Existence theory for positive solutions to one-dimensional $p$-Laplacian boundary value problems on time scales

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Abstract

In this paper we consider the one-dimensional $p$-Laplacian boundary value problem on time scales

$$(\varphi_p(u^\Delta(t)))^\Delta + h(t)f(u^{\sigma}(t)) = 0, \quad t \in [a, b],$$

$$u(a) - B_0(u^\Delta(a)) = 0, \quad u^\Delta(\sigma(b)) = 0,$$

where $\varphi_p(u)$ is $p$-Laplacian operator, i.e., $\varphi_p(u) = |u|^{p-2}u$, $p > 1$. Some new results are obtained for the existence of at least single, twin or triple positive solutions of the above problem by using Krasnosel’skii’s fixed point theorem, new fixed point theorem due to Avery and Henderson and Leggett–Williams fixed point theorem. This is probably the first time the existence of positive solutions of one-dimensional $p$-Laplacian boundary value problems on time scales has been studied.

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1. Introduction

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [24], and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, phytoremediation of metals, wound healing, and epidemic models [15, 25, 26, 38].

We begin by presenting some basic definitions which can be found in [1, 2, 14, 24, 29]. Another excellent source on dynamic equations on time scales is the book [15].

A time scale \( T \) is a nonempty closed subset of \( \mathbb{R} \). It follows that the jump operators \( \sigma, \rho : T \to T \)

\[
\sigma(t) = \inf\{\tau \in T : \tau > t\} \quad \text{and} \quad \rho(t) = \sup\{\tau \in T : \tau < t\}
\]

(supplemented by \( \inf \emptyset := \sup T \) and \( \sup \emptyset := \inf T \)) are well defined. The point \( t \in T \) is left-dense, left-scattered, right-dense, right-scattered if \( \rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t \), respectively. If \( T \) has a left-scattered maximum \( M \), define \( T^\kappa = T - \{M\} \); otherwise, set \( T^\kappa = T \). The forward graininess is \( \mu(t) := \sigma(t) - t \).

Throughout this paper, we make the blanket assumption that \( a < b \) are points in \( T \), and

\[
[a, b] = \{t \in T : a \leq t \leq b\}.
\]

For \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \), the delta derivative of \( f \) at \( t \), denoted by \( f^\Delta(t) \), is the number (provided it exists) with the property that given any \( \epsilon > 0 \), there is a neighborhood \( U \subset T \) of \( t \) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,
\]

for all \( s \in U \).
A function \( f : \mathbb{T} \rightarrow \mathbb{R} \) is rd-continuous provided it is continuous at right dense points in \( \mathbb{T} \) and its left-sided limit exists (finite) at left dense points in \( \mathbb{T} \). It is known that if \( f \) is rd-continuous, then there is a function \( F(t) \) such that \( F^{\Delta}(t) = f(t) \). In this case, we define

\[
\int_a^b f(\tau) \Delta \tau = F(b) - F(a).
\]

Recently, there is much attention paid to question of positive solutions of boundary value problems for second order dynamic equations on time scales, see [4–9,11,12,15–17,19–21,27,32,35,36]. In particular, we would like to mention some results of Agarwal and O’Regan [4], Chyan and Henderson [16], He [23], Sun and Ge [37], and Wang [40], which motivate us to consider one-dimensional \( p \)-Laplacian boundary value problem on time scales.

For convenience, throughout this paper we denote \( \varphi_p(u) \) is \( p \)-Laplacian operator, i.e., \( \varphi_p(u) = |u|^{p-2}u, \ p > 1, \ (\varphi_p)^{-1} = \varphi_{1/p}, \ 1/p + 1/q = 1, \) and

\[
f_0 = \lim_{u \to 0^+} \frac{f(u)}{\varphi_p(u)} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{\varphi_p(u)}.
\]

For the one-dimensional \( p \)-Laplacian differential equations, Wang [40] studied the existence of one positive solution of the following boundary value problem:

\[
\begin{align*}
(\varphi_p(u'))' + a(t)f(u) &= 0, \quad 0 < t < 1, \\
u(0) - A_0(u'(0)) &= 0, \quad u'(1) = 0,
\end{align*}
\]

and obtained the following theorem.

**Theorem A.** (See Wang [40].) Assume the following conditions hold:

1. **(W1)** \( A_0(v) \) is nondecreasing continuous odd function defined on \((-\infty, +\infty)\) and satisfies the condition that there exists \( b_0 > 0 \) such that \( 0 \leq A_0(v) \leq b_0v \) for all \( v \geq 0 \);
2. **(W2)** \( f(u) \) is a nonnegative, lower semi-continuous function defined on \([0, +\infty)\). Moreover, it has only a finite number of discontinuity points of the first kind in each compact subinterval of \([0, +\infty)\);
3. **(W3)** \( a(t) \) is a nonnegative measure function defined on \((0, 1)\) and satisfies \( 0 < \int_0^1 a(t) \, dt < +\infty \).

Then the problem (1.1) and (1.2) has at least one positive solution under the case (i) \( f_0 = 0 \) and \( f_\infty = \infty \), or (ii) \( f_0 = \infty \) and \( f_\infty = 0 \).

In 2001, Sun and Ge [37] considered the following boundary value problem:

\[
\begin{align*}
(\varphi_p(u'))' + a(t)f(u) &= 0, \quad 0 < t < 1, \\
u'(0) &= 0, \quad u(1) = 0,
\end{align*}
\]
where \( f \in C([0, \infty), [0, \infty)) \), \( a \in C([0, 1], [0, \infty)) \). If (i) \( f_0 = 0 \) and \( f_\infty = \infty \), or (ii) \( f_0 = \infty \) and \( f_\infty = 0 \) holds, then the above problem has at least one positive solution. If (i) \( f_0 = f_\infty = 0 \) and there exists \( \rho > 0 \) such that \( f(u) > \phi_p(\lambda \rho) \), for \( \rho/2 \leq u \leq \rho \), where \( \lambda = \left( \int_0^{1/2} \int_0^s a(r) \, dr \, ds \right)^{-1} \), or (ii) \( f_0 = \infty \) and \( f_\infty = 0 \) holds, then the above problem has at least one positive solution.

Also, Avery, Chyan and Henderson in [12] first applied the new fixed point theorem due to Avery and Henderson [13] and obtained the existence of at least two positive solutions for the right focal boundary value problem for a second order ordinary differential equation

\[
y'' + f(y) = 0, \quad 0 \leq t \leq 1, \\
y(0) = y'(1) = 0,
\]
as well as their discrete analogues

\[
\Delta^2 u(k) + g(u(k)) = 0, \quad k \in \{0, \ldots, N\}, \\
u(0) = \Delta u(N + 1) = 0.
\]

For the one-dimensional \( p \)-Laplacian difference equations, He [23] discussed the existence of at least one or two positive solutions of the following boundary value problem:

\[
\Delta \left[ \phi_p(\Delta u(k - 1)) \right] + a(k) f(u(k)) = 0, \quad k \in \{1, \ldots, T + 1\}, \quad (1.3)
\]

\[
\Delta u(0) = 0, \quad u(T + 2) = 0, \quad (1.4)
\]

and obtained the following theorems.

**Theorem B.** (See He [23].) Assume that

(P1) \( f : [0, \infty) \to [0, \infty) \) is continuous;
(P2) \( a(k) \) is a positive function defined on \( \{1, \ldots, T + 1\} \).

If (i) \( f_0 = 0 \) and \( f_\infty = \infty \), or (ii) \( f_0 = \infty \) and \( f_\infty = 0 \) holds, then there exists at least one solution of (1.3) and (1.4).

**Theorem C.** (See He [23].) Assume that the conditions (P1) and (P2) in Theorem B hold and that

(i) \( f_0 = f_\infty = \infty \);
(ii) there exists \( \rho > 0 \) such that \( f(u) < \phi_p(\xi \rho) \) for \( 0 < u \leq \rho \), where

\[
\xi = \left[ \sum_{s=0}^{T+1} \phi_q \left( \sum_{i=1}^s a(i) \right) \right]^{-1}.
\]

Then, there exist at least two solutions \( u_1 \) and \( u_2 \) of (1.3), (1.4), such that \( 0 < \|u_1\| < \rho < \|u_2\| \).
Theorem D. (See He [23].) Assume that the conditions (P1) and (P2) in Theorem B hold and that

1. \( f_0 = f_\infty = 0; \)
2. there exists \( \rho > 0 \) such that \( f(u) < \varphi_p(\lambda \rho) \) for \( \rho/2 < u \leq \rho \), where

\[
\lambda = \left[ \sum_{s \in Y} \varphi_q \left( \sum_{i=1}^{s} a(i) \right) \right]^{-1}, \quad Y = \left\{ t \in Z : 0 \leq t \leq \frac{T+2}{2} \right\}.
\]

Then, there exist at least two solutions \( u_1 \) and \( u_2 \) of (1.3) and (1.4), such that \( 0 < \|u_1\| < \rho < \|u_2\| \).

In 2003, Liu and Ge in [33] used the twin fixed point theorem [13] and obtained the existence criteria for at least two positive solutions of the boundary value problem

\[
\Delta \varphi_p(\Delta u(k)) + a(k)f(u(k)) = 0, \quad k \in \{0, \ldots, N\},
\]

\[
u(0) - B_0(\Delta u(0)) = 0, \quad \Delta u(N + 1) = 0.
\]

For the dynamic equations on time scales, by applying Leggett–Williams fixed point theorem [30], Agarwal and O’Regan [4] developed some existence criteria of three positive solutions to the following boundary value problem on time scales:

\[
u^\Delta^\Delta(t) + f(u(\sigma(t))) = 0, \quad t \in [a, b],
\]

\[
\alpha \nu(a) - \beta \nu^\Delta(a) = 0, \quad \alpha > 0, \quad \beta \geq 0, \quad \nu^\Delta(\sigma(b)) = 0. \tag{1.6}
\]

The main result of [4] is the following.

