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The Cohomology of the Regular Semisimple Variety

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We use the equivariant cohomology of hyperplane complements and their toral counterparts to give formulae for the Poincaré polynomials of the varieties of regular semisimple elements of a reductive complex Lie group or Lie algebra. As a result, we obtain vanishing theorems for certain of the Betti numbers. Similar methods, using l-adic cohomology, may be used to compute numbers of rational points of the varieties over the finite field $\mathbb{F}_q$. In the classical cases, one obtains, both for the Poincaré polynomials and for the numbers of rational points, polynomials which exhibit certain regularity conditions as the dimension increases. This regularity may be interpreted in terms of functional equations satisfied by certain power series, or in terms of the representation theory of the Weyl group. © 1998 Academic Press

In this paper we give formulae for the Poincaré and weight polynomials of the variety of regular semisimple elements of a complex reductive algebraic group and its Lie algebra. The results use the equivariant cohomology of hyperplane complements and their toral counterparts. One of the applications of our results is a vanishing theorem (see (3.4) below) for the top cohomology in the Lie algebra case. Our general formulae are applied to give explicit polynomials in the case of classical complex Lie algebras of type A, B, C, or D. These are quite manageable and easily yield the relevant Betti numbers (see (5.8), (7.6), and (7.11) below).

As a special case, we give an explicit formula for the Poincaré polynomial of the variety of $n \times n$ matrices with distinct eigenvalues. In this case we prove a stability result, which asserts that the first $[n/2] + 1$ Betti numbers of this variety are "independent of n" in the sense that there is a single universal power series whose first $[n/2] + 1$ coefficients are the first $[n/2] + 1$ Betti numbers of the regular semisimple variety of $\mathfrak{gl}_n$ for all n. The same thing applies in the case of algebras of type B or C.

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Our method may also be used to compute numbers of rational points over finite fields, which we do in Section 8 below. In this case a similar analysis to that which applies to the Poincaré polynomials shows that the proportion of regular semisimple matrices over a finite field $\mathbb{F}_q$ is a polynomial in q^{-1}, whose initial sequence of coefficients stabilizes as the size of the matrices increases. Regularity statements such as these have interpretations both in terms of functional equations for certain power series and in terms of the representation theory of the Weyl groups. We discuss both below, although we leave open many questions.

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1. INTRODUCTION AND NOTATION

Let G be a complex connected reductive algebraic group. We fix a maximal torus T and Borel subgroup $B \supseteq T$ of G and write $W = N_G(T)/T$ for the Weyl group. Then $B = TU$, where U is the unipotent radical of B. We write $\mathfrak{G}$, $\mathfrak{B}$, $\mathfrak{T}$, $\mathfrak{U}$ for the Lie algebras of G, B, T, and U, respectively. An element $x \in G$ is *regular* (cf. [K], [St]) if its centralizer $C_G(x)$ has minimal dimension (which is equal to $r = \operatorname{rank} G = \dim T$). Similarly, $X \in \mathfrak{G}$ is regular if $C_G(X)$ $(= \{g \in G | \operatorname{Ad} g(X) = X\})$ has minimal dimension r (here Ad denotes the adjoint action of G on $\mathfrak{G}$).

(1.1) DEFINITION. (i) G_{rs} (resp. $\mathfrak{G}_{rs}$) is the set of regular semisimple elements of G (resp. $\mathfrak{G}$).

(ii) T_{rs} (resp. $\mathfrak{T}_{rs}$) is the set of elements of T (resp. $\mathfrak{T}$) which are regular in G (resp. $\mathfrak{G}$).

It is known that G_{rs} (resp. $\mathfrak{G}_{rs}$) is open dense in G (resp. $\mathfrak{G}$). Moreover we have, recalling that W acts on $\mathfrak{T}$ as a finite reflection group, the following well known characterization of regular semisimple elements.

(1.2) LEMMA. *Let $x \in T$ (resp. $\xi \in \mathfrak{T}$). The following are equivalent.*

(i) *x (resp. ξ) is regular.*

(ii) *The centralizer of x (resp. ξ) in W is trivial.*

(iii) *x is not annihilated by any root of G with respect to T (resp. ξ is not on any reflecting hyperplane for the action of W on $\mathfrak{T}$).*

Our results will be formulated in terms of certain polynomials which are defined as follows. Let X be a complex algebraic variety and suppose a finite group Γ acts on X as a group of automorphisms. Denote by $H^i(X)$ the complex (singular or de Rham) cohomology of X and by $H_c^i(X)$ the complex cohomology with compact supports, both regarded as Γ-modules.

As in [DL, Sect. 1] each cohomology space $H_c^i(X)$ has a Γ-invariant weight filtration

$$0 \subset W_0 H_c^j(X) \subset W_1 H_c^j(X) \subset \cdots \subset W_{2\dim X} H_c^j(X) = H_c^j(X). \quad (1.3)$$

If we denote the graded quotients by

$$\mathrm{Gr}_m H_c^j(X) = W_m H_c^j(X) / W_{m-1} H_c^j(X)$$

then we may form the weight m equivariant Euler characteristic.

$$E_{m,c}^{\Gamma}(X) = \sum_j (-1)^j \,\mathrm{Gr}_m\, H_c^j(X) \quad (1.4)$$

This is to be thought of as an element of $R(\Gamma)$, the Grothendieck ring of Γ.

(1.5) Definition. Let S be a complex algebraic variety with a Γ-action, where Γ is a finite group. Define elements $P_X^{\Gamma}(t)$ and $Q_{X,c}^{\Gamma}(t)$ of $R(\Gamma)[t]$ by

$$P_X^{\Gamma}(t) = \sum_j H^j(X) t^j$$

$$Q_{X,c}^{\Gamma}(t) = \sum_m E_{m,c}^{\Gamma}(X) t^m.$$

The reason for considering $Q_{X,c}^{\Gamma}(t)$ is its "Boolean additivity" (cf. [DL, Section 2]) which asserts that if Y and Z are complementary Γ-invariant subvarieties of X which are respectively open and closed, then $Q_{X,c}^{\Gamma}(t) = Q_{Y,c}^{\Gamma}(t) + Q_{Z,c}^{\Gamma}(t)$.

(1.6). *We write, for $g \in \Gamma$,*

$$P_X^{\Gamma}(g,t) = \sum_j \operatorname{trace}\big(g, H^j(X)\big) t^j$$

and

$$Q_{X,c}^{\Gamma}(g,t) = \sum_m \operatorname{trace}\big(g, E_{m,c}^{\Gamma}(X)\big) t^m.$$

If $\Gamma = \{e\}$, we drop the superscript and replace all modules by their dimension. Our purpose here is to compute these polynomials when $X = G_{rs}$ or $\mathfrak{G}_{rs}$.

2. AN UNRAMIFIED COVERING

We recall in this section that the part of the "Grothendieck–Springer resolution" which lies over G_{rs} (resp. $\mathfrak{G}_{rs}$) is unramified and show that this fact reduces our problem to W-equivariant problems concerning T_{rs} (resp. $\mathfrak{T}_{rs}$) and G/T.

