom}(\delta^c) \).

"\( \delta_{\text{range}} \leq \delta_{\text{dist}}^c \)". Let \( A \in \mathcal{A} \) and \( \delta_{\text{range}}(p) = A \), i.e., \( A \) is strongly \( (\rho^n, \rho^n) \)-realized of a total function \( f : \mathbb{N} \to \mathbb{R}^n \) such that range(\( f \)) is dense in \( A \) or \( A = \emptyset \) and \( \eta^o_F(y) = \#^o \) for all \( y \in \Sigma^* \). Then

\[
d_A(\rho^n(q)) < \overline{w} \iff (\exists k) d(\rho^n(q), f(k)) < \overline{w}
\]

\[
\iff (\exists y) d(\rho^n(q), \rho^n \eta^o_F(y)) < \overline{w}
\]

\[
\iff (\exists y, u, v)(\exists u) \text{ is a subword of } q,
\]

"(v)" is a subword of \( \eta^o_F(y) \) and \( \sup_{a \in \Sigma, b \in \Sigma} d(a, b) < \overline{w} \).

By the utm-Theorem there is a computable function \( H : \subseteq \Sigma^* \times \Sigma^* \to \Sigma^* \) which with input \( p, q \) gives a list of all \( w \) such that \( d_A(\rho^n(q)) < \overline{w} \), i.e. \( \rho^n H(p, q) = d_A(\rho^n(q)) \). By the smn-theorem \( H(p, q) = \eta_F(\rho^n(q)) \) for some computable function \( F : \Sigma^* \to \Sigma^* \). Thus, \( d_A \) is strongly \((\rho^n, \rho^n)\)-realized by \( \eta_F \). Hence, \( \delta_{\text{dist}} F(p) = \delta_{\text{range}}(p) \) for all \( p \in \text{dom}(\delta_{\text{range}}) \).

"\( \delta_{\text{dist}}^c \leq \delta^c \)". Let \( A \in \mathcal{A} \) and \( \delta_{\text{dist}}^c(p) = A \), i.e., \( A \) is strongly \((\rho^n, \rho^n)\)-realized by \( \eta_F \). Then

\[
A \cap B(\nu_{\rho^n}(v), \overline{w}) \neq \emptyset \iff d_A(\nu_{\rho^n}(v)) < \overline{w}
\]
for all $v \in \text{dom}(v_{Q^v})$, $w \in \text{dom}(v_{Q^v})$ and $x \in \mathbb{R}^n$. Using the fact that $v_{Q^v} \leq \rho^n$ and the utm-Theorem for $\eta_{Q^n}$ we can define a computable function $F : \subseteq \Sigma^o \to \Sigma^o$, which transforms each $p \in \text{dom}(\delta^o_{\text{dist}})$ into a list $F(p)$ of all words $w$ such that $A \cap I_w \neq \emptyset$, i.e., $\delta^o_F(p) = \delta^o_{\text{dist}}(p)$.

(2) We prove $\delta^o > \delta^o_{\text{fiber}} \leq \delta^o_{\text{dom}} \leq \delta^o_{\text{union}} \leq \delta^o_{\text{dist}} \leq \delta^o$. 

"$\delta^o \leq \delta^o_{\text{fiber}}$": Let $A \in \mathcal{A}$ and let $\delta^o_{\text{fiber}}(p) = A$. From $p$ we can compute a list of all words $w_i, v_i, i \in \mathbb{N}$ such that $A \cap \overline{B(v_{Q^v}(w_i), v_i)} = \emptyset$. Consider the function $f_p : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f_p(x) := \sum_{i=0}^{\infty} \max\{0, \frac{d(v_{Q^v}(w_i), x)}{v_i}\} 2^{-i-1}$$

for all $x \in \mathbb{R}^n$. We obtain

$$f_p^{-1}(0) = A.$$

The last equivalence holds since for every $i$ there is some $j$ such that $\overline{B(v_{Q^v}(w_i), v_i)} \subseteq B(v_{Q^v}(w_j), v_j)$. There is a computable function $G : \subseteq \Sigma^o \times \Sigma^o \to \Sigma^o$ such that $f_p \rho^n(q) = \rho G(p, q)$. By the smn-Theorem for $\eta_{Q^n}$ there is a computable function $F : \Sigma^o \to \Sigma^o$ such that $G(p, q) = \eta^o_{F(p)}(q)$. Thus $f_p$ is $(\rho^n, \rho)$-realized by $\eta^o_{F(p)}$. Hence, $\delta^o_{\text{fiber}} F(p) = \delta^o(p)$ for all $p \in \text{dom}(\delta^o)$.

"$\delta^o_{\text{fiber}} \leq \delta^o_{\text{dom}}$": Let $A \in \mathcal{A}$ and $\delta^o_{\text{dom}}(p) = A$, i.e., $\eta^o_{p^*}$ is a $(\rho^n, \rho)$-realization of a function $f_p : \mathbb{R}^n \to \mathbb{N}$ such that $f_p^{-1}(0) = A$. Define $g_p : \subseteq \mathbb{R}^n \to \mathbb{N}$ by

$$g_p(x) := \begin{cases} 1 & \text{if } f_p(x) \neq 0, \\ \uparrow & \text{else} \end{cases}$$

for all $x \in \mathbb{R}^n$. Then $\text{dom}(g_p)^c = f_p^{-1}(0)$. Since $g_p \rho^n(q) = (1$ if $\rho^n(q) > 0,$ $\uparrow$ otherwise), by utm- and smn-theorem there is a computable function $G : \subseteq \Sigma^o \times \Sigma^o \to \Sigma^o$ such that $g_p \rho^n(q) = v_{\Sigma^o} G(p, q)$. By the smn-Theorem for $\eta^o_{p^*}$ there is a computable function $F : \Sigma^o \to \Sigma^o$ such that $g_p$ is strongly $(\rho^n, \rho)$-realized by $\eta^o_{F(p)}$, i.e., $\delta^o_{\text{dom}} F(p) = \delta^o_{\text{fiber}}(p)$ for all $p \in \text{dom}(\delta^o_{\text{fiber}})$.

