Mean convergence of derivatives of Lagrange interpolation *

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Abstract


Weighted $L^p$ convergence of derivatives of Lagrange interpolation at the union of zeros of generalized Jacobi polynomials and some additional points is investigated.

Keywords: Lagrange interpolation, orthogonal polynomials.

1. Introduction

Necessary and sufficient conditions for the weighted $L^p$ convergence of interpolating Lagrange polynomials based on the zeros of generalized Jacobi polynomials were given in [7, Theorem 6, p. 695]. In the same paper a new interpolatory procedure was introduced, the so-called “quasi-Lagrange interpolation”, which is a polynomial $L_n^{(r,s)}$ interpolating the function at the zeros of orthogonal polynomials and, in addition, at ±1, where all the derivatives of $L_n^{(r,s)}$ up to the order $r - 1$ and $s - 1$ vanish. Necessary and sufficient conditions for the weighted mean convergence of such polynomials were given and it was proved that such a procedure may converge, when the usual interpolation diverges. In this paper we give a slightly more general definition of quasi-Lagrange interpolation, and we prove necessary and sufficient conditions for the convergence in $L^p$ spaces with general weights.

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Unfortunately, quasi-Lagrange interpolation generally preserves only the constant functions. Therefore it is seldom used in approximation theory and in numerical applications. Here we consider a polynomial \( L_{n,r,s} \) interpolating the function at the zeros of orthogonal polynomials and in \( r + s \) additional points in \([-1, 1]\) near \( \pm 1 \). Theorem 3.1 gives necessary and sufficient conditions for the convergence in weighted \( L^p \) spaces. In Theorem 3.2 we state sufficient conditions for the weighted \( L^p \) convergence of the derivatives of such sequences of polynomials. A consequence of these theorems are Corollaries 3.3 and 3.4, which are interesting in numerical quadrature and in numerical solution of differential and integrodifferential equations.

2. Preliminaries and notations

In what follows \( C \) denotes positive constants which can assume different values in different formulas. If \( A \) and \( B \) are two quantities depending on some parameters, we write \( A \sim B \iff \frac{|A|}{B} \leq C \) with \( C \) independent on the parameters. We will consider functions \( f \) with domain \([-1, 1]\). The classes \( C^q([-1, 1]) \), \( L^p([-1, 1]) \) and \( (L^\log)L^p([-1, 1]) \) are defined as usual, and, for sake of simplicity, we use the notation \( \| \cdot \|_p \) also if \( 0 < p < 1 \).

Special weights

The function \( \nu^{(\alpha, \beta)} \) is called a Jacobi weight if \( \nu^{(\alpha, \beta)}(x) = (1 - x)^\alpha(1 + x)^\beta \) where \( \alpha, \beta > -1 \). When we use \( \nu^{(\alpha, \beta)} \) as a weight function in \( L^p \) spaces without introducing orthogonal polynomials associated with it, then we allow the parameters \( \alpha \) and \( \beta \) to take arbitrary real values. The function \( w \) is called a generalized Jacobi weight (\( w \in \text{GJ} \)) if

\[
W(x) = \nu(x)(1 - x)^\gamma(1 + x)^\delta,
\]

where \( \nu \) is a fixed nonnegative integer, \( \alpha > -1, \beta > -1, \gamma_k > -1, k = 1, 2, \ldots, \nu, \) and \( -1 < t_1 < \cdots < t_r < 1 \). Here the function \( \nu \) is nonnegative and \( \nu^{1/2} \in L^\infty \). If, in addition, \( \nu \) is continuous and its modulus of continuity \( \omega \) satisfies \( \int_0^\infty \omega(\nu; u)u^{-1} du < \infty \), then we call \( \nu \) a generalized smooth Jacobi weight (\( \nu \in \text{GSJ} \)).

