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Triangular Dynamical r-Matrices and Quantization

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We study some general aspects of triangular dynamical r-matrices using Poisson geometry. We show that a triangular dynamical r-matrix $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ always gives rise to a regular Poisson manifold. Using the Fedosov method, we prove that non-degenerate triangular dynamical r-matrices (i.e., those such that the corresponding Poisson manifolds are symplectic) are quantizable and that the quantization is clas-

1. INTRODUCTION

In the past two decades, the theory of quantum groups has undergone tremendous development. The classical counterparts of quantum groups are Lie bialgebras [12]. Many interesting quantum groups have been found and studied by various authors, but the proof of existence of quantization for arbitrary Lie bialgebras was obtained only recently by Etingof and Kazhdan [15]. For triangular Lie bialgebras, however, an elementary proof of quantization was given by Drinfel'd in 1983 [13]. Drinfel'd's idea can be outlined as follows. A triangular r-matrix on a Lie algebra $\mathfrak{g}$ defines a left invariant Poisson structure on its corresponding Lie group G. By restricting to a Lie subalgebra if necessary, one may in fact assume that this is symplectic. One may then quantize the r-matrix by finding a G-invariant $*$-product on G, of which there may be several. In [13], Drinfel'd identified the symplectic manifold with a coadjoint orbit of a central extension of $\mathfrak{g}$ and then applied Berezin quantization [6].

Recently, there has been growing interest in the so-called quantum dynamical Yang–Baxter equation (see Eq. (13)). This equation arises naturally from various contexts in mathematical physics. It first appeared in the work of Gervais and Neveu in their study of quantum Liouville

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theory [24]. Recently it reappeared in Felder's work on the quantum Knizhnik–Zamolodchikov–Bernard equation. It also has been found to be connected with the quantum Calogeř–Moser systems [2]. Just as the quantum Yang–Baxter equation is connected with quantum groups, the quantum dynamical Yang–Baxter equation is known to be connected with elliptic quantum groups [22], as well as with Hopf algebroids or quantum groupoids [17, 18, 39, 41].

The classical counterpart of the quantum dynamical Yang–Baxter equation was first considered by Felder [22], and then studied by Etingof and Varchenko [16]. This is the so-called classical dynamical Yang–Baxter equation, and a solution to such an equation (plus some other reasonable conditions) is called a classical dynamical r-matrix. More precisely, given a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ (or over $\mathbb{C}$) with an Abelian Lie subalgebra $\mathfrak{h}$, a classical dynamical r-matrix, is a smooth (or meromorphic) function $r(\lambda) : \mathfrak{h}^* \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ satisfying the conditions,

(i) (zero weight condition) $[h \otimes 1 + 1 \otimes h, r(\lambda)] = 0, \forall h \in \mathfrak{h}$;

(ii) (normal condition) $r^{12} + r^{21} = \Omega$, where $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is a Casimir element;

(iii) (classical dynamical Yang–Baxter equation)

$$\mathrm{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{1}$$

where $\mathrm{Alt}\, dr = \sum (h_i^{(1)} \frac{\partial r^{23}}{\partial \lambda^i} - h_i^{(2)} \frac{\partial r^{13}}{\partial \lambda^i} + h_i^{(3)} \frac{\partial r^{12}}{\partial \lambda^i})$.

A fundamental question is whether any classical dynamical r-matrix is quantizable. There have appeared many results in this direction. For the standard classical dynamical r-matrix for $\mathfrak{sl}_2(\mathbb{C})$, a quantization was obtained by Babelon [3] in 1991. For general simple Lie algebras, quantizations were recently found independently by Arnaudon *et al.* [1] and Jimbo *et al.* [25] based on the approach of Fronsdal [23]. Similar results were also found by Etingof and Varchenko [18] using intertwining operators. Recently, using a method similar to [1, 23, 25], Etingof *et al.* [19] obtained a quantization of all the classical dynamical r-matrices of semisimple Lie algebras in Schiffmann's classification list [35]. However, the general quantization problem still remains open; a recipe has yet to be found. Moreover, the problem of classification of quantizations has not yet been touched.

In this paper, we study the quantization problem for general classical triangular dynamical r-matrices. Classical triangular dynamical r-matrices are those satisfying the skew-symmetric condition $r^{12}(\lambda) + r^{21}(\lambda) = 0$. In this case, Eq. (1) is equivalent to $\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} + \frac{1}{2}[r, r] = 0$. These r-matrices are in one–one correspondence with regular Poisson structures $\pi = \sum_i \overrightarrow{h_i} \wedge \frac{\partial}{\partial \lambda^i} + \overrightarrow{r(\lambda)}$ on the manifold $\mathfrak{h}^* \times G$, which are invariant under the left G- and right H-actions. Thus one may expect to quantize a classical dynamical r-matrix by looking for a certain special type of star-products [5] on the corresponding Poisson manifold. This is exactly the route we take in the present paper. In some sense, this is also a natural generalization of the quantization method used by Drinfel'd in [13] as outlined at the beginning of the introduction. In fact, in this paper, we mainly deal with non-degenerate triangular classical dynamical r-matrices (i.e., the corresponding Poisson manifolds are in fact symplectic). Berezin quantization no longer works in this situation. However, one may use the Fedosov method to obtain the desired star-products, as we will see later. It is well known that star products on a symplectic manifold are classified by the second cohomology group of the manifold with coefficients in formal $\hbar$-power series. In light of this result, we are able to classify the quantizations of a non-degenerate triangular classical dynamical r-matrix and prove that the quantizations are parameterized by the relative Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{h})[[\hbar]]$.

For a general triangular classical dynamical r-matrix, it is natural to ask whether it is possible to reduce it to a non-degenerate one by restricting to a Lie subalgebra. This is always true in the non-dynamical case [13]. Unfortunately, in general this fails in the dynamical case, and we will study the conditions under which this is possible. In this case, these r-matrices are called splittable. Splittable triangular classical dynamical r-matrices resemble in many ways non-degenerate ones. And in particular, they can be quantized by the Fedosov method.

The outline of this paper is as follows. After Section 1 (this Introduction), in Section 2, we study general properties of triangular classical dynamical r-matrices. It is proved that triangular classical dynamical r-matrices correspond to some special Poisson structures on $\mathfrak{h}^* \times G$, which are always regular. This may seem surprising at first glance since the rank of $r(\lambda)$ may depend on the point λ. The main tool in Section 2 is the method of Lie groupoids and Lie algebroids. In particular, we show how gauge transformations, first introduced by Etingof and Varchenko [16], enter naturally from the viewpoint of Lie algebroids. The study of the tangent space of the moduli space of dynamical r-matrices naturally leads to the notion of dynamical r-matrix cohomology, which is shown to be isomorphic to the relative Lie algebra cohomology when r is non-degenerate. Section 3 is devoted to the proof of the equivalence between quantizations of triangular classical dynamical r-matrices and the so called compatible star products on their corresponding Poisson manifolds $\mathfrak{h}^* \times G$. In Section 4, we study symplectic connections on such symplectic manifolds ($M = \mathfrak{h}^* \times G$). In particular, we show that there always exists a $G \times H$-invariant (i.e., left G-invariant and right H-invariant) torsion-free symplectic connection on M such that the left invariant vector fields

$\vec{h}$, $\forall h \in \mathfrak{h}$ are all parallel. The main result of Section 5 is that the Fedosov quantization obtained via such a symplectic connection and some suitable choice of Weyl curvatures gives rise to compatible $*$-products on $M = \mathfrak{h}^* \times G$. Therefore, as a consequence, we prove the existence of a quantization of non-degenerate triangular classical dynamical r-matrices. The presentation in Section 5, however, is made in a more general setting, which is of its own interest. Section 6 is devoted to the classification of quantizations. In particular, we show that the equivalence classes of quantizations of a non-degenerate triangular dynamical r-matrix $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ are parameterized by the relative Lie algebra cohomology with coefficients in the formal $\hbar$-power series $H^2(\mathfrak{g}, \mathfrak{h})[[\hbar]]$. Some speculation on the classification of quantizations of a general triangular classical dynamical r-matrix is given as a conjecture, which is consistent with Kontesvich's formality theorem [26]. In the appendix we recall some basic ingredients of the Fedosov quantization, which are used throughout the paper.

Finally, some remarks are in order. Quantization of dynamical r-matrices is related to quantization of Lie bialgebroids as shown in [41]. However, for simplicity, we will avoid using quantum groupoids in the present paper even though many ideas are rooted from there. Also in this paper, we work in the smooth case. Namely, Lie algebras are finite dimensional Lie algebras over $\mathbb{R}$, all manifolds and maps are smooth, but our approach works for the complex category as well. For simplicity, we assume that a dynamical r-matrix is always defined on $\mathfrak{h}^*$. In reality, it may only be defined on an open submanifold $U \subset \mathfrak{h}^*$, but our results hold in this situation as well.

2. TRIANGULAR DYNAMICAL r-MATRICES

In this section, we study some general aspects of triangular dynamical r-matrices. As a useful tool, we shall utilize the method of Lie algebroids and Lie groupoids. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an Abelian Lie subalgebra of dimension l. By a *triangular dynamical r-matrix*, we mean a smooth function $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ satisfying

(i) the zero weight condition: $[h, r(\lambda)] = 0$, $\forall \lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and

(ii) the classical dynamical Yang–Baxter equation (CDYBE),

$$\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} + \frac{1}{2}[r, r] = 0, \tag{2}$$

where the bracket $[\cdot, \cdot]$ refers to the Schouten type bracket $\wedge^k \mathfrak{g} \otimes \wedge^l \mathfrak{g} \longrightarrow \wedge^{k+l-1} \mathfrak{g}$ induced from the Lie algebra bracket on $\mathfrak{g}$. Here

$\{h_1, \ldots, h_l\}$ is a basis in $\mathfrak{h}$, and $(\lambda^1, \ldots, \lambda^l)$ its induced coordinate system on $\mathfrak{h}^*$. It is known [4, 30] that the CDYBE is closely related to Lie bialgebroids. Recall that a Lie bialgebroid is a pair of Lie algebroids (A, A^*) satisfying the compatibility condition (see [27, 32, 33])

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \forall X, Y \in \Gamma(A), \tag{3}$$

where the differential d_* on $\Gamma(\wedge^* A)$ comes from the Lie algebroid structure on A^*.

Given a Lie algebroid A over P with anchor a and a section Λ of $\Gamma(\wedge^2 A)$ satisfying the condition $[\Lambda, \Lambda] = 0$, one may define a Lie algebroid structure on A^* by simply requiring the differential $d_*\colon \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k+1} A)$ to be $d_* = [\Lambda, \cdot\,]$. More explicitly, denote by $\Lambda^\#$ the bundle map $A^* \longrightarrow A$ defined by $\lambda^\#(\xi)(\eta) = \Lambda(\xi, \eta)$, $\forall \xi, \eta \in \Gamma(A^*)$. Then the bracket on $\Gamma(A^*)$ is defined by

$$[\xi, \eta] = L_{\Lambda^\#\xi}\eta - L_{\Lambda^\#\eta}\xi - d[\Lambda(\xi, \eta)], \tag{4}$$

and the anchor a_* is the composition $a \circ \Lambda^\#\colon A^* \longrightarrow TP$. It is easy to show that (A, A^*) is indeed a Lie bialgebroid, which is called a *triangular Lie bialgebroid* [32].

Now consider $A = T\mathfrak{h}^* \times \mathfrak{g}$ and equip A with the standard product Lie algebroid structure. Then the anchor $a\colon T\mathfrak{h}^* \times \mathfrak{g} \longrightarrow T\mathfrak{h}^*$ is simply the projection. The relation between triangular dynamical r-matrices and triangular Lie bialgebroids is described by the following [4, 30]:

Proposition 2.1. *Given a smooth function $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, r is a triangular dynamical r-matrix iff the Lie algebroid (A, a) together with $\Lambda = \sum_i h_i \wedge \frac{\partial}{\partial \lambda^i} + r(\lambda) \in \Gamma(\wedge^2 A)$ defines a triangular Lie bialgebroid.*

Proof. By a straightforward computation, we have $[\Lambda, \Lambda] = 2(\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} + \frac{1}{2}[r, r] + \sum_i [r, h_i] \wedge \frac{\partial}{\partial \lambda^i})$. It thus follows that $[\Lambda, \Lambda] = 0$ iff $\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} + \frac{1}{2}[r, r] = 0$ and $[r, h_i] = 0$ $(i = 1, \ldots, l)$, i.e., r is a triangular dynamical r-matrix. ∎

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ an Abelian Lie subgroup with Lie algebra $\mathfrak{h}$. Consider $M = \mathfrak{h}^* \times G$. Let G act on M from the left by left multiplication on G and H act from the right by right multiplication on G. An equivalent version of Proposition 2.1 is

Proposition 2.2. *For a smooth function $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, r is a triangular dynamical r-matrix iff $\pi = \sum_i \overrightarrow{h_i} \wedge \frac{\partial}{\partial \lambda^i} + \overrightarrow{r(\lambda)}$ defines a $G \times H$-invariant Poisson structure on $M = \mathfrak{h}^* \times G$, where $\overrightarrow{h_i} \in \mathfrak{X}(M)$ is the left invariant vector* *field on M generated by h_i and similarly $\overrightarrow{r(\lambda)} \in \Gamma(\wedge^2 TM)$ is the left invariant bivector field on M corresponding to $r(\lambda)$.*

THEOREM 2.3. *If $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is a triangular dynamical r-matrix, then $\mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, the Lie subalgebras $\mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$, $\forall \lambda \in \mathfrak{h}^*$, are all isomorphic, and the isomorphisms are given by the adjoint action of G.*

Proof. For any $\lambda \in \mathfrak{h}^*$, $A_\lambda = T_\lambda \mathfrak{h}^* \oplus \mathfrak{g} \cong \mathfrak{h}^* \oplus \mathfrak{g}$ and $A_\lambda^* \cong \mathfrak{h} \oplus \mathfrak{g}^*$. Under these identifications, the bundle map $\Lambda_\lambda^{\#}: A_\lambda^* \longrightarrow A_\lambda$ is given by

$$(h, \xi) \mapsto (i^*\xi, -h + r(\lambda)^{\#}\,\xi), \quad \forall h \in \mathfrak{h} \quad \text{and} \quad \xi \in \mathfrak{g}^*, \tag{5}$$

where $i: \mathfrak{h} \longrightarrow \mathfrak{g}$ is the inclusion. Set $B = \Lambda^{\#}(A^*) = \bigcup_{\lambda \in \mathfrak{h}^*} \Lambda_\lambda^{\#}(A_\lambda^*) \subset A$. Since (A, Λ) defines a triangular Lie bialgebroid, B is integrable; i.e., $\Gamma(B)$ is closed under the Lie algebroid bracket on $\Gamma(A)$. Hence $\ker a|_{B_\lambda}$ is a Lie subalgebra of $\ker a|_{A_\lambda}$. Now it is easy to see that $\ker a|_{B_\lambda} = \mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$ and $\ker a|_{A_\lambda} = \mathfrak{g}$. It thus follows that $\mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$ is a Lie subalgebra of $\mathfrak{g}$. On the other hand, from Eq. (5), it is easy to see that $a(B_\lambda) = T_\lambda \mathfrak{h}^*$. Hence $a: B \longrightarrow T\mathfrak{h}^*$ is surjective, which implies that B is in fact a transitive Lie algebroid (also called a gauge Lie algebroid [31]). Thus it follows that the dimension of B_λ is independent of λ, and therefore B is a subbundle of A. Moreover the isotropic Lie algebras of B at different points of $\mathfrak{h}^*$ are all isomorphic, and the isomorphisms are given by the adjoint action of G. This implies that, for any $\lambda, \mu \in \mathfrak{h}^*$, $\mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$ is isomorphic to $\mathfrak{h}+r(\mu)^{\#}\,\mathfrak{h}^{\perp}$ by the adjoint action of a group element in G. ∎

For the sake of simplicity, we denote by $\mathfrak{g}_\lambda$ the Lie subalgebra $\mathfrak{h}+r(\lambda)^{\#}\,\mathfrak{h}^{\perp}$. Define the *rank of a triangular dynamical r-matrix* r to be $\dim \mathfrak{g}_\lambda - \dim \mathfrak{h}$, which is denoted as rank r. We say a triangular dynamical r-matrix r is *non-degenerate* if rank $r = \dim \mathfrak{g} - \dim \mathfrak{h}$.

An immediate consequence of Theorem 2.3 is

COROLLARY 2.3. *Under the same hypothesis as in Theorem* 2.3, rank r *is independent of the point λ and therefore is a well-defined even number. Moreover $B = \Lambda^{\#}A^* \subset A$ is a Lie subalgebroid of rank* $2 \dim \mathfrak{h} + \text{rank}\, r$, *and (M, π) is a regular Poisson manifold of rank* $2 \dim \mathfrak{h} + \text{rank}\, r$.

In particular, we have the following

COROLLARY 2.5. *Given a triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, the following statements are all equivalent:*

(i) *r is non-degenerate;*

(ii) *the bundle map $\Lambda^{\#}: A^* \longrightarrow A$ is nondegenerate;*

(iii) $\mathfrak{g}_\lambda = \mathfrak{g}$, $\forall \lambda \in \mathfrak{h}^*$;

(iv) *(M, π) is a symplectic manifold.*

If we choose a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, and choose a basis $\{h_1, ..., h_l\}$ for $\mathfrak{h}$ and a basis $\{e_1, ..., e_m\}$ for $\mathfrak{m}$, we may write

$$r(\lambda) = \sum a^{ij}(\lambda)\, h_i \wedge h_j + \sum b^{ij}(\lambda)\, h_i \wedge e_j + \sum c^{ij}(\lambda)\, e_i \wedge e_j. \tag{6}$$

It is simple to see that $\mathfrak{g}_\lambda = \mathfrak{h} \oplus \operatorname{Span}\{\sum_j c^{ij}(\lambda)\, e_j \mid i = 1, ..., m\}$, and rank r is the rank of the matrix $(c^{ij}(\lambda))$. Therefore, we immediately know that the rank of $(c^{ij}(\lambda))$ is independent of λ. Clearly r is non-degenerate iff the matrix $(c^{ij}(\lambda))$ is non-degenerate.

A natural question arises as to whether it is possible to make an arbitrary triangular dynamical r-matrix non-degenerate by considering it to be valued in a Lie subalgebra of $\mathfrak{g}$. This is true in the non-dynamical case [13], for example. However, in the dynamical case, it is not always possible, as we will see below. Nevertheless we will single out those r-matrices possessing this property, which will be called *splittable*. Splittable triangular dynamical r-matrices contain a large class of interesting dynamical r-matrices, which in fact include almost all examples we know, e.g., those as classified in [16] when $\mathfrak{g}$ is a simple Lie algebra. More precisely,

DEFINITION 2.6. A triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is said to be splittable if for any $\lambda \in \mathfrak{h}^*$, $i^*(r(\lambda)^{\#-1}\, \mathfrak{h}) = \mathfrak{h}^*$, where $i: \mathfrak{h} \longrightarrow \mathfrak{g}$ is the inclusion.