"$\delta^o_{\text{dom}} \leq \delta^o_{\text{union}}$": Let $A \in \mathcal{A}$ and $\delta^o_{\text{dom}}(p) = A$, i.e., $\eta^o_{p^*}$ is a strong $(\rho^n, \rho)$-realization of a function $f_p : \subseteq \mathbb{R}^n \to \mathbb{N}$ such that $\text{dom}(f_p) = A^c$. Let $M$ be a TT-machine computing the universal function $G : \subseteq \Sigma^o \times \Sigma^o \to \Sigma^o$ of $\eta^o_{p^*}$. If $G(p, q)$ exists then $M$ reads only a finite prefix $w$ of $q$ during its computation. In this situation we say "$M(p, w)$ halts". For $q \in \text{dom}(\rho^n)$ we have $q \in \text{dom}(\eta^o_{p^*})$ if and only if $M(p, w)$ halts for some $w \subseteq q$. Since every $q \in \text{dom}(\rho^n)$ has infinitely many subwords "(u)", we may assume w.l.o.g. that $M(p, w)$ halts only if "(u)" is the last symbol of $w$. It turns out that $\text{dom}(\eta^o_{p^*}) = \bigcup \{w \Sigma^o \mid M(p, w) \text{ halts}\}$ and $\text{dom}(f_p) = \bigcup \{\rho^n(w \Sigma^o) \mid M(p, w) \text{ halts}\} - \bigcup \{I_v \mid \exists w(w \text{ ends with "(u)"}, I_v \subseteq \rho^n(w \Sigma^o) \text{ and } M(p, w) \text{ halts}\}$. Since $I_v \subseteq \rho^n(w \Sigma^o)$ is decidable and the set of all $w$ such that $M(p, w)$ halts is r.e. in $p$, there is a computable function $F : \subseteq \Sigma^o \to \Sigma^o$ such that for all $p \in \text{dom}(\delta^o_{\text{dom}})$, $\text{dom}(f_p) = \bigcup \{I_v \mid \text{"(u)" is a subword of } F(p)\}$. Therefore $F$ translates $\delta^o_{\text{dom}}$ to $\delta^o_{\text{union}}$. 
δunion ≤ δdist

Let A ∈ A and δunion(p) = A. From p we can compute a list p' of words w, such that A^c := ∪_{i=0}^k Iw. Since B(x, w) is compact,

\[ d_A(x) > \bar{w} \Leftrightarrow (\exists k) B(x, \bar{w}) \subseteq \bigcup_{i=0}^k Iw, \]

for all w ∈ dom(v_Q) and x ∈ R^n. Therefore, there is a computable function G: Σ^ω × Σ^ω → Σ^ω such that G(p, q) is a list of all (w) with d_A(x) > \bar{w}, i.e. d_A(x) = \bar{p} ≤ G(p, q), if δunion(p) = A and p^n(q) = x. Hence, by the smn-Theorem for η^oo there is a computable F: Σ^ω → Σ^ω, such that d_A is strongly (p^n, \bar{p} ≤)-realized by η^oo_F(p). We obtain δdist(p) = δunion(p) for all p ∈ dom(δunion).

δdist ≤ δ^≥:

Let A ∈ A and δdist(p) = A, i.e., d_A is strongly (p^n, \bar{p} ≤)-realized by η^oo_p. Then

\[ A \cap B(v_Q(v), \bar{w}) = \emptyset \Leftrightarrow d_A(v_Q(v)) > \bar{w} \]

for all v ∈ dom(v_Q), w ∈ dom(v_Q). By the utm-Theorem for η^oo we can conclude that there is a computable function F: Σ^ω → Σ^ω which works as follows: F translates each p into a list F(p) of all words w such that A ∩ Iw = ∅, i.e., δ^≥ F(p) = δdist(p) for all p ∈ dom(δdist).

(3) Since “∩” (Definition 2.4) is the greatest lower-bound operation on representations, we have δ^≤ △ ≥ δ^≤ △ δ^≥ by (1) and (2). With Propositions 3.2 and 3.8 we obtain δ^≤ △ δ^dist.

Since computable functions, in particular the translations in the above theorem, map computable points to computable points we obtain the following characterizations of computable closed sets as an immediate corollary.

Corollary 3.13 (Characterization of effective closed sets). Let A ⊆ R^n be a closed set.

(1) The following statements are equivalent:
   (a) A is recursively enumerable,
   (b) \{w | A ∩ Iw ≠ ∅\} is recursively enumerable,
   (c) d_A is upper semi-computable,
   (d) range(f) is dense in A for a computable f: N → R^n or A = ∅.

(2) The following statements are equivalent:
   (a) A is co-recursively enumerable,
   (b) \{w | A ∩ Iw = ∅\} is recursively enumerable,
   (c) d_A is lower semi-computable,
   (d) A = f'^{-1}\{0\} for some computable function f: R^n → R,
   (e) A^c = dom(f) for a strongly computable f: ⊆ R^n → N,
   (f) A^c = ∪_{w∈B} Iw for some recursively enumerable set B ⊆ Σ^ω.

(3) The following statements are equivalent:
   (a) A is recursive,
   (b) A is recursively enumerable and co-recursively enumerable,
   (c) d_A is computable.
In (2)(d) the number 0 can be replaced by any other $\rho$-computable real number $a$, c.e. by $\max\{f(x) \mid x \in \mathbb{R}^n\}$.

Our co-recursively enumerable closed sets have been called "récursivement fermé" ("recursively closed") and our recursively enumerable open sets have been called "récursivement ouvert" ("recursively open") by Lacombe. He used definition (2)(f) and proved the equivalence of (2)(d) and (f) [25]. Ko and Friedman also called these sets "recursively closed" ("recursively open", respectively). They proved the equivalence of (2)(e) and (f) [18, 17]. Nerode and Huang also proved the equivalence of (2)(d), (e) and (f) [28].

The concept of an effective distance function has been used to define "located" sets in constructive analysis by Bishop, Bridges, Richman and Yuchuan [2, 9], and in recursive analysis for the definition of "recursively located" sets by Metakides and Nerode [26] and as "Turing located" sets by Ge and Nerode [12, 13]. Recently, located sets have been investigated in reverse mathematics by Giusto and Simpson [14].