Lagrange interpolation

Let \( w \in \text{GSJ} \), and let \( \{ p_m(w) \}_{m=0}^\infty \) be the corresponding system of orthonormal polynomials, that is, \( p_m \) is a polynomial of degree \( m \) with positive leading coefficient and \( \int_{-1}^1 p_m(x) p_n(x) w(x) dx = 0 \) if \( m \neq n \). If \( \{ x_k \}_{k=1}^m \) \((x_k = x_k,m(w)) \) are the zeros of \( p_m(w) \) labelled in increasing order, and \( g \) is a bounded function, then \( L_m^w(g, g) \) denotes the Lagrange polynomial interpolating \( g \) at those points. If, together with \( \{ x_k \}_{k=1}^m \), we consider the \( r + s \) additional points \( \{ y_j = y_{j,m} \}_{j=1}^r \) and \( \{ z_i = z_{i,m} \}_{i=1}^s \) as well, such that

\[
-1 \leq y_1 < y_2 < \cdots < y_s < x_1 < \cdots < x_m < z_1 < \cdots < z_r \leq 1,
\]

where

\[
x_1 - y_s \sim m^{-2} \sim z_1 - x_m, \quad y_{j+1} - y_j \sim m^{-2} \sim z_{j+1} - z_j,
\]
then we denote by $L_{m,r,s}(w, f)$ the polynomial of degree $m + r + s - 1$ interpolating the function $f$ at the points $y_1, \ldots, y_s, x_1, \ldots, x_r, z_1, \ldots, z_r$.

If the function $f$ is not differentiable, then we assume that all the points in (2.1) are different. If $f \in C^{q}$, $q \geq 1$, then the multiplicity of each $y_i$ and $z_i$ is at most $q$. In such a case, $L_{m,r,s}(w, f)$ is an Hermite interpolating polynomial. Writing

$$A_0(x) = 1, \quad A_s(x) = \prod_{j=1}^{s} (x - y_j), \quad s > 0,$$

and

$$B_0(x) = 1, \quad B_r(x) = \prod_{i=1}^{r} (x - z_i), \quad r > 0,$$

and denoting by $[u_1, \ldots, u_p; g]$ the divided difference of $g$ at the points $u_1, \ldots, u_p$, we can write

$$L_{m,r,s}(w, f; x) = A_s(x) B_r(x) L_m\left(w, \frac{f}{A_s B_r}; x\right) + A_s(x) p_m(w, x) L_r\left(\frac{f}{A_s p_m(w)}; x\right)
+ B_r(x) p_m(w, x) L_s\left(\frac{f}{B_r p_m(w)}; x\right),$$

(2.2)

where

$$L_r\left(\frac{f}{A_s p_m(w)}; x\right) = \frac{f(z_1)}{A_s(z_1) p_m(w, z_1)}
+ \sum_{i=2}^{r} (x - z_1) \cdots (x - z_{i-1}) \left[z_1, \ldots, z_i; \frac{f}{A_s p_m(w)}\right], \quad r > 0,$$

(2.3)

and

$$L_s\left(\frac{f}{B_r p_m(w)}; x\right) = \frac{f(y_1)}{B_r(y_1) p_m(w, y_1)}
+ \sum_{i=2}^{s} (x - y_1) \cdots (x - y_{i-1}) \left[y_1, \ldots, y_i; \frac{f}{B_r p_m(w)}\right], \quad s > 0.$$

(2.4)

We complete the definition by putting $L_0 = 0$ and $L_{m,0,0}(w, f) = L_m(w)$.

**Quasi-Lagrange interpolation**

Let $c_1$ and $c_2$ be two fixed points in $[-1, 1]$, and let $f$ be a bounded function. The quasi-Lagrange interpolating polynomial $L_{m,r,s}^{(c_1, c_2)}(w, f)$ corresponding to $f$ is the unique polynomial of degree at most $m + r + s - 1$ satisfying

$$L_{m,r,s}^{(c_1, c_2)}(w, f; x_k) = f(x_k), \quad k = 1, 2, \ldots, m,$$

$$L_{m,r,s}^{(c_1, c_2)}(w, f; y_i) = f(c_1), \quad i = 1, 2, \ldots, s,$$

and

$$L_{m,r,s}^{(c_1, c_2)}(w, f; z_j) = f(c_2), \quad j = 1, 2, \ldots, r,$$
Using the previous notations, we can express $L^*_m(w, f)$ as

$$L^*_m(w, f; x) = A_s(x) B_r(x) L_m(w, f; x)$$

$$+ f(c_1) A_s(x) p_m(w, x) L_r(w, f; x)$$

$$+ f(c_2) B_r(x) p_m(w, x) L_s(w, f; x).$$

If, in particular, $c_1 = -1$, $c_2 = +1$, and $z_1 = \ldots = z_r = 1$, and $y_1 = \ldots = y_s = -1$, then, by (2.3) and (2.4) we obtain the quasi-Lagrange interpolation as defined in [7, p.672]. Though $L^*_m(w)$ preserves constant functions, it does not preserve every polynomial of degree at most $m + r + s - 1$.