(2.1) DEFINITION. Define varieties $\tilde{G}_{rs}$ and $\tilde{\mathfrak{G}}_{rs}$ by

$$\tilde{G}_{rs} = \left\{(g, xT) \in G_{rs} \times G/T \,|\, x^{-1}gx \in T\right\}$$

$$\tilde{\mathfrak{G}}_{rs} = \left\{(\xi, xT) \in \mathfrak{G}_{rs} \times G/T \,|\, \mathrm{Ad}\, x^{-1}\xi \in \mathfrak{T}\right\}.$$

(2.2) PROPOSITION (cf. [Sh]). (i) *The first projection $\tilde{G}_{rs} \to G_{rs}$ is an unramified covering with covering group W and similarly for $\tilde{\mathfrak{G}}_{rs} \to \mathfrak{G}_{rs}$.*

(ii) *We have W-equivariant isomorphisms*

$$\tilde{G}_{rs} \xrightarrow{\sim} T_{rs} \times G/T$$

$$\tilde{\mathfrak{G}}_{rs} \xrightarrow{\sim} \mathfrak{T}_{rs} \times G/T,$$

where $w \in W$ acts on G/T via $xT \mapsto xwT$, on T_{rs} and $\mathfrak{T}_{rs}$ by conjugation and Ad, *respectively, on $\tilde{G}_{rs}$ by $(g, xT) \cdot w = (g, xwT)$, and similarly on $\tilde{\mathfrak{G}}_{rs}$.*

Proof. The statement (i) follows easily from the facts that if two elements of T (resp. $\mathfrak{T}$) are in the same G-orbit, they are in the same W-orbit (see [L2, 2.6]) and that if $t \in T_{rs}$ then $C_G(t) = T$ (together with the corresponding statement for $\mathfrak{T}$). This shows that the fibers of the map $\tilde{G}_{rs} \to G_{rs}$ are of the form $\{(g, xwT) \,|\, w \in W\}$, which is a W-orbit on $\tilde{G}_{rs}$.

(ii) Consider the map $\phi : \tilde{G}_{rs} \to T_{rs} \times G/T$ defined by $\phi(g, xT) = (x^{-1}gx, xT)$. This clearly has inverse $(t, xT) \mapsto (xtx^{-1}, xT)$; moreover for $w \in W$, $\phi(g, xwT) = (w^{-1}x^{-1}gxw, xwT) = \phi(g, xT) \cdot w$, whence ϕ is W-equivariant. ∎

We may now write down the following general formulae.

(2.3) THEOREM. *Let P and Q be the polynomials defined in* (1.5). *Then with the above notation, we have*

$$P_{G_{rs}}(t) = \left(P^W_{T_{rs}}(t), P^W_{G/T}(t)\right)_W$$

and

$$P_{\mathfrak{G}_{rs}}(t) = \left(P^W_{\mathfrak{T}_{rs}}(t), P^W_{G/T}(t)\right)_W,$$

where $(-,-)_W$ *denotes the inner product* [L2, *Sect*. 1], *in the ring* $R(W)[t]$, *and similarly for* $Q_{G_{rs},c}(t)$ *and* $Q_{\mathfrak{G}_{rs},c}(t)$.

Proof. From (2.2)(ii), G_{rs} is an unramified quotient of $\tilde{G}_{rs}$ by W. Hence by the transfer theorem for cohomology, $H^j(G_{rs}) = H^j(\tilde{G}_{rs})^W$ (where M^W denotes the W-invariants in a module M), i.e., $\dim H^j(G_{rs}) = (H^j(\tilde{G}_{rs}), 1)_W$. But using (2.2)(ii), we obtain from the Künneth theorem that

$$P^W_{\tilde{G}_{rs}}(t) = P^W_{T_{rs}}(t)\,P^W_{G/T}(t). \tag{2.3.1}$$

It follows that

$$\begin{aligned} P_{G_{rs}}(t) &= \left(P_{\tilde{G}_{rs}}(t), 1\right)_W \\ &= \left(P^W_{T_{rs}}(t)\,P^W_{G/T}(t), 1\right)_W \\ &= \left(P^W_{T_{rs}}(t), P^W_{G/T}(t)\right)_W \end{aligned}$$

since all representations of W are self-dual.

The proofs of the other statements are similar, although for the formula corresponding to (2.3.1) for the weight polynomials Q, we use [DL, (6.1)]. ∎

3. SOME EQUIVARIANT POINCARÉ AND WEIGHT POLYNOMIALS

From (2.3) it is apparent that in order to compute the Poincaré and weight polynomials for G_{rs} and $\mathfrak{G}_{rs}$ one requires the corresponding W-equivariant polynomials (i.e., elements of $R(W)[t]$) for G/T, T_{rs}, and $\mathfrak{T}_{rs}$. All of these have been studied elsewhere; we collect together the results in this section. In order to state them we need the following notation.

It has been remarked above that W acts on $\mathfrak{T}$ ($= \operatorname{Lie} T$) as a (real) finite reflection group. Let $S = \mathbb{C}[\mathfrak{T}]$ be the ring of polynomial functions on $\mathfrak{T}$ and write J for the ideal of S generated by the W-invariants of positive degree. Then S/J is a graded version of the regular representation of W. Moreover if we write $2N$ for the number of roots of G with respect to T (so that $N = \dim U$) then the following facts are known (see, e.g., [B]).

(3.1). *If* $(S/J)_i$ *denotes the* ith *graded component of* S/J (*as a* W-*module*), *then*

(i) $(S/J)_i = 0$ *for* $i > N$

(ii) $(S/J)_0 = 1_W$

(iii) $(S/J)_{N-i} \cong \varepsilon \otimes (S/J)_i$ (*for* $i = 0, \dots, N$) *where* ε *is the alternating representation of* W.

To express our results, we shall use the following notation. If $A = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A_i$ is a graded Γ module (Γ being any finite group), write

$$P_A^\Gamma(t) = \sum_i A_i t^i \in R(\Gamma)[[t]].$$

Recall that a variety X is said to be *pure* if for each j, there is an integer m_j such that $\mathrm{Gr}_{m_j} H^j(X) = H^j(X)$, i.e., $H^j(X)$ is a pure Hodge structure of weight m_j (cf. [DL, Sect. 3]).

(3.2) PROPOSITION. *Let* G *be a connected reductive group over* $\mathbb{C}$, T *a maximal torus of* G, *and* S/J *the corresponding coinvariant algebra regarded as a graded* W-*module, where* W *is the Weyl group of* G *with respect to* T. *Then*

(i) $H^j(G/T)$ *is pure of weight* j ($j = 0, 1, \dots$).

(ii) *We have* $H^j(G/T) = 0$ *unless* j *is even. Moreover we have*

$$P_{G/T}^W(t) = \sum_{i=0}^N (S/J)_i t^{2i} = P_{S/J}^W(t^2).$$

(iii) *In the notation of* (1.5), *we have*

$$Q_{G/T,c}^W(t) = \sum_{i=0}^N (S/J)_{N-i} t^{2i+2N} = t^{4N} P_{S/J}^W(t^{-2}).$$

Proof. (i) We have a locally trivial vibration $G/T \to G/B$ with fiber $U \cong \mathbb{A}^N$. But G/B is non-singular and complete, whence $H^j(G/B) = H_c^j(G/B)$ is pure of weight j. The result follows (e.g., from [DL, (6.1)]).

(ii) This is a standard result, essentially due to Borel (see [Hi], [Sr], [Sp]).

(iii) This follows from (i) and (ii) using [DL, (1.6)], since G/T is smooth and connected. ∎

We now turn our attention to the varieties T_{rs} and $\mathfrak{T}_{rs}$.

(3.3) PROPOSITION. *Let notation be as in* (3.2) *above.*

(i) *Let M_W be the complexified hyperplane complement corresponding to W. Then $\mathfrak{T}_{rs} \cong M_W$ and $P^W_{\mathfrak{T}_{rs}}(t) = P_{M_W}(t)$ (in the notation of* [L2, (4.6)]).

(ii) *The varieties T_{rs} and $\mathfrak{T}_{rs}$ are minimally pure (mp) in the sense of* [DL, (3.1)]; *i.e., $H^j_c(X)$ is pure of weight $2j - 2\dim X$ for $X = \mathfrak{T}_{rs}$ or T_{rs}.*

(iii) *We have*

(a) $Q^W_{\mathfrak{T}_{rs},c}(t) = t^{2r}P_{M_W}(-t^{-2})$

(b) $Q^W_{T_{rs},c}(t) = t^{2r}P^W_{T_{rs}}(-t^{-2})$.

Proof. (i) It follows from (1.2)(iii) that $\mathfrak{T}_{rs}$ is the complement in $\mathfrak{T}$ of the union of the hyperplanes which are the kernels of the roots of $\mathfrak{G}$ with respect to $\mathfrak{T}$. Thus $\mathfrak{T}_{rs} \cong M_W$ and the result follows.