In accordance with our terminology Zhou and Ge called a closed set $A$ recursively enumerable if and only if it has Property (1)(d). They called an open set $B$ recursively enumerable if and only if its complement $A = B^c$ has Property (2)(f). Finally, Ge called a closed or open set recursive if and only if the set itself and its complement are r.e. Ge and Zhou proved a characterization which corresponds to (3) [12, 45]. Mori, Tsujii and Yasugi use the same terminology [27].

4. Representations of compact sets

In this section we will treat representations of the set $\mathcal{K} := \{K \subseteq \mathbb{R}^n \mid \text{K compact}\}$ of compact subsets of Euclidean space. Some of the results can be found in [23, 42]. Since a subset $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded we obtain $\mathcal{K} \subseteq \mathcal{A}$ and each representation $\delta : \subseteq \Sigma^0 \rightarrow \mathcal{A}$ induces a corresponding representation $\delta|_\mathcal{A} : \subseteq \Sigma^0 \rightarrow \mathcal{K}$ of the set of compact subsets. Unfortunately, no finite prefix of a $\delta^\mathcal{K}$-name of a compact set $K$ provides an upper bound of the set, more precisely, the relation $\{(K, N) \mid K \in \mathcal{K}, N \in \mathbb{N}, K \subseteq [-N; N]^n\}$ is not even $(\delta^\mathcal{K}, v_N)$-continuous (easy proof). Hence, we have to supply these bounds explicitly.

**Definition 4.1 (Representation of compact sets).** Define representations $\delta_{\mathcal{K}}^\leq, \delta_{\mathcal{K}}^\geq, \delta_{\mathcal{K}}^=$

by

1. $\delta_{\mathcal{K}}^\leq(p, w) = K : \Leftrightarrow \delta^\leq(p) = K$ and $K \subseteq \overline{T}_w$,
2. $\delta_{\mathcal{K}}^\geq(p, w) = K : \Leftrightarrow \delta^\geq(p) = K$ and $K \subseteq \overline{T}_w$,
3. $\delta_{\mathcal{K}}^=(p, w) = K : \Leftrightarrow \delta^=(p) = K$ and $K \subseteq \overline{T}_w$

for all $p \in \Sigma^0, w \in \Sigma^*$ and $K \in \mathcal{K}$.

Proposition 3.2 and its proof can be transferred easily to the representations of the compact sets; $\delta_{\mathcal{K}}^\leq \not\equiv \delta_{\mathcal{K}}^\geq \not\equiv \delta_{\mathcal{K}}^\equiv \equiv \delta_{\mathcal{K}}^\leq \cap \delta_{\mathcal{K}}^\geq \equiv \delta_{\mathcal{K}}^\leq \cap \delta_{\mathcal{K}}^\equiv$. A standard representation $\delta_\tau$ equivalent to $\delta_{\mathcal{K}}^\leq$ and a topology $\tau_\tau$ can be obtained from the computation space $(\mathcal{K}, v)$ with $v : \subseteq \Sigma^* \rightarrow 2^\mathcal{K}$ defined as follows (cf. Definition
3.1: \( v(0#w) := v^<(w) \), \( v(1#w) := \{K \in \mathcal{X} \mid K \subseteq I_w\} \). The important Theorem 2.7 holds for \( \delta_v \) and \( \tau_v \) and accordingly for every representation equivalent to \( \delta_v \). These remarks hold for \( \delta^<_{\mathcal{X}} \) and for \( \delta^>_{\mathcal{X}} \) accordingly. Let \( \tau^>_{\mathcal{X}} \) be the topology on \( \mathcal{X} \) in the last case. We will refer to it in connection with the Hausdorff distance (Definition 4.8).

Obviously, the additional information on bounds does not affect the computability of single sets, i.e., a compact set \( K \subseteq \mathbb{R}^n \) is r.e., co-r.e. or recursive if and only if it is \( \delta^<_{\mathcal{X}} \)-computable, \( \delta^>_{\mathcal{X}} \)-computable or \( \delta^=_{\mathcal{X}} \)-computable, respectively. The following instructive proposition, however, shows that information about bounds is necessary in many other cases.

**Proposition 4.2** (Minimum function). We consider the case \( n = 1 \), i.e., \( \mathcal{X} := \{K \subseteq \mathbb{R} \mid K \text{ compact}\} \). Then the function \( \min : \mathcal{X} \rightarrow \mathbb{R} \) is

1. (\( \delta^<_{\mathcal{X}}, \rho^> \)), (\( \delta^>_{\mathcal{X}}, \rho^< \)) and (\( \delta^=_{\mathcal{X}}, \rho \))-computable,
2. but neither (\( \delta^<_{\mathcal{X}}, \rho^> \))- nor (\( \delta^>_{\mathcal{X}}, \rho^< \))- nor (\( \delta^=_{\mathcal{X}}, \rho^< \))-continuous.

The corresponding properties hold for the function \( \max \) with \( \rho^> \) replaced by \( \rho^< \) and vice versa.

**Proof.** (1) We can effectively determine a list of all upper bounds of \( \min(K) \) from a list of all \( w \) such that \( K \cap I_w \neq \emptyset \), since

\[
\min(K) < q \iff (\exists w) K \cap I_w \neq \emptyset \text{ and } \max(I_w) = q
\]

for all \( K \in \mathcal{X} \) and \( q \in \mathbb{Q} \). This proves the first statement. Now, let \( K \subseteq \overline{I}_w \) be compact and \( a := \min(I_w) \). Then

\[
q < \min(K) \iff q < a \text{ or } ((\exists w) K \cap I_w = \emptyset \text{ and } \overline{I}_w = [a; q])
\]

for all \( q \in \mathbb{Q} \). Thus, we can effectively determine a list of all lower bounds of \( \min(K) \) from a list of all \( w \) such that \( K \cap \overline{I}_w = \emptyset \) and a bound \( v \) such that \( K \subseteq \overline{I}_v \). The third statement is an immediate consequence of the first two ones.