In addition, we will also consider the interpolation

$$\tilde{L}^*_m(w, f; x) = v^{(r,s)}(x) L_m(w, f; x).$$

$\tilde{L}^*_m(w, f)$ interpolates $f$ at the points $x_k$, it vanishes with its derivatives at the points $\pm 1$, and, therefore, it does not even preserve constant functions.

Finally $\mathcal{L}_{m,r,s}(w, f)$ will denote one of the three polynomials

$$L_{m,r,s}(w, f), \quad L^*_m(w, f), \quad \tilde{L}^*_m(w, f).$$

### 3. Main results

**Theorem 3.1.** Let $w \in \mathcal{G}S_j$, and let $u > 0 \in (L \log^+ L)^p([-1, 1]),$ with $0 < p < \infty$. Let $r \in \mathbb{R}$, $s \in \mathbb{R}$, and let $0 < q < \infty$. If

$$v^{(r,s)}(w, f) \in L^q([-1, 1]), \quad v^{(r,s)}(w, f) \in L^q([-1, 1]).$$

Then

$$\lim_{m \to \infty} \| f - \mathcal{L}_{m,r,s}(w, f) \|_p = 0, \quad \forall f \in C([-1, 1]),$$

if and only if

$$v^{(r,s)}(w, f) \in L^q([-1, 1]), \quad v^{(r,s)}(w, f) \in L^q([-1, 1]).$$

Moreover, there exists a positive function $u$, such that $u \in L^p \setminus (L \log^+ L)^p$ and, nevertheless, conditions (3.2) do not imply (3.1).

This theorem is a slight generalization of [7, Theorem 6, p.695]. It shows that the sequences $\{L_{m,r,s}(w)\}$, $\{L^*_m(w)\}$, and $\{\tilde{L}^*_m(w)\}$ are three essentially equivalent interpolatory procedures. However, the first one has the advantage that it preserves polynomials of degree at most $m + r + s - 1$. In addition, we have the following theorem.

**Theorem 3.2.** Let $w \in \mathcal{G}S_j$, $u \in \mathcal{G}S_j$, $f \in C^q([-1, 1])$, where $q \geq 0$, and let $0 < p < \infty$. If

$$v^{(q/2 - r, q/2 - s)}(w, f) \in L^q([-1, 1]),$$

then

$$v^{(q/2 - r, q/2 - s)}(w, f) \in L^q([-1, 1]).$$
and
\[
\begin{align*}
&u \in L^p([-1, 1]), \\
&v^{(r-1/2,s-1/2)}u \in L^p([-1, 1]), \\
&\frac{v^{(r-1/2,s-1/2)}u}{\sqrt{\mu}(1/2,1/2)} \in L^p([-1, 1]),
\end{align*}
\]
(3.4)

where \( r, s \) and \( l \) are nonnegative integers and \( l \leq q \), then
\[
\| [f - L_{m,r,s}(w, f)]^{(l)}u \|_p \leq C \omega \left( f^{(q)}; \frac{1}{m} \right) m^{l-q},
\]
(3.5)

for \( m \geq 4q + 5 \), where \( \omega \left( f^{(q)} \right) \) denotes the modulus of continuity of \( f^{(q)} \) and \( C \) is a positive constant independent of \( m \) and \( f \).

This theorem improves an analogous theorem by Nevai and Vértesi [8, Theorem 2, p.493] and generalizes some results by Szabados and Varma [11, Theorem 1].