Theorem E. (See Agarwal and O’Regan [4].) Assume that \( f : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing and there exist \( 0 < r^l < l^l < l' / k' \leq R' \) such that the following conditions hold:

1. \( f(r') < r'^l / \left( \sup_{t \in [a, \sigma^l(b)]} \int_a^{\sigma(b)} G(t, s) \Delta s \right); \)
2. \( f(l') > l'^l / \left( \min_{t \in [\eta', \sigma^l(b)]} \int_{\eta'}^{\sigma(b)} G(t, s) \Delta s \right); \)
3. \( f(R') \leq R'/ \left( \sup_{t \in [a, \sigma^l(b)]} \int_a^{\sigma(b)} G(t, s) \Delta s \right), \)

where
\[ G(t, s) = \begin{cases} 
 t - a + \beta/\alpha, & t \leq s, \\
 \sigma(s) - a + \beta/\alpha, & t \geq \sigma(s), 
\end{cases} \]
\[ \eta' = \min \{ \tau \in \mathbb{T} : \tau \geq (\sigma(b) + 3a)/4 \}, \]

and
\[ k' = (a(\sigma(b) - a) + 4\beta)/(4\alpha(\sigma^2(b) - a) + 4\beta). \]

Then the problem (1.5) and (1.6) has three nonnegative solutions \( u_1, u_2 \) and \( u_3 \) such that
\[ \sup_{t \in [a, \sigma^2(b)]} u_1(t) < r', \quad \min_{t \in [\eta', \sigma^2(b)]} u_2(t) > l', \]

and
\[ \sup_{t \in [a, \sigma^2(b)]} u_3(t) > r' \quad \text{with} \quad \min_{t \in [\eta', \sigma^2(b)]} u_3(t) < l'. \]

In 2002, Chyan and Henderson [16] obtained the existence of at least two positive solutions for right focal boundary value problem on time scale
\begin{align*}
 u^\Delta (t) + f(u(\sigma(t))) &= 0, \quad t \in [0, 1] \cap \mathbb{T}, \\
 u(0) &= 0 = u^\Delta (\sigma(1)).
\end{align*}

Growth conditions are imposed on \( f \) so that the twin fixed point theorem [13] is applicable. The main result of [16] is the following.

**Theorem F.** (See Chyan and Henderson [16].) Assume that \( \eta' = \min \{ t \in \mathbb{T} : \sigma^2(1)/2 \leq t < 1 \} \)
and an interior \( r' \in \mathbb{T} \) with \( \eta' < r' < 1 \) exist, let \( C = \int_{0}^{\eta'} \sigma(s) \Delta s + \eta'(\sigma(1) - \eta') \), \( D = \int_{\eta'}^{r'} \sigma(s) \Delta s \), \( E = \int_{0}^{r'} \sigma(s) \Delta s \), \( 0 < a' < \frac{E}{C} b' < \frac{4\sigma^2(1)}{\sigma(1)D} c' \), and further that \( f : \mathbb{R} \to [0, \infty) \) is continuous and satisfies the following conditions:

(i) \( f(w) > \frac{4\sigma^2(1)}{\sigma(1)D} c' \) if \( c' \leq w < \frac{2c'}{\sigma^2(1)} \);
(ii) \( f(w) < \frac{b'}{E} c' \) if \( 0 \leq w < \frac{2b'}{\sigma^2(1)} \);
(iii) \( f(w) > \frac{a'}{E} c' \) if \( 0 \leq w \leq a' \).

Then the problem (1.7) and (1.8) has at least two positive solutions \( u_1 \) and \( u_2 \) such that
\[ a' < \max_{t \in [0, r']} u_1(t) \quad \text{with} \quad \max_{t \in [0, \eta']} u_1(t) < b', \quad \text{and} \quad b' < \max_{t \in [0, \eta']} u_2(t) \quad \text{with} \quad \max_{t \in [\eta', r']} u_2(t) < c'. \]

However, to the best of our knowledge, there are not any results on the \( p \)-Laplacian boundary value problems (BVPs) on time scales.

In this paper we study the existence of positive solutions for the one-dimensional \( p \)-Laplacian BVP on time scales.
\[
(\varphi_p(u^\Delta(t)))^\Delta + h(t)f(u^\sigma(t)) = 0, \quad t \in [a, b],
\]
(1.9)
\[
u(a) - B_0(u^\Delta(a)) = 0, \quad u^\Delta(\sigma(b)) = 0.
\]
(1.10)

Some new and more general results are obtained for the existence of at least one, two and three positive solutions for the above problem by using Krasnosel'skii's fixed point theorem [18,22, 28], new fixed point theorems due to Avery and Henderson [13] and Leggett–Williams fixed point theorem [30]. The results are even new for the special cases of difference equations and differential equations, as well as in the general time scale setting. Our results include and extend many results of Avery, Chyan and Henderson [12] (\(p = 2\)), Li and Ge [31], Lü et al. [34] and Wang [40] in the case \(T = \mathbb{R}\); Liu and Ge [33], Avery, Chyan and Henderson [12] (\(p = 2\)) in the case \(T = \mathbb{Z}\). For the general time scale \(T\), the results in the Sections 4 and 5 improve and generalize some known works by Chyan and Henderson [16], Agarwal and O’Regan [4] for the case \(p = 2\), respectively. That is to say, for the case \(T = \mathbb{R}\), if \(a = 0, b = 1\), \(B_0(u)\) is nondecreasing odd function, then our results in Section 3 reduce to those of Li and Ge [31] and Wang [40]; if \(B_0(u) \equiv 0\), \(a = 0, b = 1\), then the results in [16] \((p = 2)\) is a special case of our Theorem 4.1, the results in [34] is a special case of ours in Section 5. For the case \(T = \mathbb{Z}\), if \(a = 0, b = T\), our criteria in Section 4 include and extend the main results of Liu and Ge [33]. For the general time scale \(T\), if \(p = 2\), \(B_0(x) = \beta x\), \(f\) is nondecreasing, the main results of Agarwal and O’Regan [4] is a special case of our Theorem 5.1.

The rest of the paper is organized as follows. In Section 2, we first give two lemmas which are needed throughout this paper and then state several fixed point results: Krasnosel’skii’s fixed point theorem in a cone, new fixed point theorem due to Avery and Henderson and Leggett–Williams fixed point theorem. In Section 3 we use Krasnosel’skii’s fixed point theorem to obtain the existence of at least one or two solutions of problem (1.9) and (1.10). Section 4 will further discuss the existence of twin positive solutions of problem (1.9) and (1.10). Two new results and some corollaries will be presented by new fixed point theorem due to Avery and Henderson. Section 5 is due to develop existence criteria for (at least) three positive and arbitrary odd positive solutions of problem (1.9) and (1.10). In particular, our results in this section are new when \(T = \mathbb{R}\) (the continual case) and \(T = \mathbb{Z}\) (the discrete case).

For convenience, we list the following hypotheses:

(A1) \(\eta = \min\{t \in T; \frac{a + \sigma^2(b)}{2} \leq t < b\}\) and an interior \(r \in T\) with \(\eta < r < b\) exist;

(A2) \(h(t)\) is a nonnegative rd-continuous function defined in \([a, b]\) and satisfies that there exists \(t_0 \in [a, b]\) such that \(h(t_0) > 0\);

(A3) \(f(u)\) is a nonnegative continuous function defined on \([0, +\infty)\);

(A4) \(B_0: \mathbb{R} \to \mathbb{R}\) is continuous and satisfies there are constants \(0 \leq B_1 \leq B_2\) such that \(B_1 x \leq B_0(x) \leq B_2 x\) for \(x \in [0, +\infty)\).

2. Some lemmas

Integrating (1.9) from \(t\) to \(\sigma(b)\), one gets

\[
\varphi_p(u^\Delta(\sigma(b))) - \varphi_p(u^\Delta(t)) = - \int_t^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s.
\]
Thus, in view of (1.10), we have

\[ u^{\Delta}(t) = \varphi_q \left( \int_t^\sigma(b) h(s)f\left( u^{\sigma}(s) \right) \Delta s \right). \tag{2.1} \]

Again integrating (2.1) from \( a \) to \( t \), it follows that

\[ u(t) - u(a) = \int_a^t \varphi_q \left( \int_\tau^\sigma(b) h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \Delta \tau. \tag{2.2} \]

Since

\[ u^{\Delta}(a) = \varphi_q \left( \int_a^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right), \]

one gets

\[ u(t) = B_0 \left( \varphi_q \left( \int_a^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \right) + \int_a^t \varphi_q \left( \int_\tau^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \Delta \tau. \tag{2.3} \]

Now, let the Banach space \( E = \{ u : [a, \sigma^2(b)] \rightarrow \mathbb{R} \} \) be endowed with the norm, \( \| u \| = \sup_{t \in [a, \sigma^2(b)]} | u(t) | \), and choose the cone \( P \subset E \) defined by

\[ P = \left\{ u \in E : u(t) \geq 0 \text{ for } t \in [a, \sigma^2(b)] \text{ and } u^{\Delta\Delta}(t) \leq 0, \ u^{\Delta}(t) \geq 0 \text{ for } t \in [a, b], \ u^{\Delta}(\sigma(b)) = 0 \right\}. \]

Clearly, \( \| u \| = u(\sigma^2(b)) \) for \( u \in P \). Define the operator \( A : P \rightarrow E \) by

\[ Au(t) = B_0 \left( \varphi_q \left( \int_a^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \right) + \int_a^t \varphi_q \left( \int_\tau^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \Delta \tau. \tag{2.4} \]

Obviously, \( Au(t) \geq 0 \) for \( t \in [a, \sigma^2(b)] \).