(2) These properties can be proved straightforwardly by using standard arguments. We omit details. \( \square \)

We can derive some properties of minima and maxima of effective compact sets.

**Corollary 4.3** (Minima and maxima of compact sets). Let \( K \subseteq \mathbb{R} \) be a non-empty compact set. If \( K \) is r.e. then \( \min(K) \) is upper and \( \max(K) \) is lower semi-computable. If \( K \) is co-r.e. then \( \min(K) \) is lower and \( \max(K) \) is upper semi-computable. If \( K \) is recursive then \( \min(K) \) and \( \max(K) \) are computable.

Next, we characterize our representations of the set of compact subsets by means of coverings. By the classical Heine–Borel Theorem a set \( K \subseteq \mathbb{R}^n \) is compact if and only if each open cover of \( K \) has a finite subcover. This characterization of compactness leads us to some further representations. The first two are defined via computation spaces.
Definition 4.4 (Representation of compact sets by covers). Define computation spaces \((\mathcal{K}, \nu_{\text{cover}})\) and \((\mathcal{K}, \nu_{\text{min-cover}})\) as follows:

1. \(K \in \nu_{\text{cover}}\left(#w_1\# \ldots \#w_k\#\right) : \iff K \subseteq \bigcup_{i=1}^{k} I_{w_i},\)
2. \(K \in \nu_{\text{min-cover}}\left(#w_1\# \ldots \#w_k\#\right) : \iff K \subseteq \bigcup_{i=1}^{k} I_{w_i}\) and \(K \cap I_{w_i} \neq \emptyset\) for \(i = 1, \ldots, k\) for all \(k \in \mathbb{N}, w_1, \ldots, w_k \in \text{dom}(I),\) and \(K \in \mathcal{K}.

Let \(\delta_{\text{cover}}, \delta_{\text{min-cover}} : \subseteq \Sigma^\omega \rightarrow \mathcal{K}\) be the corresponding standard representations. (We assume tacitly that \(\text{dom}(I) \subseteq (\Sigma \setminus \{\#\})^*\).

Now we will introduce a further ad hoc representation of the set of compact subsets via the Heine-Borel property. We will call a function \(f : \subseteq \Sigma^\omega \rightarrow \Sigma^*\) a **Heine-Borel function** of a compact set \(K \in \mathcal{K}\) if and only if it proves the compactness of \(K\) in the following sense: whenever \(p = wo\#w_1\# \ldots\) is a sequence such that \(w_i \in \text{dom}(I)\) and \(K \subseteq \bigcup_{i=0}^{\infty} I_{w_i}\) then \(k := |f(p)|\) exists and \(K \subseteq \bigcup_{i=0}^{k} I_{w_i}\). In all other cases \(p \notin \text{dom}(f)\).

This definition is due to Kreitz and Weihrauch [23].

Definition 4.5 (Heine-Borel representation). Define a representation \(\delta_{\text{Heine-Borel}} : \subseteq \Sigma^\omega \rightarrow \mathcal{K}\) by

\[
\delta_{\text{Heine-Borel}}(p) = K : \iff \eta_p^{\omega*} : \subseteq \Sigma^\omega \rightarrow \Sigma^* \text{ is a Heine-Borel function of } K
\]

for all \(p \in \Sigma^\omega\) and \(K \in \mathcal{K}\).

The following theorem is a uniformly computable version of the classical Heine-Borel theorem: A set \(A \subseteq \mathbb{R}^n\) is closed and bounded if and only if any open covering has a finite subcovering.

Theorem 4.6 (Effective Heine-Borel Theorem). \(\delta_{\text{Heine-Borel}} \equiv \delta_{\text{cover}} \equiv \delta_{\gamma}^\omega\).

Proof. We prove \(\delta_{\text{Heine-Borel}} \leq \delta_{\text{cover}} \leq \delta_{\gamma}^\omega \leq \delta_{\text{Heine-Borel}}\).

“\(\delta_{\text{Heine-Borel}} \leq \delta_{\text{cover}}\)”: Let \(K \in \mathcal{K}\) and \(\delta_{\text{Heine-Borel}}(p) = K\). Then

\[
K \subseteq \bigcup_{i=0}^{k} I_{w_i} \iff w_0\#w_1\# \ldots \#w_k\# \in \text{dom}(\eta_p^{\omega*})
\]

for all \(k \in \mathbb{N}, w_0, \ldots, w_k \in \text{dom}(I)\). By using the utm-Theorem for \(\eta_p^{\omega*}\) we can effectively determine a list \(F(p)\) of all finite covers of \(K\) from a name \(p\) of a Heine-Borel function of \(K\), i.e., \(\delta_{\text{cover}}(p) = \delta_{\text{Heine-Borel}}(p)\) for all \(p \in \text{dom}(\delta_{\text{Heine-Borel}})\).

“\(\delta_{\text{cover}} \leq \delta_{\gamma}^\omega\)”: Let \(K \in \mathcal{K}\) and \(\delta_{\text{cover}}(p) = K\). From the first open “cover” \(#w_1\# \ldots \#w_k\#\) listed in \(p\) it is easy to find a bound \(v\) such that \(K \subseteq \bigcup_{i=1}^{k} I_{w_i} \subseteq \overline{I}_v\). Furthermore,

\[
K \cap \overline{I}_v = \emptyset \iff (\exists w_1, \ldots, w_k \in \text{dom}(I)) K \subseteq \bigcup_{i=1}^{k} I_{w_i}
\]

and \(\overline{I}_w \cap I_{w_i} = \emptyset\) for all \(i = 1, \ldots, k\).
for all \( w \in \text{dom}(f) \). Hence, from \( p \) we can compute a list \( F(p) \) of all \( w \) with \( K \cap w = \emptyset \). Thus, \( \delta_{\text{Heine-Borel}}(f) = \delta_{\text{cover}} (p) \) for all \( p \in \text{dom}(\delta_{\text{cover}}) \).