If, in particular,
\[
\lim_{m \to \infty} \| [f - L_m(w, f)]u \|_p = 0,
\]
(3.6)

then by Theorem 3.1 applied with \( r = s = 0 \) (or by [7, Theorem 6, p.695])
\[
\frac{u}{\sqrt{\mu}(1/2,1/2)} \in L^1([-1, 1]) \quad \text{and} \quad \frac{u}{\sqrt{\mu}(1/2,1/2)} \in L^p([-1, 1]).
\]

Therefore, if (3.6) is true and \( r = \frac{1}{2}(q + 1) \), then (3.5) holds. This agrees with a result given in [3, Corollary 2]. However, (3.5) and (3.6) are not equivalent. Consider the two weight functions
\[
w(x) = \phi(x)(1-x)^{\alpha} \prod_{k=1}^{\mu} |t_k - x|^{\gamma_k} (1+x)^{\beta}, \quad w \in GSJ,
\]
(3.7)

and
\[
u(x) = g(x)(1-x)^{\gamma} \prod_{k=1}^{\nu} |\tau_k - x|^{\lambda_k} (1+x)^{\delta}, \quad u \in GJ,
\]
(3.8)

with the condition
\[
\prod_{k=1}^{\nu} |\tau_k - x|^{\lambda_k} \in L^p([-1, 1]),
\]
(3.9)

Obviously, (3.6) does not follow from (3.9). Applying Theorem 3.2 we get the following corollary.

**Corollary 3.3.** Let \( f \in C^{(q)}([-1, 1]), \) where \( q \geq 0 \). Let \( u \) and \( w \) be defined by (3.7) and (3.8). Assume \( u \in L^p([-1, 1]) \) and (3.9) holds. Let \( 0 \leq l \leq q \) and \( p \in (0, \infty) \). Then there exist two integers \( r \) and \( s \), defined by
\[
\frac{l}{2} + \frac{\alpha}{2} + \frac{1}{4} - \gamma - \frac{1}{p} < r < \frac{q}{2} + \frac{\alpha}{2} + \frac{1}{4} + 1
\]
such that
\[
\left[ \int_{-1}^{1} \left| f^{(l)}(x) - L^{(l)}_{m,r,s}(w, f; x) \right|^p u^p(x) \, dx \right]^{1/p} \leq C m^{l-q} \omega \left( f^{(q)}; \frac{1}{m} \right),
\]
for \( m \geq 4q + 5 \).

Corollary 3.3 says that, if \( L_m(w, f) \) diverges because of the choice of the exponents \( \alpha, \beta, \gamma \) and \( \delta \) in the weights \( u \) and \( w \) (i.e., (3.9) holds), then, by adding additional points near \( \pm 1 \), we can transform \( L_m(w, f) \) into \( L_{m,r,s}(w, f) \) which will already approximate the function and its derivatives in \( L_p \).

It also follows from Theorem 3.2 that if (3.9) does not hold (i.e., the divergence of \( L_m(w, f) \) depends on the exponents \( \gamma_k \) and \( \lambda_k \)), then adding additional points near \( \pm 1 \) is useless.

A useful consequence of Corollary 3.3 is the following corollary.

Corollary 3.4. Let \( f \in C^{(q)}([-1, 1]), \) where \( q \geq 0, 0 \leq l \leq q, w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \) and let \( u \) be defined by (3.8). Let \( 0 \leq l \leq q \) and \( p \in (0, \infty) \). Then there exist two integers \( r \) and \( s \) defined by
\[
\frac{1}{2}(l + \alpha) + \frac{1}{4} - \gamma - 1 < r < \frac{1}{2}(q + \alpha) + \frac{1}{4} + 1
\]
and
\[
\frac{1}{2}(l + \beta) + \frac{1}{4} - \delta - 1 < s < \frac{1}{2}(q + \beta) + \frac{1}{4} + 1,
\]
such that
\[
\left[ \int_{-1}^{1} \left| f^{(l)}(x) - L^{(l)}_{m,r,s}(w, f; x) \right|^p u^p(x) \, dx \right]^{1/p} \leq C m^{l-q} \omega \left( f^{(q)}; \frac{1}{m} \right),
\]
for \( m \geq 4q + 5 \).