From the definition of \( A \), we claim that for each \( u \in P \), \( Au \in P \) and satisfies (1.10), and \( Au(\sigma^2(b)) \) is the maximum value of \( Au(t) \) on \( [a, \sigma^2(b)] \).

In fact,

\[ (Au)^{\Delta}(t) = \varphi_q \left( \int_t^{\sigma(b)} h(s)f\left( u^{\sigma}(s) \right) \Delta s \right) \geq 0, \quad t \in [a, b], \]

is continuous and nonincreasing in \( [a, b] \); moreover, \( \varphi_q(x) \) is a monotone increasing continuously differentiable function,
\[
\left( \int_{\sigma(\tau)}^{\sigma(b)} h(s) f(u^{\sigma}(s)) \Delta s \right)_{\Delta} = -h(t) f(u^{\sigma}(t)) \leq 0,
\]
then by the chain rule [14, Theorem 1.87, p. 31], we obtain
\[
(Au)^{\Delta\Delta}(t) \leq 0,
\]
so, \(A : P \rightarrow P\).

**Lemma 2.1.** \(A : P \rightarrow P\) is completely continuous.

**Proof.** First, we show that \(A\) maps bounded set into bounded set.

Assume \(c > 0\) is a constant and \(u \in \overline{P}_c = \{x \in P : \|x\| \leq c\}\). Note that the continuity of \(f\) guarantees that there is \(C > 0\) such that \(f(u) \leq \varphi_p(C)\). So
\[
\|Au\| = Au(\sigma^2(b)) = B_0 \left( \varphi_q \left( \int_{\sigma(b)}^{\sigma(\tau)} h(s) f(u^{\sigma}(s)) \Delta s \right) \right) + \int_{\tau}^{\sigma(b)} \varphi_q \left( \int_{\sigma(b)}^{\tau} h(s) f(u^{\sigma}(s)) \Delta s \right) \Delta \tau
\]
\[
\leq C \left( B_2 \varphi_q \left( \int_{\sigma(b)}^{\sigma(\tau)} h(s) \Delta s \right) + \int_{\sigma(b)}^{\sigma(\tau)} \varphi_q \left( \int_{\sigma(b)}^{\tau} h(s) \Delta s \right) \Delta \tau \right).
\]
That is, \(A \overline{P}_c\) is uniformly bounded.

In addition, notice that
\[
|Au(t_1) - Au(t_2)| = \left| \int_{t_1}^{t_2} \varphi_q \left( \int_{\sigma(b)}^{\sigma(\tau)} h(s) f(u^{\sigma}(s)) \Delta s \right) \Delta \tau \right|
\]
\[
\leq C \left| \int_{t_1}^{t_2} \varphi_q \left( \int_{\sigma(b)}^{\sigma(\tau)} h(s) \Delta s \right) \Delta \tau \right|
\]
\[
\leq C |t_1 - t_2| \varphi_q \left( \int_{\sigma(b)}^{\sigma(\tau)} h(s) \Delta s \right).
\]
So, by applying Arzela–Ascoli Theorem on time scales [3] we obtain that \(A \overline{P}_c\) is relatively compact.

We next claim that \(A : \overline{P}_c \rightarrow P\) is continuous. Suppose that \(\{u_n\}_{n=1}^{\infty} \subset \overline{P}_c\) and \(u_n(t)\) converges to \(u_0(t)\) uniformly on \([a, \sigma^2(b)]\). Hence, \(\{Au_n(t)\}_{n=1}^{\infty}\) is uniformly bounded and equicontinuous on \([a, \sigma^2(b)]\). The Arzela–Ascoli Theorem on time scales [3] tells us that there exists uniformly
convergent subsequence in \( \{A u_n(t)\}_{n=1}^{\infty} \). Let \( \{A u_{n(m)}(t)\}_{m=1}^{\infty} \) be a subsequence which converges to \( v(t) \) uniformly on \([a, \sigma^2(b)]\). In addition,

\[
0 \leq A u_n(t) \leq C \left( B_2 \varphi_q \left( \int_a^t h(s) \Delta s \right) + \int_a^t \varphi_q \left( \int_a^s h(s) \Delta s \right) \Delta \tau \right).
\]

Observe that

\[
A u_n(t) = B_0 \left( \varphi_q \left( \int_a^t h(s) f \left( u_{n(1)}^\sigma(s) \right) \Delta s \right) \right) + \int_a^t \varphi_q \left( \int_a^s h(s) f \left( u_{n(1)}^\sigma(s) \right) \Delta s \right) \Delta \tau.
\]

Inserting \( u_{n(m)} \) into the above and then letting \( m \to \infty \), we obtain

\[
v(t) = B_0 \left( \varphi_q \left( \int_a^t h(s) f \left( u_0^\sigma(s) \right) \Delta s \right) \right) + \int_a^t \varphi_q \left( \int_a^s h(s) f \left( u_0^\sigma(s) \right) \Delta s \right) \Delta \tau,
\]

here we have used the Lebesgue’s dominated convergence theorem on time scales [10]. From the definition of \( A \), we know that \( v(t) = A u_0(t) \) on \([a, \sigma^2(b)]\). This shows that each subsequence of \( \{A u_n(t)\}_{n=1}^{\infty} \) uniformly converges to \( A u_0(t) \). Therefore, the sequence \( \{A u_n(t)\}_{n=1}^{\infty} \) uniformly converges to \( A u_0(t) \). This means that \( A \) is continuous at \( u_0 \in P_c \). So, \( A \) is continuous on \( P_c \) since \( u_0 \) is arbitrary. Thus, \( A \) is completely continuous. The proof is complete. \( \square \)

**Lemma 2.2.** Let \( u \in P \), then \( u(t) \geq \frac{t-a}{\sigma^2(b)-a} \| u \| \) for \( t \in [a, \sigma^2(b)] \).

**Proof.** Since \( u^{\Delta \Delta}(t) \leq 0 \), it follows that \( u^\Delta(t) \) is nonincreasing. Hence, for \( a < t < \sigma^2(b) \),

\[
u(t) - u(a) = \int_a^t u^\Delta(s) \Delta s \geq (t-a) u^\Delta(t)
\]

and

\[
u(\sigma^2(b)) - u(t) = \int_t^{\sigma^2(b)} u^\Delta(s) \Delta s \leq (\sigma^2(b)-t) u^\Delta(t),
\]

from which we have

\[
u(t) \geq \frac{(t-a) u(\sigma^2(b)) + (\sigma^2(b)-t) u(a)}{\sigma^2(b)-a} \geq \frac{t-a}{\sigma^2(b)-a} \| u \|.
\]

The proof is complete. \( \square \)
For each \( u \in P \), \( u(\eta) \geq \frac{\eta - a}{\sigma^2(b) - a} \| u \| \), where \( \eta \) is defined in (A1), thus we have
\[
\| u \| \leq \frac{\sigma^2(b) - a}{\eta - a} u(\eta) \quad \text{for all} \quad u \in P.
\]

In the rest of this section, we provide some background material from the theory of cones in Banach spaces, and we then state several fixed point theorems which we needed later.

Let \( E \) be a Banach space, and \( P \) be a cone in \( E \). A map \( \psi : P \to [0, +\infty) \) is said to be a nonnegative continuous increasing functional provided \( \psi \) is nonnegative and continuous and satisfies \( \psi(x) \leq \psi(y) \) for all \( x, y \in P \) and \( x \leq y \).

Given a nonnegative continuous functional \( \psi \) on a cone \( P \) of a real Banach space \( E \), we define, for each \( d > 0 \), the set
\[
P(\psi, d) = \{ x \in P : \psi(x) < d \}.
\]

**Lemma 2.3.** (See [18,22,28].) Let \( P \) be a cone in a Banach space \( E \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \). If
\[
A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P
\]
is a completely continuous operator such that either

(i) \( \| Ax \| \leq \| x \|, \forall x \in P \cap \partial \Omega_1 \) and \( \| Ax \| \geq \| x \|, \forall x \in P \cap \partial \Omega_2 \), or

(ii) \( \| Ax \| \geq \| x \|, \forall x \in P \cap \partial \Omega_1 \) and \( \| Ax \| \leq \| x \|, \forall x \in P \cap \partial \Omega_2 \),

then \( A \) has a fixed point in \( P \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

**Lemma 2.4.** (See [13].) Let \( P \) be a cone in a real Banach space \( E \). Let \( \alpha \) and \( \gamma \) be increasing nonnegative continuous functional on \( P \), and let \( \theta \) be a nonnegative continuous functional on \( P \) with \( \theta(0) = 0 \) such that, for some \( c > 0 \) and \( H > 0 \),
\[
\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \| x \| \leq H \gamma(x)
\]
for all \( x \in \overline{P(\gamma, c)} \). Suppose there exist a completely continuous operator \( A : \overline{P(\gamma, c)} \to P \) and \( 0 < a < b < c \) such that
\[
\theta(\lambda x) \leq \lambda \theta(x) \quad \text{for} \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \partial P(\theta, b),
\]
and

(i) \( \gamma(Ax) > c \) for all \( x \in \partial P(\gamma, c) \);

(ii) \( \theta(Ax) < b \) for all \( x \in \partial P(\theta, b) \);

(iii) \( P(\alpha, a) \neq \emptyset \) and \( \alpha(Ax) > a \) for \( x \in \partial P(\alpha, a) \).