"\( \delta_{\text{Heine-Borel}} \leq \delta_{\text{Heine-Borel}} \)" Let \( K \in \mathcal{K} \) and \( \delta_{\text{Heine-Borel}} (p, v) = K \). By Theorem 3.12(2) from \( p \) we can effectively find some \( q \) with \( \delta_{\text{union}} (q) = K \). Let \( u_i, \; i \in \mathbb{N} \), be a list of all words such that \( "(u_i)" \) is a subword of \( q \). Moreover, let \( r := w_0 \# w_1 \# \cdots \) with \( w_i \in \text{dom}(f) \) be such that \( K \subseteq \bigcup_{i=0}^{\infty} I_{w_i} \). By the classical Heine-Borel Theorem there is a \( k \in \mathbb{N} \) such that \( K \subseteq \bigcup_{i=0}^{k} I_{w_i} \). We can find such a \( k \) effectively from \( q, r \) and \( v \), since

\[
K \subseteq \bigcup_{i=0}^{k} I_{w_i} \iff \bigcap_r \subseteq \bigcup_{i=0}^{k} I_{w_i} \iff (\exists j) \bigcap_r \subseteq \bigcup_{i=0}^{j} I_{w_i} \]

for all \( k \in \mathbb{N} \). On the other hand, if \( K \nsubseteq \bigcup_{i=0}^{\infty} I_{w_i} \) we will never find such a \( k \). Hence, \( f : \subseteq \Sigma^o \to \Sigma^* \), \( r \mapsto \emptyset^k \) is a Heine-Borel function of \( K \). Thus, by the smn-Theorem there is a computable function \( F : \Sigma^o \to \Sigma^o \) such that \( f = \eta_{F(p,v)} \), i.e., \( \delta_{\text{Heine-Borel}}(F(p,v)) = \delta_{\text{Heine-Borel}}(p,v) \) for all \( (p,v) \in \text{dom}(\delta_{\text{Heine-Borel}}) \).

The equivalence \( \delta_{\text{Heine-Borel}} = \delta_{\text{Heine-Borel}} \) of the Effective Heine-Borel Theorem has first been proved by Kreitz and Weihrauch ([23]). Since \( \delta_{\Sigma^o} = \delta_{\Sigma^o} \cap \delta_{\text{Heine-Borel}} \) (analog Proposition 3.2), we obtain an easy corollary on minimal covers.

**Corollary 4.7.** \( \delta_{\text{min-cover}} = \delta_{\Sigma^o} \).

Let \( (M, d) \) be a metric space and let \( \alpha : \subseteq \Sigma^* \to D \) be a notation of a dense subset \( D \subseteq M \). Then the computation space \( (M, v) \) with \( v : \subseteq \Sigma^* \to 2^M \) defined by \( v(u \# w) := B(\alpha(u), w) \) for all \( u \in \text{dom}(\alpha) \) and \( w \in \text{dom}(v_D) \) with \( w > 0 \) induces a natural computability concept on the space \( (M, d, \alpha) \). An easy proof shows that the derived standard representation \( \delta \) (Definition 2.6) is equivalent to the Cauchy representation \( \delta : \subseteq \Sigma^o \to M \) derived from \( v \) which is defined as follows:

\[
\delta(p) = x : \iff p = w_0 \# w_1 \# \cdots, \quad \lim_{i \to \infty} \alpha(w_i) = x, \quad \text{and} \quad d(\alpha(w_i), \alpha(w_j)) < 2^{-i} \quad \text{for all} \quad j > i
\]

(see [22, 36, 37, 39, 40]).

In the following we apply this concept to the space of the non-empty compact subsets with Hausdorff metric. Let \( \mathcal{X} := \mathcal{X} \setminus \{\emptyset\} \) denote the set of non-empty compact subsets of \( \mathbb{R}^n \). If we equip \( \mathcal{X}^* \) with the Hausdorff distance \( d_H : \mathcal{X}^* \times \mathcal{X}^* \to \mathbb{R} \), defined by

\[
d_H(A,B) := \max \left\{ \sup_{a \in A} d_B(a), \sup_{b \in B} d_A(b) \right\}
\]

for all \( A,B \in \mathcal{X}^* \), then we obtain a complete separable metric space \( (\mathcal{X}^*, d_H) \). A very simple countable dense subset of this space is the set \( \mathcal{Z} := \{ A \subseteq \mathbb{Q}^n | A \text{ finite and non-empty} \} \) of finite rational subsets of \( \mathbb{R}^n \). Let \( v_\mathcal{Z} : \subseteq \Sigma^* \to \mathcal{Z} \), defined by

\[
v_\mathcal{Z}("(w_0 \# w_1 \# \cdots \# w_k)") := \{ v_{\mathcal{Q}^*}(w_0), v_{\mathcal{Q}^*}(w_1), \ldots, v_{\mathcal{Q}^*}(w_k) \}
\]
for all $w_0, \ldots, w_k \in \text{dom}(v_{2^k})$, $k \in \mathbb{N}$, be a standard notation of $\mathcal{B}$. For $w \in \text{dom}(v_{2})$ let $\bar{w} := v_{2}(w)$ be the set denoted by $w$. We introduce the Cauchy representation of $\mathcal{K}^*$ induced by $v_{2}$ explicitly.

**Definition 4.8 (Hausdorff representation of compact subsets).** Define a representation $\delta_{\text{Haus}} : \subseteq \Sigma^\omega \rightarrow \mathcal{K}^*$ by

$$\delta_{\text{Haus}}(p) = K :\iff p = w_0 \# w_1 \# \cdots (w_i \in \text{dom}(v_{2})), \quad \lim_{i \rightarrow \infty} \bar{w}_i = K, \quad \text{and} \quad d_H(\bar{w}_i, \bar{w}_j) < 2^{-i} \quad \text{for all} \quad j > i$$

for all $p \in \Sigma^\omega$ and $K \in \mathcal{K}^*$.

Since $(a, b) \mapsto d_H(a, b)$ for $a, b \in \mathcal{B}$ is $(v_{2}, v_{2}, v_{Q})$-computable, the metric $d_H : \mathcal{K}^* \times \mathcal{K}^* \rightarrow \mathbb{R}$ is $(\delta_{\text{Haus}}, \delta_{\text{Haus}}, p)$-computable [37].