The last corollary has interesting applications. In the theory of numerical quadratures, in integral equations and in differential equations one frequently needs to evaluate integrals of the form \( \int_{-1}^{1} f^{(q)}(t) u(t) \, dt \), where \( q \geq 0 \) and \( u \in GJ \). An often used numerical procedure consists of replacing the function \( f \) by a Lagrange interpolating polynomial based on the zeros of Jacobi polynomials; this is called a "product rule". Corollary 3.4 guarantees the existence of convergent "product rules". In order to evaluate such integrals, sometimes it is necessary to use values of the functions and its derivatives at additional points as well. Corollary 3.4 shows that such a choice is feasible by choosing the parameters of the corresponding Jacobi weight appropriately. Corollary 3.4 is a generalization of some results of Sloan and Smith (cf. [9,10]).

4. Proofs of the theorems

We need some preliminary results. The following lemma is due to Gopengauz and Teliakovskii (cf. [2]).
Lemma 4.1. Given \( q = 0, 1, \ldots, n = 1, 2, \ldots, \) and \( f \in C^{(q)}([-1, 1]), \) there exists an algebraic polynomial \( Q_n \) of degree \( n \geq 4q + 5, \) such that
\[
\left| \left( f(x) - Q_n(x) \right)^{(l)} \right| \leq C \left( \frac{\sqrt{1-x^2}}{n} \right)^{q-l} \omega \left( f^{(q)}; \frac{\sqrt{1-x^2}}{n} \right), \quad -1 \leq x \leq 1, \tag{4.1}
\]
for \( l = 0, 1, \ldots, q, \) where \( C \) is a positive constant independent of \( f \) and \( n. \)

Lemma 4.2. Given \( q = 0, 1, \ldots, m \geq 4q + 5, \) and \( f \in C^{(q)}([-1, 1]), \) define \( r_m \) by \( r_m = f - Q_m, \) where \( Q_m = Q_m(f) \) is a polynomial satisfying (4.1). Let \( w \in \text{GSJ} \) be defined by (3.7), and let \( L_r \) and \( L_s \) be defined by (2.3) and (2.4), respectively. If \( v^{(q/2-r/2-s)}(w) \in L^1([-1, 1]), \) then the inequalities
\[
\left| \frac{r_m}{A_s p_m(w)} \right| \leq C m^{-q} \omega \left( f^{(q)}; \frac{1}{m^2} \right) \left( \frac{1}{1-x} \right)^{q+\alpha+1/2} \tag{4.2}
\]
and
\[
\left| \frac{r_m}{B_r p_m(w)} \right| \leq C m^{-q} \omega \left( f^{(q)}; \frac{1}{m^2} \right) \left( \frac{1}{1+x} \right)^{q+\beta+1/2} \tag{4.3}
\]
hold for every \( x \in [-1, 1] \) and \( f \in C^{(q)}([-1, 1]) \) where the constant \( C \) is independent of \( f \) and \( m. \)

Proof. We will only prove (4.2), whereas (4.3) can be proved analogously. Notice that
\[
\left| r_m^{(k)}(z_i) \right| \leq C m^{2k-2q} \omega \left( f^{(q)}; \frac{1}{m^2} \right), \quad k = 0, 1, \ldots, q, \ i = 1, \ldots, r, \ r \leq q + 1.
\]
Moreover, if the points \( \{ z_k \} \) satisfy (2.1), then
\[
\left[ z_1, \ldots, z_i; \frac{r_m}{A_s p_m(w)} \right] = \frac{1}{(i-1)!} \left( \frac{r_m(\xi_i)}{A_s(\xi_i) p_m(w, \xi_i)} \right)^{(i-1)},
\]
where
\[
z_1 \leq \xi_i \leq z_i, \quad i = 1, 2, \ldots, r.
\]
By Leibnitz's rule,
\[
\left( \frac{r_m(\xi_i)}{A_s(\xi_i) p_m(w, \xi_i)} \right)^{(i-1)} \leq \sum_{k=0}^{i-1} \binom{i-1}{k} \left| r_m^{(k)}(\xi_i) \right| \left( \frac{1}{A_s(\xi_i) p_m(w, \xi_i)} \right)^{(i-k)} \leq C \omega \left( f^{(q)}; \frac{1}{m^2} \right) \sum_{k=0}^{i-1} \binom{i-1}{k} m^{2k-2q} \left( \frac{1}{A_s(\xi_i) p_m(w, \xi_i)} \right)^{(i-k)}.
\]
Since all derivatives of $A_s^{-1}(x)$ are bounded for $x > 0$, we have

$$\left( \frac{r_m(\xi_i)}{A_s(\xi_i)p_m(w; \xi_i)} \right)^{(i-1)} \leq C\omega(f^{(q)}; \frac{1}{m^2}) \sum_{k=0}^{i-1} \binom{i-1}{k} m^{2k-2q} \sum_{l=0}^{i-1-k} \binom{i-1-k}{l} \left| \frac{1}{p_m(w; \xi_i)} \right|^{(l)}.$$