PROPOSITION 2.7. *Suppose that r is a triangular dynamical r-matrix. Then the following statements are equivalent:*

(i) *r is splittable;*

(ii) *for any $\lambda \in \mathfrak{h}^*$, $r(\lambda)^{\#}\, \mathfrak{g}^* \subset \mathfrak{g}_\lambda$;*

(iii) *if $r(\lambda)$ is given as in Eq.* (6) *under a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then for any i, $\sum_j b^{ij}(\lambda)\, e_j \in \operatorname{Span}\{\sum_j c^{ij}(\lambda)\, e_j \mid i = 1, ..., m\}$;*

(iv) *for any fixed $\lambda \in \mathfrak{h}^*$, there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, under which*

$$r(\lambda) = \sum a^{ij}(\lambda)\, h_i \wedge h_j + \sum c^{ij}(\lambda)\, e_i \wedge e_j; \tag{7}$$

(v) $T\mathfrak{h}^* \times \{0\} \subset B$.

Let us first prove the following simple lemma from linear algebra.

LEMMA 2.8. *Let $V=\mathfrak{h}\oplus\mathfrak{m}$ be a decomposition of vector spaces, and let $\{h_1, \dots, h_l\}$ be a basis of $\mathfrak{h}$, and $\{e_1, \dots, e_m\}$ a basis of $\mathfrak{m}$. Let $r\in\bigwedge^2 V$ be any element such that*

$$r=\sum a^{ij}h_i\wedge h_j+\sum h_i\wedge x^i+\sum c^{ij}e_i\wedge e_j,$$

where $x^i\in\mathfrak{m}$, and a_{ij}, c_{ij} are skew-symmetric, i.e., $a_{ij}=-a_{ji}$ and $c_{ij}=-c_{ji}$. If $I\subset\{1, \dots, l\}$ is a subset of indexes such that for any $i_0\in I$, $x^{i_0}\in\operatorname{Span}\{\sum_j c^{ij}e_j \mid i=1, \dots, m\}$. Then one can change the decomposition $V=\mathfrak{h}\oplus\tilde{\mathfrak{m}}$ so that under a suitable basis $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ of $\tilde{\mathfrak{m}}$, r can be written as

$$r=\sum \tilde{a}^{ij}h_i\wedge h_j+\sum_{i\notin I} h_i\wedge x^i+\sum c^{ij}\tilde{e}_i\wedge\tilde{e}_j.$$

Proof. $\forall i_0\in I$, by assumption, there are constants $\gamma_i^{i_0}$, $i=1, \dots, m$, such that $x^{i_0}=2\sum_{ij}\gamma_i^{i_0}c^{ij}e_j$. Let $\tilde{e}_i=e_i+\sum_{i_0\in I}\gamma_i^{i_0}h_{i_0}$, $\forall i=1, \dots, m$. Then

$$\begin{aligned}\sum c^{ij}\tilde{e}_i\wedge\tilde{e}_j&=\sum c^{ij}\Big(e_i+\sum_{i_0\in I}\gamma_i^{i_0}h_{i_0}\Big)\wedge\Big(e_j+\sum_{i_0\in I}\gamma_j^{i_0}h_{i_0}\Big)\\&=\sum c^{ij}e_i\wedge e_j+2\sum c^{ij}\gamma_i^{i_0}h_{i_0}\wedge e_j \qquad (\operatorname{mod} \textstyle\bigwedge^2\mathfrak{h})\\&=\sum c^{ij}e_i\wedge e_j+\sum_{i_0\in I}h_{i_0}\wedge x^{i_0} \qquad (\operatorname{mod} \textstyle\bigwedge^2\mathfrak{h}).\end{aligned}$$

Hence $r=\sum c^{ij}\tilde{e}_i\wedge\tilde{e}_j+\sum_{i\notin I}h_i\wedge x^i \ (\operatorname{mod} \bigwedge^2\mathfrak{h})$. This concludes the proof. ∎

Proof of Proposition 2.7.

(i) ⇒ (ii) Let us fix a basis $\{h_1, \dots, h_l\}$ of $\mathfrak{h}$, and let $\{h_*^1, \dots, h_*^l\}$ be its dual basis in $\mathfrak{h}^*$. By assumption, for any $1\leqslant j\leqslant l$, there is a $\xi^j\in\mathfrak{g}^*$ such that $i^*\xi^j=h_*^j$ and $r(\lambda)^{\#}\xi^j\in\mathfrak{h}$. Given any $\xi\in\mathfrak{g}^*$, take $a_j=\langle\xi, h_j\rangle$ and $\eta=\xi-\sum a_j\xi^j$. Then it is easy to see that $\eta\in\mathfrak{h}^\perp$. Hence $r(\lambda)^{\#}\xi=\sum a_j r(\lambda)^{\#}\xi^j+r(\lambda)^{\#}\eta\in\mathfrak{h}+r(\lambda)^{\#}\mathfrak{h}^\perp=\mathfrak{g}_\lambda$.

(ii) ⇒ (iii) Let $\{h_1, \dots, h_l\}$ be a basis of $\mathfrak{h}$, $\{e_1, \dots, e_m\}$ a basis of $\mathfrak{m}$, and $\{h_*^1, \dots, h_*^l, e_*^1, \dots, e_*^m\}$ the dual basis of $\{h_1, \dots, h_l, e_1, \dots, e_m\}$ in $\mathfrak{g}^*$. It is trivial to see that $r(\lambda)^{\#}e_*^i=-\sum_j b^{ji}(\lambda)\,h_j+2\sum_j c^{ij}(\lambda)\,e_j$. Hence we have

$$\mathfrak{g}_\lambda=\mathfrak{h}\oplus\operatorname{Span}\Big\{\sum_j c^{ij}(\lambda)\,e_j \mid i=1, \dots, m\Big\}.$$

Now $r(\lambda)^{\#} h_*^i = \sum_j 2a^{ij}(\lambda) h_j + \sum_j b^{ij}(\lambda) e_j$. Since $r(\lambda)^{\#} h_*^i \in \mathfrak{g}_\lambda$ by assumption, it follows that $\sum_j b^{ij}(\lambda) e_j \in \mathrm{Span}\{\sum_j c^{ij}(\lambda) e_j \mid i = 1, \ldots, m\}$.

(iii) $\Rightarrow$ (iv) This follows from Lemma 2.8.

(iv) $\Rightarrow$ (v) If $r(\lambda) = \sum a^{ij}(\lambda) h_i \wedge h_j + \sum c^{ij}(\lambda) e_i \wedge e_j$, then $r(\lambda)^{\#} h_*^i = 2\sum_j a^{ij}(\lambda) h_j$. Thus according to Eq. (5), $\Lambda_\lambda^{\#}(2\sum_j a^{ij}(\lambda) h_j, h_*^i) = (h_*^i, 0)$. Hence, $(h_*^i, 0) \in B_\lambda$. This implies that $T_\lambda \mathfrak{h}^* \times \{0\} \subset B_\lambda$.

(v) $\Rightarrow$ (i) Given any $\varphi \in \mathfrak{h}^*$, we know that $(\varphi, 0) \in B_\lambda$ by assumption. Therefore there exist $h \in \mathfrak{h}$ and $\xi \in \mathfrak{g}^*$ such that $\Lambda_\lambda^{\#}(h, \xi) = (\varphi, 0)$, i.e., $(i^*\xi, -h + r(\lambda)^{\#} \xi) = (\varphi, 0)$ according to Eq. (5). This implies that $\varphi = i^*\xi$ and $r(\lambda)^{\#} \xi = h$. Hence $\varphi \in i^*(r(\lambda)^{\#-1} \mathfrak{h})$. Therefore, we conclude that $\mathfrak{h}^* \subset i^*(r(\lambda)^{\#-1} \mathfrak{h})$. ∎

Remark. In the proof above, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and the choice of the basis $\{e_1, \ldots, e_m\}$ in (iv) depend on a particular point λ. It is not clear whether it is possible to find a decomposition so that Eq. (7) holds uniformly for all points in $\mathfrak{h}^*$. On the other hand, if there exists such a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ so that a triangular dynamical r-matrix is of the form in Eq. (7), it is always splittable.

An immediate consequence of Proposition 2.7 is the following:

COROLLARY 2.9. *If $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is a splittable triangular dynamical r-matrix, then*

(i) *$\mathfrak{g}_\lambda$ is independent of λ, i.e., $\mathfrak{g}_\lambda = \mathfrak{g}_\mu$, $\forall \lambda, \mu \in \mathfrak{h}^*$. We will denote $\mathfrak{g}_\lambda$ by $\mathfrak{g}_1$.*

(ii) *r can be considered a non-degenerate triangular dynamical r-matrix valued in $\wedge^2 \mathfrak{g}_1$.*

Proof. By Proposition 2.7, $T\mathfrak{h}^* \times \{0\}$ is a Lie subalgebroid of B. Hence for any $X \in \mathfrak{X}(\mathfrak{h}^*)$, $(X, 0) \in \Gamma(B)$. Let φ_t be the (local) flow on $\mathfrak{h}^*$ generated by X. The bisection $\exp t(X, 0)$ on the groupoid $\Gamma = \mathfrak{h}^* \times \mathfrak{h}^* \times G$ generated by the section $(X, 0) \in \Gamma(A)$ is $\{(\lambda, \varphi_t(\lambda), 1) \mid \lambda \in \mathfrak{h}^*\}$. Hence its induced isomorphism between Γ_λ and $\Gamma_{\varphi_t(\lambda)}$ is the identity map, when both of them are naturally identified with G. Here Γ_λ and $\Gamma_{\varphi_t(\lambda)}$ denote the isotropic groups of Γ at the points λ and $\varphi_t(\lambda)$, respectively. Therefore, $Ad_{\exp t(X,0)}$ is an identity map between their corresponding isotropic Lie algebras. On the other hand, since $(X, 0) \in \Gamma(B)$; hence $Ad_{\exp t(X,0)}$, when being restricted to B, is exactly the map which establishes the isomorphism between $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\varphi_t(\lambda)}$. Hence, $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\varphi_t(\lambda)}$ are equal as Lie subalgebras of $\mathfrak{g}$.

For the second part, since r is splittable, we have $r(\lambda)^{\#}\mathfrak{g}^* \subset \mathfrak{g}_1$ according to Proposition 2.7. Hence $\forall \lambda \in \mathfrak{h}^*$, $r(\lambda) \in \wedge^2 (r(\lambda)^{\#}\mathfrak{g}^*) \subset \wedge^2 \mathfrak{g}_1$. By dimension counting, one easily sees that r is non-degenerate when being considered as a dynamical r-matrix valued in $\wedge^2 \mathfrak{g}_1$. ∎

Let $g: \mathfrak{h}^* \longrightarrow G^H$ be a smooth map, where G^H denotes the centralizer of H in G with its Lie algebra being denoted by $\mathfrak{g}^H$. Then g can be naturally considered as a bisection of the groupoid $\Gamma = \mathfrak{h}^* \times \mathfrak{h}^* \times G$, and hence we can talk about the induced automorphism Ad_g of the corresponding Lie algebroid. In particular, we have a Gerstenhaber algebra automorphism Ad_g on $\bigoplus \Gamma(\wedge^* A)$ [40].

Given a smooth function $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, let $\Lambda_r = \sum_i h_i \wedge \frac{\partial}{\partial \lambda^i} + r(\lambda) \in \Gamma(\wedge^2 A)$ as in Proposition 2.1. Then

$$\begin{aligned}
Ad_g \Lambda_r &= Ad_g \left(\sum_i h_i \wedge \frac{\partial}{\partial \lambda^i} + r \right) \\
&= \sum_i Ad_g h_i \wedge \left(\frac{\partial}{\partial \lambda^i} - \frac{\partial g}{\partial \lambda^i} g^{-1} \right) + Ad_g r \\
&= \sum_i h_i \wedge \left(\frac{\partial}{\partial \lambda^i} - \frac{\partial g}{\partial \lambda^i} g^{-1} \right) + Ad_g r \\
&= \sum_i h_i \wedge \frac{\partial}{\partial \lambda^i} + \left(Ad_g r - \sum_i h_i \wedge \frac{\partial g}{\partial \lambda^i} g^{-1} \right).
\end{aligned}$$

Here in the second from the last equality, we used $Ad_g h_i = h_i$ since $g \in G^H$. Let

$$r_g = Ad_g r - \sum_i h_i \wedge \frac{\partial g}{\partial \lambda^i} g^{-1}. \tag{8}$$

Combining with Proposition 2.1, we thus have proved the following:

Proposition 2.10. *Assume that $g: \mathfrak{h}^* \longrightarrow G^H$ is a smooth map. Then*

(i) $\Lambda_{r_g} = Ad_g \Lambda_r$;

(ii) *r is a triangular dynamical r-matrix iff r_g is a triangular dynamical r-matrix.*

(iii) rank r_g=rank r; *in particular, if r is non-degenerate, so is r_g.*

This proposition naturally leads us to the notion of gauge transformations on dynamical r-matrices, which was first introduced by Etingof and

Varchenko [16]. Recall that triangular dynamical r-matrices r_1 and r_2 are said to be *gauge equivalent* if there exists a smooth function $g: \mathfrak{h}^* \longrightarrow G^H$ such that $r_2 = (r_1)_g$.

Remark. Although non-degenerate triangular dynamical r-matrices are preserved by gauge transformations, splittable dynamical r-matrices in general are not. For example, the trivial triangular dynamical r-matrix $r = 0$ is always splittable. However, $r_g = -\sum h_i \wedge \frac{\partial g}{\partial \lambda^i} g^{-1}$ is never splittable unless $G^H = H$.

By $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$, we denote the quotient space of the space of all triangular dynamical r-matrices $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ by gauge transformations, which is called the *moduli space* of triangular dynamical r-matrices.

Next we will introduce the dynamical r-matrix cohomology $H_r^*(\mathfrak{g}, \mathfrak{h})$, whose second cohomology group describes the tangent space of the moduli space $\mathscr{M}(\mathfrak{g}, \mathfrak{h})$. As we will see in Section 6, the second cohomology group $H_r^2(\mathfrak{g}, \mathfrak{h})$ is connected with the classification of quantizations of r when it is non-degenerate.

Consider $C^k = C^\infty(\mathfrak{h}^*, (\wedge^k \mathfrak{g})^H)$ (or equivalently denoted as $C^\infty(\mathfrak{h}^*, (\wedge^k \mathfrak{g})^{\mathfrak{h}})$), and define a differential $\delta_r: C^k \longrightarrow C^{k+1}$ by

$$\delta_r \tau = \sum_i h_i \wedge \frac{\partial \tau}{\partial \lambda^i} \tau + [r, \tau], \quad \forall \tau \in C^k. \tag{9}$$

PROPOSITION 2.11. *$\delta_r: C^k \longrightarrow C^{k+1}$ is well defined and $\delta_r^2 = 0$.*

Proof. It is clear that $\delta_r \tau$ is in $C^\infty(\mathfrak{h}^*, (\wedge^{k+1} \mathfrak{g})^H)$ provided that $\tau \in C^\infty(\mathfrak{h}^*, (\wedge^k \mathfrak{g})^H)$. For any $\tau \in C^k = C^\infty(\mathfrak{h}^*, (\wedge^k \mathfrak{g})^H)$, τ can be naturally considered as a section of $\wedge^k A$, and

$$\begin{aligned} [\Lambda, \tau] &= \left[\sum_i h_i \wedge \frac{\partial}{\partial \lambda^i} + r, \tau \right] \\ &= \sum_i h_i \wedge \frac{\partial \tau}{\partial \lambda^i} + [r, \tau] \\ &= \delta_r \tau. \end{aligned}$$

Since $[\Lambda, \Lambda] = 0$, it thus follows that $\delta_r^2 = 0$. ∎

Hence the cochain complex $\delta_r: C^k \longrightarrow C^{k+1}$ defines a cohomology, called *the dynamical r-matrix cohomology*, and denoted by $H_r^*(\mathfrak{g}, \mathfrak{h})$. Two remarks are in order.

Remarks. (1) The cochain complex $\delta_r\colon C^k \longrightarrow C^{k+1}$ is in fact a subcomplex of the Lie algebroid cohomology cochain complex $d_*\colon \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k+1} A)$, $d_* X = [\Lambda, X]$. Therefore it is easy to see that such a cochain complex is always defined for an arbitrary dynamical r-matrix, which is not necessary triangular.

(2) When r is triangular, $H^*_r(\mathfrak{g}, \mathfrak{h})$ can be naturally identified with a "special" $G \times H$-invariant Poisson cohomology of the Poisson manifold (M, π), i.e., the cohomology obtained by restricting the Poisson cochain complex to $G \times H$-invariant multi-vector fields tangent to the fibers of the fibration: $\mathfrak{h}^* \times G \longrightarrow \mathfrak{h}^*$.

PROPOSITION 2.12. *If $g\colon \mathfrak{h}^* \longrightarrow G^H$ is a smooth map, then*

(i) $\delta_{r_g} \circ Ad_g = Ad_g \circ \delta_r$;

(ii) $Ad_g\colon (C^*, \delta_r) \longrightarrow (C^*, \delta_{r_g})$ *induces an isomorphism* $H^*_r(\mathfrak{g}, \mathfrak{h}) \cong H^*_{r_g}(\mathfrak{g}, \mathfrak{h})$.

Proof. For any $\tau \in C^\infty(\mathfrak{h}^*, (\wedge^k \mathfrak{g})^H)$,

$$
\begin{aligned}
(Ad_g \circ \delta_r)\, \tau &= Ad_g[\Lambda, \tau] \\
&= [Ad_g\, \Lambda, Ad_g\, \tau] \\
&= [\Lambda_{r_g}, Ad_g\, \tau] \\
&= (\delta_{r_g} \circ Ad_g)\, \tau.
\end{aligned}
$$

The conclusion thus follows immediately. ∎

As a consequence, we conclude that $H^*_r(\mathfrak{g}, \mathfrak{h})$ only depends on the gauge equivalence class of the dynamical r-matrix. For this reason, we also denote this group by $H^*_{[r]}(\mathfrak{g}, \mathfrak{h})$.

PROPOSITION 2.13. *For any triangular dynamical r-matrix $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, $T_{[r]}\mathcal{M}(\mathfrak{g}, \mathfrak{h}) \cong H^2_{[r]}(\mathfrak{g}, \mathfrak{h})$.*

Proof. In Eq. (2), replace r by $r + t\tau$ and take the derivative at $t = 0$, one obtains the linearization equation $\sum_i h_i \wedge \frac{\partial \tau}{\partial \lambda^i} + [r, \tau] = 0$; i.e., $\delta_r \tau = 0$. It is clear that τ is of zero weight since $r + t\tau$ is of zero weight.