It should be mentioned that the Hausdorff topology on $\mathcal{K}^*$ induced by the Hausdorff metric $d_H$ coincides with the Fell topology and the Vietoris topology on $\mathcal{K}^*$ (cf. [1]).

With $\delta_{\text{range}}$ and $\delta_{\text{union}}$ we have two representations of the set of closed subsets which describe closed sets by approximations: $\delta_{\text{range}}$ by an inner approximation and $\delta_{\text{union}}$ by an outer approximation. With the help of the Hausdorff distance we can quantify these approximations. This quantification leads to two further representations of the set of compact subsets. These representations are inspired by Zhou who characterized recursive compact sets in a similar way [45].

**Definition 4.9 (Representation of compact sets by range and union).** Define representations $\delta_{\text{range}}', \delta_{\text{union}}' : \subseteq \Sigma^\omega \rightarrow \mathcal{K}^*$ by

1. \[\delta_{\text{range}}'(p, q) = K :\iff \eta_{p}^{**}, \eta_{q}^{**} \text{ are realizations of functions} \ f : \mathbb{N} \rightarrow \mathbb{R}^n, \ g : \mathbb{N} \rightarrow \mathbb{N}, \ \text{respectively, such that} \ \text{range}(f) \ \text{is dense in} \ K, \ \text{and} \ d_H(K_i, K_j) < 2^{-i} \ \text{for all} \ j > i, \ \text{where} \ K_i := \{f(0), \ldots, f(g(i))\} \ \text{for all} \ i \in \mathbb{N},\]

2. \[\delta_{\text{union}}'(p, q, w) = K :\iff \eta_{p}^{**}, \eta_{q}^{**} \text{ are realizations of functions} \ f : \mathbb{N} \rightarrow \Sigma^*, \ g : \mathbb{N} \rightarrow \mathbb{N}, \ \text{respectively, such that} \ K^{w} = \bigcup_{k=0}^{\infty} I_f(k), \ K \subseteq I_w, \ \text{and} \ d_H(K_i, K_j) < 2^{-i} \ \text{for all} \ j > i, \ \text{where} \ K_i := \bigcup_{k=0}^{\infty} I_f(k) \ \text{for all} \ i \in \mathbb{N},\]

for all $p, q, w \in \Sigma^\omega$, $w \in \Sigma^*$, and $K \in \mathcal{K}^*$.

The representation $\delta_{\text{range}}'$ can be considered as a Cauchy representation of $\mathcal{K}^*$ by sequences $(K_i)_{i \in \mathbb{N}}$ of finite sets with the restriction $K_i \subseteq K_{i+1}$. Of course, we have $K_i \subseteq K := \lim_{i \rightarrow \infty} K_i$ and therefore cannot require $K_i \in \mathcal{B}$ but have $K_i \subseteq \mathbb{R}^n$ in this case. The representation $\delta_{\text{union}}'$ can be considered as a Cauchy representation by decreasing sequences $(K_i)_{i \in \mathbb{N}}$ of simple compact sets. Again, we prove that our new representations are equivalent to one of the former ones.

**Theorem 4.10 (Effective Hausdorff Approximation Theorem).** $\delta_{\text{Haus}}^\infty | \mathcal{K}^* \equiv \delta_{\text{Haus}} \equiv \delta_{\text{range}}' \equiv \delta_{\text{union}}'.$
Proof. We will prove \( \delta\|_{\mathcal{X}} \leq \delta_{\text{Haus}} \leq \delta_{\text{range}} \leq \delta_{\text{union}} \leq \delta_{\|_{\mathcal{X}}} \).

"\( \delta_{\|_{\mathcal{X}}} \leq \delta_{\text{Haus}} \)"- Since \( \delta_{\text{cover}} \equiv \delta_{\|_{\mathcal{X}}} \) it suffices to prove \( \delta := \left( \delta \cap \delta_{\text{cover}} \right)_{\mathcal{X}} \leq \delta_{\text{Haus}} \). Let \( K \in \mathcal{X} \) and \( \delta(p, q) = K \). Then \( q \) is a list of all finite coverings of \( K \) and since \( K \) is totally bounded we can effectively find for each \( i \in \mathbb{N} \) a finite "cover" \( \#w_{1}\# \cdots \#w_{k} \), i.e., \( K \subseteq \bigcup_{j=1}^{k} I_{w_{j}} \) such that \( \text{diam}(I_{w_{j}}) < 2^{-i} \). Since \( p \) is a list of all \( w \) such that \( K \cap I_{w} \neq \emptyset \), we can assume \( K \cap I_{w_{j}} \neq \emptyset \) for \( j = 1, \ldots, k \). Let \( c_{j} \) be the set of all centers of \( I_{w_{1}}, \ldots, I_{w_{k}} \). Then

\[
\delta'_{\text{Haus}}(K, c_{j}) = \max \left\{ \sup_{x \in K} \inf_{y \in c_{j}} d(x, y), \sup_{y \in c_{j}} \inf_{x \in K} d(x, y) \right\} < 2^{-i-1},
\]

i.e., \( \lim_{i \to \infty} c_{j} = K \) and \( d_{\text{Haus}}(c_{j}, c_{i}) < 2^{-j} \) for all \( i > j \). Thus, we can effectively find a \( F(p, q) = c_{0}\#c_{1}\# \cdots \) such that \( \delta_{\text{Haus}}(p, q) = \delta(p, q) \) for all \( \langle p, q \rangle \in \text{dom}(\delta) \).