Then, \( A \) has at least two fixed points \( x_1 \) and \( x_2 \) belonging to \( \overline{P(\gamma, c)} \) satisfying
\[
a < \alpha(x_1) \quad \text{with} \quad \theta(x_1) < b, \quad \text{and} \quad b < \theta(x_2) \quad \text{with} \quad \gamma(x_2) < c.
\]
The following lemma is similar to Lemma 2.4.

**Lemma 2.5.** (See [33].) Let $P$ be a cone in a real Banach space $E$. Let $\alpha$ and $\gamma$ be increasing nonnegative continuous functional on $P$, and let $\theta$ be a nonnegative continuous functional on $P$ with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq H\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda \theta(x) \quad \text{for} \ 0 \leq \lambda \leq 1 \ \text{and} \ x \in \partial P(\theta, b),$$

and

(i) $\gamma(Ax) < c$ for all $x \in \partial P(\gamma, c)$;
(ii) $\theta(Ax) > b$ for all $x \in \partial P(\theta, b)$;
(iii) $P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) < a$ for $x \in \partial P(\alpha, a)$.

Then, $A$ has at least two fixed points $x_1$ and $x_2$ belonging to $\overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \quad \text{with} \ \theta(x_1) < b, \quad \text{and} \quad b < \theta(x_2) \quad \text{with} \ \gamma(x_2) < c.$$

Let $0 < a < b$ be given and let $\alpha$ is a nonnegative continuous concave functional on the cone $P$. Define the convex sets $P_a, P(\alpha, a, b)$ by

$$P_a = \{x \in P : \|x\| < a\},$$

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \ \|x\| \leq b\}.$$

Finally we state the Leggett–Williams fixed point theorem [30].

**Lemma 2.6.** Let $P$ be a cone in a real Banach space $E$, $A : \overline{P} \to \overline{P}$ be completely continuous and $\alpha$ be a nonnegative continuous concave functional on $P$ with $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}$. Suppose there exist $0 < d < a < b \leq c$ such that

(i) $\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;
(ii) $\|Ax\| < d$ for $\|x\| \leq d$;
(iii) $\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.

Then $A$ has at least three fixed points $x_1, x_2, x_3$ satisfying

$$\|x_1\| < d, \quad a < \alpha(x_2), \quad \|x_3\| > d \quad \text{and} \quad \alpha(x_3) < a.$$
3. Single or twin solutions

For the sake of convenience, we define as $i_0 = \text{number of zeros in the set } \{ f_0, f_\infty \}$ and $i_\infty = \text{number of infinities in the set } \{ f_0, f_\infty \}$. Clearly, $i_0, i_\infty = 0, 1, \text{ or } 2$ and there are six possible cases: (i) $i_0 = 0$ and $i_\infty = 0$; (ii) $i_0 = 0$ and $i_\infty = 1$; (iii) $i_0 = 0$ and $i_\infty = 2$; (iv) $i_0 = 1$ and $i_\infty = 0$; (v) $i_0 = 1$ and $i_\infty = 1$; and (vi) $i_0 = 2$ and $i_\infty = 0$. By using Krasnosel’skii’s fixed point theorem in a cone, some results are obtained for the existence of at least one or two positive solutions of the BVP (1.9) and (1.10) under the above six possible cases.

3.1. For the case $i_0 = 1$ and $i_\infty = 1$

In this subsection, we discuss the existence of single positive solution for the BVP (1.9) and (1.10) under $i_0 = 1$ and $i_\infty = 1$.

**Theorem 3.1.** BVP (1.9) and (1.10) has at least one positive solution in the case $i_0 = 1$ and $i_\infty = 1$.

**Proof.** First, we consider the case $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, then there exists $H_1 > 0$ such that $f(u) \leq \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$ for $0 < u \leq H_1$, where $\varepsilon$ satisfies

$$
\varepsilon (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \Delta s \right) \leq 1.
$$

If $u \in P$ with $\|u\| = H_1$, then

$$
\|Au\| = Au(\sigma^2(b))
$$

$$
= B_0 \left( \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^b \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
$$

$$
\leq B_2 \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) + \int_a^b \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
$$

$$
\leq (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \varphi_p(\varepsilon u^\sigma(s)) \Delta s \right)
$$

$$
= \varepsilon \|u\| (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \Delta s \right)
$$

$$
\leq \|u\|.
$$

It follows that if

$$
\Omega_{H_1} = \{ u \in E: \|u\| < H_1 \},
$$

then $\|Au\| \leq \|u\|$ for $u \in P \cap \partial \Omega_{H_1}$. 

Since \( f_\infty = \infty \), there exists \( H'_2 > 0 \) such that \( f(u) \geq \varphi_p(k)\varphi_p(u) = \varphi_p(ku) \) for \( u \geq H'_2 \), where \( k > 0 \) is chosen such that

\[
\frac{k(\eta - a)}{\sigma^2(b) - \frac{a}{\eta - a}} \left( B_1 \varphi_q \left( \int_\eta^\sigma h(s) \Delta s \right) + \int_\eta^\tau \varphi_q \left( \int_\tau^\sigma h(s) \Delta s \right) \Delta \tau \right) \geq 1.
\]

Set \( H_2 = \max\{2H_1, \frac{\sigma^2(b) - a}{\eta - a} H'_2 \} \) and

\[ \Omega_{H_2} = \{ u \in E : \|u\| < H_2 \}. \]

If \( u \in P \) with \( \|u\| = H_2 \), then

\[
\min_{t \in [\eta, \sigma^2(b)]} u(t) = u(\eta) \geq \frac{\eta - a}{\sigma^2(b) - \frac{a}{\eta - a}} \|u\| \geq H'_2.
\]

So that

\[
\|Au\| = Au(\sigma^2(b))
\]

\[
= B_0 \left( \varphi_q \left( \int_a^{\sigma^2(b)} h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma^2(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
\]

\[
\geq B_1 \varphi_q \left( \int_\eta^{\sigma^2(b)} h(s) f(u^\sigma(s)) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma^2(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
\]

\[
\geq B_1 \varphi_q \left( \int_\eta^{\sigma^2(b)} h(s) \varphi_p(ku^\sigma(s)) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma^2(b)} h(s) \varphi_p(ku^\sigma(s)) \Delta s \right) \Delta \tau
\]

\[
\geq \frac{k(\eta - a)}{\sigma^2(b) - \frac{a}{\eta - a}} \|u\| \left( B_1 \varphi_q \left( \int_\eta^{\sigma^2(b)} h(s) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma^2(b)} h(s) \Delta s \right) \Delta \tau \right)
\]

\[
\geq \|u\|.
\]

In other words, if \( u \in P \cap \partial \Omega_{H_2} \), then \( \|Au\| \geq \|u\| \). Thus by (i) of Lemma 2.3, it follows that \( A \) has a fixed point \( u \) in \( P \cap (\overline{\Omega_{H_2}} \setminus \Omega_{H_1}) \) with \( H_1 \leq \|u\| \leq H_2 \).

Now we consider the case \( f_0 = \infty \) and \( f_\infty = 0 \). Since \( f_0 = \infty \), there exists \( H_3 > 0 \) such that \( f(u) \geq \varphi_p(m)\varphi_p(u) = \varphi_p(mu) \) for \( 0 < u \leq H_3 \), where \( m \) is such that

\[
\frac{m(\eta - a)}{\sigma^2(b) - \frac{a}{\eta - a}} \left( B_1 \varphi_q \left( \int_\eta^{\sigma^2(b)} h(s) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma^2(b)} h(s) \Delta s \right) \Delta \tau \right) \geq 1.
\]

If \( u \in P \) with \( \|u\| = H_3 \), then we have
\[ \|Au\| = Au(\sigma^2(b)) \]
\[ = B_0 \left( \varphi_q \left( \int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \geq B_1 \varphi_q \left( \int_\eta^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \geq B_1 \varphi_q \left( \int_\eta^{\sigma(b)} h(s) \varphi_p(\mu u^\sigma(s)) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) \varphi_p(\mu u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \geq \frac{m(\eta - a)}{\sigma^2(b) - a} \|u\| \left( B_1 \varphi_q \left( \int_\eta^{\sigma(b)} h(s) \Delta s \right) + \int_\eta^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) \Delta s \right) \Delta \tau \right) \]
\[ \geq \|u\|. \]

Thus we let
\[ \Omega_{H_3} = \{ u \in E: \|u\| < H_3 \}, \]
so that \( \|Au\| \geq \|u\| \) for \( u \in P \cap \partial \Omega_{H_3} \).