(ii) The minimal purity of the hyperplane complements is proved in [L3]; moreover T_{rs} is the complement in T of a "toral arrangement," i.e., the union of a finite set of codimension 1 subtori of T. This is shown in [DL, (4.2)] to be mp (see also Looijenga [La, (2.4.3)]).

(iii) Let $X = \mathfrak{T}_{rs}$ or T_{rs}. Then X is smooth of dimension r ($=$ rank G). It follows from (ii) and Poincaré duality that $H^j(X)$ is pure of weight $2j$, so that $Q^W_{X,c}(t) = t^{2r}P^W_X(-t^{-2})$. The statements (a) and (b) follow. ∎

As an immediate application we obtain

(3.4) THEOREM. *Let G be as above (i.e., a connected reductive algebraic group over $\mathbb{C}$). If s is the semisimple rank of G (i.e., the rank of G') and N is the number of positive roots of G (with respect to T) then $H^i(\mathfrak{G}_{rs}) = 0$ for $i > 2N + s - 2 = \dim G' - 2$, where G' is the derived group of G.*

Proof. It follows from Theorem (2.3), together with (3.2)(ii) and (3.3)(i) that

$$P_{\mathfrak{G}_{rs}}(t) = \left(P^W_{S/J}(t^2), P_{M_W}(t)\right)_W. \tag{3.4.1}$$

Now $P_{M_W}(t)$ is a polynomial of degree s, while $P^W_{S/J}(t^2)$ has degree $2N$. Moreover the term of highest degree of $P^W_{S/J}(t^2)$ is $\varepsilon_W t^{2N}$, where ε_W is the alternating character of W. But from [L2, Theorem (4.8)] (see also [L7]) we have

$$\left(\varepsilon_W, P_{M_W}(t)\right)_W = 0. \tag{3.4.2}$$

It follows that the highest possible power of t occurring in $P_{\mathfrak{G}_{rs}}(t)$ is t^{2N-2+s}. ∎

4. EXPLICIT FORMULAE

(4.1) LEMMA. *With notation as in Section* 3, *we have*

$$P^W_{S/J}(w,t) = \prod_{j=1}^{r}(1-t^{d_j})\det(1-wt)^{-1},$$

where $w \in W$, det *is the determinant of the action of W on $\mathfrak{T}$, and $d_1, \ldots, d_r$ are the basic degrees of W.*

Proof. (cf. [L2, Sect. 1]). It is well known that by Chevalley's theorem, $S \cong S/J \otimes S^W$ as graded W-modules (where S^W is the set of W-invariants in S), so that

$$P^W_S(t,w) = P^W_{S/J}(t,w)P_{S^W}(t).$$

It is also well known that (cf. [L2]) $P^W_S(t,w) = \det(1-wt)^{-1}$; the result is now clear. ∎

This enables us to write explicit formulae for the required polynomials, which reduce their computation to that of the corresponding polynomials for T_{rs} and $\mathfrak{T}_{rs}$.

(4.2) THEOREM. *Let G, T, W, $\mathfrak{T}$, $\mathfrak{T}_{rs}$, etc. be as above* (*so that $\mathfrak{G}_{rs}$ is the variety of regular semisimple elements in the Lie algebra $\mathfrak{G}$ of the complex connected reductive algebraic group G*). *Then with notation as in* (1.5) *we have*

(i) $$P_{\mathfrak{G}_{rs}}(t) = |W|^{-1}\prod_{j=1}^{r}(1-t^{2d_j})\sum_{w\in W}\frac{P_{M_W}(w,t)}{\det(1-wt^2)},$$

where $P_{M_W}(t)$ is the Poincaré polynomial of W acting on the cohomology of its complex hyperplane complement.

(ii) $$Q_{\mathfrak{G}_{rs},c}(t) = |W|^{-1}t^{2r+2N}\prod_{j=1}^{r}(t^{2d_j}-1)\sum_{w\in W}\frac{P_{M_W}(w,-t^{-2})}{\det(t^2-w)}.$$

(4.3) THEOREM. *With notation as in* (4.2), *we have*

(i) $$P_{G_{rr}}(t) = |W|^{-1}\prod_{j=1}^{r}(1-t^{2d_j})\sum_{w\in W}\frac{P_{T_{rs}}(w,t)}{\det(1-wt^2)}$$

(ii) $$Q_{G_{rs},c}(t) = |W|^{-1}t^{2r+2N}\prod_{j=1}^{r}(t^{2d_j}-1)\sum_{w\in W}\frac{P_{T_{rs}}(w,-t^{-2})}{\det(t^2-w)}.$$

Proof. Both Theorems (4.2) and (4.3) follow from (2.3), (3.2), (3.3), and (4.1) after a little calculation. ∎

Now the polynomials $P_{M_W}(t)$ have all been computed (see [L1, L4, FJ, B1]) and general formulae exist for $P_{M_W}(w,t)$ for many (but not all) $w \in W$ (see [L3, L6]). Thus Theorem (4.2) does give an explicit solution to the problem of computing the polynomials $P_{\mathfrak{G}_{rs}}$ and $Q_{\mathfrak{G}_{rs},c}$. For the group case, $P^W_{T_{rs}}(t)$ has only hitherto been computed for G of type A [L5].

5. THE CASE OF GL_n

In this section we give closed formulae for the four polynomials P and Q in the case $G = GL_n(\mathbb{C})$, using the results of [L1, L5].

(5.1) DEFINITION. (i) For any natural number i $(= 1, 2, \dots)$ define the polynomials

$$p_i(t) = \sum_{d|i} \mu\left(\frac{i}{d}\right)(-t)^{i-d}$$

$$q_i(t) = \frac{1}{i(-t)^i} p_i(t)$$

(where $\mu(j)$ is the arithmetical Möbius function).

(ii) For rational integers $i \geq 1$ and $m \geq 0$ define the polynomial

$$\begin{aligned} P_i^{(m)}(t) &= p_i(t)\left(p_i(t) - i(-t)^i\right)\left(p_i(t) - 2i(-t)^i\right)\cdots \\ &\qquad \times\left(p_i(t) - (m-1)i(-t)^i\right) \\ &= \left(i(-t)^i\right)^m m!\binom{q_i(t)}{m}, \end{aligned}$$

where for any non-negative integer m, we write

$$\binom{a}{m} = \frac{a(a-1)\cdots(a-m+1)}{m!}. \tag{5.1.1}$$

When $G = GL_n$, W is the symmetric group S_n. The conjugacy class of $w \in S_n$ is described by a partition λ of n, which we write in the form $\lambda = (i^{m_i}), \sum_i im_i = n$. The integer m_i is called the multiplicity of the part i in λ. With this notation we have

(5.2) PROPOSITION [L1, (5.5)]. *Suppose $W = S_n$ and that $w \in S_n$ is of type (i.e., belongs to the conjugacy class of type) $\lambda = (i^{m_i})$. Then $P_{M_W}(w, t) = \prod_i P_i^{(m_i)}(t)$ where $P_i^{(m)}(t)$ is as defined in* (5.1)(ii).

Now the number $n(\lambda)$ of elements $w \in S_n$ of type $\lambda = (i^{m_i})$ is given by

$$n(\lambda) = \frac{n!}{\prod_i m_i! i^{m_i}}. \tag{5.3}$$

Moreover the eigenvalues of $w(\sim (i^{m_i}))$ acting on $\mathfrak{T}$ form a complete set of ith roots of unity, each occurring with multiplicity m_i. It is therefore apparent that

$$\det(1 - wt) = \prod_i (1 - t^i)^{m_i}. \tag{5.4}$$

(5.5) THEOREM. *Let $\mathfrak{G}_{rs}$ be the variety of $n \times n$ matrices over $\mathbb{C}$ with distinct eigenvalues. The Poincaré and weight polynomials* (*see* (1.5)) *of $\mathfrak{G}_{rs}$ are given by*

$$\text{(i)}\quad P_{\mathfrak{G}_{rs}}(t) = \prod_{j=1}^{n}(1 - t^{2j}) \sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}} \prod_i \begin{pmatrix} q_i(t) \\ m_i \end{pmatrix} \left(\frac{(-t)^i}{1 - t^{2i}} \right)^{m_i}$$

$$\text{(ii)}\quad Q_{\mathfrak{G}_{rs},c}(t) = t^{n^2+n} \prod_{j=1}^{n}(t^{2j} - 1) \sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}} \prod_i \begin{pmatrix} q_i(-t^2) \\ m_i \end{pmatrix} \left(\frac{t^{-2i}}{t^{2i} - 1} \right)^{m_i},$$

where the sums are over all partitions $\lambda = (i^{m_i})$ such that $\sum_i im_i = n$ and the polynomials $q_i(t)$ are defined in (5.1).