To compute the tangent space to the gauge orbit at r, one needs to compute $\frac{d}{dt}\big|_{t=0}(r_{\exp tf})$, for $f \in C^\infty(\mathfrak{h}^*, \mathfrak{g}^H)$. Now $r_{\exp tf} = Ad_{\exp tf}\, r - \sum_i h_i \wedge \frac{\partial \exp tf}{\partial \lambda^i} (\exp tf)^{-1}$. It is thus simple to see that $\frac{d}{dt}\big|_{t=0}(r_{\exp tf}) = [f, r] - \sum_i h_i \wedge \frac{\partial f}{\partial \lambda^i} = -\delta_r f$. The conclusion thus follows immediately. ∎

Given a Lie algebra $\mathfrak{g}$, one may also consider classical triangular dynamical r-matrices $r_\hbar\colon \mathfrak{h}^* \longrightarrow (\wedge^2 \mathfrak{g})[\![\hbar]\!]$ valued in $\mathfrak{g}[\![\hbar]\!]$ such that $r_\hbar(\lambda) = r(\lambda) + \hbar r_1(\lambda) + \cdots$. The gauge transformation can be defined formally in an obvious way. Thus one can form the moduli space $\mathscr{M}(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$. Assume that $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is a classical triangular dynamical r-matrix. From Proposition 2.13, it follows that $T_{[r]}\mathscr{M}(\mathfrak{g}[\![\hbar]\!], \mathfrak{h}) \cong H^2_{[r]}(\mathfrak{g}, \mathfrak{h})[\![\hbar]\!]$. By *a formal neighbourhood of* r in $\mathscr{M}(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$, denoted by $\mathscr{M}_r(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$, we mean the subset in $\mathscr{M}(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$ consisting of the classes of those elements $r + O(\hbar)$. Then $H^2_{[r]}(\mathfrak{g}, \mathfrak{h})[\![\hbar]\!]$ can be considered as a linearization of $\mathscr{M}_r(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$. In general, these two spaces are different. However, when r is non-degenerate, they expect to be isomorphic, which should follow from Moser lemma.

In fact, as we will see in the next theorem, when r is non-degenerate, $H^*_{[r]}(\mathfrak{g}, \mathfrak{h})$ is isomorphic to the relative Lie algebra cohomology.

THEOREM 2.14. *If* $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ *is a non-degenerate dynamical r-matrix, then* $H^*_{[r]}(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to* $H^*(\mathfrak{g}, \mathfrak{h})$*, the relative Lie algebra cohomology of the pair* $(\mathfrak{g}, \mathfrak{h})$.

Proof. Since r is non-degenerate, (M, π) is a symplectic manifold. As it is well known, $\pi^\#\colon \Omega^*(M) \longrightarrow \mathfrak{X}^*(M)$ induces an isomorphism between the de Rham cohomology cochain complex and the Poisson cohomology cochain complex. Now a k-mutivector field $P \in \mathfrak{X}^k(M)$ is in C^k iff (i) P is left G-invariant and right H-invariant; and (ii) $d\lambda^i \lrcorner P = 0$, $\forall i = 1, \ldots, l$. This, however, is equivalent to that (i) $(\pi^\#)^{-1} P$ is both left G-invariant and right H-invariant; and (ii) $\overrightarrow{h_i} \lrcorner (\pi^\#)^{-1} P = 0$, because $\pi^\#(d\lambda^i) = \overrightarrow{h_i}$, $\forall i = 1, \ldots, l$, and π is $G \times H$-invariant. Note that a k-form $\omega \in \Omega^k(M)$ is H-invariant and satisfies $\overrightarrow{h_i} \lrcorner \omega = 0$, $\forall i = 1, \ldots, l$, iff ω is the pull back of a k-form on the quotient space M/H, i.e., $\omega = p^*\omega'$, where $p\colon M \longrightarrow M/H$ is the projection and $\omega' \in \Omega^k(M/H)$. Moreover, ω is left G-invariant iff ω' is left G-invariant since the left G-action on M commutes with the right H-action. In summary, we have proved that the space $(\pi^\#)^{-1}(C^k)$ can be naturally identified with the space of left G-invariant k-forms on $M/H \cong \mathfrak{h}^* \times G/H$. Under such an identification, the differential δ_r goes to the de Rham differential. Hence $H^k_{[r]}(\mathfrak{g}, \mathfrak{h})$ is isomorphic to the invariant de Rham cohomology $H^k(\mathfrak{h}^* \times G/H)^G$. Since G does not act on the first factor $\mathfrak{h}^*$, the latter is isomorphic to $H^k(G/H)^G$, which is in turn isomorphic to the relative Lie algebra cohomology $H^k(\mathfrak{g}, \mathfrak{h})$ [10]. ∎

3. QUANTIZATION AND STAR PRODUCTS

In this section, we investigate the relation between quantizations of a triangular dynamical r-matrix and star products on its associated Poisson manifold (M, π). The main theme is to show that quantizing r is equivalent to finding a certain special type of star products on M. Let us first introduce the precise definition of a quantization.

DEFINITION 3.1. Let $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ be a triangular dynamical r-matrix. A quantization of r is an element $F(\lambda) = 1 + \hbar F_1(\lambda) + O(\hbar^2) \in C^\infty(\mathfrak{h}^*, U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]]$ satisfying

(i) the zero weight condition, $[1 \otimes h + h \otimes 1, F(\lambda)] = 0$, $\forall h \in \mathfrak{h}$;

(ii) the shifted cocycle condition,

$$(\Delta \otimes id)\, F(\lambda)\, F^{12}(\lambda - \tfrac{1}{2}\hbar h^{(3)}) = (id \otimes \Delta)\, F(\lambda)\, F^{23}(\lambda + \tfrac{1}{2}\hbar h^{(1)}); \tag{10}$$

(iii) the normal condition,

$$(\varepsilon \otimes id)\, F(\lambda) = 1; \; (id \otimes \varepsilon)\, F(\lambda) = 1; \text{ and} \tag{11}$$

(iv) the quantization condition, $F_1^{12}(\lambda) - F_1^{21}(\lambda) = r(\lambda)$,

where $\Delta\colon U\mathfrak{g} \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}$ is the standard comultiplication, $\varepsilon\colon U\mathfrak{g} \longrightarrow \mathbb{C}$ is the counit map, and $F^{12}(\lambda - \frac{1}{2}\hbar h^{(3)})$, $F^{23}(\lambda + \frac{1}{2}\hbar h^{(1)})$ are $U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$-valued functions on $\mathfrak{h}^*$ defined by

$$\begin{aligned} F^{12}\left(\lambda - \frac{1}{2}\hbar h^{(3)}\right) &= F(\lambda) \otimes 1 - \frac{\hbar}{2} \sum_i \frac{\partial F}{\partial \lambda^i} \otimes h_i + \frac{1}{2!}\left(-\frac{\hbar}{2}\right)^2 \sum_{i_1 i_2} \frac{\partial^2 F}{\partial \lambda^{i_1} \partial \lambda^{i_2}} \otimes h_{i_1} h_{i_2} \\ &\quad + \cdots + \frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \sum \frac{\partial^k F}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \otimes h_{i_1} \cdots h_{i_k} + \cdots, \end{aligned} \tag{12}$$

and similarly for $F^{23}(\lambda + \frac{1}{2}\hbar h^{(1)})$.

The relation between this definition of quantizations and the well-known quantum dynamical Yang–Baxter equation (QDYBE) is explained by the following proposition, which can be proved by a straightforward verification.

PROPOSITION 3.2. *If $F(\lambda)$ is a quantization of a triangular dynamical r-matrix $r(\lambda)\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, then $R(\lambda) = F^{21}(\lambda)^{-1} F^{12}(\lambda)$ can be written as $R(\lambda) = 1 + \hbar r(\lambda) + O(\hbar^2)$ and satisfies the quantum dynamical Yang–Baxter equation (QDYBE):*

$$\begin{aligned} &R^{12}(\lambda - \tfrac{1}{2}\hbar h^{(3)})\, R^{13}(\lambda + \tfrac{1}{2}\hbar h^{(2)})\, R^{23}(\lambda - \tfrac{1}{2}\hbar h^{(1)}) \\ &\quad = R^{23}(\lambda + \tfrac{1}{2}\hbar h^{(1)})\, R^{13}(\lambda - \tfrac{1}{2}\hbar h^{(2)})\, R^{12}(\lambda + \tfrac{1}{2}\hbar h^{(3)}). \end{aligned} \tag{13}$$

Remark. This is a symmetrized version of QDYBE, which is known [19] to be equivalent to the non-symmetrized QDYBE:

$$R^{12}(\lambda+\hbar h^{(3)})\, R^{13}(\lambda)\, R^{23}(\lambda+\hbar h^{(1)}) = R^{23}(\lambda)\, R^{13}(\lambda+\hbar h^{(2)})\, R^{12}(\lambda).$$

The reason for us to choose the symmetrized QDYBE in this paper is that it is related to the Weyl quantization, while the non-symmetrized QDYBE is related to the normal ordering quantization, as indicated in [41]. Since we will use the Fedosov method later on, the Weyl quantization is obviously of some advantage.

To proceed, we need some preparation on notations. Let $\mathscr{A} = \mathscr{D} \otimes U\mathfrak{g}[\![\hbar]\!]$, where $\mathscr{D}$ is the algebra of smooth differential operators on $\mathfrak{h}^*$. Then $\mathscr{D} \otimes U\mathfrak{g}$ can be naturally identified with the algebra of left G-invariant differential operators on M. Hence $\mathscr{A}$ becomes a Hopf algebroid [41] with base algebra $R = C^\infty(\mathfrak{h}^*)[\![\hbar]\!]$. The comultiplication

$$\Delta\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes_R \mathscr{A} \cong \mathscr{D} \otimes_{C^\infty(\mathfrak{h}^*)} \mathscr{D} \otimes U\mathfrak{g} \otimes U\mathfrak{g}[\![\hbar]\!]$$

is a natural extension of the comultiplications on $\mathscr{D}$ and on $U\mathfrak{g}$,

$$\Delta(D \otimes u) = \Delta D \otimes \Delta u, \quad \forall D \in \mathscr{D}, \quad \text{and} \quad u \in U\mathfrak{g},$$

where ΔD is the bidifferential operator on $\mathfrak{h}^*$ given by $(\Delta D)(f, g) = D(fg)$, $\forall f, g \in C^\infty(\mathfrak{h}^*)$, and $\Delta u \in U\mathfrak{g} \otimes U\mathfrak{g}$ is the usual comultiplication on $U\mathfrak{g}$. Let us fix a basis in $\mathfrak{h}$, say $\{h_1, \ldots, h_l\}$, and let $\{\xi_1, \ldots, \xi_l\}$ be its dual basis, which in turn defines a coordinate system $(\lambda^1, \ldots, \lambda^l)$ on $\mathfrak{h}^*$.

Set

$$\theta = \frac{1}{2} \sum_{i=1}^{l} \left(h_i \otimes \frac{\partial}{\partial \lambda^i} - \frac{\partial}{\partial \lambda^i} \otimes h_i \right) \in \mathscr{A} \otimes \mathscr{A}, \qquad \text{and} \qquad \Theta = \exp \hbar\theta \in \mathscr{A} \otimes \mathscr{A}. \tag{14}$$

Note that θ, and hence Θ, is independent of the choice of a basis in $\mathfrak{h}$.

For each $D \in \mathscr{D} \otimes U\mathfrak{g}$, we denote by $\vec{D}$ its corresponding left G-invariant differential operator on $M = \mathfrak{h}^* \times G$. We also use a similar notation to denote multi-differential operators on M as well. Now let $r(\lambda)\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ be a triangular dynamical r-matrix, and $M = \mathfrak{h}^* \times G$ its associated (regular) Poisson manifold with Poisson tensor $\pi = \sum_i \vec{h_i} \wedge \frac{\partial}{\partial \lambda^i} + \overrightarrow{r(\lambda)}$. It is simple to see that the Poisson brackets on $C^\infty(M)$ can be described as follows:

(i) for any $f, g \in C^\infty(\mathfrak{h}^*)$, $\{f, g\} = 0$;

(ii) for any $f \in C^\infty(\mathfrak{h}^*)$ and $g \in C^\infty(G)$, $\{f, g\} = -\sum_i \left(\frac{\partial f}{\partial \lambda^i}\right) (\vec{h_i}\, g)$;

(iii) for any $f, g \in C^\infty(G)$, $\{f, g\} = \overrightarrow{r(\lambda)}(f, g)$.

This Poisson bracket relation naturally motivates the following theorem, which is indeed the main theorem of this section.

THEOREM 3.3. *Let (M, π) be the Poisson manifold associated to a triangular dynamical r-matrix as in Proposition 2.2. Assume that $*_\hbar$ is a $G \times H$-invariant star product on (M, π) satisfying the properties*

(i) *for any $f, g \in C^\infty(\mathfrak{h}^*)$,*

$$f(\lambda) *_\hbar g(\lambda) = f(\lambda)\, g(\lambda);$$

(ii) *for any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g(x) \in C^\infty(G)$,*

$$f(\lambda) *_\hbar g(x) = \vec{\Theta}(f, g) = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \vec{h}_{i_1} \cdots \vec{h}_{i_k} g,$$

$$g(x) *_\hbar f(\lambda) = \vec{\Theta}(g, f) = \sum_{k=0}^{\infty} \left(\frac{\hbar}{2}\right)^k \frac{1}{k!} \vec{h}_{i_1} \cdots \vec{h}_{i_k} g \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}};$$

(iii) *there is a smooth map $F: \mathfrak{h}^* \to U\mathfrak{g} \otimes U\mathfrak{g}[[\hbar]]$ such that for any $f(x), g(x) \in C^\infty(G)$,*

$$f *_\hbar g = \overrightarrow{F(\lambda)}(f, g). \tag{15}$$

Then $F(\lambda)$ is a quantization of the dynamical r-matrix $r(\lambda)$. Conversely, any quantization of $r(\lambda)$ corresponds to a $G \times H$-invariant star product on M satisfying the properties (i)–(iii).

A $G \times H$-invariant star product on M with properties (i)–(iii) is called a *compatible* star product. In other words, Theorem 3.3 can be stated as that a quantization of $r(\lambda)$ is equivalent to a compatible star-product on M.

To prove Theorem 3.3, we need several lemmas.

LEMMA 3.4. *Θ satisfies the equation*

$$[(\Delta \otimes id)\, \Theta]\, \Theta^{12} = [(id \otimes \Delta)\, \Theta]\, \Theta^{23} \qquad in \quad \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}. \tag{16}$$

Proof. Note that both sides of Eq. (16) normally are elements in $\mathscr{A} \otimes_R \mathscr{A} \otimes_R \mathscr{A}$. In our situation, however, they indeed can be considered as elements in $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$.

Now

$$\begin{aligned}
[(\Delta\otimes id)\,\Theta]\,\Theta^{12} &= [(\Delta\otimes id)\exp\hbar\theta]\exp\hbar\theta^{12}\\
&= \exp\hbar[(\Delta\otimes id)\,\theta+\theta^{12}]\\
&= \exp\frac{1}{2}\hbar\sum_{i=1}^{k}\Big(h_i\otimes 1\otimes\frac{\partial}{\partial\lambda^i}+1\otimes h_i\otimes\frac{\partial}{\partial\lambda^i}+h_i\otimes\frac{\partial}{\partial\lambda^i}\otimes 1\\
&\quad -\frac{\partial}{\partial\lambda^i}\otimes 1\otimes h_i-1\otimes\frac{\partial}{\partial\lambda^i}\otimes h_i-\frac{\partial}{\partial\lambda^i}\otimes h_i\otimes 1\Big).
\end{aligned}$$

Here in the second equality we used the fact that $(\Delta\otimes id)\,\theta$ and θ^{12} commute in $\mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A}$.

A similar computation leads to the same expression for $[(id\otimes\Delta)\,\Theta]\,\Theta^{23}$. This proves Eq. (16). ▮

LEMMA 3.5. *$\forall D_1, D_2, D_3\in\mathscr{A}$, and $\forall f_1(\lambda)\in C^\infty(\mathfrak{h}^*)$, $f_2(x)\in C^\infty(G)$, and $g(\lambda,x)\in C^\infty(\mathfrak{h}^*\times G)$,*

$$\begin{aligned}
&\overrightarrow{[(\Delta\otimes id)\,F(\lambda)(D_1\otimes D_2\otimes D_3)]}(f_1(\lambda), f_2(x), g(\lambda,x))\\
&\quad=\overrightarrow{[F^{23}(\lambda)(D_1\otimes D_2\otimes D_3)]}(f_1(\lambda), f_2(x), g(\lambda,x)).
\end{aligned}$$

Proof. Write $F(\lambda)=\sum a_{\alpha\beta}(\lambda)\,u_\alpha\otimes u_\beta$, with $u_\alpha, u_\beta\in U\mathfrak{g}$ and $a_{\alpha\beta}(\lambda)\in C^\infty(\mathfrak{h}^*)[\![\hbar]\!]$. Then

$$((\Delta\otimes id)\,F(\lambda))(D_1\otimes D_2\otimes D_3)=\sum a_{\alpha\beta}(\lambda)\,\Delta u_\alpha(D_1\otimes D_2)\otimes u_\beta D_3.$$

Hence

$$\begin{aligned}
&\overrightarrow{[(\Delta\otimes id)\,F(\lambda)(D_1\otimes D_2\otimes D_3)]}(f_1(\lambda), f_2(x), g(\lambda,x))\\
&\quad=\sum a_{\alpha\beta}(\lambda)\,\overrightarrow{\Delta u_\alpha(D_1\otimes D_2)}(f_1(\lambda), f_2(x))(\overrightarrow{u_\beta D_3}\,g)(\lambda,x)\\
&\quad=\sum a_{\alpha\beta}(\lambda)\,\overrightarrow{u_\alpha}[(\overrightarrow{D_1}\,f_1)(\lambda)(\overrightarrow{D_2}\,f_2)(x)](\overrightarrow{u_\beta D_3}\,g)(\lambda,x)\\
&\quad=\sum a_{\alpha\beta}(\lambda)(\overrightarrow{D_1}\,f_1)(\lambda)((\overrightarrow{u_\alpha D_2})\,f_2)(x)(\overrightarrow{u_\beta D_3}\,g)(\lambda,x)\\
&\quad=\overrightarrow{D_1\otimes F(\lambda)(D_2\otimes D_3)}(f_1(\lambda), f_2(x), g(\lambda,x))\\
&\quad=\overrightarrow{F^{23}(\lambda)(D_1\otimes D_2\otimes D_3)}(f_1(\lambda), f_2(x), g(\lambda,x)).
\end{aligned}$$ ▮

COROLLARY 3.6. *$\forall f_1(\lambda)\in C^\infty(\mathfrak{h}^*)$, $f_2(x)\in C^\infty(G)$ and $g(\lambda,x)\in C^\infty(\mathfrak{h}^*\times G)$,*

$$\overrightarrow{F(\lambda)\,\Theta}(f_1(\lambda)*_\hbar f_2(x), g(\lambda,x))=\vec{\Theta}(f_1(\lambda), \overrightarrow{F(\lambda)\,\Theta}(f_2(x), g(\lambda,x))).$$

Proof.

$$
\begin{aligned}
&\overrightarrow{F(\lambda)\,\Theta}(f_1(\lambda) *_\hbar f_2(x), g(\lambda, x)) \\
&\quad = \overrightarrow{F(\lambda)\,\Theta}(\vec{\Theta}(f_1(\lambda), f_2(x)), g(\lambda, x)) \\
&\quad = \overrightarrow{(\Delta \otimes id)(F(\lambda)\,\Theta)\,\Theta^{12}}(f_1(\lambda), f_2(x), g(\lambda, x)) \\
&\quad = \overrightarrow{(\Delta \otimes id)\,F(\lambda)(\Delta \otimes id)\,\Theta\Theta^{12}}(f_1(\lambda), f_2(x), g(\lambda, x)) \quad \text{(by Lemma 3.4)} \\
&\quad = \overrightarrow{(\Delta \otimes id)\,F(\lambda)(id \otimes \Delta)\,\Theta\Theta^{23}}(f_1(\lambda), f_2(x), g(\lambda, x)) \quad \text{(by Lemma 3.5)} \\
&\quad = \overrightarrow{F^{23}(\lambda)(id \otimes \Delta)\,\Theta\Theta^{23}}(f_1(\lambda), f_2(x), g(\lambda, x)).
\end{aligned}
$$