"\( \delta_{\text{Haus}} \leq \delta_{\text{range}} \)": Let \( K \in \mathcal{X} \) and \( \delta_{\text{Haus}}(p) = K \) with \( p = c_{0}\#c_{1}\# \cdots \) and let \( c_{i} = \{ v_{Q}(c_{0}), \ldots, v_{Q}(c_{k}) \} \) and \( I_{ij} = B(v_{Q}(c_{j}), 2^{-i+1}) \) for all \( i \in \mathbb{N}, j = 0, \ldots, k_{i} \). Then \( \lim_{i \to \infty} c_{j} = K \) and \( d_{\text{Haus}}(c_{i}, c_{j}) < 2^{-i} \) for all \( j > i \), thus \( d_{\text{Haus}}(K, c_{i}) < 2^{-i} \) and hence \( K \subseteq \bigcup_{j=0}^{\infty} I_{ij} \) for all \( i \in \mathbb{N} \). We obtain

\( K \cap I_{w} \neq \emptyset \iff (\exists i, j) I_{ij} \subseteq I_{w} \)

for all \( w \in \text{dom}(I) \). Hence, we can effectively compute from \( p \) a list \( q \) of all \( w \) such that \( K \cap I_{w} \neq \emptyset \), i.e., \( \delta_{\text{range}}(q) = K \). Since \( \delta \leq \delta_{\text{range}} \) we can effectively compute from \( q \) an \( r \) such that \( \delta_{\text{range}}(r) = K \). Let \( K_{i} := \{ f(0), \ldots, f(i) \} \) for all \( i \in \mathbb{N} \). Since \( f : \mathbb{R}^{n} \to \mathcal{X}, x \mapsto \{ x \} \) is \( (\rho_{n}, \delta_{\text{Haus}}) \)-computable and \( d_{\text{Haus}} : \mathcal{X}^{*} \times \mathcal{X}^{*} \to \mathbb{R}^{+} \) is \( (\delta_{\text{Haus}}, \delta_{\text{Haus}}, \rho) \)-computable, and by the utm-Theorem for \( \eta_{r}^{\omega} \) we can compute \( d_{\text{Haus}}(K_{i}, K) = \min_{j=0, \ldots, i} d_{\text{Haus}}(\{ f(j) \}, K) \), hence we can find a \( g(i) \) such that \( d_{\text{Haus}}(K_{g(i)}, K) < 2^{-i-1} \). Thus, by the smn-Theorem for \( \eta_{r}^{\omega} \) there is a computable function \( F : \Sigma^{0} \to \Sigma^{0} \) such that \( F(p) = \langle r, s \rangle \) and \( g \) is realized by \( \eta_{r}^{\omega} \), i.e., \( \delta_{\text{range}}(F(p)) = \delta_{\text{range}}(p) \) for all \( p \in \text{dom}(\delta_{\text{Haus}}) \).

"\( \delta_{\text{Haus}} \leq \delta_{\text{union}} \)": Let \( K \in \mathcal{X} \) and \( \delta_{\text{range}}(p, q) = K \). Then there is a function \( f : \mathbb{N} \to \mathbb{R}^{n} \) which is realized by \( \eta_{r}^{\omega} \) and a function \( g : \mathbb{N} \to \mathbb{N} \) which is realized by \( \eta_{g}^{\omega} \) such that range\((f)\) is dense in \( K \). Let \( K_{i} := \{ f(0), \ldots, f(i) \} \) for all \( i \in \mathbb{N} \). By the utm-Theorem for \( \eta_{r}^{\omega} \) and \( \eta_{g}^{\omega} \) we can effectively find for each \( i \in \mathbb{N} \) words \( c_{i_{0}}, \ldots, c_{i_{g(i)+4}} \) such that \( d(f(j), v_{Q}(c_{i})) < 2^{-i-4} \) for each \( j = 0, \ldots, g(i + 4) \). Let \( I_{ij} = B(v_{Q}(c_{j}), 2^{-i+3}) \) for all \( i \in \mathbb{N} \) and \( j = 0, \ldots, g(i + 4) \). Then \( d_{\text{Haus}}(K_{i}, K_{i+4}) < 2^{-i-4} \) and hence \( K \subseteq \bigcup_{j=0}^{g(i+4)} I_{ij} =: U_{i} \) and \( d_{\text{Haus}}(K, U_{i}) < 2^{-i-2} \) for all \( i \in \mathbb{N} \). It is easy to find a \( w \) such that \( K \subseteq \bigcup_{j=0}^{g(4)} I_{0j} \subseteq I_{w} \). Moreover, we can inductively find a number \( g'(i) \) and words \( f'(0), \ldots, f'(g'(i)) \) such that \( v_{i} := \bigcup_{j=0}^{g'(i)} f'(j) \) is disjoint from \( U_{i} \) and \( d_{\text{Haus}}(K_{i}', U_{i}) < 2^{-i-2} \), where \( K_{i}' := \overline{I_{w}} \setminus V_{i} \) for all \( i \in \mathbb{N} \). Then \( d_{\text{Haus}}(K_{i}', K_{j}') < 2^{-i} \) for all \( j > i \) follows. By the smn-Theorem for \( \eta_{r}^{\omega} \) there is a computable function \( F : \Sigma^{0} \to \Sigma^{0} \) such that \( F(p, q) = \langle r, s, w \rangle \) and \( f' \) is realized by \( \eta_{r}^{\omega} \) and \( g' \) is realized by \( \eta_{g}^{\omega} \), i.e., \( \delta_{\text{union}}(F(p, q)) = \delta_{\text{range}}(p, q) \).
"$\delta'_{\text{union}} \equiv \delta'_{\text{Heine-Borel}}$": It suffices to prove $\delta'_{\text{union}} \equiv (\delta'_{\text{Heine-Borel}} | \Sigma^*) \equiv \delta'_{\text{union}}$.

Let $K \in \Sigma^*$ and $\delta'_{\text{union}}(p, q, w) = K$. Then there is a function $f : \mathbb{N} \to \Sigma^*$ which is realized by $\eta_{p}^{**}$ and a function $g : \mathbb{N} \to \mathbb{N}$ which is realized by $\eta_{q}^{**}$ such that $K^c = \bigcup_{k=0}^{\infty} A_{f(k)}$, $K \subseteq I_w$, and $d_H(K_i, K_j) < 2^{-i}$ for all $j > i$, where $K_i := I_w \setminus \bigcup_{k=0}^{p(i)} A_{f(k)}$ for all $i \in \mathbb{N}$. By the utm-Theorem for $\eta^{**}$ we can effectively find words $c_0, \ldots, c_k$ for each $i \in \mathbb{N}$ such that $d_H(K_i, \overline{c_i}) < 2^{-i}$ and

$$K \cap I_w \neq \emptyset \iff (\exists i, j) B(v_{c_i}(c_{ij}), 2^i) \subseteq I_w$$

for all $w \in \text{dom}(I)$. Hence, we can effectively compute from $p, q$ a list $r$ of all $w$ with $K \cap I_w \neq \emptyset$, i.e., $\delta'(r) = K$. On the other hand, by the utm-Theorem for $\eta^{**}$ and since $\delta_{\text{union}} \equiv \delta'_{\text{union}}$ we can compute a $s$ from $p$ such that $\delta'(s) = K$. Thus, there is a computable function $F : \subseteq \Sigma^* \to \Sigma^*$ such that $F(p, q, w) = \langle r, \langle s, w \rangle \rangle$, i.e., $\delta'(p, q, w) = \delta'_{\text{union}}(p, q, w)$ for all $(p, q, w) \in \text{dom}(\delta'_{\text{union}})$. \qed

The following corollary summarizes the results on the introduced representations.