(4.4)

Recall that $z_1$ is chosen in such a way that $x_m + C/m^2 \leq z_1 \leq 1$. We have

$$|p_m(w, t)| - p_m(w, 1) = m^{a+1/2}$$

(cf. [5, Theorem 9.33, p.171]) and

$$\left( \frac{1}{p_m(w, t)} \right)^{(i)} \sim \frac{m^2}{p_m(w, 1)},$$

(cf. [7, proof of formula (22), p.674]) for $m = 1, 2, \ldots$ and $z_1 \leq t \leq 1$. Therefore, using the identity

$$\left( \frac{1}{p_m(w, t)} \right)^{(i)} = -\frac{1}{p_m(w, t)} \sum_{j=0}^{i-1} \binom{i}{j} \left( \frac{1}{p_m(w, t)} \right)^{(j)} p_m^{(i-j)}(w, t),$$

we obtain

$$\left| \left( \frac{1}{p_m(w, t)} \right)^{(i)} \right| \leq C m^{2l-a-1/2}, \quad z_1 \leq t \leq 1,$$

for $m = 1, 2, \ldots$. Hence, by (4.4),

$$\frac{1}{(i-1)!} \left( \frac{r_m(\xi_i)}{A_s(\xi_i)p_m(w; \xi_i)} \right)^{(i-1)} \leq C m^{2i-2-2q-a-1/2} \omega(f^{(q)}; \frac{1}{m^2}).$$

Using (2.4) and the inequality $|(x-z_1) \cdots (x-z_{i-1})| \leq (\sqrt{1-x} + 1/m)^{2l-2}$, we obtain

$$\left| L_r \left( \frac{r_m}{A_s p_m(w); x} \right) \right| \leq C m^{-2q-a-1/2} \omega(f^{(q)}; \frac{1}{m^2}) \sum_{i=0}^{r-1} \left[ 1 + m\sqrt{1-x} \right]^{2i}, \quad -1 \leq x \leq 1,$$

(4.4')

for $m \geq 4q + 5$. On the other hand, if $q = 0$,

$$L_r \left( \frac{r_m}{A_s p_m(w); x} \right) = \sum_{k=1}^{r} \prod_{i \neq k}^r \frac{x - z_i}{z_k - z_i} \frac{r_m(z_k)}{A_s(z_k)p_m(w; z_k)}.$$

Since

$$\left| \prod_{i \neq k}^r \frac{x - z_i}{z_k - z_i} \right| \leq \text{const.}(m\sqrt{1-x} + 1)^{2r-2}, \quad x \leq 1.$$

If $1 - x \leq Cm^{-2}$, then this proves (4.2). On the other hand, if $1 - x > Cm^{-2}$, then

$$\left| L_r \left( \frac{r_m}{A_s p_m(w); x} \right) \right| \leq C m^{-2q-a-5/2+2r} \omega(f^{(q)}; \frac{1}{m^2})(1-x)^{r-1}.$$
By the assumptions we have $\frac{1}{2} q - r + \frac{1}{2} \alpha + \frac{\beta}{2} > 0$. Therefore,

$$
L_r \left( \frac{r_m}{A_s p_m(w)} ; x \right) \leq C m^{-q} \omega \left( f^{(q)} ; \frac{1}{m^2} \right) \left( \sqrt{1 - x} \right)^{q + \alpha + 1/2}, \quad 1 - x > C m^{-2},
$$

for $m \geq 4q + 5$, which proves (4.2) for $1 - x > C m^{-2}$ as well. \( \square \)

**Lemma 4.3.** Let $w$, $L_r$ and $L_s$, be defined as in Lemma 4.2. If $v^{(-r, -s)} \| w v^{(1/2, 1/2)} \in L^1([-1, 1])$, then

$$
\left| L_r \left( \frac{1}{A_s p_m(w)} ; x \right) \right| \leq C \left( \sqrt{1 - x} + \frac{1}{m} \right)^{\alpha + 1/2}, \quad -1 < x < 1, \quad (4.5)
$$

$$
\left| L_s \left( \frac{1}{B_r p_m(w)} ; x \right) \right| \leq C \left( \sqrt{1 + x} + \frac{1}{m} \right)^{\beta + 1/2}, \quad -1 < x < 1, \quad (4.6)
$$

for $m = 1, 2, \ldots$, where the constant $C$ is independent of $m$.