Let us write $\Theta = \sum D_\alpha \otimes D_\beta$. Then $(id \otimes \Delta)\,\Theta = \sum D_\alpha \otimes \Delta D_\beta$ and

$$
\begin{aligned}
&\overrightarrow{F^{23}(\lambda)(id \otimes \Delta)\,\Theta\Theta^{23}}(f_1(\lambda), f_2(x), g(\lambda, x)) \\
&\quad = \sum \overrightarrow{[D_\alpha \otimes F(\lambda)\,\Delta D_\beta \Theta]}(f_1(\lambda), f_2(x), g(\lambda, x)) \\
&\quad = \sum (\overrightarrow{D_\alpha} f_1)(\lambda)\,\overrightarrow{F(\lambda)\,\Delta D_\beta \Theta}(f_2(x), g(\lambda, x)).
\end{aligned}
$$

Using the expansion $\Theta = \sum_{k=0}^{\infty} (\frac{\hbar}{2})^k \frac{1}{k!} (\sum_{i=1}^{l} (h_i \otimes \frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_i} \otimes h_i))^k$, one obtains that

$$
\begin{aligned}
&\overrightarrow{F(\lambda)\,\Theta}(f_1(\lambda) *_\hbar f_2(x), g(\lambda, x)) \\
&\quad = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \frac{\partial^k f_1(\lambda)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{F(\lambda)\,\Delta(h_{i_1} \cdots h_{i_k})\,\Theta}(f_2(x), g(\lambda, x)) \\
&\quad = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \frac{\partial^k f_1(\lambda)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{\Delta(h_{i_1} \cdots h_{i_k})\,F(\lambda)\,\Theta}(f_2(x), g(\lambda, x)) \\
&\quad = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \frac{\partial^k f_1(\lambda)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} [\overrightarrow{F(\lambda)\,\Theta}(f_2(x), g(\lambda, x))] \\
&\quad = \vec{\Theta}(f_1(\lambda), \overrightarrow{F(\lambda)\,\Theta}(f_2(x), g(\lambda, x))).
\end{aligned}
$$

Here the second equality follows from the fact that $F(\lambda)$ is of zero weight; i.e., $F(\lambda)(\Delta h) = (\Delta h)\,F(\lambda)$, $\forall h \in \mathfrak{h}$. This concludes the proof. ∎

Proposition 3.7. *Under the same hypothesis as in Theorem* 3.3, *we have*

(1) *for any* $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ *and* $g(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$,

$$f(\lambda) *_\hbar g(\lambda, x) = \vec{\Theta}(f, g) = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \vec{h}_{i_1} \cdots \vec{h}_{i_k} g, \quad (17)$$

$$g(\lambda, x) *_\hbar f(\lambda) = \vec{\Theta}(g, f) = \sum_{k=0}^{\infty} \left(\frac{\hbar}{2}\right)^k \frac{1}{k!} \vec{h}_{i_1} \cdots \vec{h}_{i_k} g \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}; \quad (18)$$

(2) *for any* $f(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$ *and* $g(x) \in C^\infty(G)$,

$$\begin{aligned} f(\lambda, x) *_\hbar g(x) &= (\overrightarrow{F(\lambda)\,\Theta})(f, g) \\ &= \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} \overrightarrow{F(\lambda)} \left(\frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}, \vec{h}_{i_1} \cdots \vec{h}_{i_k} g\right), \quad (19) \\ g(x) *_\hbar f(\lambda, x) &= (\overrightarrow{F(\lambda)\,\Theta})(g, f) \\ &= \sum_{k=0}^{\infty} \left(\frac{\hbar}{2}\right)^k \frac{1}{k!} \overrightarrow{F(\lambda)} \left(\vec{h}_{i_1} \cdots \vec{h}_{i_k} g, \frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}\right). \quad (20) \end{aligned}$$

Proof. We will prove Eq. (17) first. For that, it suffices to show this for $g(\lambda, x) = g_1(\lambda) *_\hbar g_2(x)$, $\forall g_1(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g_2(x) \in C^\infty(G)$, since, at each point, the C^∞-jet space of $C^\infty(\mathfrak{h}^* \times G)[[\hbar]]$ is spanned by the C^∞-jets of this type of functions. Now

$$\begin{aligned} f(\lambda) *_\hbar g(\lambda, x) &= f(\lambda) *_\hbar (g_1(\lambda) *_\hbar g_2(x)) \\ &= (f(\lambda) *_\hbar g_1(\lambda)) *_\hbar g_2(x) \\ &= (f(\lambda)\, g_1(\lambda)) *_\hbar g_2(x) \\ &= \vec{\Theta}(f(\lambda)\, g_1(\lambda), g_2(x)) \\ &= \vec{\Theta}(\vec{\Theta}(f(\lambda), g_1(\lambda)), g_2(x)) \\ &= \overrightarrow{[(\Delta \otimes id)\, \Theta]\, \Theta^{12}}(f(\lambda), g_1(\lambda), g_2(x)) \quad \text{(by Lemma 3.4)} \\ &= \overrightarrow{[(id \otimes \Delta)\, \Theta]\, \Theta^{23}}(f(\lambda), g_1(\lambda), g_2(x)) \\ &= \vec{\Theta}(f(\lambda), \vec{\Theta}(g_1(\lambda), g_2(x))) \\ &= \vec{\Theta}(f(\lambda), g_1(\lambda) *_\hbar g_2(x)) \\ &= \vec{\Theta}(f(\lambda), g(\lambda, x)). \end{aligned}$$

Equation (18) can be proved similarly.

To prove Eq. (19), similarly we may assume that $f(\lambda, x) = f_1(\lambda) *_\hbar f_2(x)$, for $f_1(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $f_2(x) \in C^\infty(G)$. Then

$$\begin{aligned}
f(\lambda, x) *_\hbar g(x) &= (f_1(\lambda) *_\hbar f_2(x)) *_\hbar g(x) \\
&= f_1(\lambda) *_\hbar (f_2(x) *_\hbar g(x)) \quad \text{(using Eq. (17))} \\
&= \vec{\Theta}(f_1(\lambda), f_2(x) *_\hbar g(x)) \\
&= \vec{\Theta}(f_1(\lambda), \overrightarrow{F(\lambda)}(f_2(x), g(x))) \\
&= \vec{\Theta}(f_1(\lambda), \overrightarrow{F(\lambda)\,\Theta}(f_2(x), g(x))) \quad \text{(by Corollary 3.6)} \\
&= \overrightarrow{F(\lambda)\,\Theta}(f_1(\lambda) *_\hbar f_2(x), g(x)) \\
&= \overrightarrow{F(\lambda)\,\Theta}(f(\lambda, x), g(x)).
\end{aligned}$$

Equation (20) can also be proved similarly. ∎

We are now ready to prove the main theorem of the section.

THEOREM 3.8. *Under the same hypothesis as in Theorem* 3.3, $\overrightarrow{F(\lambda)\,\Theta}$ *is the formal bidifferential operator defining the star product* $*_\hbar$ *; i.e., for any* $f(\lambda, x), g(\lambda, x) \in C^\infty(\mathfrak{h}^* \times G)$,

$$f(\lambda, x) *_\hbar g(\lambda, x) = \overrightarrow{F(\lambda)\,\Theta}(f, g).$$

Proof. We may assume that $f(\lambda, x) = f_1(\lambda) *_\hbar f_2(x)$, for $f_1(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $f_2(x) \in C^\infty(G)$. Then

$$\begin{aligned}
&f(\lambda, x) *_\hbar g(\lambda, x) \\
&\quad = (f_1(\lambda) *_\hbar f_2(x)) *_\hbar g(\lambda, x) \\
&\quad = f_1(\lambda) *_\hbar (f_2(x) *_\hbar g(\lambda, x)) \quad \text{(by Proposition 3.7)} \\
&\quad = \vec{\Theta}(f_1(\lambda), \overrightarrow{F(\lambda)\,\Theta}(f_2(x), g(\lambda, x))) \quad \text{(by Corollary 3.6)} \\
&\quad = \overrightarrow{F(\lambda)\,\Theta}(f_1(\lambda) *_\hbar f_2(x), g(\lambda, x)) \\
&\quad = \overrightarrow{F(\lambda)\,\Theta}(f(\lambda, x), g(\lambda, x)).
\end{aligned}$$

This concludes the proof. ∎

Finally, before proving Theorem 3.3, we need the following result, which connects the shifted cocycle condition with the associativity of a star-product.

PROPOSITION 3.9. *Under the same hypothesis as in Theorem* 3.3, $\forall f_1(x), f_2(x), f_3(x) \in C^\infty(G)$,

(i) $\overrightarrow{(\Delta \otimes id)\, F(\lambda)\, F^{12}(\lambda - \frac{1}{2}\hbar h^{(3)})}(f_1(x), f_2(x), f_3(x))$
$= (f_1(x) *_\hbar f_2(x)) *_\hbar f_3(x)$;

(ii) $\overrightarrow{(id \otimes \Delta)\, F(\lambda)\, F^{23}(\lambda + \frac{1}{2}\hbar h^{(1)})}(f_1(x), f_2(x), f_3(x))$
$= f_1(x) *_\hbar (f_2(x) *_\hbar f_3(x))$.

Proof. From Eq. (12), it follows that

$$(\Delta \otimes id)\, F(\lambda)\, F^{12}\left(\lambda - \frac{1}{2}\hbar h^{(3)}\right)$$
$$= \sum \frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k [(\Delta \otimes id)\, F(\lambda)]\left(\frac{\partial^k F}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} \otimes h_{i_1} \cdots h_{i_k}\right).$$

Hence

$$\overrightarrow{(\Delta \otimes id)\, F(\lambda)\, F^{12}\left(\lambda - \frac{1}{2}\hbar h^{(3)}\right)}(f_1(x), f_2(x), f_3(x))$$
$$= \sum \frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \overrightarrow{F(\lambda)}\left[\overrightarrow{\frac{\partial^k F(\lambda)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}}(f_1(x), f_2(x)), (\overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} f_3)(x)\right]$$
$$= \sum \frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \overrightarrow{F(\lambda)}\left[\frac{\partial^k (f_1 *_\hbar f_2)}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}, (\overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} f_3)(x)\right] \quad \text{(using Eq. (19))}$$
$$= (f_1(x) *_\hbar f_2(x)) *_\hbar f_3(x).$$

The second identity can be proved similarly. ∎

Proof of Theorem 3.3. Since $*_\hbar$ is invariant under the right H-action, $\overrightarrow{F(\lambda)}$ is right H-invariant. This implies that $F(\lambda)$ is Ad_H-invariant, and therefore is of zero weight. The normal condition follows from the fact that 1 is the unit of the star algebra, i.e., $1 *_\hbar f = f *_\hbar 1 = f$. And the shifted cocycle condition follows from the associativity of the star product together with Proposition 3.9. Finally, let us write $F(\lambda) = 1 + \hbar F_1(\lambda) + O(\hbar^2)$. Since $*_\hbar$ is a star product quantizing π, it follows that $\overrightarrow{(F_1(\lambda) - F_1^{21}(\lambda))}(f, g) = \{f, g\} = \overrightarrow{r(\lambda)}(f, g)$, $\forall f, g \in C^\infty(G)$. Hence it follows that $F_1(\lambda) - F_1^{21}(\lambda) = r(\lambda)$.

Conversely, if $F(\lambda)$ is a quantization of $r(\lambda)$, according to Theorem 7.5 in [41], $\overrightarrow{F(\lambda)\, \Theta}$ is indeed an associator and therefore defines a star product on $M = \mathfrak{h}^* \times G$. It is simple to see that this star product is a quantization of π and satisfies Properties (i)–(iii) in Theorem 3.3. ∎

We end this section by the following

Remark. Bordemann *et al.* found an explicit formula for a star-product on $\mathbb{R}\times SU(2)$ [9] using a quantum analogue of Marsden–Weinstein reduction. It would be interesting to investigate if this is a compatible star-product.

4. SYMPLECTIC CONNECTIONS

From now on, we will confine ourselves mostly to non-degenerate triangular dynamical r-matrices. In this case, the corresponding Poisson manifolds are in fact symplectic, and therefore can be quantized by Fedosov method [20, 21]. As is well known, Fedosov quantization relies on the choice of a symplectic connection. Serving as a preliminary, this section is devoted to the discussion on symplectic connections. We will start with some general notations and constructions.

Let ∇ be a torsion-free symplectic connection on a symplectic manifold (M, ω). Define the symplectic curvature [20] by

$$R(X, Y, Z, W) = \omega(X, R(Z, W)\, Y), \quad \forall X, Y, Z, W \in \mathfrak{X}(M), \tag{21}$$

where $R(Z, W)\, Y = \nabla_Z \nabla_W Y - \nabla_W \nabla_Z Y - \nabla_{[Z, W]} Y$ is the usual curvature tensor of ∇.

PROPOSITION 4.1. (i) *$R(X, Y, Z, W)$ is skew symmetric with respect to Z and W and symmetric with respect to X and Y; i.e.,*

$$R(X, Y, Z, W) = -R(X, Y, W, Z), \quad R(X, Y, Z, W) = R(Y, X, Z, W). \tag{22}$$

(ii) *The following Bianchi's identity holds:*

$$R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0. \tag{23}$$

Proof. It is clear by definition that $R(X, Y, Z, W)$ is skew symmetric with respect to Z and W. Now since ∇ is a symplectic connection,

$$\begin{aligned}\omega(X, \nabla_Z \nabla_W Y) &= Z(\omega(X, \nabla_W Y)) - \omega(\nabla_Z X, \nabla_W Y)\\ &= Z(W\omega(X, Y)) - Z\omega(\nabla_W X, Y)\\ &\quad - W\omega(\nabla_Z X, Y) + \omega(\nabla_W \nabla_Z X, Y).\end{aligned}$$

Similarly,

$$\omega(X, \nabla_W \nabla_Z Y) = W(Z\omega(X, Y)) - W\omega(\nabla_Z X, Y) \\ - Z\omega(\nabla_W X, Y) + \omega(\nabla_Z \nabla_W X, Y).$$

Hence

$$\omega(X, \nabla_{[Z,W]} Y) = [Z, W](\omega(X, Y)) - \omega(\nabla_{[Z,W]} X, Y).$$

Thus

$$\begin{aligned} R(X, Y, Z, W) &= \omega(X, R(Z, W)\, Y) \\ &= \omega(X, \nabla_Z \nabla_W Y - \nabla_W \nabla_Z Y - \nabla_{[Z,W]} Y) \\ &= -\omega(\nabla_Z \nabla_W X - \nabla_W \nabla_Z X - \nabla_{[Z,W]} X, Y) \\ &= \omega(Y, R(Z, W)\, X) \\ &= R(Y, X, Z, W). \end{aligned}$$

This concludes the proof of (i). Finally, (ii) follows from the usual Bianchi identity for a torsion-free connection. ▮

Symplectic connections always exist on any symplectic manifold. In fact, there is a standard procedure to construct a torsion-free symplectic connection from an arbitrary torsion-free linear connection [20, 28]. Since such a construction is essential to our discussion here, let us recall it briefly below.

Assume that ∇^0 is a torsion-free linear connection on a symplectic manifold M. Then any linear connection on M can be written as

$$\nabla_X Y = \nabla^0_X Y + S(X, Y), \forall X, Y \in \mathfrak{X}(M), \tag{24}$$

where S is a (2, 1)-tensor on M. Clearly, ∇ is torsion-free iff S is symmetric; i.e., $S(X, Y) = S(Y, X)$, $\forall X, Y \in \mathfrak{X}(M)$. And ∇ is symplectic iff $\nabla_X \omega = 0$. The latter is equivalent to

$$\omega(S(X, Y), Z) - \omega(S(X, Z), Y) = (\nabla^0_X \omega)(Y, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M). \tag{25}$$

LEMMA 4.2. *If ∇^0 is a torsion-free linear connection, and S is a (2, 1)-tensor defined by the equation:*

$$\omega(S(X, Y), Z) = \tfrac{1}{3}\,[(\nabla^0_X \omega)(Y, Z) + (\nabla^0_Y \omega)(X, Z)], \tag{26}$$

then $\nabla_X Y = \nabla^0_X Y + S(X, Y)$ is a torsion-free symplectic connection. Moreover, if M is a symplectic G-space and ∇^0 is a G-invariant connection, then ∇ is also G-invariant.

Proof. Clearly, $S(X, Y)$, defined in this way, is symmetric with respect to X and Y. Now

$$\begin{aligned}
&\omega(S(X, Y), Z) - \omega(S(X, Z), Y)\\
&\quad = \tfrac{1}{3}[(\nabla^0_X\omega)(Y, Z) + (\nabla^0_Y\omega)(X, Z)] - \tfrac{1}{3}[(\nabla^0_X\omega)(Z, Y) + (\nabla^0_Z\omega)(X, Y)]\\
&\quad = \frac{1}{3}[(\nabla^0_X\omega)(Y, Z) + (\nabla^0_Y\omega)(X, Z) + (\nabla^0_X\omega)(Y, Z) + (\nabla^0_Z\omega)(Y, X)]\\
&\quad = (\nabla^0_X\omega)(Y, Z),
\end{aligned}$$

where the last step follows from the identity

$$(\nabla^0_X\omega)(Y, Z) + (\nabla^0_Y\omega)(Z, X) + (\nabla^0_Z\omega)(X, Y) = 0.$$

This means that ∇ is a torsion-free symplectic connection. The second statement is obvious according to Eq. (26). ▮

Now we retain to the case that $M = \mathfrak{h}^* \times G$, the symplectic manifold associated with a non-degenerate triangular dynamical r-matrix r, which is our main subject of interest in the present paper. The main result is the following

THEOREM 4.3. *Assume that $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ is a non-degenerate triangular dynamical r-matrix. Let $M = \mathfrak{h}^* \times G$ be equipped with the symplectic structure as in Corollary 2.5. Then M admits a $G \times H$-invariant torsion-free symplectic connection ∇ satisfying the property that $\nabla_X \vec{h} = 0$, $\forall X \in \mathfrak{X}(M)$, $h \in \mathfrak{h}$.*

We need a couple of lemmas first.