Next consider \( f_\infty = 0 \). By definition there exists \( H'_4 > 0 \) such that \( f(u) \leq \varphi_p(\delta) \varphi_p(u) = \varphi_p(\delta u) \) for \( u \geq H'_4 \), where \( \delta > 0 \) satisfies
\[ \delta(B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^{\sigma(b)} h(s) \Delta s \right) \leq 1. \quad (3.1) \]

Suppose \( f \) is bounded. Then \( f(u) \leq \varphi_p(K) \) for all \( u \in [0, \infty) \) for some constant \( K > 0 \). Pick
\[ H_4 = \max \left\{ 2H_3, K(B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^{\sigma(b)} h(s) \Delta s \right) \right\}. \]

If \( u \in P \) with \( \|u\| = H_4 \), then
\[ \|Au\| = Au(\sigma^2(b)) \]
\[ = B_0 \left( \varphi_q \left( \int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \leq B_2 \varphi_q \left( \int_a^{\sigma(b)} h(s) \varphi_p(K) \Delta s \right) + \int_a^{\sigma^2(b)} \varphi_q \left( \int_\tau^{\sigma(b)} h(s) \varphi_p(K) \Delta s \right) \Delta \tau \]
\[ K \left( B_2 + \sigma^2(b) - a \right) \varphi_q \left( \int_a^b h(s) \Delta s \right) \leq H_4 = \|u\|. \]

Now suppose \( f \) is unbounded. From \( f \in C([0, \infty), [0, \infty)) \) it is easy to know that there exists 
\[ H_4 \geq \max \{ 2H_3, \frac{\sigma^2(b) - a}{\eta-a} H_4' \} \] 
such that \( f(u) \leq f(H_4) \) for \( 0 < u \leq H_4 \). If \( u \in P \) with \( \|u\| = H_4 \), then by using (3.1) we have
\[
\|Au\| = B_0 \left( \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_{\tau}^b \varphi_q \left( \int_a^\tau h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau 
\leq B_2 \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) + \int_{\tau}^b \varphi_q \left( \int_a^\tau h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau 
\leq \left( B_2 + \sigma^2(b) - a \right) \varphi_q \left( \int_a^b h(s) f(H_4) \Delta s \right) 
\leq \left( B_2 + \sigma^2(b) - a \right) \varphi_q \left( \int_a^b h(s) \varphi_p(\delta H_4) \Delta s \right) 
\leq \delta H_4 \left( B_2 + \sigma^2(b) - a \right) \varphi_q \left( \int_a^b h(s) \Delta s \right) 
\leq H_4 = \|u\|. 
\]

Consequently, in either case we take
\[ \Omega_{H_4} = \{ u \in E : \|u\| < H_4 \}, \]
so that for \( u \in P \cap \partial \Omega_{H_4} \) we have \( \|Au\| \leq \|u\| \). Thus by (ii) of Lemma 2.3 it follows that \( A \) has a fixed point \( u \) in \( P \cap (\overline{\Omega}_{H_4} \setminus \Omega_{H_3}) \) with \( H_3 \leq \|u\| \leq H_4 \). The proof is complete.

3.2. For the case \( i_0 = 0 \) and \( i_\infty = 0 \)

In this subsection, we discuss the existence for the positive solutions of BVP (1.9) and (1.10) under \( i_0 = 0 \) and \( i_\infty = 0 \).

Now, we shall state and prove the following main result:
Theorem 3.2. Suppose that the following conditions hold:

(C1) there exists constant $p' > 0$ such that $f(u) \leq \varphi_p(p' \Lambda_1)$, for $0 \leq u \leq p'$, where

$$
\Lambda_1 = \left( (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \Delta s \right) \right)^{-1};
$$

(C2) there exists constant $q' > 0$ such that $f(u) \geq \varphi_p(q' \Lambda_2)$, for $u \in [\frac{\eta-a}{\sigma^2(b)-a} q', q']$, where

$$
\Lambda_2 = \left( B_1 \varphi_q \left( \int_{\eta}^b h(s) \Delta s \right) + \int_{\eta}^\tau \varphi_q \left( \int_a^\tau h(s) \Delta s \right) \Delta \tau \right)^{-1},
$$

furthermore, $p' \neq q'$.

Then BVP (1.9) and (1.10) has at least one positive solution $u$ such that $\|u\|$ lies between $p'$ and $q'$.

Proof. Without loss of generality, we may assume that $p' < q'$.

Let $\Omega_{p'} = \{u \in E : \|u\| < p'\}$, for any $u \in P \cap \partial \Omega_{p'}$. In view of (C1), we have

$$
\|Au\| = Au(\sigma^2(b))
$$

$$
= B_0 \left( \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^\tau \varphi_q \left( \int_a^\tau h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
$$

$$
\leq B_2 \varphi_q \left( \int_a^b h(s) f(u^\sigma(s)) \Delta s \right) + \int_a^\tau \varphi_q \left( \int_a^\tau h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
$$

$$
\leq (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \varphi_p(p' \Lambda_1) \Delta s \right)
$$

$$
= p' \Lambda_1 (B_2 + \sigma^2(b) - a) \varphi_q \left( \int_a^b h(s) \Delta s \right)
$$

$$
= p',
$$

which yields

$$
\|Au\| \leq \|u\| \quad \text{for } u \in P \cap \partial \Omega_{p'}.
$$

(3.2)

Now, set $\Omega_{q'} = \{u \in E : \|u\| < q'\}$ for $u \in P \cap \partial \Omega_{q'}$, we have $\frac{\eta-a}{\sigma^2(b)-a} q' \leq u(t) \leq q'$ for $t \in [\eta, \sigma^2(b)]$. Hence, by (C2) we can get
\[ \|Au\| = Au(\sigma^2(b)) \]
\[
= B_0 \left( \varphi_q \left( \int_a^\sigma f(u^\sigma(s)) \Delta s \right) \right) + \int_a^\sigma \varphi_q \left( \int_\tau f(u^\sigma(s)) \Delta s \right) \Delta \tau
\]
\[
\geq B_1 \varphi_q \left( \int_\eta h(s)f(u^\sigma(s)) \Delta s \right) + \int_\eta \varphi_q \left( \int_\tau h(s)f(u^\sigma(s)) \Delta s \right) \Delta \tau
\]
\[
\geq B_1 \varphi_q \left( \int_\eta h(s) \varphi_p(q'\Lambda_2) \Delta s \right) + \int_\eta \varphi_q \left( \int_\tau h(s) \varphi_p(q'\Lambda_2) \Delta s \right) \Delta \tau
\]
\[
= q'\Lambda_2 \left( B_1 \varphi_q \left( \int_\eta h(s) \Delta s \right) + \int_\eta \varphi_q \left( \int_\tau h(s) \Delta s \right) \Delta \tau \right)
\]
\[
= q'.
\]

So, if we take \( \Omega_{q'} = \{u \in E: \|u\| < q'\} \), then
\[\|Au\| \geq \|u\|, \quad u \in P \cap \partial \Omega_{q'}. \] 

Consequently, in view of \( p' < q' \), (3.2) and (3.3), it follows from Lemma 2.3 that \( A \) has a fixed point \( u \in P \cap (\overline{\Omega}_{q'} \setminus \Omega_{p'}) \). Moreover, it is a positive solution of (1.9) and (1.10) and \( p' < \|u\| < q' \). The proof is complete. \( \Box \)

3.3. For the case \( i_0 = 1 \) and \( i_\infty = 0 \) or \( i_0 = 0 \) and \( i_\infty = 1 \)

In this subsection, we discuss the existence for the positive solutions of BVP (1.9) and (1.10) under \( i_0 = 1 \) and \( i_\infty = 0 \) or \( i_0 = 0 \) and \( i_\infty = 1 \).

**Theorem 3.3.** Suppose that \( f_0 \in [0, \varphi_p(\Lambda_1)) \) and \( f_\infty \in (\varphi_p(\sigma^2(b)-a \Lambda_2), \infty) \) hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Proof.** It is easy to see that under the assumptions, the conditions (C1) and (C2) in Theorem 3.2 are satisfied. So the proof is easy and we omit it here. \( \Box \)

**Theorem 3.4.** Suppose that \( f_0 \in (\varphi_p(\sigma^2(b)-a \Lambda_2), \infty) \) and \( f_\infty \in [0, \varphi_p(\Lambda_1)) \) hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Proof.** First, in view of \( f_0 \in (\varphi_p(\sigma^2(b)-a \Lambda_2), \infty) \), for \( \varepsilon = f_0 - \varphi_p(\sigma^2(b)-a \Lambda_2) > 0 \), there exists a sufficiently small \( q' > 0 \) such that
\[ \frac{f(u)}{\varphi_p(u)} \geq f_0 - \varepsilon = \varphi_p \left( \frac{\sigma^2(b)-a}{\eta-a} \Lambda_2 \right) \quad \text{for } u \in (0, q').\]
Thus, if \( u \in \left[ \eta - a \sigma^2(b) - a q', q' \right] \), then one has

\[
f(u) \geq \varphi_p \left( \frac{\sigma^2(b) - a}{\eta - a} \Lambda_2 \right) \varphi_p(u) \geq \varphi_p(\Lambda_2 q'),
\]

which yields the condition (C2) in Theorem 3.2.

Next, by \( f_\infty \in [0, \varphi_p(\Lambda_1)) \), for \( \varepsilon = \varphi_p(\Lambda_1) - f_\infty > 0 \), there exists a sufficiently large \( p'' \) (> \( q' \)) such that

\[
\frac{f(u)}{\varphi_p(u)} \leq f_\infty + \varepsilon = \varphi_p(\Lambda_1) \quad \text{for} \quad u \in [p'', \infty).
\]

(3.4)

We consider two cases:

Case (i). Suppose that \( f \) is bounded, say

\[
f(u) \leq \varphi_p(K) \quad \text{for} \quad u \in [0, \infty).
\]

(3.5)

In this case, take sufficiently large \( p' \) such that \( p' \geq \max \{ K/\Lambda_1, p'' \} \), then from (3.5), we know

\[
f(u) \leq \varphi_p(K) \leq \varphi_p(\Lambda_1 p') \quad \text{for} \quad u \in [0, p'].
\]

Then from the above inequality, the condition (C1) of Theorem 3.2 is satisfied.