Proof. We substitute the expressions (5.2) and (5.3) into (4.2), taking into account that $\{d_1, \ldots, d_n\} = \{1, \ldots, n\}$ here and after some rearrangement, obtain the stated formulae. ∎

We turn now to the group case.

(5.6) PROPOSITION [L5]. *Let $G = GL_n$ and suppose $w \in W = S_n$ is of type $\lambda = (i^{m_i})$. Then we have $P_{T_{rs}}^W(w, t) = P_1^{(m_1+1)}(t)\prod_{i>1} P_i^{(m_i)}(t)$, where $P_i^{(m)}(t)$ is as above.*

Given (5.6), the next result follows in analogous fashion to (5.5).

(5.7) THEOREM. *Let G_{rs} be the regular semisimple variety in $GL_n(\mathbb{C})$. The Poincaré and weight polynomials of G_{rs} are given by*

$$\text{(i)}\quad P_{G_{rs}}(t) = \prod_{j=1}^{n}(1 - t^{2j}) \sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}} \frac{P_1^{(m_1+1)}(t)}{m_1!(1 - t^2)^{m_1}} \prod_{i>1} \frac{P_i^{(m_i)}(t)}{m_i! i^{m_i}(1 - t^{2i})^{m_i}}$$

$$\text{(ii)}\quad Q_{G_{rs},c}(t) = t^{n^2+n} \prod_{j=1}^{n}(t^{2j} - 1) \sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}} \frac{P_1^{(m_1+1)}(-t^{-2})}{m_1!(t^2 - 1)}$$

$$\times \prod_{i>1} \frac{P_i^{(m_i)}(-t^{-2})}{m_i! i^{m_i}(t^{2i} - 1)^{m_i}},$$

where the polynomials $P_i^{(m)}(t)$ are defined in (5.1).

(5.8) EXAMPLE. If we write $P_n(t)$ for $P_{(gl_n)_{rs}}(t)$, we list here the first few values

n	$P_n(t)$
2	$1+t$
3	$1+t+t^3+t^4+t^5+t^6=(1+t)(1+t^3+t^5)$
4	$1+t+t^3+2t^4+3t^5+3t^6+2t^7+3t^8+3t^9+2t^{10}+t^{11}+t^{12}+t^{13}$

The Betti numbers of $\mathfrak{G}_{rs}$ are thus easily obtainable from this formula.

(5.9) PROPOSITION. *Let* $\mathfrak{G}_{rs}$ *be as in* (5.5). *Then* $H^i(\mathfrak{G}_{rs})=0$ *for* $i>n^2-3$ *while* $\dim H^{n^2-3}(\mathfrak{G}_{rs})=1$.

Proof. The first statement follows from (3.4) since $2N+s-2=n^2-3$ here. Next, observe that the term of degree $2N-2$ in $P^W_{S/J}(t^2)$ is $\varepsilon_W\rho_W t^{2N-2}$, where ρ_W is the reflection character of W. But a simple calculation using Frobenius reciprocity, together with [L1, (5.5)] shows that in this case

$$\left(\varepsilon_W\rho_W, H^{n-1}(M_W)\right)_W = 1. \tag{5.9.1}$$

This, together with (2.3), (3.2)(ii), and (3.3)(i) shows that $H^{n^2-3}(\mathfrak{G}_{rs})$ has dimension one, as required. ∎

6. A STABILITY RESULT FOR THE REGULAR SEMISIMPLE VARIETY OF $\mathfrak{GL}_n$

In this section we shall prove

(6.1) THEOREM. *let* $\mathfrak{G}$ *be the Lie algebra of* $GL_n(\mathbb{C})$. *If* $\beta_i(n)=\dim H^i(\mathfrak{G}_{rs},\mathbb{C})$ *is the ith Betti number of its variety of regular semisimple elements, then* $\beta_i(n)=\beta_i(2i)$ *for all* $n\geq 2i$.

The proof will involve the explicit formula of (5.5). Notice that the theorem implies that the Betti numbers of $(\mathfrak{gl}_n)_{rs}$ stabilize (with respect to n) and that we may therefore speak of *universal Betti numbers* of $(\mathfrak{gl}_n)_{rs}$, or Betti numbers of $\mathfrak{gl}_{\infty rs}$.

In the proof of (6.1) we shall use

(6.2) LEMMA ("Cyclotomic Identity"). *For any indeterminates x and t over (say)* $\mathbb{C}$, *we have the identity*

$$\prod_{i\geq 1}(1-x^i)^{q_i(t)} = 1+\frac{x}{t}$$

in the ring $\mathbb{C}[[x,t^{-1}]]$, *where for elements* h, k *of* $\mathbb{C}[[x,t^{-1}]]$, $(1+h)^k$ *denotes* $\sum_{r\geq 0}\binom{k}{r}h^r$.

Although this is well known, we give here the following short proof. Note that in the definition of $(1+h)^k$ above, h and k must be such that the definition makes sense, e.g., $h \in x\mathbb{C}[[x,t^{-1}]]$ suffices, for then the coefficient of $x^i t^j$ involves only finitely many terms in the sum.

Proof. For any prime power q, it is well known that $q_i(-q^{-1})$ is the number of irreducible polynomials of the form $t^i + a_1 t^{i-1} + \cdots + a_i$ with $a_j \in \mathbb{F}_q$. Consider the set $\mathscr{S}_n$ of all polynomials over $\mathbb{F}_q$ of the above form, with $i = n$. Since each such polynomial has a unique (up to order) factorization as a product of irreducible polynomials of the same form, the cardinality of $\mathscr{S}_n$ is clearly the coefficient of x^n in $\prod_{i\geq 1}(1-x^i)^{-q_i(-q^{-1})}$. Using the fact that this number is q^n, we obtain that for any prime power q, $\prod_{i\geq 1}(1-x^i)^{-q_i(-q^{-1})} = \sum_{n\geq 0} q^j x^j = (1-qx)^{-1}$. The result is now clear. ∎

Proof of (6.1). For any integer $n \geq 1$, write $P_n(t) = \sum_{i\geq 0}\beta_i(n)t^i \in \mathbb{C}[t]$; also $P_0(t) = 1$. The stated result is clearly equivalent to

$$\text{For } n \geq 2,\ t^{[n+1/2]} \text{ divides } P_n(t) - P_{n-1}(t) \text{ in } \mathbb{C}[t]. \tag{6.1.1}$$

To prove (6.1.1) we write $P_n(t) = f_n(t)g_n(t)$, where for $n \geq 1$, $f_n(t) = \prod_{j=1}^n(1-t^{2j})$ and $g_n(t) = \sum_{\lambda=(i^{m_i})/\lambda\vdash n}\prod_i\binom{q_i(t)}{m_i}((-t)^i/(1-t^{2i}))^{m_i}$. Also write $f_0(t) = g_0(t) = 1$. Note that $f_n(t) \in \mathbb{C}[t]$, while $g_n(t) \in \mathbb{C}[t]_M$, the localization of $\mathbb{C}[t]$ at M, where M is the multiplicative set of polynomials in $\mathbb{C}[t]$, all of whose roots are roots of unity.

Now

$$P_n(t) - P_{n-1}(t) = f_n(t)\big(g_n(t) - g_{n-1}(t)\big) + g_{n-1}(t)\big(f_n(t) - f_{n-1}(t)\big). \tag{6.1.2}$$

Moreover $f_n(t) - f_{n-1}(t) = f_{n-1}(t)\cdot(-t^{2n})$, so that the second summand in (6.1.2) lies in $t^{2n}\mathbb{C}[t]$.