LEMMA 4.4. *Assume that $\mathfrak{g}$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$; i.e., $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Then, the equations*

$$\begin{aligned}
&\nabla^0_X \frac{\partial}{\partial \lambda^i} = 0, \qquad \nabla^0_X \vec{h} = 0, \qquad \nabla^0_X \vec{e} = 0;\\
&\nabla^0_{\vec{h}} \frac{\partial}{\partial \lambda^i} = 0, \qquad \nabla^0_{\vec{h}} \vec{h_1} = 0, \qquad \nabla^0_{\vec{h}} \vec{e} = \overrightarrow{[h, e]};\\
&\nabla^0_{\vec{e}} \frac{\partial}{\partial \lambda^i} = 0, \qquad \nabla^0_{\vec{e}} \vec{h} = 0, \qquad \nabla^0_{\vec{e_1}} \vec{e_2} = \frac{1}{2}\overrightarrow{[e_1, e_2]},
\end{aligned} \tag{27}$$

where $X \in \mathfrak{X}(\mathfrak{h}^*)$, $h, h_1 \in \mathfrak{h}$, and $e, e_1, e_2 \in \mathfrak{m}$, define a biinvariant torsion-free linear connection ∇^0 on M.

Proof. This follows from a straightforward verification. ∎

LEMMA 4.5. *Given a Lie algebra $\mathfrak{g}$, if there exists a non-degenerate triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, then $\mathfrak{g}$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ so that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.*

Proof. Fixing any $\lambda \in \mathfrak{h}^*$, we take $\mathfrak{m} = r(\lambda)^{\#} \mathfrak{h}^{\perp}$. Since $r(\lambda)$ is non-degenerate, by definition, we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. On the other hand, it is clear that $\dim \mathfrak{m} \leqslant \dim \mathfrak{h}^{\perp} = \dim \mathfrak{g} - \dim \mathfrak{h}$. Hence, $\dim \mathfrak{h} + \dim \mathfrak{m} \leqslant \dim \mathfrak{g}$. Therefore $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ must be a direct sum. For any $h \in \mathfrak{h}$ and $\xi \in \mathfrak{g}^*$, since $r(\lambda)$ is of zero weight, we have $[h, r(\lambda)^{\#} \xi] = r(\lambda)^{\#} (ad_h^* \xi)$. Since $ad_h^* \xi \in \mathfrak{h}^{\perp}$ for any $\xi \in \mathfrak{g}^*$, it follows that $\mathfrak{m} = r(\lambda)^{\#} \mathfrak{h}^{\perp}$ is stable under the adjoint action of $\mathfrak{h}$. ∎

Remark. Note that, in our proof above, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ depends on the choice of a particular point $\lambda \in \mathfrak{h}^*$. It is not clear if $\mathfrak{m} = r(\lambda)^{\#} \mathfrak{h}^{\perp}$ is independent of λ.

Proof of Theorem 4.3. According to Lemma 4.5, we may find a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Let ∇^0 be the G-biinvariant torsion-free connection on M as in Lemma 4.4. According to Lemma 4.2, one can construct a torsion-free symplectic connection ∇ on M. Since the symplectic structure is $G \times H$-invariant, the resulting symplectic connection ∇ is $G \times H$-invariant. It remains to show that ∇ still satisfies the condition that $\nabla_X \vec{h} = 0$, $\forall X \in \mathfrak{X}(M)$ and $h \in \mathfrak{h}$. The latter is equivalent to that $S(\vec{h}, X) = 0$. To show this identity, first note that $\forall X \in \mathfrak{X}(M)$, $\nabla^0_{\vec{h}} X = L_{\vec{h}} X$, since ∇^0 is torsion-free and $\nabla^0_X \vec{h} = 0$. Hence $\nabla^0_{\vec{h}} \omega = L_{\vec{h}} \omega$. However, $L_{\vec{h}} \omega = 0$ since ω is invariant under the right H-action. Thus, we have $\nabla^0_{\vec{h}} \omega = 0$. According to Eq. (26), $\forall Y \in \mathfrak{X}(M)$, $\omega(S(\vec{h}, X), Y) = \frac{1}{3} [(\nabla^0_{\vec{h}} \omega)(X, Y) + (\nabla^0_X \omega)(\vec{h}, Y)] = \frac{1}{3} (\nabla^0_X \omega)(\vec{h}, Y)$. This implies that $\omega^b(S(\vec{h}, X)) = \frac{1}{3} (\vec{h} \lrcorner \nabla^0_X \omega) = \frac{1}{3} \nabla^0_X (\vec{h} \lrcorner \omega)$ since $\nabla^0_X \vec{h} = 0$. Finally, for any i, $\vec{h_i} \lrcorner \omega = d\lambda^i$ and from the table in Lemma 4.4, it is easy to check that $\nabla^0_X (d\lambda^i) = 0$, $\forall i = 1, \ldots, l$. It thus follows that $S(\vec{h_i}, X) = 0$, $\forall i = 1, \ldots, l$. This concludes the proof. ∎

In the case that $r(\lambda) \in \wedge^2 \mathfrak{m}$, the symplectic connection can be described more explicitly.

PROPOSITION 4.6. *Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition, $\{h_1, \ldots, h_l\}$ is a basis of $\mathfrak{h}$, and $\{e_1, \ldots, e_m\}$ is a basis of $\mathfrak{m}$. Suppose that $r(\lambda) = \sum_{ij} r^{ij}(\lambda)\, e_i \wedge e_j$ is a non-degenerate triangular dynamical r-matrix.*

Then the symplectic connection on M obtained from ∇^0, using the standard construction as in Lemma 4.4, has the form

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial\lambda^i}} \frac{\partial}{\partial\lambda^j}=0, && \nabla_{\frac{\partial}{\partial\lambda^i}} \vec{h_j}=0, && \nabla_{\frac{\partial}{\partial\lambda^i}} \vec{e_j}=\sum_k d_{ij}^k(\lambda)\, \vec{e_k};\\
&\nabla_{\vec{h_i}} \frac{\partial}{\partial\lambda^j}=0, && \nabla_{\vec{h_i}} \vec{h_j}=0, && \nabla_{\vec{h_i}} \vec{e_j}=\overrightarrow{[h_i, e_j]};\\
&\nabla_{\vec{e_i}} \frac{\partial}{\partial\lambda^j}=\sum_k d_{ij}^k(\lambda)\, \vec{e_k}, && \nabla_{\vec{e_i}} \vec{h_j}=0, && \nabla_{\vec{e_i}} \vec{e_j}=\tfrac{1}{2}\,\overrightarrow{[e_i, e_j]}+\sum_k f_{ij}^k(\lambda)\, \vec{e_k},
\end{aligned}
\tag{28}
$$

where $d_{ij}^k(\lambda)$ and $f_{ij}^k(\lambda)$ are smooth functions on $\mathfrak{h}^*$.

Proof. The proof is essentially a straightforward computation. We omit it here. ▮

COROLLARY 4.7. *Under the same hypothesis as in Proposition 4.6, if $\{h_*^1, ..., h_*^l, e_*^1, ..., e_*^m\}$ denotes the dual basis of $\{h_1, ..., h_l, e_1, ..., e_*^m\}$, then*

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial\lambda^i}} d\lambda^j=0, && \nabla_{\frac{\partial}{\partial\lambda^i}} \overrightarrow{h_*^j}=0, && \nabla_{\frac{\partial}{\partial\lambda^i}} \overrightarrow{e_*^j}=\sum_k d_{ik}^j(\lambda)\, \overrightarrow{e_*^k};\\
&\nabla_{\vec{h_i}} d\lambda^j=0, && \nabla_{\vec{h_i}} \overrightarrow{h_*^j}=0, && \nabla_{\vec{h_i}} \overrightarrow{e_*^j}=\overrightarrow{ad_{h_i}^* e_*^j};\\
&\nabla_{\vec{e_i}} d\lambda^j=0, && \nabla_{\vec{e_i}} \overrightarrow{h_*^j}=-\tfrac{1}{2}\sum_k a_{ik}^j \overrightarrow{e_*^k}, && \nabla_{\vec{e_i}} \overrightarrow{e_*^j}=-\sum_k d_{ik}^j(\lambda)\, d\lambda^k\\
&&&&&\quad -\left(\tfrac{1}{2}\, a_{ik}^j+f_{ik}^j(\lambda)\right) \overrightarrow{e_*^k},
\end{aligned}
\tag{29}
$$

where the coadjoint action is defined by $\langle ad_u^*\xi, v\rangle=-\langle\xi, [u, v]]\rangle$, $\forall u, v\in\mathfrak{g}$ and $\xi\in\mathfrak{g}^*$, and the constants a_{ij}^k are defined by the equation $[e_i, e_j]=\sum_k a_{ij}^k h_k (mod\, \mathfrak{m})$.

We end this section by generalizing Theorem 4.3 to the splittable triangular dynamical r-matrix case. According to Corollary 2.9, one may reduce a splittable triangular dynamical r-matrix to a non-degenerate one by considering the Lie subalgebra $\mathfrak{g}_1\subset\mathfrak{g}$. Thus immediately we obtain the following

COROLLARY 4.8. *Assume that $r\colon \mathfrak{h}^*\to\bigwedge^2\mathfrak{g}$ is a splittable triangular dynamical r-matrix. Let $M=\mathfrak{h}^*\times G$ be its associated Poisson manifold as in Proposition 2.2, which admits a (regular) symplectic foliation. Then there* *exists a $G\times H$-invariant torsion-free leafwise Poisson connection ∇ satisfying $\nabla_X\vec{h}=0$, for any $h\in\mathfrak{h}$ and any vector field $X\in\mathfrak{X}(M)$ tangent to the symplectic foliation.*

However, when a triangular dynamical r-matrix r is not splittable, such a Poisson connection may not exist. We give a counterexample below.

EXAMPLE 4.9. Consider a two dimensional Lie algebra $\mathfrak{g}$ with basis $\{h, e\}$ satisfying the bracket relation $[h, e]=ah$, where a is a fixed constant. Let $\mathfrak{h}=\mathbb{R}h$ and $r(\lambda)=f(\lambda)\,h\wedge e$, where $f(\lambda)$ is a smooth function. It is simple to see that $r(\lambda)$ is a triangular dynamical r-matrix of rank zero, and it is not splittable unless $a=0$. Nevertheless, $r(\lambda)$ defines a regular rank 2 Poisson structure on the three dimensional space $M=\mathbb{R}\times G$ with the Poisson tensor $\pi=\vec{h}\wedge\frac{d}{d\lambda}+f(\lambda)\,\vec{h}\wedge\vec{e}$, where G is a 2-dimensional Lie group integrating the Lie algebra $\mathfrak{g}$. It is simple to see that the symplectic foliation of M is spanned by the vector fields $\vec{h}$ and $\frac{d}{d\lambda}+f(\lambda)\,\vec{e}$. Let us denote $X=\frac{d}{d\lambda}+f(\lambda)\,\vec{e}$. Then, we have $[\vec{h}, X]=af(\lambda)\,\vec{h}$. Now suppose that ∇ is a $G\times H$-invariant torsion-free leafwise Poisson connection on M satisfying the condition that $\nabla_{\vec{h}}\vec{h}=0$ and $\nabla_X\vec{h}=0$. Since ∇ is torsion-free, it follows that $\nabla_{\vec{h}}X=[\vec{h}, X]=af(\lambda)\,\vec{h}$. Assume that $\nabla_X X=b(\lambda, x)\,\vec{h}+c(\lambda, x)\,X$, where $b(\lambda, x)$ and $c(\lambda, x)$ are smooth functions on M. Then, $\nabla_X\pi=\nabla_X(\vec{h}\wedge X)=\vec{h}\wedge\nabla_X X=c(\lambda, x)\,\vec{h}\wedge X$. Since ∇ is a Poisson connection, it follows that $c(\lambda, x)=0$. Finally, we still need to check that ∇ is $G\times H$-invariant. It is clear that ∇ is G-invariant iff the function $b(\lambda, x)$ is independent of $x\in G$ (which will be denoted by $b(\lambda)$). For it to be invariant under the right H-action, one needs the following condition:

$$\nabla_{[\vec{h}, X]}X+\nabla_X[\vec{h}, X]=[\vec{h}, \nabla_X X]=[\vec{h}, b(\lambda)\,\vec{h}]=0.$$

It thus follows that $\nabla_{(af(\lambda)\,\vec{h})}X+\nabla_X(af(\lambda)\,\vec{h})=0$, which implies that $f^2(\lambda)\,a^2\vec{h}+a(-\frac{df}{d\lambda})\,\vec{h}=0$. Therefore, we arrive at the following equation (under the assumption that $a\neq 0$):

$$\frac{df}{d\lambda}=af^2(\lambda). \tag{30}$$

In conclusion, we have proved that such a connection does not exist unless $f(\lambda)$ is a solution of the above equation. It would be interesting to find out what is the geometric meaning of this equation.

Remark. Our quantization method does not work for this particular example. It is thus very natural to ask whether this dynamical r-matrix is still quantizable. Etingof and Nikshych recently have given an affimative answer to this question using the so-called vertex-IRF transformation method [14]. Their method indeed works for a large class of dynamical r-matrices called "completely degenerate," which somehow are opposite to the non-degenerate ones considered in this paper. It would be very interesting to see whether one could combine these two methods together to completely solve the quantization problem for arbitary triangular dynamical r-matrices.

5. COMPATIBLE FEDOSOV STAR PRODUCTS

In this section, we consider Fedosov star products on a symplectic Hamiltonian H-space M, where H is an Abelian group. For the reader's convenience, we give a brief account of the general construction of Fedosov star products in the Appendix. Readers may refer to that section for various notations and formulas that are used here. What is eventually relevant to our situation is the case where M is the symplectic manifold $\mathfrak{h}^* \times G$ corresponding to a nondegenerate dynamical r-matrix. However, we believe that our general presentation will be of interest on its own right. We can now state the main result of this section.

THEOREM 5.1. *Let H be an Abelian group and M a symplectic Hamiltonian H-space with an equivariant momentum map $J: M \longrightarrow \mathfrak{h}^*$. Assume that J is a submersion, and there exists a H-invariant symplectic connection ∇ such that $\vec{h}$ is parallel for any $h \in \mathfrak{h}$; i.e., $\nabla_X \vec{h} = 0$, $\forall X \in \mathfrak{X}(M)$. Let $*_\hbar$ be the corresponding Fedosov star product on M with Weyl curvature $\Omega = \omega + \hbar\omega_1 + \cdots + \hbar^i\omega_i + \cdots \in Z^2(M)[[\hbar]]$, which satisfies the condition that $i_{\vec{h}}\omega_i = 0$, $\forall i \geqslant 1$, $\forall h \in \mathfrak{h}$. Then for any $f(\lambda) \in C^\infty(\mathfrak{h}^*)$ and $g(x) \in C^\infty(M)$, we have*

$$(J^*f) *_\hbar g(x) = \sum_{k=0}^{\infty} \left(-\frac{\hbar}{2}\right)^k \frac{1}{k!} J^*\left(\frac{\partial^k f}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\right) \vec{h}_{i_1}\cdots\vec{h}_{i_k} g;$$

$$g(x) *_\hbar (J^*f) = \sum_{k=0}^{\infty} \left(\frac{\hbar}{2}\right)^k \frac{1}{k!} (\vec{h}_{i_1}\cdots\vec{h}_{i_k} g)\, J^*\left(\frac{\partial^k f}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\right).$$

Here $\vec{h}$ denotes the corresponding Hamiltonian vector field on M generated by $h \in \mathfrak{h}$.

Remark. From Theorem 5.1, it follows that $J^*: C^\infty(\mathfrak{h}^*)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ is an algebra homomorphism, where $C^\infty(\mathfrak{h}^*)[[\hbar]]$ is equipped with pointwise multiplication. In other words, J^* is a quantum momentum map [37]. It would be interesting to see how to generalize this result to the case where H is not Abelian [42].

Applying Theorem 5.1 to the symplectic manifold $M = \mathfrak{h}^* \times G$ associated to a nondegenerate triangular dynamical r-matrix, and using Theorem 4.3, we obtain

COROLLARY 5.2. *Let $r\colon \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ be a nondegenerate triangular dynamical r-matrix and $M = \mathfrak{h}^* \times G$ its associated symplectic manifold. Let ∇ be the symplectic connection on M as in Theorem 4.3. Suppose that $\Omega = \omega + \hbar\omega_1 + \cdots + \hbar^i\omega_i + \cdots \in Z^2(M)^G[\![\hbar]\!]$ satisfies the condition that $i_{\vec{h}}\omega_i = 0$, $\forall i \geqslant 1$, $h \in \mathfrak{h}$. Then the Fedosov star product on M corresponding to (∇, Ω) is a compatible star product.*

Combining with Theorem 3.3, we are led to the following main result of the paper.

THEOREM 5.3. *Any nondegenerate triangular dynamical r-matrix is quantizable.*

More generally, if r is a splittable triangular dynamical r-matrix, according to Corollary 4.8, the corresponding Poisson manifold $M = \mathfrak{h}^* \times G$ admits a $G \times H$-invariant leafwise (w.r.t. the symplectic foliation) Poisson connection such that $\nabla_X \vec{h} = 0$, $\forall h \in \mathfrak{h}$. Applying Theorem 5.1 leafwisely, we thus have the following

THEOREM 5.4. *Any splittable triangular dynamical r-matrix is quantizable.*

The rest of the section is devoted to the proof of Theorem 5.1. We will start with the following

PROPOSITION 5.5. *Under the same hypothesis as in Theorem 5.1, we have*

(i) *For any (r, s)-type tensor $S \in \mathcal{T}^{(r,s)}M$ and $h \in \mathfrak{h}$, we have $\nabla_{\vec{h}} S = L_{\vec{h}} S$.*

(ii) *$\forall X \in \mathfrak{X}(M)$ and $i \geqslant 1$, $\nabla_X (J^* \, d\lambda^i) = 0$.*

(iii) *Given any $\theta \in \Omega^1(M)$, if $\vec{h} \lrcorner \theta = 0$, then $\vec{h} \lrcorner \nabla_X \theta = 0$, $\forall X \in \mathfrak{X}(M)$.*

(iv) *$R(X, Y, Z, W) = 0$, if any of the vectors X, Y, Z, W is tangent to the H-orbits.*

(v) *$\nabla_{\vec{h}} R = 0$, $\forall h \in \mathfrak{h}$.*

Proof. (i) Since ∇ is torsion-free, for any vector field $X \in \mathfrak{X}(M)$, we have

$$\nabla_{\vec{h}} X = \nabla_X \vec{h} + [\vec{h}, X] = [\vec{h}, X] = L_{\vec{h}} X.$$

This implies that $\nabla_{\vec{h}} \theta = L_{\vec{h}} \theta$ for any one form $\theta \in \Omega^1(M)$. Therefore $\nabla_{\vec{h}} S = L_{\vec{h}} S$ for any (r, s)-type tensor $S \in \mathcal{T}^{(r,s)}M$.

(ii) Since $J\colon M \longrightarrow \mathfrak{h}^*$ is a momentum map, it follows that $J^* d\lambda^i = \omega^b \overrightarrow{h_i}$, where $\omega^b\colon \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism induced by the symplectic structure ω. Hence $\nabla_X(J^* d\lambda^i) = \nabla_X(\omega^b \overrightarrow{h_i}) = \omega^b(\nabla_X \overrightarrow{h_i}) = 0$, since ∇ is a symplectic connection.

(iii) We have $\nabla_X(\vec{h} \lrcorner \theta) = (\nabla_X \vec{h}) \lrcorner \theta + \vec{h} \lrcorner \nabla_X \theta = \vec{h} \lrcorner \nabla_X \theta$. The claim thus follows.