Case (ii). Suppose that \( f \) is unbounded. Then from \( f \in C([0, \infty), [0, \infty)) \), we know that there is \( p' > p'' \) such that

\[
f(u) \leq f(p') \quad \text{for} \quad u \in [0, p'].
\]

(3.6)

Since \( p' > p'' \), then from (3.4), (3.6), one has

\[
f(u) \leq f(p') \leq \varphi_p(\Lambda_1 p') \quad \text{for} \quad u \in [0, p'].
\]

Thus, the condition (C1) of Theorem 3.2 is satisfied.

Hence, from Theorem 3.2, the conclusion of this theorem holds. The proof is complete. \( \square \)

From Theorems 3.3 and 3.4, we have the following two results.

**Corollary 3.1.** Suppose that \( f_0 = 0 \) and the condition (C2) in Theorem 3.2 hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Corollary 3.2.** Suppose that \( f_\infty = 0 \) and the condition (C2) in Theorem 3.2 hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Theorem 3.5.** Suppose that \( f_0 \in (0, \varphi_p(\Lambda_1)) \) and \( f_\infty = \infty \) hold. Then BVP (1.9) and (1.10) has at least one positive solution.
**Proof.** First, in view of $f_\infty = \infty$, similar to the first part of Theorem 3.1, we have
\[
\|Au\| \geq \|u\| \quad \text{for } u \in P \cap \partial \Omega_{H_2}.
\]
Next, by $f_0 \in (0, \varphi_p(A_1))$, for $\varepsilon = \varphi_p(A_1) - f_0 > 0$, there exists a sufficiently small $p' \in (0, H_2)$ such that
\[
f(u) \leq (f_0 + \varepsilon)\varphi_p(u) = \varphi_p(A_1u) \leq \varphi_p(A_1p') \quad \text{for } u \in [0, p'].
\]
Similar to the proof of Theorem 3.2, we obtain
\[
\|Au\| \leq \|u\| \quad \text{for } u \in P \cap \partial \Omega_{p'}.
\]
The result is obtained and the proof is complete. \(\square\)

**Theorem 3.6.** Suppose that $f_0 = \infty$ and $f_\infty \in (0, \varphi_p(A_1))$ hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Proof.** First, in view of $f_0 = \infty$, similar to the second part of Theorem 3.2, we have
\[
\|Au\| \geq \|u\|, \quad u \in P \cap \partial \Omega_{H_3}.
\]
Next, by $f_\infty \in (0, \varphi_p(A_1))$, similar to the second part of the proof of Theorem 3.4 and Theorem 3.2, we have
\[
\|Au\| \leq \|u\|, \quad u \in P \cap \partial \Omega_{p'},
\]
where $p' > H_3$. Thus, (1.9) and (1.10) has at least one positive solution and the proof is complete. \(\square\)

From Theorems 3.5 and 3.6, the following corollaries are easily obtained.

**Corollary 3.3.** Suppose that $f_\infty = \infty$ and the condition (C1) in Theorem 3.2 hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**Corollary 3.4.** Suppose that $f_0 = \infty$ and the condition (C1) in Theorem 3.2 hold. Then BVP (1.9) and (1.10) has at least one positive solution.

**3.4.** For the case $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$

In this subsection we study the existence of multiple positive solutions for the BVP (1.9) and (1.10) under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$.

Combining the proof of Theorems 3.1 and 3.2, the following two theorems are easily proved.

**Theorem 3.7.** Suppose that $i_0 = 0$ and $i_\infty = 2$ and the condition (C1) of Theorem 3.2 hold, then (1.9) and (1.10) has at least two positive solutions $u_1, u_2 \in P$ such that $0 < \|u_1\| < p' < \|u_2\|$.

**Theorem 3.8.** Suppose that $i_0 = 2$ and $i_\infty = 0$ and the condition (C2) of Theorem 3.2 hold, then (1.9) and (1.10) has at least two positive solutions $u_1, u_2 \in P$ such that $0 < \|u_1\| < q' < \|u_2\|$.
4. Further results on twin solutions

In the previous section, we have obtained some results on the existence of at least two positive solutions of (1.9) and (1.10). In this section, we will further discuss the existence of two positive solutions of (1.9) and (1.10) under the more general case.

For $u \in P$, we define the nonnegative increasing continuous functionals $\gamma$, $\theta$, and $\alpha$ by

$$
\gamma(u) = \min_{t \in [\eta, r]} u(t) = u(\eta),
$$

$$
\theta(u) = \max_{t \in [a, \eta]} u(t) = u(\eta),
$$

and

$$
\alpha(u) = \max_{t \in [a, r]} u(t) = u(r).
$$

We observe that, for each $u \in P$,

$$
\gamma(u) = \theta(u) \leq \alpha(u). \quad (4.1)
$$

In addition, for each $u \in P$, $\gamma(u) = u(\eta) \geq \frac{\eta - a}{\sigma^2(b) - a} \|u\|$. Thus

$$
\|u\| \leq \frac{\sigma^2(b) - a}{\eta - a} \gamma(u) \quad \text{for all } u \in P. \quad (4.2)
$$

Finally, we also note that

$$
\theta(\lambda u) = \lambda \theta(u), \quad 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, b').
$$

For the notational convenience, we denote

$$
M = (B_1 + \eta - a)\varphi_q \left( \int_{\eta}^{(b)} h(s) \Delta s \right),
$$

$$
N = (B_2 + \eta - a)\varphi_q \left( \int_{a}^{(b)} h(s) \Delta s \right),
$$

and

$$
L = (B_1 + r - a)\varphi_q \left( \int_{r}^{(b)} h(s) \Delta s \right).
$$

We now present the results in this section.
**Theorem 4.1.** Suppose that there are positive numbers \(a' < b' < c'\) such that

\[
0 < a' < \frac{L}{N} b' < \frac{(\eta - a)L}{(\sigma^2(b) - a)N} c'.
\]

Assume \(f(u)\) satisfies the following conditions:

(i) \(f(u) > \varphi_p\left(\frac{c'}{M}\right)\) for \(c' \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} c'\);

(ii) \(f(u) < \varphi_p\left(\frac{b'}{N}\right)\) for \(0 \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} b'\);

(iii) \(f(u) > \varphi_p\left(\frac{a'}{L}\right)\) for \(a' \leq u \leq \frac{\sigma^2(b) - a}{r - a} a'\).

Then BVP (1.9) and (1.10) has at least two positive solutions \(u_1\) and \(u_2\) such that

\[
a' < \max_{t \in [a, r]} u_1(t) \quad \text{with} \quad \max_{t \in [a, \eta]} u_1(t) < b', \quad \text{and} \quad b' < \max_{t \in [a, \eta]} u_2(t) \quad \text{with} \quad \max_{t \in [\eta, r]} u_2(t) < c'.
\]

**Proof.** By the definition of operator \(A\) and its properties, it suffices to show that the conditions of Lemma 2.4 hold with respect to \(A\).

We first show that if \(u \in \partial P(\gamma, c')\), then \(\gamma(Au) > c'\).

Indeed, if \(u \in \partial P(\gamma, c')\), then \(\gamma(u) = \min_{t \in [\eta, r]} u(t) = u(\eta) = c'\), since \(u \in P\), one gets \(u(t) \geq c', t \in [\eta, \sigma^2(b)]\). If we recall that \(\|u\| \leq \frac{\sigma^2(b) - a}{\eta - a} \gamma(u) = \frac{\sigma^2(b) - a}{\eta - a} c'\), we have \(c' \leq u(t) \leq \frac{\sigma^2(b) - a}{\eta - a} c', t \in [\eta, \sigma^2(b)]\). As a consequence of (i),

\[
f(u(t)) > \varphi_p\left(\frac{c'}{M}\right) \quad \text{for} \quad t \in [\eta, \sigma^2(b)].
\]

Also, \(Au \in P\), so we get

\[
\gamma(Au) = Au(\eta)
\]

\[
= B_0\left(\varphi_q\left(\int_a^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right)\right) + \int_a^{\eta} \varphi_q\left(\int_{\tau}^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right) \Delta \tau
\]

\[
\geq B_1 \varphi_q\left(\int_a^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right) + \int_a^{\eta} \varphi_q\left(\int_{\tau}^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right) \Delta \tau
\]

\[
\geq B_1 \varphi_q\left(\int_{\eta}^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right) + \int_{\eta}^{a} \varphi_q\left(\int_{\eta}^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right) \Delta \tau
\]

\[
= (B_1 + \eta - a) \varphi_q\left(\int_{\eta}^{\sigma(b)} h(s) f\left(u^{\sigma}(s)\right) \Delta s\right)
\]
\[ > (B_1 + \eta - a) \varphi_q \left( \sigma(b) \int_{\eta} h(s) \varphi_p \left( \frac{c'}{M} \right) \Delta s \right) \]

\[ = \frac{c'}{M} (B_1 + \eta - a) \varphi_q \left( \sigma(b) \int_{\eta} h(s) \Delta s \right) = c'. \]

Next, we verify that \( \theta(Au) < b' \) for \( u \in \partial P(\theta, b') \).