Now consider the first summand $f_n(t)(g_n(t) - g_{n-1}(t))$. Define $\Gamma(z,t) \in \mathbb{C}[t]_M[[z]]$ by $\Gamma(z,t) = \sum_{n\geq 0} g_n(t)z^n$. A simple calculation then shows that

$$\begin{aligned}\Gamma(z,t) &= \prod_{i\geq 1}\left(1 + \frac{(-tz)^i}{1-t^{2i}}\right)^{q_i(t)} \\ &= \prod_{i\geq 1}\left(1 - t^{2i} + (-tz)^i\right)^{q_i(t)}\prod_{i\geq 1}\left(1-t^{2i}\right)^{-q_i(t)}.\end{aligned} \tag{6.1.3}$$

Note that although $q_i(t) \in t^{-i}\mathbb{C}[t]$, each term of the sums $(1+h)^{q_i(t)}$ occurring in (6.1.3) lies in $\mathbb{C}[t]_M[[z]]$

But by (6.2), $\prod_{i\geq 1}(1-t^{2i})^{q_i(t)} = 1+t$, whence

$$\Gamma(z,t) = (1+t)^{-1}\prod_{i\geq 1}\left(1-t^{2i}+(-tz)^i\right)^{q_i(t)}. \tag{6.1.4}$$

If we write $g_{-1}(t)=0$ then $(1-z)\Gamma(z,t)=\sum_{n\geq 0}(g_n(t)-g_{n-1}(t))z^n$. Moreover applying (6.2) again, we have

$$1-z = \prod_{i\geq 1}\left(1-(-zt)^i\right)^{q_i(t)}. \tag{6.1.5}$$

Combining (6.1.4) and (6.1.5) we obtain

$$(1-z)\Gamma(z,t) = (1+t)^{-1}\prod_{i\geq 1}\left(1-(zt)^{2i}-t^{2i}+(-t^3z)^i\right)^{q_i(t)}. \tag{6.1.6}$$

Now $f_n(t)(g_n(t)-g_{n-1}(t))$, which we know to be in $\mathbb{C}[t]$, is $f_n(t)$ times the coefficient of z^n in the right hand side of (6.1.6). We show that this coefficient is divisible by $t^{[(n+1)/2]}$ in $\mathbb{C}[t]_M$, from which it will follow that $f_n(t)(g_n(t)-g_{n-1}(t))$ is divisible by $t^{[(n+1)/2]}$ in $\mathbb{C}[t]$.

Now

$$\begin{aligned}
&\left(1-(zt)^{2i}-t^{2i}+(-t^3z)^i\right)^{q_i(t)}\\
&\quad=\sum_{k\geq 0}(-1)^k\binom{q_i(t)}{k}\left((zt)^{2i}+t^{2i}-(-t^3z)^i\right)^k\\
&\quad=\sum_{k\geq 0}(-1)^k\binom{q_i(t)}{k}t^{ik}\left(z^{2i}t^i+t^i-(-t^2z)^i\right)^k.
\end{aligned} \tag{6.1.7}$$

Recalling that $\binom{q_i(t)}{k}t^{ik}\in\mathbb{C}[t]$, it is clear from (6.1.7) that in the expansion of the right side of (6.1.6), the coefficient of z^n is divisible (in $\mathbb{C}[t]_M$) by t^m, where m is an integer satisfying $m\geq n/2$. This completes the proof of (6.1). ∎

We remark that there is strong empirical evidence that the statement (6.1) is not the best possible. In fact we have

(6.3) CONJECTURE. *In the notation of* (6.1), *we have* $\beta_i(n)=\beta_i(i)$ *for all* $n\geq i$. *Moreover* $\beta_i(i-1)=\beta_i(i)-1$ *for all* $i\geq 1$.

(6.4) EXAMPLE. Let β_i be the ith Betti number of $(\mathfrak{gl}_n)_{rs}$ for all sufficiently large n; by (6.1), $n\geq 2i$ suffices. We give a list of the first 15 values of β_i in the table below.

i	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
β_i	:	1	1	0	1	2	4	6	9	17	30	47	75	131	221	358

(6.5) COROLLARY. *Let $(S/J)^{(n)}$ and M_n respectively be the coinvariant algebra and complex hyperplane complement associated with the symmetric group S_n of degree n. Then for any $i \geq 0$ the inner product of characters*

$$\sum_{2j+k=i} \left((S/J)_j^{(n)}, H^k(M_n)\right)_{S_n}$$

is independent of n for $n \geq 2i$.

This is simply a restatement of (6.1) taking into account (3.4).

7. LIE ALGEBRAS OF TYPE B, C, OR D

In this section we use the results of [L4] to give an explicit formula for the Poincaré polynomial $P_{\mathfrak{G}_{rs}}(t)$ when $\mathfrak{G}$ is of type B, C, or D. Recall that in the hyperoctahedral case (type B or C), a conjugacy class in the Weyl group W is given (see [L4, Sect. 1]) by a pair $\lambda = (\lambda^+, \lambda^-)$ of partitions such that $|\lambda^+| + |\lambda^-| = n$. We shall write $\lambda = (i^{m_i}, j^{n_j})$, where λ^+ has m_i cycles of length i, λ^- has n_j cycles of length j, and $\sum_i im_i + \sum_j jn_j = n$.

(7.1) DEFINITION. (i) For any natural number $i \geq 1$ (i.e., $i = 1, 2, 3, \ldots$) define the polynomials

$$p_i^+(t) = \begin{cases} 1+t & \text{if } i = 1 \\ p_i(t) & \text{if } i \text{ is odd, } i \neq 1 \\ 2p_i(t) - p_{i_1}(-t^{2^k}) & \text{if } i = 2^k i_1, \text{ with } i_1 \text{ odd, } i_1 \neq 1, \text{ and } k \neq 0 \\ 2p_i(t) + t^i - 1 & \text{if } i = 2^k,\ k \neq 0, \end{cases}$$

and

$$p_j^-(t) = \begin{cases} 1+t & \text{if } j = 1 \\ p_{j_1}\left(-(-t)^{2^k}\right) & \text{if } j = 2^k j_1, \text{ with } j_1 \text{ odd and } j_1 \neq 1. \\ 1 - t^j & \text{if } j = 2^k,\ k \neq 0, \end{cases}$$

where $p_i(t)$ is the polynomial defined in (5.1).

(ii) With notation as in (i), define, for any positive integer i,

$$q_i^{\pm}(t) = \frac{p_i^{\pm}(t)}{2i(-t)^i}.$$

(iii) For natural numbers i and m, define the polynomials

$$P_i^{(m)\pm}(t) = \left(2i(-t)^i\right)^m m! \binom{q_i^{\pm}(t)}{m}$$

notation being as in (5.1.1).

(7.2) PROPOSITION [L4, (5.6)]. *Let W be a Weyl group of type B_n or C_n, i.e., a hyperoctahedral group. Suppose $w \in W$ is of type $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ = (i^{m_i})$ and $\lambda^- = (j^{n_j})$. Then in the notation of* (7.1)(iii),

$$P_{M_W}(w, t) = \prod_i P_i^{(m_i)+}(t) \prod_j P_j^{(n_j)-}(t).$$

The number $n(\lambda)$ of elements of W of type λ (notation as in (7.2)) is given by

$$n(\lambda) = \frac{2^n n!}{\prod_i (2i)^{m_i} \prod_j (2j)^{n_j} m_i! n_j!}. \tag{7.3}$$

Further, the explicit description of $w \in W$ of type λ as a product of positive and negative cycles (cf. [L4, (1.4)]) yields easily that

$$\det(1 - wt) = \prod_i (1 - t^i)^{m_i} \prod_j (1 + t^j)^{n_j}. \tag{7.4}$$

We are now able to state

(7.5) THEOREM. *Let $\mathfrak{G}$ be the Lie algebra of a semisimple complex Lie group of type B_n or C_n. The Poincaré polynomial of its regular semisimple variety is given by*

$$P_{\mathfrak{G}_{rs}}(t) = \prod_{k=1}^{n} (1 - t^{4k}) \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ \lambda^+ = (i^{m_i}),\ \lambda^- = (j^{n_j})}} \prod_i \binom{q_i^+(t)}{m_i} \left(\frac{(-t)^i}{1 - t^{2i}} \right)^{m_i}$$

$$\times \prod_j \binom{q_j^-(t)}{n_j} \left(\frac{(-t)^j}{1 + t^{2j}} \right)^{n_j},$$

where the sum is over pairs $\lambda = (\lambda^+, \lambda^-)$ *of partitions* $\lambda^+ = (i^{m_i})$, $\lambda^- = (j^{n_j})$ *such that* $\sum_i im_i + \sum_j jn_j = n$ *and the polynomials* $q_i^{\pm}(t)$ *are defined in* (7.1)(ii).