(iv) Let Φ denote the H-action on M. For any $h \in \mathfrak{h}$, since ∇ is H-invariant, it follows that $\forall W, Y \in \mathfrak{X}(M)$, $\nabla_{(\Phi_{\exp th*} W)}(\Phi_{\exp th*} Y) = \Phi_{\exp th*}(\nabla_W Y)$. Taking the derivative at $t = 0$, one obtains that

$$\nabla_{[\vec{h}, W]} Y + \nabla_W [\vec{h}, Y] = [\vec{h}, \nabla_W Y].$$

Hence,

$$\begin{aligned} R(\vec{h}, W)\, Y &= \nabla_{\vec{h}} \nabla_W Y - \nabla_W \nabla_{\vec{h}} Y - \nabla_{[\vec{h}, W]} Y \\ &= [\vec{h}, \nabla_W Y] - \nabla_W [\vec{h}, Y] - \nabla_{[\vec{h}, W]} Y \\ &= 0. \end{aligned}$$

On the other hand, we know that $R(Z, W)\, \vec{h} = 0$, since $\vec{h}$ is parallel by assumption. This means that $R(X, Y, Z, W) = 0$ if $Y = \vec{h}$ or $Z = \vec{h}$. Since $R(X, Y, Z, W)$ is antisymmetric with respect to W, Z, and symmetric with respect to X, Y according to Proposition 4.1, the conclusion thus follows.

(v) Since both the connection ∇ and the symplectic structure ω are H-invariant, the symplectic curvature R, as defined by Eq. (21), is also H-invariant. Hence, for any $h \in \mathfrak{h}$, according to (i), $\nabla_{\vec{h}} R = L_{\vec{h}} R = 0$.

This completes the proof of the proposition. ∎

By $K \subset TM$, we denote the integrable distribution on M corresponding to the H-orbits, and $K^\perp$ its conormal subbundle. That is, a covector θ is in $K^\perp$ iff $\langle \theta, \vec{h} \rangle = 0$, $\forall h \in \mathfrak{h}$. For any $x \in M$, by $\mathrm{pol}(K_x^\perp)$, we denote the polynomials on $T_x M$ generated by those linear functions corresponding to covectors in $K_x^\perp$. By $W_x^\perp$, we denote the formal power series in $\hbar$ with coefficients in $\mathrm{pol}(K_x^\perp)$. Clearly $W_x^\perp$ is a subalgebra of the Weyl algebra W_x. Let $W^\perp = \bigcup_{x \in M} W_x^\perp$ be the subbundle of W. We also consider $W^\perp \otimes \bigwedge^q K^\perp$, a subbundle of $W \otimes \bigwedge^q T^*M$, whose space of sections is denoted by $\Gamma W^\perp \otimes (\Lambda^\perp)^q$. As before, let us fix a basis $\{h_1, \ldots, h_l\}$ of $\mathfrak{h}$, and denote by $(\lambda^1, \ldots, \lambda^l)$ its induced coordinate system on $\mathfrak{h}^*$. Since $J\colon M \longrightarrow \mathfrak{h}^*$ is a momentum map, we have $X_{J^*\lambda^i} = \overrightarrow{h_i}$, $\forall i = 1, \ldots, l$. It thus follows that $J_* \overrightarrow{h_i} = J_* X_{J^*\lambda^i} = 0$, $\forall i = 1, \ldots, l$, since $\mathfrak{h}$ is Abelian. Next we need to extend $\{\overrightarrow{h_i}, \ldots, \overrightarrow{h_l}\}$ to a set of (local) vector fields which constitutes a basis of tangent fibers of M. For this purpose, let $\{u_1, \ldots, u_m\}$ be (local) vector fields on M tangent to the J-fibers such that $\{\overrightarrow{h_i}, \ldots, \overrightarrow{h_l}, u_1, \ldots, u_m\}$ constitutes a basis of the tangent spaces of the J-fibers. Choose (local) vector fields $\{v_1, \ldots, v_l\}$ on M such that $J_* v_i = \frac{\partial}{\partial \lambda^i}$, $\forall i = 1, \ldots, l$, which is always possible since J is a submersion. It is easy to see that locally $\{\overrightarrow{h_i}, \ldots, \overrightarrow{h_l}, v_1, \ldots, v_l, u_1, \ldots, u_m\}$ constitutes a basis of the tangent fibers of M. Let $\{\overrightarrow{h^1_*}, \ldots, \overrightarrow{h^l_*}, v^1_*, \ldots, v^l_*, u^1_*, \ldots, u^m_*\}$ be its dual basis. Then any section of $W \otimes \Lambda$ can be written as

$$a = \sum \hbar^k a_{k, i_1 \cdots i_p, j_1 \cdots j_q} y^{i_1}_* \cdots y^{i_p}_* x^{j_1}_* \wedge \cdots \wedge x^{j_q}_*, \tag{31}$$

where all y^i_*'s and x^i_*'s are either $\overrightarrow{h^i_*}$, v^i_* or u^i_*, and the coefficients $a_{k, i_1 \cdots i_p, j_1 \cdots j_q}$ are covariant tensors symmetric with respect to $i_1 \cdots i_p$ and antisymmetric in $j_1 \cdots j_q$. It is simple to see that a section a belongs to $\Gamma W^\perp \otimes (\Lambda^\perp)^q$ iff there are no terms involving explicit h^i_*'s in the above expression.

LEMMA 5.6. (i) *For any* $i = 1, \ldots, l$, $J^* d\lambda^i = v^i_*$;

(ii) *for any* i, j, $\nabla_{\overrightarrow{h_i}} v^j_* = 0$, *and* $\nabla_{\overrightarrow{h_i}} h^j_*$ *and* $\nabla_{\overrightarrow{h_i}} u^j_*$ *belong to* $\Gamma K^\perp$;

(iii) *for any* i, j, $\pi(v^i_*, h^j_*) = \delta_{ij}$, $\pi(v^i_*, v^j_*) = 0$, $\pi(v^i_*, u^j_*) = 0$;

(iv) *the commutatant of* $\{v^1_*, \ldots, v^l_*\}$ *in* ΓW *is* $\Gamma W^\perp$.

Proof. (i) $\langle J^* d\lambda^i, v_j \rangle = \langle d\lambda^i, J_* v_j \rangle = \langle d\lambda^i, \frac{\partial}{\partial \lambda^j} \rangle = \delta_{ij}$. Similarly, we have $\langle J^* d\lambda^i, u_j \rangle = 0$ and $\langle J^* d\lambda^i, h_j \rangle = 0$. Therefore, $J^* d\lambda^i = v^i_*$.

(ii) According to Proposition 5.5, $\nabla_{\overrightarrow{h_i}} v^j_* = \nabla_{\overrightarrow{h_i}} (J^* d\lambda^j) = 0$. Also, $\forall k$, $\langle \nabla_{\overrightarrow{h_i}} h^j_*, \overrightarrow{h_k} \rangle = \nabla_{\overrightarrow{h_i}} \langle h^j_*, \overrightarrow{h_k} \rangle - \langle h^j_*, \nabla_{\overrightarrow{h_i}} \overrightarrow{h_k} \rangle = 0$. Hence it follows that $\nabla_{\overrightarrow{h_i}} h^j_* \in \Gamma K^\perp$. Similarly, we can prove that $\nabla_{\overrightarrow{h_i}} u^j_* \in \Gamma K^\perp$.

(iii) We have $\pi(v^i_*, h^j_*) = \langle \pi^\#(J^* d\lambda^i), h^j_* \rangle = \langle \overrightarrow{h_i}, h^j_* \rangle = \delta_{ij}$. Similarly, we can show that $\pi(v^i_*, v^j_*) = 0$ and $\pi(v^i_*, u^j_*) = 0$.

(iv) Assume that $a \in \Gamma W$ such that $[a, v^i_*] = 0$, $\forall i = 1, \ldots, l$. It thus follows that $\{a, v^i_*\} = 0$, where the Poisson bracket refers to the one corresponding to the fiberwise symplectic structure on TM. Thus $a \in \Gamma W^\perp$ according to (iii). ∎

LEMMA 5.7. (i) $\Gamma W^\perp \otimes \Lambda^\perp$ *is closed under the multiplication* $\circ$ *as defined by Eq.* (46).

(ii) $\Gamma W^\perp \otimes \Lambda^\perp$ *is closed under both the operators* δ *and* δ^{-1}.

(iii) $\Gamma W^\perp \otimes \Lambda^\perp$ *is invariant under the covariant derivative* ∇_X, $\forall X \in \mathfrak{X}(M)$.

Proof. (i) and (ii) are obvious. For (iii), note that $\Gamma(K^\perp)$ is invariant under the covariant derivative ∇_X according to Proposition 5.5(iii). Hence $\Gamma W^\perp \otimes \Lambda^\perp$ is also invariant. ∎

As an immediate consequence, we have the following

COROLLARY 5.8. *If* $a \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$ *and* $\nabla_{\vec{h}} a = 0, \forall h \in \mathfrak{h}$, *then* $\partial a \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$.

To prove Theorem 5.1, we start with the following

LEMMA 5.9. *Under the same hypthesis as in Theorem* 5.1, *we have* $\gamma_0 = \delta^{-1}\tilde{\Omega} \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$ *and* $\nabla_{\vec{h}}\gamma_0 = 0, \forall h \in \mathfrak{h}$.

Proof. According to Eq. (57), we know that $\tilde{\Omega} = \Omega - \omega + R = R + \hbar\omega_1 + \hbar^2\omega_2 + \cdots$. By assumption, we have $\omega_i \in \Gamma W^{\perp} \otimes \Lambda^{\perp}, \forall i \geqslant 1$. On the other hand, according to Proposition 5.5(iv), we know that $R \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$. Therefore, $\tilde{\Omega} \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$. Hence $\gamma_0 \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$ by Lemma 5.7.

Finally, note that for any $h \in \mathfrak{h}$, $L_{\vec{h}}\omega_i = i_{\vec{h}}(d\omega_i) + d(i_{\vec{h}}\omega_i) = d(i_{\vec{h}}\omega_i) = 0$. According to Proposition 5.5, we have $L_{\vec{h}}R = \nabla_{\vec{h}}R = 0$. Hence $L_{\vec{h}}\tilde{\Omega} = 0$. It thus follows that $\nabla_{\vec{h}}\gamma_0 = L_{\vec{h}}\gamma_0 = L_{\vec{h}}\delta^{-1}\tilde{\Omega} = \delta^{-1}L_{\vec{h}}\tilde{\Omega} = 0$. ∎

PROPOSITION 5.10. *Under the same hypothesis as in Theorem* 5.1, *the element* γ, *defined as in Theorem* A.2, *belongs to* $\Gamma W^{\perp} \otimes \Lambda^{\perp}$ *and satisfies* $\nabla_{\vec{h}}\gamma = 0, \forall h \in \mathfrak{h}$.

Proof. We prove this proposition by induction. Assume that $\gamma_n \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$ and $\nabla_{\vec{h}}\gamma_n = 0, \forall h \in \mathfrak{h}$. It suffices to show that γ_{n+1} satisfies the same conditions. By Eq. (59), γ_{n+1} and γ_n are related by the following equation:

$$\gamma_{n+1} = \gamma_0 + \delta^{-1}\left(\partial\gamma_n + \frac{i}{\hbar}\gamma_n^2\right), \forall n \geqslant 0. \tag{32}$$

According to Corollary 5.8, we have $\partial\gamma_n \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$. On the other hand, by Lemma 5.7, $\gamma_n^2 \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$. Hence $\gamma_{n+1} \in \Gamma W^{\perp} \otimes \Lambda^{\perp}$ according to Lemma 5.7 and Lemma 5.9.

Now

$$\begin{aligned}
\nabla_{\vec{h}}(\gamma_{n+1}) &= \nabla_{\vec{h}}\gamma_0 + \nabla_{\vec{h}}\delta^{-1}\left(\partial\gamma_n + \frac{i}{\hbar}\gamma_n^2\right)\\
&= L_{\vec{h}}\,\delta^{-1}\left(\partial\gamma_n + \frac{i}{\hbar}\gamma_n^2\right)\\
&= \delta^{-1}\left(L_{\vec{h}}\partial\gamma_n + \frac{i}{\hbar}L_{\vec{h}}\gamma_n^2\right)\\
&= 0.
\end{aligned}$$

Here, in the last step, we used the relation $L_{\vec{h}}\partial = \partial L_{\vec{h}}$, which follows from the fact that the symplectic connection is H-invariant. This concludes the proof. ∎

As in the Appendix, for any $a \in C^\infty(M)$, we denote by $\tilde{a} \in W_D$ its parallel lift; i.e., $D\tilde{a} = 0$ and $\tilde{a}|_{y=0} = a$. Theorem 5.1 is in fact an immediate consequence of the following

PROPOSITION 5.11. *Under the same hypothesis as in Theorem* 5.1,

(i) *if* $a = J^* f$ *for* $f \in C^\infty(\mathfrak{h}^*)$, *then*

$$\tilde{a} = \sum_{k=0}^{\infty} \frac{1}{k!} J^*\left(\frac{\partial^k f}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}}\right) v_*^{i_1} \cdots v_*^{i_k}; \tag{33}$$

(ii) *for any* $a \in C^\infty(M)$,

$$\tilde{a} = \sum_{k=0}^{\infty} \frac{1}{k!} (\vec{h}_{i_1} \cdots \vec{h}_{i_k} a)\, h_*^{i_1} \cdots h_*^{i_k} + T,$$

where the remainder T *does not contain any terms which are pure polynomials of* h_*^i*'s.*

Proof. For (i), it suffices to prove that $\tilde{a}$, given by Eq. (33), is a parallel section. According to Proposition 5.10 and Lemma 5.6, we have $[\gamma, \tilde{a}] = 0$. Thus it follows that $D\tilde{a} = -\delta\tilde{a} + \partial\tilde{a}$, which clearly vanishes since $\partial v_*^i = 0$ by Proposition 5.5(ii) and Lemma 5.6(ii).

For (ii), recall that $\tilde{a}$ is determined by the iteration formula

$$a_{n+1} = a + \delta^{-1}\left(\partial a_n + \left[\frac{i}{\hbar}\gamma, a_n\right]\right). \tag{34}$$

So it suffices to prove that

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} (\vec{h}_{i_1} \cdots \vec{h}_{i_k} a_0)\, h_*^{i_1} \cdots h_*^{i_k} + T_n,$$

where each term in the remainder T_n is not a pure polynomial of h_*^i's. This can be proved by induction again.

Assume that this assertion holds for a_n. To show that it still holds for a_{n+1}, we need to analyze which terms in a_n would produce pure polynomials of h_*^i's out of Eq. (34). Since $\gamma \in \Gamma W^\perp \otimes \Lambda^\perp$, we may ignore

$\delta^{-1}[\frac{i}{\hbar}\gamma, a_n]$ and only consider $\delta^{-1}\partial a_n = \delta^{-1}(\sum_i \nabla_{\vec{h_i}} a_n \wedge h^i_* + \sum_i \nabla_{v_i} a_n \wedge v^i_* + \sum_i \nabla_{u_i} a_n \wedge u^i_*)$. From this, it is clear that those terms containing pure polynomials of h^i_*'s arise only from $\delta^{-1}(\sum_i \nabla_{\vec{h_i}} a_n \wedge h^i_*)$. Now a general term in a_n has the form $\hbar^k a_{\alpha\beta\gamma}(x)\, v^\alpha_* h^\beta_* u^\gamma_*$, where α, β, and γ are multi-indices. However,

$$\begin{aligned}
&\nabla_{\vec{h_i}}(a_{\alpha\beta\gamma}(x)\, v^\alpha_* h^\beta_* u^\gamma_*)\\
&\quad = (\vec{h_i}\, a_{\alpha\beta\gamma}(x))\, v^\alpha_* h^\beta_* u^\gamma_* + a_{\alpha\beta\gamma}(x)(\nabla_{\vec{h_i}} v^\alpha_*)\, h^\beta_* u^\gamma_*\\
&\qquad + a_{\alpha\beta\gamma}(x)\, v^\alpha_*(\nabla_{\vec{h_i}} h^\beta_*)\, u^\gamma_* + a_{\alpha\beta\gamma}(x)\, v^\alpha_* h^\beta_*(\nabla_{\vec{h_i}} u^\gamma_*)\\
&\quad = (\vec{h_i}\, a_{\alpha\beta\gamma}(x))\, v^\alpha_* h^\beta_* u^\gamma_* + a_{\alpha\beta\gamma}(x)\, v^\alpha_*(\nabla_{\vec{h_i}} h^\beta_*)\, u^\gamma_* + a_{\alpha\beta\gamma}(x)\, v^\alpha_* h^\beta_*(\nabla_{\vec{h_i}} u^\gamma_*).
\end{aligned}$$

According to Lemma 5.6, neither $\nabla_{\vec{h_i}} h^\beta_*$ nor $\nabla_{\vec{h_i}} u^\gamma_*$ will be a pure polynomial of h^i_*'s. Hence to produce a pure h^i_*-polynomial term, one needs that $\alpha = \gamma = 0$. And in this case, the resulting pure h^i_*-polynomial term is $\hbar^k(\vec{h_i}\, a_{0\beta 0}(x))\, h^\beta_*$. In conclusion, only pure h^i_*-polynomial terms in a_n can give rise to pure h^i_*-polynomial terms in $\delta^{-1}\partial a_n$. Hence the pure h^i_*-polynomial terms in a_{n+1} is $a_0 + \sum_{k=0}^{n} \frac{1}{k!}\frac{1}{k+1}\vec{h_i}(\vec{h}_{i_1}\cdots\vec{h}_{i_k} a_0)\, h^i_* h^{i_1}_* \cdots h^{i_k}_*$, which clearly equals to $\sum_{k=0}^{n+1}\frac{1}{k!}(\vec{h}_{i_1}\cdots\vec{h}_{i_k} a_0)\, h^{i_1}_* \cdots h^{i_k}_*$. This concludes the proof. ∎

6. CLASSIFICATION

This section is devoted to the classification of quantization of a nondegenerate triangular dynamical r-matrix. Our method relies heavily on the classification result for star products on a symplectic manifold. First, let us introduce the following:

DEFINITION 6.1. Two quantizations $F(\lambda)$ and $E(\lambda)$ of a triangular dynamical r-matrix are said to be equivalent if there exists a $T(\lambda)$: $\mathfrak{h}^* \to (U\mathfrak{g})^{\mathfrak{h}}[\![\hbar]\!]$ satisfying the condition that $T(\lambda) = 1 (\mathrm{mod}\, \hbar)$ and $\varepsilon(T(\lambda)) = 1$ such that

$$E(\lambda) = \Delta T(\lambda)^{-1} F(\lambda)\, T_1(\lambda - \tfrac{1}{2}\hbar h^{(2)})\, T_2(\lambda + \tfrac{1}{2}\hbar h^{(1)}). \tag{35}$$

To justify this definition, we need the following result, which interprets this equivalence in terms of star products.