The proof of Lemma 4.3 is similar to the proof of Lemma 4.2.

**Proof of Theorem 3.1.** For $\mathcal{L}_{m, r, s}(w, f) = L_m^{(r, s)}(w, f)$, Theorem 3.1 is [7, Theorem 6, p.695]. Next let $\mathcal{L}_{m, r, s}(w, f) = L_m^{(r, s)}(w, f)$. First we prove that the conditions in (3.2) imply (3.1). By the uniform boundedness principle, it is sufficient to prove that (3.2) imply

$$
\sup_{m \geq 1} \left\| L_m^{(r, s)}(w, f) u \right\|_p \leq C \| f \|_\infty, \quad \forall f \in C([-1, 1]), \quad (4.7)
$$

with a suitable constant $C$ independent of $f$. By (2.5) we have

$$
\left\| L_m^{(r, s)}(w, f) u \right\|_p \leq C \left\| v^{(r, s)} L_m(w, f) u \right\|_p
$$

$$
+ C \| f \|_\infty \left\{ \left\| p_m(w) v^{(0, s)} L_r \left( \frac{1}{A_s p_m(w)} \right) u \right\|_p
$$

$$
+ \left\| p_m(w) v^{(r, 0)} L_s \left( \frac{1}{B_r p_m(w)} \right) u \right\|_p \right\},
$$

for all $p \in (0, \infty)$. In view of Badkov's pointwise estimate for the generalized Jacobi polynomials [1, Theorem 1.1, p.226],

$$
| p_m(w, x) | \leq C \left[ 1 + \frac{1}{\sqrt{w(x)/1 - x^2}} \right], \quad -1 < x < 1, \quad (4.8)
$$

$m = 1, 2, \ldots$ (cf. [5, Lemma 9.29 and Theorem 3.33, pp. 170, 171]), we can use Lemma 4.3 and (3.2) to obtain

$$
\left\| p_m(w) v^{(0, s)} L_r \left( \frac{1}{A_s p_m(w)} \right) u \right\|_p \leq C
$$
Finally, \[7, \text{Theorem 1, p.6801, (3.2)}\) guarantees
\[
\|v^{(r,s)}L_m(w, f^{(r,-r,-s)})u\|_p \leq C \|f\|_\infty,
\]
that is, (3.1) is satisfied. Conversely, let us assume that (3.1) holds. Then (4.7) is true for every continuous function \(f\) such that \(f(c_1) = f(c_2) = 0\). Therefore,
\[
\sup_{m \geq 1} \left\| A_s B_r L_m^{(r,s)}(w, f)u \right\|_p \leq C \|f\|_\infty,
\]
from which
\[
\sup_{m \geq 1} \left\| v^{(r,s)}L_m^{(r,s)}(w, f)u \right\|_p \leq C \|f\|_\infty
\]
follows. Therefore, (3.2) follows from \[7, \text{Theorem 2, p.6861}\]. The statement regarding the existence of a positive function \(u\), such that \(u \in L^p \setminus (L \log L)^p\) and conditions (3.2) do not imply (3.1), also follows \[7, \text{Theorem 2, p.6861}\]. For \(\mathcal{L}_{m,r,s}(w, f) = L_{m,r,s}(w, f)\) the proof is analogous. \(\square\)