So, let us choose \( u \in \partial P(\theta, b') \), then \( \theta(u) = \max_{r \in [a, \eta]} u(t) = u(\eta) = b' \). This implies \( 0 \leq u(t) \leq b', \ t \in [a, \eta] \), and since \( u \in P \), we also have

\[ b' \leq u(t) \leq \|u\| \leq \frac{\sigma^2(b) - a}{\eta - a} u(\eta) = \frac{\sigma^2(b) - a}{\eta - a} b'. \]

So

\[ 0 \leq u(t) \leq \frac{\sigma^2(b) - a}{\eta - a} b', \quad t \in [a, \sigma^2(b)]. \]

Using (ii),

\[ f \left( u(t) \right) < \varphi_p \left( \frac{b'}{N} \right), \quad t \in [a, \sigma^2(b)]. \]

\( Au \in P \), and so

\[ \theta(Au) = (Au)(\eta) \]

\[ = B_0 \left( \varphi_q \left( \int_{a}^{\sigma(b)} h(s) f \left( u^\sigma(s) \right) \Delta s \right) + \int_{a}^{\eta} \varphi_q \left( \int_{\tau}^{\sigma(b)} h(s) f \left( u^\sigma(s) \right) \Delta s \right) \Delta \tau \right) \]

\[ \leq B_2 \varphi_q \left( \int_{a}^{\sigma(b)} h(s) f \left( u^\sigma(s) \right) \Delta s \right) + \int_{a}^{\eta} \varphi_q \left( \int_{a}^{\sigma(b)} h(s) f \left( u^\sigma(s) \right) \Delta s \right) \Delta \tau \]

\[ < (B_2 + \eta - a) \varphi_q \left( \int_{a}^{\sigma(b)} h(s) f \left( u^\sigma(s) \right) \Delta s \right) \]

\[ < (B_2 + \eta - a) \varphi_q \left( \int_{a}^{\sigma(b)} h(s) \varphi_p \left( \frac{b'}{N} \right) \Delta s \right) \]

\[ = \frac{b'}{N} (B_2 + \eta - a) \varphi_q \left( \int_{a}^{\sigma(b)} h(s) \Delta s \right) = b'. \]

Finally, we prove that \( P(\alpha, a') \neq \emptyset \) and \( \alpha(Au) > a' \) for all \( u \in \partial P(\alpha, a') \).
In fact, the constant function \( \frac{a'}{2} \in P(\alpha, a') \). Moreover, for \( u \in \partial P(\alpha, a') \), we have \( \alpha(u) = \max_{t \in [a, r]} u(t) = u(r) = a' \). This implies
\[
a' \leq u(t) \leq \frac{\sigma^2(b) - a}{r - a'} a', \quad t \in [r, \sigma^2(b)].
\]

Using assumption (iii),
\[
f(u(t)) > \varphi_p \left( \frac{a'}{L} \right), \quad t \in [r, \sigma^2(b)].
\]

As before \( Au \in P \), and so
\[
\alpha(Au) = (Au)(r)
= B_0 \left( \varphi_q \left( \int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \right) + \int_a^r \varphi_q \left( \int_\tau^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
\geq B_1 \varphi_q \left( \int_r^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) + \int_a^r \varphi_q \left( \int_r^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s \right) \Delta \tau
\geq (B_1 + r - a) \varphi_q \left( \int_r^{\sigma(b)} h(s) \varphi_p \left( \frac{a'}{L} \right) \Delta s \right)
= \frac{a'}{L} (B_1 + r - a) \varphi_q \left( \int_r^{\sigma(b)} h(s) \Delta s \right)
= a'.
\]

Thus, by Lemma 2.4, there exist at least two fixed points of \( A \) which are positive solutions \( u_1 \) and \( u_2 \), belonging to \( P(\gamma, c') \), of the BVP (1.9) and (1.10) such that
\[
a' < \alpha(u_1) \quad \text{with} \quad \theta(u_1) < b', \quad \text{and} \quad b' < \theta(u_2) \quad \text{with} \quad \gamma(u_2) < c'.
\]

The proof is complete. \( \Box \)

For convenience, we denote \( M', N' \) and \( L' \) by
\[
M' = (B_2 + \eta - a) \varphi_q \left( \int_a^{\sigma(b)} h(s) \Delta s \right),
\]
\[
N' = (B_1 + \eta - a) \varphi_q \left( \int_\eta^{\sigma(b)} h(s) \Delta s \right),
\]
\[ L' = (B_2 + r - a)\varphi_q \left( \int_a^{\sigma(b)} h(s) \Delta s \right). \]

**Theorem 4.2.** Assume that there are positive numbers \( a' < b' < c' \) such that

\[ 0 < a' < \frac{r - a}{\sigma^2(b) - a} b' < \frac{(r - a)N'}{(\sigma^2(b) - a)M'} c'. \]

Suppose \( f(u) \) satisfies the following conditions:

(i) \( f(u) < \varphi_p \left( \frac{c'}{M'} \right) \) for \( 0 \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} c' \);

(ii) \( f(u) > \varphi_p \left( \frac{b'}{N'} \right) \) for \( b' \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} b' \);

(iii) \( f(u) < \varphi_p \left( \frac{a'}{L'} \right) \) for \( 0 \leq u \leq \frac{\sigma^2(b) - a}{r - a} a' \).

Then BVP (1.9) and (1.10) has at least two positive solutions \( u_1 \) and \( u_2 \) such that

\[ a' < \max_{t \in [a, r]} u_1(t) \quad \text{with} \quad \max_{t \in [a, \eta]} u_1(t) < b', \quad \text{and} \quad b' < \max_{t \in [\eta, r]} u_2(t) \quad \text{with} \quad \max_{t \in [\eta, r]} u_2(t) < c'. \]

Using Lemma 2.5, the proof is similar to that of Theorem 4.1 and we omit it here.

**Corollary 4.1.** Assume that \( f \) satisfies conditions

(i) \( f_0 = \infty, f_\infty = \infty \);

(ii) there exists \( u_0 > 0 \) such that

\[ f(u) < \varphi_p \left( \frac{1}{N} \right) \varphi_p \left( \frac{\eta - a}{\sigma^2(b) - a} u_0 \right) \quad \text{for} \quad 0 \leq u \leq u_0. \]

Then BVP (1.9) and (1.10) has at least two positive solutions.

**Proof.** At first, by (ii), choose \( b' = \frac{\eta - a}{\sigma^2(b) - a} u_0 \), one gets

\[ f(u) < \varphi_p \left( \frac{1}{N} \right) \varphi_p (b') = \varphi_p \left( \frac{b'}{N} \right) \quad \text{for} \quad 0 \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} b'. \]

Now, choose \( K_1 \) sufficiently large such that

\[ K_1 L = K_1 (B_1 + r - a) \varphi_q \left( \int_r^{\sigma(b)} h(s) \Delta s \right) > 1. \]

Since \( f_0 = \infty \), there exists \( r_1 > 0 \) sufficiently small such that

\[ f(u) \geq \varphi_p (K_1) \varphi_p (u) = \varphi_p (K_1 u) \quad \text{for} \quad 0 \leq u \leq r_1. \]
Without loss of generality, suppose
\[ r_1 \leq \frac{L}{N} \frac{\sigma^2(b) - a}{r - a} b'. \]

Choose \( a' > 0 \) so that \( a' < \frac{r-a}{\sigma^2(b) - a} r_1 \). For \( a' \leq u \leq \frac{\sigma^2(b) - a}{r - a} a' \), we have \( u \leq r_1 \) and \( a' < \frac{L}{N} b' \). Thus
\[ f(u) \geq \varphi_p(K_1u) \geq \varphi_p(K_1a') > \varphi_p \left( \frac{a'}{L} \right) \text{ for } a' \leq u \leq \frac{\sigma^2(b) - a}{r - a} a'. \]

Thirdly, choose \( K_2 \) sufficiently large such that
\[ K_2 M = K_2(B_1 + \eta - a) \varphi_q \left( \int_{\eta}^{\eta} h(s) \Delta s \right) > 1. \]

Since \( f_\infty = \infty \), there exists \( r_2 > 0 \) sufficiently large such that
\[ f(u) \geq \varphi_p(K_2u) = \varphi_p(K_2u) \text{ for } u \geq r_2. \]

Without loss of generality, suppose \( r_2 > \frac{\sigma^2(b) - a}{\eta - a} b' \). Choose \( c' \geq r_2 \). Then
\[ f(u) \geq \varphi_p(K_2u) \geq \varphi_p(K_2c') > \varphi_p \left( \frac{c'}{M} \right) \text{ for } c' \leq u \leq \frac{\sigma^2(b) - a}{\eta - a} c'. \]

We get now
\[ 0 < a' < \frac{L}{N} b' < \frac{(\eta - a)L}{(\sigma^2(b) - a) N} c', \]
and then the conditions in Theorem 4.1 are all satisfied. By Theorem 4.1, BVP (1.9) and (1.10) has at least two positive solutions. \( \square \)

**Corollary 4.2.** Assume that \( f \) satisfies conditions
\begin{enumerate}
  \item \( f_0 = 0, \ f_\infty = 0; \)
  \item there exists \( u_0 > 0 \) such that
    \[ f(u) > \varphi_p \left( \frac{1}{N'} \right) \varphi_p \left( \frac{\eta - a}{(\sigma^2(b) - a) u_0} \right) \text{ for } \frac{\eta - a}{(\sigma^2(b) - a) u_0} u_0 \leq u \leq u_0. \]
\end{enumerate}

Then BVP (1.9) and (1.10) has at least two positive solutions.

Similar to that of Corollary 4.3 and applying Theorem 4.2, the proof is easy and we omitted it.
5. Triple solutions

In this section, we shall study the existence of three positive solutions of (1.9) and (1.10). Let the nonnegative continuous concave functional \( \Psi : P \to [0, \infty) \) be defined by

\[
\Psi(u) = \min_{t \in [\eta, \tau_2(b)]} u(t) = u(\eta), \quad u \in P.
\]

Note that for \( u \in P \), \( \Psi(u) \leq \|u\| \).