THEOREM 6.2. *Given a compatible star product $*_\hbar$ on the Poisson manifold (M, π) associated to a triangular dynamical r-matrix $r(\lambda)$, assume* *that* $T(\lambda)\colon \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^{\mathfrak{h}}[\![\hbar]\!]$ *satisfies the condition that* $T(\lambda) = 1 (\mathrm{mod}\, \hbar)$ *and* $\varepsilon(T(\lambda)) = 1$. *Then the* $*$*-product:*

$$f \,\widetilde{*_\hbar}\, g = \vec{T}^{-1}(\vec{T} f *_\hbar \vec{T} g), \forall f, g \in C^\infty(M) \tag{36}$$

is still a compatible star-product. Moreover, if $f, g \in C^\infty(G)$, *then*

$$f \,\widetilde{*_\hbar}\, g = \overrightarrow{E(\lambda)}(f, g), \tag{37}$$

where $E(\lambda)$ *is given by Eq.* (35).

Thus we are led to the following

DEFINITION 6.3. Compatible star-products $*_\hbar$ and $\widetilde{*_\hbar}$ are said to be strongly equivalent iff they are related by Eq. (36) for some $T(\lambda)\colon \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^{\mathfrak{h}}[\![\hbar]\!]$ satisfying the property that $T(\lambda) = 1 (\mathrm{mod}\, \hbar)$ and $\varepsilon(T(\lambda)) = 1$.

An immediate consequence of Theorem 6.2 is the following:

COROLLARY 6.4. *If* $F(\lambda)$ *is a quantization of a triangular dynamical* r*-matrix* $r\colon \mathfrak{h}^* \longrightarrow \bigwedge^2 \mathfrak{g}$ *and* $T(\lambda)\colon \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^{\mathfrak{h}}[\![\hbar]\!]$ *satisfies the condition that* $T(\lambda) = 1 (\mathrm{mod}\, \hbar)$ *and* $\varepsilon(T(\lambda)) = 1$, *then*

$$E(\lambda) = \Delta T(\lambda)^{-1} F(\lambda)\, T_1(\lambda - \tfrac{1}{2}\hbar h^{(2)})\, T_2(\lambda + \tfrac{1}{2}\hbar h^{(1)})$$

is also a quantization of $r(\lambda)$.

Due to this fact, Definition (6.1) is well justified. Indeed, Theorem 6.2 allows us to reduce the classification problem of quantizations of a triangular dynamical r-matrix to that of strongly equivalent star products on M.

Remark. Let $R_E(\lambda) = E^{21}(\lambda)^{-1} E^{12}(\lambda)$ and $R_F(\lambda) = F^{21}(\lambda)^{-1} F^{12}(\lambda)$. It is easy to see that they are related by

$$R_E(\lambda) = T_2(\lambda - \tfrac{1}{2}\hbar h^{(1)})^{-1}\, T_1(\lambda + \tfrac{1}{2}\hbar h^{(2)})^{-1}\, R_F(\lambda)\, T_1(\lambda - \tfrac{1}{2}\hbar h^{(2)})\, T_2(\lambda + \tfrac{1}{2}\hbar h^{(1)}). \tag{38}$$

Alternatively, we may define a quantization of a triangular dynamical r-matrix $r(\lambda)$ to be an element $R(\lambda) = 1 + \hbar r(\lambda) + \cdots \in U\mathfrak{g} \otimes U\mathfrak{g}[\![\hbar]\!]$ satisfying the QDYBE, and define an equivalence of quantizations by Eq. (38). This definition sounds weaker than our original one. We do not know, however, at this moment whether these two definitions are equivalent. It would be interesting to have this clarified.

To prove Theorem 6.2, we need a lemma.

LEMMA 6.5. *Assume that* $T(\lambda)\colon \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^{\mathfrak{h}}[\![\hbar]\!]$ *is as in Theorem 6.2, then*

(i) $(ad_\theta)^n\,(T\otimes 1) = (-\frac{1}{2})^n \sum_{i_1\cdots i_n} \frac{\partial^n T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_n}} \otimes h_{i_1}\cdots h_{i_n}$;

(ii) $\Theta(T\otimes 1)\,\Theta^{-1} = T_1(\lambda - \frac{1}{2}\hbar h^{(2)})$;

(iii) $\Theta(1\otimes T)\,\Theta^{-1} = T_2(\lambda + \frac{1}{2}\hbar h^{(1)})$;

(iv) $\Theta(T\otimes T)\,\Theta^{-1} = T_1(\lambda - \frac{1}{2}\hbar h^{(2)})\, T_2(\lambda + \frac{1}{2}\hbar h^{(1)})$.

Proof. (i) We prove this equation by induction. Obviously, it holds for $n=0$. Assume that it holds for $n=k$. Now

$$
\begin{aligned}
&(ad_\theta)^{k+1}\,(T\otimes 1)\\
&\quad = ad_\theta\left[\left(-\frac{1}{2}\right)^k \sum_{i_1\cdots i_k} \frac{\partial^k T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} \otimes h_{i_1}\cdots h_{i_k}\right]\\
&\quad = \left(-\frac{1}{2}\right)^k \frac{1}{2} \sum_{i_1\cdots i_k} \sum_i \left(\left[h_i \otimes \frac{\partial}{\partial\lambda^i}, \frac{\partial^k T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} \otimes h_{i_1}\cdots h_{i_k}\right]\right.\\
&\quad\quad \left. - \left[\frac{\partial}{\partial\lambda^i} \otimes h_i, \frac{\partial^k T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} \otimes h_{i_1}\cdots h_{i_k}\right]\right)\\
&\quad = \left(-\frac{1}{2}\right)^{k+1} \sum_{i_1\cdots i_{k+1}} \frac{\partial^{k+1} T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_{k+1}}} \otimes h_{i_1}\cdots h_{i_{k+1}}.
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
\Theta(T\otimes 1)\,\Theta^{-1} &= \exp(\hbar\, ad_\theta)(T\otimes 1)\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\,(\hbar\, ad_\theta)^k\,(T\otimes 1)\\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \frac{\partial^k T}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}} \otimes h_{i_1}\cdots h_{i_k}\\
&= T_1\left(\lambda - \frac{1}{2}\hbar h^{(2)}\right).
\end{aligned}
$$

(iii) is proved similarly, and (iv) follows from (ii) and (iii). ∎

Proof of Theorem 6.2. If $f, g \in C^\infty(\mathfrak{h}^*)$, then $\vec{T} f = f$ and $\vec{T} g = g$ since $\varepsilon(T) = 1$. Hence

$$\vec{T}^{-1}(\vec{T} f *_\hbar \vec{T} g) = fg.$$

Now if $f \in C^\infty(\mathfrak{h}^*)$ and $g \in C^\infty(G)$,

$$\begin{aligned}
\vec{T} f *_\hbar \vec{T} g &= f *_\hbar \vec{T} g \\
&= \vec{\Theta}(f, \vec{T} g) \\
&= \overrightarrow{\Theta(1 \otimes T)}(f, g) \quad \text{(by Lemma 6.5)} \\
&= \overrightarrow{T_2\left(\lambda + \frac{1}{2}\hbar h^{(1)}\right)\Theta}(f, g) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\hbar}{2}\right)^k \overrightarrow{\left(h_{i_1}\cdots h_{i_k} \otimes \frac{\partial^k T}{\partial \lambda^{i_1}\cdots\partial\lambda^{i_k}}\right)\Theta}(f, g) \\
&= \overrightarrow{(1 \otimes T)}\, \vec{\Theta}(f, g) \\
&= \vec{T}(f *_\hbar g).
\end{aligned}$$

Here in the last equality, we used the fact that $\vec{T}$ does not involve any derivative $\frac{\partial}{\partial \lambda^i}$. So we have proved that $\vec{T}^{-1}(\vec{T} f *_\hbar \vec{T} g) = \vec{\Theta}(f, g)$.

Finally, assume that $f, g \in C^\infty(G)$. According to Theorem 3.8,

$$\begin{aligned}
\vec{T} f *_\hbar \vec{T} g &= \overrightarrow{(F(\lambda)\,\Theta)}(\vec{T} f, \vec{T} g) \\
&= \overrightarrow{F(\lambda)\,\Theta(T \otimes T)}(f, g) \quad \text{(by Lemma 6.5)} \\
&= \overrightarrow{F(\lambda)\,T_1(\lambda - \tfrac{1}{2}\hbar h^{(2)})\,T_2(\lambda + \tfrac{1}{2}\hbar h^{(1)})\,\Theta}(f, g).
\end{aligned}$$

It thus follows that

$$\begin{aligned}
\vec{T}^{-1}(\vec{T} f *_\hbar \vec{T} g) &= \overrightarrow{\Delta T(\lambda)^{-1}\,F(\lambda)\,T_1(\lambda - \tfrac{1}{2}\hbar h^{(2)})\,T_2(\lambda + \tfrac{1}{2}\hbar h^{(1)})\,\Theta}(f, g) \\
&= \overrightarrow{E(\lambda)}\,\vec{\Theta}(f, g).
\end{aligned}$$

This conclude the proof. ∎

The rest of the section is devoted to the classification of strongly equivalent classes of compatible star products on $M = \mathfrak{h}^* \times G$. The classification of star products on a general symplectic manifold was studied by many authors, for example, see [8, 11, 34, 36, 37]. Here we follow the elementary approach due to Bertelson *et al.* [7] concerning invariant star products.

First we prove

THEOREM 6.6. *Let $M = \mathfrak{h}^* \times G$ be the symplectic manifold corresponding to a non-degenerate dynamical r-matrix $r(\lambda)$. Two compatible Fedosov $*$-products are strongly equivalent iff their Weyl curvatures Ω_* and Ω are strongly cohomologous; i.e., $\Omega_* - \Omega = d\theta$, where $\theta \in \Omega^1(M)[\![\hbar]\!]$ is $G \times H$-invariant and satisfies $i_{\vec{h}}\theta = 0$, $\forall h \in \mathfrak{h}$.*

From now on, in this section, by M we always mean the symplectic manifold $\mathfrak{h}^* \times G$ associated with a non-degenerate dynamical r-matrix. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition as in Lemma 4.5, and $\{h_1, \ldots, h_l\}$ a basis in $\mathfrak{h}$, and $\{e_1, \ldots, e_m\}$ a basis of $\mathfrak{m}$. If we choose $v_i = \frac{\partial}{\partial \lambda^i}$ and $u_i = \vec{e_i}$, then $\{\vec{h_i}, \ldots, \vec{h_l}, v_1, \ldots, v_l, u_1 \cdots, u_m\}$ constitutes a local (in fact global in this case) basis of tangent fibers of M, which satisfies all the required properties as in the construction preceding Lemma 5.6. In what follows, we will fix such a choice and denote by $\{\vec{h}^1_*, \ldots, \vec{h}^l_*, v^1_*, \ldots, v^l_*, u^1_*, \ldots, u^m_*\}$ its dual basis.

LEMMA 6.7. *Assume that D is an Abelian connection defining a compatible $*$-product on M as in Corollary 5.2. For any $a \in C^\infty(M)$, let*

$$\tilde{a} = \sum \hbar^k D_{k,\alpha\beta\gamma}(a)\, v^\alpha_* h^\beta_* u^\gamma_* \in \Gamma(W) \tag{39}$$

be its parallel lift, where α, β, and γ are multi-indices and $D_{k,\alpha\beta\gamma}$ are certain differential operators on M. If an operator $D_{k,\alpha\beta\gamma}$ involves a derivative of $\lambda \in \mathfrak{h}^*$, then the corresponding term satisfies $|\alpha| > 0$.

Proof. As is known, $\tilde{a}$ is given by the iteration formula

$$a_{n+1} = a_0 + \delta^{-1}\left(\partial a_n + \left[\frac{i}{\hbar}\gamma, a_n\right]\right),$$

so it suffices to show that a_n possesses such a property for any n, which we shall prove by induction.

Assume that a_n possesses this property, and we need to show that so does a_{n+1}. Let $\hbar^k D_{k,\alpha\beta\gamma}(a)\, v^\alpha_* h^\beta_* u^\gamma_*$ be a term in a_{n+1}, where $D_{k,\alpha\beta\gamma}$ involves a derivative of λ. There are two possible sources that this term may come from. One is from $\delta^{-1}[\frac{i}{\hbar}\gamma, a_n]$. Since this operation does not affect the part involving derivatives on a, so it must come from a term having the form

$$\delta^{-1}\left[\frac{i}{\hbar}\gamma, \hbar^{k'} D_{k,\alpha\beta\gamma}(a)\, v^{\alpha'}_* h^{\beta'}_* u^{\gamma'}_*\right], \tag{40}$$

where $\hbar^{k'} D_{k,\alpha\beta\gamma}(a)\, v^{\alpha'}_* h^{\beta'}_* u^{\gamma'}_*$ is one of the terms in a_n. By assumption, we know that $|\alpha'| > 0$. Since $\gamma \in \Gamma W^\perp \otimes \Lambda^\perp$, it follows from Lemma 5.6 that any resulting term in Eq. (40) has at least a factor $v^\alpha_*{}'$.

Another possible source is from $\delta^{-1}(\partial a_n)$. Now

$$\delta^{-1}\,\partial a_n = \sum_i (\nabla_{\frac{\partial}{\partial\lambda^i}} a_n)\, v^i_* + \sum_i (\nabla_{\vec{h_i}} a_n)\, h^i_* + \sum_i (\nabla_{\vec{e_i}} a_n)\, u^i_*.$$

If it arises from the first term, we are done. Assume that it comes from the second term: $(\nabla_{\vec{h}_i} a_n)\, h_*^i$. Let $\hbar^k D_{k,\alpha\beta\gamma}(a)\, v_*^\alpha h_*^\beta u_*^\gamma$ be a general term in a_n; then

$$\nabla_{\vec{h}_i}(\hbar^k D_{k,\alpha\beta\gamma}(a)\, v_*^\alpha h_*^\beta u_*^\gamma) = \hbar^k(\vec{h}_i D_{k,\alpha\beta\gamma})(a)\, v_*^\alpha h_*^\beta u_*^\gamma + \hbar^k D_{k,\alpha\beta\gamma}(a)\, v_*^\alpha (\nabla_{\vec{h}_i} h_*^\beta)\, u_*^\gamma + \hbar^k D_{k,\alpha\beta\gamma}(a)\, v_*^\alpha h_*^\beta (\nabla_{\vec{h}_i} u_*^\gamma).$$

From this equation, it is clear that $D_{k,\alpha\beta\gamma}$ must already contain some derivative of $\lambda \in \mathfrak{h}^*$. The conclusion thus follows from the inductive assumption. A similar argument applies when it arises from the last term $(\nabla_{\vec{e}_i} a_n)\, u_*^i$. This concludes the proof. ∎

Proof of Theorem 6.6. Our proof here is essentially a modification of the proof of Corollary 5.5.4 in [21].

"Necessity." Let

$$D = -\delta + \partial + \frac{i}{\hbar}[\gamma, \cdot], \qquad \text{and}$$

$$D_* = -\delta + \partial + \frac{i}{\hbar}[\gamma_*, \cdot]$$

be the Abelian connections with Weyl curvatures Ω and Ω_*, respectively, and $A: W_D \longrightarrow W_{D_*}$ an isomorphism of algebras. It is standard that A lifts to an automorphism of the Weyl bundle W, which will be denoted by the same symbol $A: W \longrightarrow W$. Then A is $G \times H$-equivariant. As in [21], we may assume that $A(a) = U \circ a \circ U^{-1}$ for some $U \in \Gamma W_+$, $\forall a \in W$. We may also assume that U is $G \times H$-invariant since A is $G \times H$-equivariant. By assumption, we also know that $Aa = a$ if $a = \sum_0^\infty \frac{1}{k!} \frac{\partial^k a_0}{\partial \lambda^{i_1} \cdots \partial \lambda^{i_k}} v_*^{i_1} \cdots v_{*k}^{i}$, $\forall a_0 \in C^\infty(\mathfrak{h}^*)$, which is the parallel lift of a_0 according to Proposition 5.11. This implies that U commutes with v_*^i, $i = 1, \ldots, l$, and therefore $U \in \Gamma W_+^\perp$ according to Lemma 5.6. Consider another Abelian connection: $\tilde{D}a = (A \circ D \circ A^{-1})(a) = U \circ D(U^{-1} a U) \circ U^{-1} = Da - [DU \circ U^{-1}, a]$. Then $\tilde{D}$ has the same Weyl curvature as D (see Theorem 5.5.3 and Corollary 5.5.4 in [21]), which is assumed to be Ω. On the other hand,

$$\begin{aligned} D_* a - \tilde{D}a &= \frac{i}{\hbar}[\gamma_* - \gamma - i\hbar(DU \circ U^{-1}), a] \\ &= \frac{i}{\hbar}[\Delta\gamma, a]. \end{aligned} \tag{41}$$

Since $U \in \Gamma W_+^\perp$ and $\gamma_*, \gamma \in \Gamma W^\perp \otimes \Lambda^\perp$, it follows that $\Delta\gamma \in \Gamma W^\perp \otimes \Lambda^\perp$. It is also clear that $\Delta\gamma$ is $G \times H$-invariant. Moreover, from Eq. (41), it follows that $[\Delta\gamma, a] = 0$, if $a \in W_{D_*}$. Hence $\Delta\gamma$ is a scalar form. Thus $\Omega_* - \Omega = d\Delta\gamma$. Clearly, $\Delta\gamma$ is $G \times H$-invariant and $i_{\vec{h}}\Delta\gamma = 0$, $\forall h \in \mathfrak{h}$.

"Sufficiency". Assume that $\Omega_* - \Omega = d\theta$, $\theta \in \Omega^1(M)[\![\hbar]\!]$ is $G \times H$-invariant and $i_{\vec{h}}\theta = 0$, $\forall h \in \mathfrak{h}$. Let $\Omega(t) = \Omega + t d\theta$ and $D_t = -\delta + \partial + \frac{i}{\hbar}[\gamma(t), \cdot]$ be the Abelian connection with Weyl curvature $\Omega(t)$, where $\gamma(t)$ is as in Theorem A.2, satisfying $\delta^{-1}\gamma(t) = 0$.

Let $H(t) \in \Gamma W$ be the solution of the equation $D_t H(t) = -\theta + \dot{\gamma}(t)$ satisfying $H(t)|_{y=0} = 0$. Then $H(t)$ is $G \times H$-invariant since $D_t, \theta, \gamma(t)$ are all $G \times H$-invariant. On the other hand, since $\gamma(t) \in \Gamma W^\perp \otimes \Lambda^\perp$ according to Proposition 5.10, and $\theta \in \Gamma \Lambda^\perp$ by assumption, it follows that $H(t) \in \Gamma W^\perp$.

According to Theorem 5.5.3 [21], the solution of the Heisenberg equation

$$\frac{d\tilde{a}}{dt} + [H(t), \tilde{a}] = 0 \tag{42}$$

establishes an isomorphism $W_D \longrightarrow W_{D_*}$, which is given by $\tilde{a}(0) \longrightarrow \tilde{a}(1)$. In fact, $D_t\tilde{a}(t) = 0$ if $D\tilde{a}(0) = 0$.

Clearly, this correspondence is $G \times H$-equivariant since $H(t)$ is $G \times H$-invariant. So its corresponding formal differential operator $T: C^\infty(M)[\![\hbar]\!] \longrightarrow C^\infty(M)[\![\hbar]\!]$ is $G \times H$-invariant. Finally it remains to show that T, as a formal differential operator, does not involve any derivative of $\lambda \in \mathfrak{h}^*$.