**Proof of Theorem 3.2.** Let \(Q_m = Q_m(f)\) be a polynomial satisfying (4.1) and let \(r_m = f - Q_m\). Applying the weighted Markov–Bernstein inequality \[6, \text{Theorem 5, p.2421}\], (cf. \[4, \text{Theorem 3, p.100}\]) with generalized Jacobi weights, we obtain
\[
\left\| [f - L_{m,r,s}(w, f)]^{(l)}u \right\|_p \leq \|r_m^{(l)}u\|_p + \left\| L_{m,r,s}(w, r_m) \right\|_p
\leq C_m l^{-q} \left\| v^{(l)}(f^{(q)}, \frac{1}{m}) \right\|_p + C_m l^{-l/2,q} \left\| L_{m,r,s}(w, r_m) v^{(-l/2,-l/2)}u \right\|_p,
\]
where \(0 < p < \infty\). Therefore, by (2.2) we have
\[
\left\| L_{m,r,s}(w, r_m) v^{(-l/2,-l/2)}u \right\|_p
\leq C \left[ \left\| A_s B_r L_m(w, \frac{r_m}{A_s B_r}) u^{(-l/2,-l/2)} \right\|_p + A_s p_m(w) L_r \left( \frac{r_m}{A_s p_m(w)} \right) u^{(-l/2,-l/2)} \right]_p + B_r p_m(w) L_r \left( \frac{r_m}{B_r p_m(w)} \right) u^{(-l/2,-l/2)} \right\|_p = C[I_1 + I_2 + I_3].
\]
To estimate \(I_1, I_2\) and \(I_3\), we observe that \(u \in GJ\) and therefore it has a representation (3.8). For every \(C > 0\), we put
\[
\Delta_m(C) = \left[ -1 + \frac{C}{m^2}, 1 - \frac{C}{m^2} \right] \setminus \bigcup_{k=1}^{\mu} \left[ y_k - \frac{C}{m}, y_k + \frac{C}{m} \right],
\]
and we denote by $1^C_m$ the characteristic function of $\Delta_m(C)$. Then there exists $C > 0$ such that for every polynomial $R$ of degree at most $m$ the inequality
\[ \| R \|_p \leq C \| R \|_p 1^C_m, \quad 0 < p < \infty, \]
holds (cf. [5, Theorem 6.3.14, p.113, and Theorem 6.3.28 and Remark 6.3.29, p.120]). Then, since $| A_s(x) B_r(x) | \sim v^{(r-s)}(x)$ for $|x| \leq 1 - C m^{-2}$, we have
\[ I_1 \leq \sup_{\| f \|_x = 1} \| v^{(r-s)} L_m(w, f_p q^{2-r} q^{2-s}) (uw(-l/2,-l/2)) \| p m^{-q} \omega \left( f(q); \frac{1}{m} \right). \]
In view of the assumptions and by [7, Theorem 1, p.680] we have
\[ I_1 \leq C m^{-q} \omega \left( f(q); \frac{1}{m} \right). \quad (4.11) \]
Moreover, by (4.8),
\[ | p_m(w, x) | \leq \frac{C}{w(x) \sqrt{1 - x^2}}, \quad |x| \leq 1 - C m^{-2}, \]
so that there exist $C > 0$ such that
\[ I_2 \leq C \left\| v^{(-l/2,-l/2)} A_s L_r \left( \frac{r_m}{A_s p_m(w)} \right) 1^C_m p_m(w) u \right\|_p \leq C \left\| v^{(-l/2,-l/2)} L_r \left( \frac{r_m}{A_s p_m(w)} \right) \frac{u}{w^{(1/2,1/2)}} \right\|_p, \]
and by Lemma 4.2
\[ I_2 \leq C m^{-q} \omega \left( f(q); \frac{1}{m} \right) \left\| v^{(q/2-l/2,-l/2)} w^{(a/2+1/4,0)} \right\|_{w^{(1/2,1/2)}}. \]
Therefore by the assumptions (3.3) and (3.4),
\[ I_2 \leq C m^{-q} \omega \left( f(q); \frac{1}{m} \right). \quad (4.12) \]
Similarly,
\[ I_3 \leq C m^{-q} \omega \left( f(q); \frac{1}{m} \right). \quad (4.13) \]
Combining (4.10)–(4.13) we obtain
\[ \| L_{m,r,s}(w, r_m) v^{(-l/2,-l/2)} u \|_p \leq C m^{-q} \omega \left( f(q); \frac{1}{m} \right), \]
and now the theorem follows from (4.9).

**References**