**Corollary 4.11** (Equivalent representations of compact sets).

1. $\delta'_{\text{Heine-Borel}} \equiv \delta_{\text{cover}} \equiv \delta_{\text{Heine-Borel}}$,
2. $\delta'_{\text{min-cover}}$, and
3. $\delta'_{\text{range}} \equiv \delta'_{\text{union}}$.

We obtain the following characterizations of effective compact sets. Here, we will call a sequence $(K_i)_{i \in \mathbb{N}}$ of finite rational subsets $K_i \subseteq \mathbb{R}^n$ computable if and only if $f : \mathbb{N} \to \mathbb{R}, i \mapsto K_i$ is $(\nu_n, \nu_2)$-computable.

**Corollary 4.12** (Characterization of effective compact sets). Let $K \subseteq \mathbb{R}^n$ be a compact set.

1. The following statements are equivalent:
   a) $K$ is co-recursively enumerable,
   b) the set \{#w_1# \ldots #w_k# | K \subseteq \bigcup_{i=1}^{k} I_{w_i} \} \subseteq \Sigma^* “of finite coverings” is recursively enumerable,
   c) $K$ admits a computable Heine–Borel function $f : \subseteq \Sigma^* \to \Sigma^*$, i.e., whenever $p = w_0##w_1## \ldots$ is a sequence such that $K \subseteq \bigcup_{i=0}^{\infty} I_{w_i}$, then $k := |f(p)|$ exists and $K \subseteq \bigcup_{i=0}^{k} I_{w_i}$. In all other cases $p \notin \text{dom}(f)$.

2. The following statements are equivalent:
   a) $K$ is recursive,
   b) the set \{#w_1# \ldots #w_k# | K \subseteq \bigcup_{i=1}^{k} I_{w_i} and K \cap I_{w_i} \neq \emptyset for i = 1, \ldots, k \} \subseteq \Sigma^* “of minimal finite coverings” is recursively enumerable,
   c) there is a computable sequence $(K_i)_{i \in \mathbb{N}}$ of finite rational subsets $K_i \subseteq \mathbb{R}^n$ such that $\lim_{i \to \infty} K_i = K$ and $d_H(K_i, K_j) \lt 2^{-i}$ for all $j > i$ or $K = \emptyset$.
   d) there are computable functions $f : \mathbb{N} \to \mathbb{R}^n$ and $g : \mathbb{N} \to \mathbb{N}$ such that range$(f)$ is dense in $K$ and $d_H(K_i, K_j) \lt 2^{-i}$ for all $j > i$, where $K_i := \{f(0), \ldots, f(g(i))\}$ for all $i \in \mathbb{N}$ or $K = \emptyset.$
(e) there are computable functions \( f : \mathbb{N} \rightarrow \Sigma^* \) and \( g : \mathbb{N} \rightarrow \mathbb{N} \) and there is a \( w \in \Sigma^* \) such that \( K^c = \bigcup_{k=0}^{\infty} I_{f(k)}, \) \( K \subseteq I_w \) and \( d_{H}(K_i,K_j)<2^{-i} \) for all \( j > i, \) where \( K_i := \overline{I_{f(k)}} \) for all \( i \in \mathbb{N} \) or \( K = \emptyset. \)

With (c) and (d) we have two different effective approximations of recursive compact sets \( K \) by finite sets \( K_i: \) while in (c) the sets \( K_i \) are rational, the sets \( K_i \) in (d) consist of computable points; on the other hand \( K_i \subseteq K \) in (d) while in general this cannot be achieved in (c).

Characterizations corresponding to (2)(d) and (e) can be found in [45]. It is worth noticing that different to Zhou we do not need the additional condition \( I_{f(k)} \subseteq K^c \) in (e) since we consider rational balls \( I_{f(k)} \) instead of computable ones.

5. Conclusion

In this paper we have discussed computability concepts on the closed and on the compact subsets of Euclidean space. The corresponding notions of recursive, recursively enumerable and co-recursively enumerable subsets have been shown to fit well into the framework of recursive analysis.

It should be mentioned that computability of subsets of Euclidean space has also been investigated in the real random access machine model (real RAM for short) by Blum, Shub and Smale [3, 4]. However, the corresponding notions are quite different from ours: while each r.e. open set \( A \subseteq \mathbb{R}^n \) is easily seen to be r.e. on a real RAM using only rational constants, there is an open set \( B \subseteq \mathbb{R}^n, \) r.e. with a real RAM using rational constants, which is not r.e. open in our sense. While the Turing machines used for computations in TTE can be realized by digital computers, real RAMs cannot be realized by physical devices. Consequently, many results obtained for the real RAM model are not significant for computations on physical computers, e.g., the theorem stating that the Mandelbrot set is r.e. but not recursive in the real RAM model (cf. [5]). For further comparisons of computable sets in different approaches, cf. [6, 34, 44].

So far we have laid a sound foundation of some important computability concepts on subsets of Euclidean space. There are some further interesting computability concepts based on stronger conditions, on measure or on properties like convexity, which have to be studied and compared in detail. This foundational work has to be extended also to concepts for computational complexity (cf. [10, 17, 19]). Beyond this there are a lot of interesting and promising subjects related to computable sets, among which the investigation of dynamical systems and Julia sets is only one example.

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