The proof is an application of (4.2)(i), taking into account (7.2), (7.3), and (7.4).

(7.6) EXAMPLE. If we write $P_{B_n}(t)$ for $P_{\mathfrak{G}_{rs}}(t)$ with $\mathfrak{G}$ as in (7.5), we list here the first few values

n	$P_{B_n}(t)$
2	$2t^6 + 2t^5 + t^2 + 2t + 1$
3	$t^{19} + t^{18} + 4t^{15} + 5t^{14} + t^{13} + 5t^{11} + 8t^{10} + 3t^9 + 4t^7 + 7t^6 + 3t^5 + t^3 + 2t^2 + 2t + 1$
4	$3t^{32} + 4t^{31} + t^{30} + 12t^{28} + 16t^{27} + 4t^{26} + 19t^{24} + 31t^{23} + 13t^{22} + t^{21}$ $+ 26t^{20} + 42t^{19} + 18t^{18} + 2t^{17} + 22t^{16} + 40t^{15} + 22t^{14} + 4t^{13} + 16t^{12} + 28t^{11}$ $+ 16t^{10} + 4t^9 + 6t^8 + 13t^7 + 10t^6 + 3t^5 + t^4 + 2t^3 + 2t^2 + 2t + 1$

This list, together with other evidence, points to the hypothesis that there is a stability result like (6.1) for this case. We now prove such a result. We start with a variation on the cyclotomic identity (cf. (6.2)).

(7.7) LEMMA. *For any indeterminates x and t over (say)* $\mathbb{C}$, *we have the identity*

$$\prod_{i\geq 1}(1-x^i)^{q_i^+(t)}\prod_{j\geq 1}(1-x^j)^{q_j^-(t)} = (1-x)^{-1}\left(1+\frac{x}{t}\right)$$

in the ring $\mathbb{C}[[x, t^{-1}]]$, *where for elements* h, k *of* $\mathbb{C}[[x, t^{-1}]]$, $(1+h)^k$ *denotes* $\sum_{r\geq 0}\binom{k}{r}h^r$.

Proof. This follows immediately from (6.2) upon observing that $q_i^+(t) + q_i^-(t) = q_i(t)$ for $i \geq 2$ while $q_q^+(t) + q_1^-(t) = q_1(t) - 1$. The latter equations may be checked from definitions (5.1) and (7.1), although they can also be proved directly. ∎

(7.8) THEOREM. *Let* $\mathfrak{G}$ *be the Lie algebra of a simple Lie group of type* B_n *or* C_n. *If* $\beta_i(n) = \dim H^i(\mathfrak{G}_{rs}, \mathbb{C})$ *is the ith Betti number of its variety of regular semisimple elements, then* $\beta_i(n) = \beta_i(2i)$ *for all* $n \geq 2i$.

Proof. This is similar to that of (6.1), so we suppress the details. Denote the Poincaré polynomial given in (7.5) by $P_n(t)$ and write $P_n(t) = h_n(t)k_n(t)$, where

$$h_n(t) = \prod_{k=1}^{n}(1-t^{4k})$$

and

$$k_n(t) = \sum_{\substack{\lambda=(\lambda^+,\lambda^-)\\ \lambda^+=(i^{m_i}),\ \lambda^-=(j^{n_j})}} \prod_i \binom{q_i^+(t)}{m_i}\left(\frac{(-t)^i}{1-t^{2i}}\right)^{m_i} \prod_j \binom{q_j^-(t)}{n_j}\left(\frac{(-t)^j}{1+t^{2j}}\right)^{n_j}.$$

As in the proof of (6.1), we reduce the proof to showing that if $\Gamma(t,z) = \sum_{n\geq 0} k_n(t)z^n$ ($\in \mathbb{C}[t]_M[[z]]$), then the coefficient of z^n in $(1-z)\Gamma(t,z)$ lies in $t^m\mathbb{C}[t]_M$, where $m \geq n/2$. Using (7.7), we have

$$1 - z = (1+zt)\prod_{i\geq 1}\left(1-(-zt)^i\right)^{q_i^+(t)}\prod_{j\geq 1}\left(1-(-zt)^j\right)^{q_j^-(t)}. \quad (7.8.1)$$

Moreover one easily sees that

$$\Gamma(z,t) = \prod_{i\geq 1}\left(1+\frac{(-tz)^i}{1-t^{2i}}\right)^{q_i^+(t)}\prod_{j\geq 1}\left(1+\frac{(-tz)^j}{1+t^{2j}}\right)^{q_j^-(t)}. \quad (7.8.2)$$

Hence $(1-z)\Gamma(t,z)$ is the product of $1+zt$ with two factors, each of which is similar to the right hand side of (6.1.6). But each of these factors has the property that the coefficient of z^n lies in $t^m\mathbb{C}[t]_M$, where $m \geq n/2$. This is obvious for $1+zt$ and proved for the other two factors in the same way as in (6.1), the relevant point being that both $q_i^+(t)t^{ik}$ and $q_i^-(t)t^{ik}$ lie in $\mathbb{C}[t]$. This completes the proof. ∎

As in the case of type A_n, this result may be restated in terms of the graded representations of the hyperoctahedral group on its coinvariant algebra and the cohomology ring of its hyperplane complement.

(7.9) COROLLARY. *Let $(S/J)^{(n)}$ and K_n respectively be the coinvariant algebra and complex hyperplane complement associated with the hyperoctahedral Weyl group of type B_n (or C_n). Then for any $i \geq 0$ the inner product of characters*

$$\sum_{2j+k=i}\left((S/J)_j^{(n)}, H^k(K_n)\right)_{S_n}$$

is independent of n for $n \geq 2i$.

(7.10) EXAMPLE. Let β_i be the ith Betti number of $\mathfrak{G}_{rs}$, where $\mathfrak{G}$ is as in (7.8), for all sufficiently large n; by (7.8), $n \geq 2i$ suffices. We give a list of the first 14 values of β_i in the table below.

i	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
β_i	:	1	2	2	2	2	5	12	20	29	45	75	127	213	349

We now turn to the case of type D. The Weyl group $W(D_n)$ has index two in $W(B_n)$. It is therefore a union of conjugacy classes of $W(B_n)$ and it is standard that, in the above notation, the conjugacy class (λ^+, λ^-) of $W(B_n)$ is contained in $W(D_n)$ precisely when $\sum_j n_j$ is even, i.e., when the number of negative cycles is even.

The method of [L3] may be applied to obtain the analogue for $W(D_n)$ of (7.2) (see [B1, (5.5.4v)]).

(7.11) PROPOSITION. *Let W be a Weyl group of type D_n. Suppose $w \in W$ is of type $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ = (i^{m_i})$ and $\lambda^- = (j^{n_j})$. Then in the notation of* (7.1)(iii),

$$P_{M_{W(D_n)}}(w,t) = \left(1 - \frac{n_1 t}{1+(2n_1-1)t} - \frac{m_1 t}{1+(2m_1-1)t}\right) \times \prod_i P_i^{(m_i)+}(t) \prod_j P_j^{(n_j)-}(t),$$

where $M_{W(D_n)}$ is the hyperplane complement of type D_n.