**Theorem 5.1.** Suppose that there exist constants \( 0 < d' < a' < \frac{d'(\sigma^2(b) - a)}{\eta - a} \leq c' \) such that

(i) \( f(u) < \varphi_p\left(\frac{d'}{D'}\right) \) for \( u \in [0, d'] \), where

\[
D' = (B_2 + \sigma^2(b) - a)\varphi_q\left(\int_a^{\sigma(b)} h(s) \Delta s\right);
\]

(ii) \( f(u) \geq \varphi_p\left(\frac{a'}{C'}\right) \) for \( u \in [a', \frac{\sigma^2(b) - a}{\eta - a} a'] \), where

\[
C' = (B_1 + \eta - a)\varphi_q\left(\int_{\eta}^{\sigma(b)} h(s) \Delta s\right);
\]

(iii) \( f(u) \leq \varphi_p\left(\frac{c'}{D'}\right) \) for \( u \in [0, c'] \).

Then BVP (1.9) and (1.10) has at least three positive solutions.

**Proof.** By the definition of operator \( A \) and its properties, it suffices to show that the conditions of Lemma 2.6 hold with respect to \( A \).

For convenience, we denote \( b' = \frac{\sigma^2(b) - a}{\eta - a} a' \).

We first show that if there exists a positive number \( r' \) such that \( f(u) < \varphi_p(r'/D') \) for \( u \in [0, r'] \), then \( A : P_{r'} \to P_{r'} \).

Indeed, if \( u \in P_{r'} \), then

\[
\|Au\| = Au(\sigma^2(b)) \leq B_2\varphi_q\left(\int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s\right) + \varphi_q\left(\int_{\tau}^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s\right) \Delta \tau \leq B_2\varphi_q\left(\int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s\right) + \varphi_q\left(\int_a^{\sigma(b)} h(s) f(u^\sigma(s)) \Delta s\right) \Delta \tau
\]
\[ (B_2 + \sigma^2(b) - a)\phi_q \left( \int_a^{\sigma(b)} h(s)\phi_p \left( \frac{r'}{D} \right) \Delta s \right) \]
\[ = \frac{r'}{D} (B_2 + \sigma^2(b) - a)\phi_q \left( \int_a^{\sigma(b)} h(s) \Delta s \right) \]
\[ = r', \]
thus, \( Au \in P_{r'} \).

Hence, we have shown that if (i) and (iii) hold, then \( A \) maps \( \overline{P}_{d'} \) into \( P_{d'} \) and \( \overline{P}_{c'} \) into \( \overline{P}_{c'} \).

Next, we verify that \( \{ u \in P(\Psi, a', b') : \Psi(u) > a' \} \neq \emptyset \) and \( \Psi(Au) > a' \) for all \( u \in P(\Psi, a', b') \).

In fact,
\[ u = \frac{a' + b'}{2} \in \{ u \in P(\Psi, a', b') : \Psi(u) > a' \}. \]

For \( u \in P(\Psi, a', b') \), we have
\[ a' \leq \min_{t \in [\eta, \sigma^2(b)]} u(t) = u(\eta) \leq b', \]
for all \( t \in [\eta, \sigma^2(b)] \). Then, in view of (ii), we know that
\[ \Psi(Au) = \min_{t \in [\eta, \sigma^2(b)]} Au(t) = Au(\eta) \]
\[ = B_0 \left( \phi_q \left( \int_a^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) \right) + \int_{\eta}^{\sigma(b)} \phi_q \left( \int_{\tau}^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \geq B_1\phi_q \left( \int_a^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) + \int_{\eta}^{\sigma(b)} \phi_q \left( \int_{\tau}^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ \geq B_1\phi_q \left( \int_{\eta}^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) + \int_{\eta}^{\sigma(b)} \phi_q \left( \int_{\eta}^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) \Delta \tau \]
\[ = (B_1 + \eta - a)\phi_q \left( \int_{\eta}^{\sigma(b)} h(s)f(u^\sigma(s)) \Delta s \right) \]
\[ > (B_1 + \eta - a)\phi_q \left( \int_{\eta}^{\sigma(b)} h(s)\phi_p \left( \frac{a'}{C'} \right) \Delta s \right) \]
\[ = \frac{a'}{C'}(B_1 + \eta - a)\varphi_q \left( \int_{\eta}^{\sigma(b)} h(s) \Delta s \right) = a'. \]

Finally, we assert that if \( u \in P(\Psi, a', c') \) and \( \|Au\| > b' \), then \( \Psi(Au) > a' \).

Suppose \( u \in P(\Psi, a', c') \) and \( \|Au\| > b' \), then

\[ \Psi(Au) = \min_{t \in [\eta, \sigma^2(b)]} Au(t) = Au(\eta) \geq \frac{\eta - a}{\sigma^2(b) - a} \|Au\| > \frac{\eta - a}{\sigma^2(b) - a} b' = a'. \]

To sum up, the hypotheses of Lemma 2.6 are satisfied, hence BVP (1.9) and (1.10) has at least three positive solutions \( u_1, u_2, u_3 \) such that

\[ \|u_1\| < d', \quad a' < \min_{t \in [\eta, \sigma^2(b)]} u_2(t), \quad \text{and} \quad \|u_3\| > d' \quad \text{with} \quad \min_{t \in [\eta, \sigma^2(b)]} u_3(t) < a'. \]

The proof is complete. \( \square \)

We remark that the condition (iii) in Theorem 5.1 can be replaced by the following condition (iii'):

\[ \lim_{u \to \infty} \frac{f(u)}{\varphi_p(u)} < \varphi_p \left( \frac{1}{D'} \right), \]

which is a special case of (iii).

**Corollary 5.1.** If the condition (iii) in Theorem 5.1 is replaced by (iii'), then the conclusion of Theorem 5.1 also holds.

**Proof.** By Theorem 5.1, we only need to prove that (iii') implies that (iii) holds, that is, if (iii') holds, then there is a number \( c' \geq \frac{a'(\sigma^2(b) - a)}{\eta - a} \) such that \( f(u) < \varphi_p \left( \frac{c'}{D'} \right) \) for \( u \in [0, c'] \).

Suppose on the contrary that for any \( c' \geq \frac{a'(\sigma^2(b) - a)}{\eta - a} \), there exists \( u_c \in [0, c'] \) such that \( f(u_c) \geq \varphi_p \left( \frac{c'}{D'} \right) \). Hence, if we choose \( c'_n > \frac{a'(\sigma^2(b) - a)}{\eta - a} \) \((n = 1, 2, \ldots)\) with \( c'_n \to \infty \), then there exist \( u_n \in [0, c'_n] \) such that

\[ f(u_n) \geq \varphi_p \left( \frac{c'_n}{D'} \right), \quad (5.1) \]

and so

\[ \lim_{n \to \infty} f(u_n) = \infty. \quad (5.2) \]

Since the condition (iii') holds, then there exists \( \tau > 0 \) such that

\[ f(u) < \varphi_p \left( \frac{u}{D'} \right), \quad u > \tau. \quad (5.3) \]
Hence we have \( u_n \leq \tau \). Otherwise, if \( u_n > \tau \), then it follows from (5.3) that
\[
f(u_n) < \varphi_p \left( \frac{u_n}{D} \right) \leq \varphi_p \left( \frac{c'_n}{D} \right),
\]
which contradicts (5.1).

Let \( W = \max_{u \in [0, \tau]} f(u) \), then \( f(u_n) \leq W \) \((n = 1, 2, \ldots)\), which also contradicts (5.2). The proof is complete. \( \square \)

From Theorem 5.1, we see that, when assumptions as (i)–(iii) are imposed appropriately on \( f \), we can establish the existence of an arbitrary odd number of positive solutions of (1.9) and (1.10).

**Theorem 5.2.** Suppose that there exist constants
\[
0 < d'_1 < a'_1 < \frac{a'_1 (\sigma^2 (b) - a)}{\eta - a} < d'_2 < a'_2 < \frac{a'_2 (\sigma^2 (b) - a)}{\eta - a} < d'_3 < \cdots < d'_n, \quad n \in \mathbb{N},
\]
such that the following conditions are satisfied:

(i) \( f(u) < \varphi_p \left( \frac{d'_1}{D} \right) \) for \( u \in [0, d'_1] \);

(ii) \( f(u) \geq \varphi_p \left( \frac{d'_1}{C} \right) \) for \( u \in [d'_1, a'_1 (\sigma^2 (b) - a) / (\eta - a)] \).

Then, BVP (1.9) and (1.10) has at least \( 2n - 1 \) positive solutions.

**Proof.** When \( n = 1 \), it is immediate from condition (i) that \( A : \overline{P}_{d'_1} \to \overline{P}_{d'_1} \subset \overline{P}_{d'_1} \), which means that \( A \) has at least one fixed point \( u_1 \in \overline{P}_{d'_1} \) by the Schauder fixed point theorem. When \( n = 2 \), it is clear that Theorem 5.1 holds (with \( c_1 = d'_2 \)). Then we can obtain at least three positive solutions \( u_1, u_2, \) and \( u_3 \) satisfying
\[
\|u_1\| < d'_1, \quad \min_{t \in [\eta, \sigma^2 (b)]} u_2(t) > a'_1, \quad \text{and} \quad \|u_3\| > d'_1 \quad \text{with} \quad \min_{t \in [\eta, \sigma^2 (b)]} u_3(t) < a'_1.
\]

Following this way, we finish the proof by induction. The proof is complete. \( \square \)

**References**