To show this, for any $a \in C^\infty(M)$, let $\tilde{a} \in W_D$ be its parallel lift, and $\tilde{a}(t)$ the solution of Eq. (42) satisfying the initial condition $\tilde{a}(0) = \tilde{a}$. Then $D_t\tilde{a}(t) = 0$. Also, let $a(t) = \tilde{a}(t)|_{y=0}$. Write

$$\tilde{a}(t) = \sum \hbar^k D_{t,k,\alpha\beta\gamma}(a(t))\, v_*^\alpha h_*^\beta u_*^\gamma.$$

If an operator $D_{t,k,\alpha\beta\gamma}$ involves a derivative to $\lambda \in \mathfrak{h}^*$, we know that $\alpha \neq 0$ according to Lemma 6.7. Since $H(t) \in \Gamma W^\perp$, it thus follows that $[H(t), D_{t,k,\alpha\beta\gamma}(a(t))\, v_*^\alpha h_*^\beta u_*^\gamma]|_{y=0} = 0$. This implies that $[H(t), \tilde{a}(t)]|_{y=0} = \mathscr{D}_t a(t)$, where $\mathscr{D}_t$ is a formal differential operator on M involving no derivatives of $\lambda \in \mathfrak{h}^*$. Now Eq. (42) implies that

$$\frac{da(t)}{dt} + \mathscr{D}_t(a(t)) = 0.$$

Therefore the equivalence operator $T: C^\infty(M)[\![\hbar]\!] \longrightarrow C^\infty(M)[\![\hbar]\!]$, which sends $a(0)$ to $a(1)$, does not involve any derivative of $\lambda \in \mathfrak{h}^*$. This concludes the proof. ∎

As in [7], by $C^k_{diff,0}(M)$, we denote the space of differential Hochschild k-cochains on $C^\infty(M)$ (i.e., k-multidifferential operators on M) vanishing on constants and denote by $b\colon C^k_{diff,0}(M) \longrightarrow C^{k+1}_{diff,0}(M)$ the Hochschild co-boundary operator.

PROPOSITION 6.8. *Suppose that $*_\hbar$ and $*'_\hbar$ are two compatible star-products on M:*

$$u *_\hbar v = \sum_{k=0}^{\infty} \hbar^k C_k(u, v),\ u *'_\hbar v = \sum_{k=0}^{\infty} \hbar^k C'_k(u, v),\ \forall u, v \in C^\infty(M).$$

Assume that $*_\hbar$ and $*'_\hbar$ coincide with each other up to order n; i.e., $C_k = C'_k$, $0 \leqslant k \leqslant n$. Then

(i) $(C_{n+1} - C'_{n+1})(u, v) = \vec{B}(u, v) + (b\vec{E})(u, v)$, *where $B \in C^2(\mathfrak{h}^*, (\wedge^2 \mathfrak{g})^{\mathfrak{h}})$ is a δ_r 2-cocycle (i.e., $\delta_r B = 0$) and $E\colon \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^{\mathfrak{h}}$. Here δ_r denotes the coboundary operator defined by Eq.* (9).

(ii) $C_1 = \frac{1}{2}\{\cdot, \cdot\} + b\vec{c}_1$ *for some $c_1 \in C^\infty(\mathfrak{h}^*, (U\mathfrak{g})^{\mathfrak{h}})$.*

(iii) *If $B = \delta_r X$, $X \in C^\infty(\mathfrak{h}^*, \mathfrak{g}^{\mathfrak{h}})$, then the formal operator $T = 1 + \hbar^n \vec{X} + \hbar^{n+1}\vec{E_1}$ transforms $*_\hbar$ to another star-product, which coincides with $*'_\hbar$ up to order $n+1$. Here $E_1 = E(u) - [X, c_1]$.*

Proof. We use an argument similar to that in [7].

(i) By definition, if either u or v is in $C^\infty(\mathfrak{h}^*)$, we have $u *_\hbar v = u *'_\hbar v = \vec{\Theta}(u, v)$, which implies that $(C_{n+1} - C'_{n+1})(u, v) = 0$.

On the other hand, as is well known, $C_{n+1} - C'_{n+1}$ is a Hochschild 2-cocycle [7, 37]. Hence we may write

$$C_{n+1} - C'_{n+1} = S + bT,$$

where $S \in \Gamma(\wedge^2 TM)$ and T is a Hochschild 1-cochain. Since S and bT are, respectively, the skew-symmetric and symmetric parts of $C_{n+1} - C'_{n+1}$, they share many common properties as $C_{n+1} - C'_{n+1}$. In particular, both of them are $G \times H$-invariant and vanish when one of the argument u or v belongs to $C^\infty(\mathfrak{h}^*)$. This implies that $S = \vec{B}$, for some $B \in C^\infty(\mathfrak{h}^*, (\wedge^2 \mathfrak{g})^{\mathfrak{h}})$. It is also standard [7, 37] that S satisfies the equation $[\pi, S] = 0$, which is equivalent to $\delta_r B = 0$ according to the remark following Proposition 2.11.

Now $M = \mathfrak{h}^* \times G$ clearly admits a $G \times H$-invariant (in fact G-biinvariant) connection. Since bT is $G \times H$-invariant, according to Proposition 2.1 in [7], we can assume that T is a $G \times H$-invariant 1-cochain. Since $(bT)(u, v) = 0$, $\forall u, v \in C^\infty(\mathfrak{h}^*)$, we have $u(Tv) - T(uv) + (Tu)\, v = 0$. On the other hand, since Tu is G-invariant, it must be a function of $\lambda \in \mathfrak{h}^*$ only; i.e.,

$Tu \in C^\infty(\mathfrak{h}^*)$. Hence the restriction of the operator T to $C^\infty(\mathfrak{h}^*)$ defines a vector field Y on $\mathfrak{h}^*$. Now since $(bT)(u, v) = 0$, $\forall u \in C^\infty(\mathfrak{h}^*)$, it follows that

$$(T-Y)(uv) = u(T-Y)(v), \qquad \forall u \in C^\infty(\mathfrak{h}^*), \quad v \in C^\infty(M).$$

Hence $T-Y$ does not involve any derivative with respect to $\lambda \in \mathfrak{h}^*$. Since $T-Y$ is $G \times H$-invariant, it follows that $T-Y = \vec{E}$ for some $E: \mathfrak{h}^* \longrightarrow (U\mathfrak{g})^\flat$. Therefore, $bT = b\vec{E}$.

(ii) It is standard that $C_1 = \frac{1}{2}\{\cdot, \cdot\} + bc_1'$, where c_1' is a Hochschild 1-cochain. By repeating an argument similar to that in (i), we can prove that c_1' can be chosen so that $c_1' = \vec{c}_1$ for some $c_1 \in C^\infty(\mathfrak{h}^*, (U\mathfrak{g})^\flat)$.

(iii) If $B = \delta_r X$, then $\vec{B} = [\pi, \vec{X}]$ according to the remark following Proposition 2.11. It is easy to check that the operator $T = 1 + \hbar^n \vec{X} + \hbar^{n+1}\vec{E_1}$ transforms $*_\hbar$ to another star-product, which coincides with $*'_\hbar$ up to order $n+1$. ∎

As a consequence, we have

COROLLARY 6.9. *If r is a non-degenerate triangular dynamical r-matrix and $M = \mathfrak{h}^* \times G$ its associated symplectic manifold, then every compatible $*$-product on M is strongly equivalent to a Fedosov $*$-product as constructed in Corollary* 5.2.

Proof. This follows essentially from the same argument as in the proof of Proposition 4.1 in [7]. We will omit it here. ∎

Combining with Theorem 6.6, we thus have proved:

THEOREM 6.10. *Let $M = \mathfrak{h}^* \times G$ be the symplectic manifold associated with a non-degenerate triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$. Then the equivalent classes of compatible $*$-products on M are classified by the relative Lie algebra cohomology (with coefficients being formal power series of $\hbar$)* $H^2(\mathfrak{g}, \mathfrak{h})[\![\hbar]\!]$.

Using Theorem 6.2, we are thus led to the following

THEOREM 6.11. *The equivalence classes of quantization of a non-degenerate triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$ are classified by the relative Lie algebra cohomology (with coefficients being formal power series of $\hbar$)* $H^2(\mathfrak{g}, \mathfrak{h})[\![\hbar]\!]$.

Remark. It would be interesting to see if this theorem can be proved by directly applying the usual classification theorem of star products on a symplectic manifold. One of the difficulties is that the characteristic class of a star product is usually difficult to computer. Recently, Tsygan comes up a nice way of redefining the characteristic class using the jet bundle. This may shed some new light on our problem.

Inspired by Kontesvich's formality theorem, we end this section with the following:

Conjecture. For an arbitrary classical triangular dynamical r-matrix $r: \mathfrak{h}^* \longrightarrow \wedge^2 \mathfrak{g}$, the quantization is classified by $\mathscr{M}_r(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$, the formal neighbourhood of r in the moduli space $\mathscr{M}(\mathfrak{g}[\![\hbar]\!], \mathfrak{h})$.

APPENDIX

In this section, we recall some basic ingredients of the Fedosov construction of $*$-products on a symplectic manifold, as well as some useful notations, which are used throughout the paper. For details, readers should consult [20, 21].

Let (M, ω) be a symplectic manifold of dimension $2n$. Then, each tangent space T_xM is equipped with a linear symplectic structure, which can be quantized using the standard Moyal–Weyl product. The resulting space is denoted by W_x. More precisely,

DEFINITION A.1. A formal Weyl algebra W_x associated to T_xM is an associative algebra with a unit over $\mathbb{C}$, whose elements consist of formal power series in $\hbar$ with coefficients being formal polynomials in T_xM. In other words, each element has the form

$$a(y, \hbar) = \sum \hbar^k a_{k,\alpha} y^\alpha, \tag{43}$$

where $y = (y^1, \ldots, y^{2n})$ is a linear coordinate system on T_xM, $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index, $y^\alpha = (y^1)^{\alpha_1} \cdots (y^{2n})^{\alpha_{2n}}$, and $a_{k,\alpha}$ are constants. The product is defined according to the Moyal–Weyl rule:

$$a * b = \sum_{k=0}^{\infty} \left(\frac{\hbar}{2}\right)^k \frac{1}{k!} \pi^{i_1 j_1} \cdots \pi^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}. \tag{44}$$

Let $W = \bigcup_{x \in M} W_x$. Then W is a bundle of algebras over M, called the Weyl bundle. Its space of sections ΓW forms an associative algebra with unit under the fiberwise multiplications. One may think of W as a "quantum tangent bundle" of M, whose space of sections ΓW gives rise to a deformation quantization for the tangent bundle TM, considered as a Poisson manifold with fiberwise linear symplectic structures. As in [20], by

W^+ we denote the extension of the algebra W consisting of those elements described as follows:

(i) elements $a \in W^+$ are given by series (43), but the powers of $\hbar$ can be both positive and negative;

(ii) the total degree $2k + |\alpha|$ of any term of the series is nonnegative;

(iii) there exists a finite number of terms with a given nonnegative total degree.

The center $Z(W)$ of ΓW consists of sections not containing y's, thus can be naturally identified with $C^\infty(M)[[\hbar]]$. By assigning degrees to y's and $\hbar$ with $\deg y^i = 1$ and $\deg \hbar = 2$, there is a natural filtration

$$C^\infty(M) \subset \Gamma(W_1) \subset \cdots \Gamma(W_i) \subset \Gamma(W_{i+1}) \cdots \subset \Gamma(W)$$

with respect to the total degree (e.g., any individual term in the summation of the RHS of Eq. (43) has degree $2k + |\alpha|$.)

A differential q-form with values in W is a section of the bundle $W \otimes \wedge^q T^*M$, which can be expressed locally as

$$a(x, y, \hbar, dx) = \sum \hbar^k a_{k, i_1 \cdots i_p, j_1 \cdots j_q} y^{i_1} \cdots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q}. \tag{45}$$

Here the coefficient $a_{k, i_1 \cdots i_p, j_1 \cdots j_q}$ is a covariant tensor symmetric with respect to $i_1 \cdots i_p$ and antisymmetric in $j_1 \cdots j_q$. For short, we denote the space of these sections by $\Gamma W \otimes \Lambda^q$. There is an associative product $\circ$ on $\Gamma W \otimes \Lambda^*$, which naturally extends the multiplication $*$ on ΓW and the wedge product on Λ^*:

$$(a \otimes \theta) \circ (b \otimes \omega) = (a * b) \otimes (\theta \wedge \omega), \qquad \forall a, b \in \Gamma W, \text{ and } \theta, \omega \in \Lambda^*. \tag{46}$$

The usual exterior derivative on differential forms extends, in a straightforward way, to an operator δ on W-valued differential forms:

$$\delta a = \sum_i dx^i \wedge \frac{\partial a}{\partial y^i}, \qquad \forall a \in \Gamma W \otimes \Lambda^*. \tag{47}$$

By δ^{-1}, we denote its "inverse" operator defined by

$$\delta^{-1} a = \sum_i \frac{1}{p+q} y^i \left(\frac{\partial}{\partial x^i} \lrcorner a \right) \tag{48}$$

when $p + q > 0$ and $\delta^{-1} a = 0$ when $p + q = 0$, where $a \in \Gamma W \otimes \Lambda^q$ is homogeneous of degree p in y.

There is a "Hodge"-decomposition,

$$a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00}, \qquad \forall a \in \Gamma W \otimes \Lambda^*, \tag{49}$$

where $a_{00}(x)$ is the constant term of a, i.e., the 0-form term of $a|_{y=0}$ or $a_{00}(x) = a(x, 0, 0, 0)$. The operator δ possesses most of the basic properties of the usual exterior derivatives. For example,

$$\delta^2 = 0 \qquad \text{and} \qquad (\delta^{-1})^2 = 0.$$

It is also clear that both δ and δ^{-1} commute with the Lie derivative; i.e., $\forall X \in \mathfrak{X}(M)$,

$$L_X \circ \delta = \delta \circ L_X, \qquad \text{and} \qquad L_X \circ \delta^{-1} = \delta^{-1} \circ L_X. \tag{50}$$

Let ∇ be a torsion-free symplectic connection on M and

$$\partial\colon \Gamma W \longrightarrow \Gamma W \otimes \Lambda^1$$

be its induced covariant derivative.

Consider a connection on W of the form

$$D = -\delta + \partial + \frac{i}{\hbar}[\gamma, \cdot\,], \tag{51}$$

with $\gamma \in \Gamma W \otimes \Lambda^1$.

Clearly, D is a derivation with respect to the Moyal–Weyl product; i.e.,

$$D(a * b) = a * Db + Da * b, \qquad \forall a, b \in \Gamma W. \tag{52}$$

A simple calculation yields that

$$D^2 a = -\left[\frac{i}{\hbar}\Omega, a\right], \qquad \forall a \in \Gamma W, \tag{53}$$

where

$$\Omega = \omega - R + \delta\gamma - \partial\gamma - \frac{i}{\hbar}\gamma^2. \tag{54}$$

Here $R = \frac{1}{4} R_{ijkl}\, y^i y^j\, dx^k \wedge dx^l$, and $R_{ijkl} = \omega_{im} R^m_{jkl}$ is the curvature tensor of the symplectic connection as defined by Eq. (21).

A connection of the form (51) is called *Abelian* if Ω is a scalar 2-form; i.e., $\Omega \in \Omega^2(M)[\![\hbar]\!]$. It is called a *Fedosov connection* if it is Abelian and in addition $\gamma \in \Gamma W_3 \otimes \Lambda^1$. For an Abelian connection, the Bianchi identity implies that $d\Omega = D\Omega = 0$; i.e., $\Omega \in Z^2(M)[\![\hbar]\!]$. In this case, Ω is called the Weyl curvature.

THEOREM A.2 (Fedosov [21]). *Let ∇ be a torsion-free symplectic connection and $\Omega = \omega + \hbar\omega_1 + \cdots \in Z^2(M)[\![\hbar]\!]$ a perturbation of the symplectic form in the space $Z^2(M)[\![\hbar]\!]$. There exists a unique $\gamma \in \Gamma W_3 \otimes \Lambda^1$ such that D, given by Eq.* (51)*, is a Fedosov connection, which has Ω as the Weyl curvature and satisfies*

$$\delta^{-1}\gamma = 0.$$

Proof. It suffices to solve the equation

$$\omega - R + \delta\gamma - \partial\gamma - \frac{i}{\hbar}\gamma^2 = \Omega. \tag{55}$$

This is equivalent to

$$\delta\gamma = \tilde{\Omega} + \partial\gamma + \frac{i}{\hbar}\gamma^2, \tag{56}$$

where

$$\tilde{\Omega} = \Omega - \omega + R. \tag{57}$$

Applying the operator δ^{-1} to Eq. (56) and using the Hodge decomposition (Eq. (49)), we obtain

$$\gamma = \delta^{-1}\tilde{\Omega} + \delta^{-1}\left(\partial\gamma + \frac{i}{\hbar}\gamma^2\right). \tag{58}$$

Note that $\gamma_{00} = 0$ since γ is a 1-form.

Take $\gamma_0 = \delta^{-1}\tilde{\Omega}$, and consider the iteration equation

$$\gamma_{n+1} = \gamma_0 + \delta^{-1}\left(\partial\gamma_n + \frac{i}{\hbar}\gamma_n^2\right), \qquad \forall n \geqslant 0. \tag{59}$$

Since the operator ∂ preserves the filtration and δ^{-1} raises it by 1, γ_n defined by Eq. (59) converges to a unique $\gamma \in \Gamma W \otimes \Lambda^1$, which is clearly a solution to Eq. (58). Moreover since γ_0 is at least of degree 3, γ is indeed an element in $\Gamma W_3 \otimes \Lambda^1$. ∎

Theorem A.2 indicates that a Fedosov connection D is uniquely determined by a torsion-free symplectic connection ∇ and a Weyl curvature $\Omega = \sum_{i=0}^{\infty} \hbar^i \omega_i \in Z^2(M)[\![\hbar]\!]$. For this reason, we will say that D is a Fedosov connection corresponding to the pair (∇, Ω).

If D is a Fedosov connection, the space of all parallel sections W_D automatically becomes an associative algebra. Let σ denote the projection from W_D to its center $C^\infty(M)[\![\hbar]\!]$ defined by $\sigma(a) = a|_{y=0}$.

THEOREM A.3 (Fedosov [21]). *For any $a_0(x, \hbar) \in C^\infty(M)[\![\hbar]\!]$ there is a unique section $a \in W_D$ such that $\sigma(a) = a_0$. Therefore, σ establishes an isomorphism between W_D and $C^\infty(M)[\![\hbar]\!]$ as vector spaces.*

Proof. The equation $Da = 0$ can be written as

$$\delta a = \partial a + \left[\frac{i}{\hbar}\gamma, a\right].$$

Applying the operator δ^{-1}, it follows from the Hodge decomposition (Eq. (49)) that

$$a = a_0 + \delta^{-1}\left(\partial a + \left[\frac{i}{\hbar}\gamma, a\right]\right). \tag{60}$$

In analogy to the proof of Theorem A.2, we can solve this equation by the iteration formula:

$$a_{n+1} = a + \delta^{-1}\left(\partial a_n + \left[\frac{i}{\hbar}\gamma, a_n\right]\right). \quad \blacksquare \tag{61}$$

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