Note that although we only need (7.11) for $w \in W(D_n)$, the larger group $W(B_n)$ acts on the hyperplane complement $M_{W(D_n)}$ and the formula in (7.11) holds for any $w \in W(B_n)$.

The same analysis as in (7.5) may now be carried through. The result is

(7.12) THEOREM. *Let $\mathfrak{G}$ be the Lie algebra of a semisimple complex Lie group of type D_n. The Poincaré polynomial of its regular semisimple variety is given by*

$$P_{\mathfrak{G}_{rs}}(t) = 2(1-t^{2n}) \prod_{k=1}^{n-1} (1-t^{4k}) \times \sum_{\substack{\lambda = (\lambda^+, \lambda^-) \\ \lambda^+ = (i^{m_i}),\ \lambda^- = (j^{n_j})}} \left\{ \left(1 - \frac{n_1 t}{1+(2n_1-1)t} - \frac{m_1 t}{1+(2m_1-1)t}\right) \times \prod_i \binom{q_i^+(t)}{m_i} \left(\frac{(-t)^i}{1-t^{2i}}\right)^{m_i} \prod_j \binom{q_j^-(t)}{n_j} \left(\frac{(-t)^j}{1+t^{2j}}\right)^{n_j} \right\},$$

where the sum is over pairs $\lambda = (\lambda^+, \lambda^-)$ of partitions $\lambda^+ = (i^{m_i})$, $\lambda^- = (j^{n_j})$ with $\sum_j n_j \equiv 0$ (mod 2), $\sum_i im_i + \sum_j jn_j = n$, and the polynomials $q_i^{\pm}(t)$ are defined in (7.1)(ii).

(7.13) EXAMPLE. If we write $P_{D_n}(t)$ for $P_{\mathfrak{G}_{rs}}(t)$ with $\mathfrak{G}$ as in (7.12), we give here the coefficients of $P_{D_n}(t)$ for $n = 4$, 5, and 6. They are listed below in increasing order of degree (i.e., with the constant term first, etc.)

n	Coefficients of $P_{D_n}(t)$
4	1, 1, 0, 1, 1, 3, 9, 9, 3, 4, 14, 22, 12, 3, 16, 27, 14, 1, 8, 19, 12, 0, 3, 6, 3
5	1, 1, 0, 0, 0, 4, 9, 9, 9, 15, 26, 31, 32, 41, 62, 66, 62, 71, 95, 101, 91, 92, 114, 114, 94, 91, 103, 100, 78, 67, 71, 66, 45, 36, 35, 31, 18, 12, 10, 9, 4, 2, 1, 1
6	1, 1, 0, 0, 0, 3, 6, 8, 16, 24, 23, 36, 78, 103, 75, 99, 222, 288, 180, 182, 453, 605, 349, 277, 730, 996, 547, 343, 971, 1359, 728, 360, 1087, 1563, 822, 316, 1033, 1531, 799, 238, 831, 1268, 661, 143, 557, 885, 463, 69, 301, 501, 265, 24, 123, 218, 119, 5, 33, 65, 37, 0, 4, 10, 6

8. RATIONAL POINTS OVER FINITE FIELDS

Suppose that X is a variety defined over $\mathbb{F}_q$ (the Galois field of q elements), with corresponding Frobenius endomorphism $F : X \to X$. The $\mathbb{F}_q$-rational points of X are the fixed points X^F of X under F; they may be counted by Grothendieck's formula [D]

$$|X^F| = \sum_j (-1)^j \operatorname{trace}\big(F, H_c^j\big(X, \overline{Q}_l\big)\big), \tag{8.1}$$

where $H_c^j(-, Q_l)$ denotes l-adic cohomology (l a prime not dividing q) with compact supports.

The methods above may be used to compute $|G_{rs}^F|$ and $|\mathfrak{G}_{rs}^F|$ where G is a connected reductive group defined over $\mathbb{F}_q$, F is the corresponding Frobenius endomorphism, and G_{rs} and $\mathfrak{G}_{rs}$ (etc.) are defined as for the G of Section 2. The key results (analogous to (3.2) and (3.3) above) are

(8.2) PROPOSITION. (i) *$H_c^j(G/T, \overline{Q}_l) = 0$ unless $j = 2s$ ($s \in \mathbb{N}$) and F acts on $H_c^{2s}(G/T, \overline{Q}_l)$ with all eigenvalues equal to q^s.*

(ii) *F acts on $H_c^t(\mathfrak{T}_{rs})$ and on $H_c^t(T_{rs})$ with all eigenvalues equal to q^{t-r}.*

Proof. Part (ii) is proved in [L3] while (i) is well known (see, e.g., [Sp] or [Sr]). ∎

(8.3) PROPOSITION [L2, (4.12)]. *Suppose that in the notation above, the lattice of intersections of the reflecting hyperplanes of W is the same in $\mathfrak{T}(\mathbb{C})$ as in $\mathfrak{G}(\mathbb{F}_q)$, i.e., that in the notation of* [L2], *the characteristic p is regular for W. Then*

$$|\mathfrak{G}_{rs}^F| = |\mathfrak{G}^F|\big(P_{M_W}^W(-q^{-1}), P_{S/J}^W(q^{-1})\big)_W.$$

Proof. We have, by the Künneth theorem, using (2.2)

$$H_c^j(\mathfrak{G}_{rs}) \cong \bigoplus_{s=0}^{[j/2]} \left\{H_c^{2s}(G/T) \otimes H_c^{j-2s}(\mathfrak{T}_{rs})\right\}_{1_W}, \tag{8.3.1}$$

where H_c^j denotes l-adic cohomology with compact supports and $\{A\}_{1_W}$ denotes the 1_W-isotypic component of the W-module A. Moreover the isomorphism (8.3.1) respects the action of F, so that F acts on $\{H_c^{2s}(G/T) \otimes H_c^{j-2s}(\mathfrak{T}_{rs})\}_{1_W}$ with all eigenvalues equal to $q^{j-(r+s)}$. It follows that

$$\begin{aligned}|\mathfrak{G}_{rs}^F| &= \sum_j (-1)^j \operatorname{trace}\left(F, H_c^j(\mathfrak{G}_{rs})\right)\\ &= \sum_{j,s} (-1)^j q^{j-r-s}\left(H_c^{2s}(G/T), H_c^{j-2s}(\mathfrak{T}_{rs})\right)_W.\end{aligned}$$

But similarly to (3.2)(ii), we have $H_c^{2s}(G/T) \cong (S/J)_{2N-s}$ for all s. Changing variables in the above sum to $a = 2N - s$ and $b = j - 2s$ we then obtain, taking into account that $|\mathfrak{G}^F| = q^{2N+r}$,

$$\begin{aligned}|\mathfrak{G}_{rs}^F| &= |\mathfrak{G}^F| \sum_{a,b} q^{-2r}\left(q^{-a}(S/J)_a, (-q)^b H_c^b(\mathfrak{T}_{rs})\right)_W\\ &= |\mathfrak{G}^F|\left(P_{S/J}(q^{-1}), q^{-2r}P_{\mathfrak{T}_{rs},c}(-q)\right),\end{aligned}$$

where $P_{\mathfrak{T}_{rs},c}(t) = \sum_b H_c^b(\mathfrak{T}_{rs}, \overline{\mathbb{Q}}_l)t^b$.

But the assumption on the lattices of hyperplane intersections implies, by [L3, Theorem 1.4] that $t^{-2r}P_{\mathfrak{T}_{rs},c}(-t) = P_{M_W}(-t^{-1})$, whence the result. ∎

The corresponding result for G_{rs} is

(8.4) THEOREM. *Let G be a connected reductive group defined over $\mathbb{F}_q$, with corresponding Frobenius endomorphism F. Then in the notation above*

$$|G_{rs}^F| = q^{2N-r}\left(P^W_{T_{rs},c}(-q), P^W_{S/J}(q^{-1})\right)_W,$$

where

$$P^W_{T_{rs},c}(t) = \sum_j H_c^j\left(T_{rs}, \overline{\mathbb{Q}}_l\right)t^j.$$

Proof. This is entirely analogous to that of (8.3). ∎

Note that $P^W_{T_{rs},c}(t)$ may generically be computed with T_{rs} replaced by its analogue over $\mathbb{C}$ (cf. [DL, (5.5)]).

In analogy with (5.5), these formulae may be given explicitly for the case of type A. For $\mathfrak{gl}_n$ the statement is

(8.5) COROLLARY. *If the characteristic p is regular for* $\mathfrak{gl}_n$ *(p does not divide n suffices) the number of regular semisimple elements in* $\mathfrak{gl}_n(\mathbb{F}_q)$ *is*

$$q^{n(n-1)/2}\prod_{j=1}^{n}(q^j-1)\sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}}\prod_i\binom{q_i(-q^{-1})}{m_i}(q^i-1)^{-m_i},$$

where $q_i(t)$ *is the polynomial (in* $\mathbb{C}[t^{-1}]$*) defined in* (5.1).

We shall next prove a stability result like (6.1) for the proportion of regular semisimple matrices in $\mathfrak{gl}_n(\mathbb{F}_q)$. Dividing the expression in (8.5) by q^{n^2}, we see that this proportion is given by

$$K_n(q)=\prod_{j=1}^{n}(1-q^{-j})\sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}}\prod_i\binom{q_i(-q^{-1})}{m_i}(q^i-1)^{-m_i}.$$

Since the degree in t of $q_i(t^{-1})$ is i, $K_n(q)$ is a polynomial in $\mathbb{Z}[q^{-1}]$ (of course we know this already from (8.3)).

(8.6) PROPOSITION. The proportion of regular semisimple matrices in $\mathfrak{gl}_n(\mathbb{F}_q)$ is a polynomial

$$K_n(q)=\sum_{i=0}^{\frac{1}{2}(n-1)(n+2)-1}\alpha_i(n)q^{-i}.$$

Moreover we have $\alpha_i(n)=\alpha_i(2i)$ for all $n\geq 2i$.

Proof. The first part of the statement is clear from the above remarks, (8.3), and (3.4.2). For the second part, write

$$\begin{aligned}H_n(t)=K_n(t^{-1})&=\prod_{j=1}^{n}(1-t^j)\sum_{\substack{\lambda=(i^{m_i})\\ \lambda\vdash n}}\prod_i\binom{q_i(-t)}{m_i}\left(\frac{t^i}{1-t^i}\right)^{m_i}\\&=c_n(t)d_n(t),\end{aligned}$$

where $c_n(t)=\prod_{j=1}^n(1-t^j)\in\mathbb{C}[t]$ and $d_n(t)=\sum_{\lambda=(i^{m_i})/\lambda\vdash n}\prod_i\binom{q_i(-t)}{m_i}(t^i/(1-t^i))^{m_i}$, an element of $\mathbb{C}[t]_M$, which we recall is the localization of $\mathbb{C}[t]$ at the set M of polynomials all of whose roots are roots of unity. The proof now follows the lines of that of (6.1). We have

$$H_n(t) - H_{n-1}(t) = c_n(t)\big(d_n(t) - d_{n-1}(t)\big) + d_{n-1}(t)\big(c_n(t) - c_{n-1}(t)\big).$$

Since $c_n(t) - c_{n-1}(t)$ is divisible by t^n, it will suffice to show that

$$\text{For } n \geq 2,\ t^{[(n+1)/2]} \text{ divides } d_n(t) - d_{n-1}(t) \text{ in } \mathbb{C}[t]_M. \tag{8.6.1}$$

Write $\theta(z,t) = \sum_{n\geq 0} d_n(t) z^n \in \mathbb{C}[t]_M[[z]]$. Then in analogy with (6.1.3), we have

$$\theta(z,t) = \prod_{i\geq 1}\left(1 + \frac{(tz)^i}{1-t^i}\right)^{q_i(-t)}. \tag{8.6.2}$$

Moreover, replacing t by $-t$ in (6.1.5), we have

$$1 - z = \prod_{i\geq 1}\big(1 - (-zt)^i\big)^{q_i(-t)}. \tag{8.6.3}$$

Combining (8.6.3) and (8.6.2), it follows that

$$(1-z)\theta(z,t) = \prod_{i\geq 1}\left(1 + \frac{t^i}{1-t^i}\big((tz)^i - (tz^2)^i\big)\right)^{q_i(-t)}. \tag{8.6.4}$$

Using the fact that $t^i q_i(t) \in \mathbb{C}[t]$, (8.6.1) follows immediately, proving the proposition. ∎

The next statement is a rephrasing of (8.6), taking (8.3) into account.

(8.7) COROLLARY. *In the notation of* (6.5), *we have for any* i, *the inner product*

$$\alpha_i(n) = \sum_{j+k=i} (-1)^k \Big((S/J)_j^{(n)}, H^k(M_n)\Big)_{S_n}$$

is independent of n *for* $n \geq 2i$

We give below a list of values of the constants in (8.7). Let α_i be the common value of $\alpha_i(n)$ for $n \geq 2i$.

(8.8) EXAMPLE. The first fifteen values of the α_i are given in the table below.

i	:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
α_i	:	1	-1	-1	0	0	1	0	1	0	0	0	0	-1	0	0

There appears to be empirical evidence that $\alpha_i = 0, \pm 1$ for all i. We have verified this by computer for $0 \le i \le 20$.

The next result is an explicit statement of the result corresponding to (8.5) for Lie algebras of type B_n or C_n.

(8.9) PROPOSITION. *Let $\mathfrak{G}$ be the Lie algebra of a connected semisimple group of type B_n or C_n which is defined and split over $\mathbb{F}_q$. Assume that the characteristic is regular for $\mathfrak{G}$. The number of $\mathbb{F}_q$-rational points of the regular semisimple variety in $\mathfrak{G}$ is*

$$q^{n^2}\prod_{k=1}^{n}(q^{2k}-1)\sum_{\substack{\lambda=(\lambda^+,\lambda^-)\\ \lambda^+=(i^{m_i}),\ \lambda^-=(j^{n_j})}}\prod_i\binom{q_i^+(-q^{-1})}{m_i}(q^i-q)^{-m_i}$$
$$\times\prod_j\binom{q_j^-(-q^{-1})}{n_j}(q^j+1)^{-n_j},$$

where the sum is over pairs $\lambda = (\lambda^+, \lambda^-)$ of partitions $\lambda^+ = (i^{m_i})$, $\lambda^- = (j^{n_j})$ such that $\sum_i im_i + \sum_j jn_j = n$, and the polynomials $q_i^{\pm}(t)$ are defined in (7.1)(ii).

This is (8.3) applied to the cases described.

We conclude with some specific values of the polynomials above.

(8.10 EXAMPLE. Write $L_n(q)$ for the number of (8.9), i.e., $L_n(q)$ is the number of $\mathbb{F}_q$-rational points of the regular semisimple variety in $\mathfrak{G}$, where $\mathfrak{G}$ is of type B_n or C_n; we list here the first few values of $L_n(q)$. Recall that $|\mathfrak{G}_{rs}^F| = q^{2n^2+n}$.

n	$L_n(q)$
2	$q^{10}(1-2q^{-1}+q^{-2}-2q^{-3}+2q^{-4})$
3	$q^{21}(1-2q^{-1}+2q^{-2}-4q^{-3}+7q^{-4}-7q^{-5}+8q^{-6}-6q^{-7}+5q^{-8}-4q^{-9}+q^{-10}-q^{-11})$
4	$q^{36}(1-2q^{-1}+2q^{-2}-5q^{-3}+11q^{-4}-17q^{-5}+22q^{-6}-32q^{-7}+38q^{-8}-42q^{-9}+40q^{-10}-43q^{-11}+39q^{-12}-31q^{-13}+23q^{-14}-16q^{-15}+13q^{-16}-4q^{-17}+3q^{-18})$

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