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MODULES OVER THE STEENROD ALGEBRA

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§1. INTRODUCTION

IN [1], Adams and Margolis prove the following theorem about modules over the mod 2 Steenrod algebra: Let M be a module which is bounded below, then M is a free module if and only if $H(M, a_i) = 0$, where $\{a_i\}$ is a collection of elements in the Steenrod algebra with $(a_i)^2 = 0$.

In this paper we generalize this theorem to the mod p Steenrod algebra. To do this there are several problems to be overcome. Firstly, a correct theorem must be formulated. This is done by finding elements a_i in the Steenrod algebra with $(a_i)^p = 0$ and defining $H(M, a_i) = \operatorname{Ker} a_i/\operatorname{Im}(a_i)^{p-1}$. Secondly, the proof given in [1] does not seem to generalize to the mod p Steenrod algebra and hence we had to find a proof that did. Finally, the reduced power operations only generate a subalgebra of the mod p Streenrod algebra and we had to extend our results from this subalgebra to the full Steenrod algebra.

In Section 2 we state our results precisely and give the proofs in the following sections.

§2. STATEMENTS OF RESULTS

Let $\mathscr{A}$ denote the mod p Steenrod algebra (including $p = 2$). Let $'\mathscr{A} \subset \mathscr{A}$ be the subalgebra generated by the reduced power operations. Let $P_t(r) = \mathscr{P}^{(0, \ldots, r, 0, \ldots)}$, with the r in the tth place, using Milnor's notation [4]. Let $P_t^s = P_t(p^s)$. It is easy to check that $(P_t^s)^p = 0$ if $s < t$. We can now state our first main theorem, which is the same as the theorem of Adams and Margolis when $p = 2$.

THEOREM 2.1. *Let M be a graded left $'\mathscr{A}$-module which is bounded below. Then M is a free $'\mathscr{A}$-module if and only if $H(M, P_t^s) = 0$ for all $s < t$.*

Let $p > 2$. Let $Q_i \in \mathscr{A}$ be defined by $Q_0 = \beta$ and $Q_{i+1} = [\mathscr{P}^{p^i}, Q_i]$. Let $E \subset \mathscr{A}$ be the exterior algebra on the Q_i's. $\mathscr{A}$ is generated by the subalgebras $'\mathscr{A}$ and E (see [4]). Our second main theorem is the following one.

THEOREM 2.2. *Let M be a graded left $\mathscr{A}$-module which is bounded below. Then M is a free $\mathscr{A}$-module if and only if M is a free $'\mathscr{A}$-module and a free E-module if and only if $H(M, P_t^s) = 0$ for all $s < t$ and $H(M, Q_i) = 0$, all $i \geq 0$.*

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As an application of our results, we note that the analogue of the main theorem of [6] concerning coalgebras over the mod p Steenrod algebra is an immediate corollary of Theorem 2.1 or 2.2 and the proof given in [6].

Let $\Lambda' = \Lambda_t'$ be the subalgebra of $\mathscr{A}$ generated by P_t^0, P_t^1, $\ldots$, P_t^{t-1}, and P_{2t}^0. Let $\Lambda = \Lambda_t$ be the subalgebra of $\mathscr{A}$ generated by Λ' and P_t^t. The structure of Λ' and Λ is given by the following proposition.

PROPOSITION 2.3.

(a) *If* $r_1 < p^t$, *then* $P_t(r_1)P_t(r_2) = [(r_1 + r_2)!/r_1!r_2!]P_t(r_1 + r_2)$,

(b) *if* r_1 *and* $r_2 < p^t$, *then* $[P_t(r_1), P_t(r_2)] = 0$,

(c) *if* $r < p^t$, *then* $(P_t(r))^p = 0$,

(d) *if* $r \leq p^t$, *then* $[P_t(r), P_{2t}^0] = 0$,

(e) *if* $1 \leq r < p^t$, *then* $[P_t^t, P_t(r)] = P_t(r-1)P_{2t}^0$,

and

(f) $(P_t^t)^p = P_t(p^t - (p-1))(P_{2t}^0)^{p-1}$.

Relation (f) for p odd will be proved in §6; the rest of Proposition 2.3 is an easy exercise in manipulation with Milnor matrices [4].

It follows immediately from Proposition 2.3 that $\Lambda_t' \approx Z_p[P_t^0, \ldots, P_t^{t-1}, P_{2t}^0]/(P_t^0)^p, \ldots, (P_t^{t-1})^p, (P_{2t}^0)^p$, and that the extension $0 \to \Lambda'' \to \Lambda_t$ is determined by relations (d), (e), and (f). Note that $\Lambda_t//\Lambda_t' \approx Z_p[P_{tt}]/(P_t^t)^p$. The algebra Λ_1 has been studied by Liulevicius [2]. The following theorem is the main step in the proof of Theorem 2.1.

THEOREM 2.4. *Let M be a left Λ_t-module which is bounded below. If M is free as a Λ_t'-module, then M is free as a Λ_t-module.*

D. W. Anderson (unpublished) has given a general method for proving theorems like 2.4. This can be used to prove Theorem 2.4 for $p = 2$, but as yet we have not been able to use this method to prove 2.4 for odd primes.

Let K be a finite dimensional Hopf algebra over a field. Let K′ be a Hopf subalgebra. Assume K′ is $(n-1)$-connected and that K is generated by K′ and $z \in$ K. Assume $z^q = 0$, and $z^{q-1} \neq 0$. The following theorem is the main step in the proof of Theorem 2.2.

THEOREM 2.5. *Assume $|z^{q-1}| < n$. Let M be a* K*-module which is bounded below. If M is free as a* K′*-module and $H(M, z) = 0$, then M is free as a* K*-module.*

In §3 we give some elementary properties of differentials of height q, in §4 we prove Theorem 2.4, in §5 we prove Theorems 2.5, 2.1, and 2.2, and in §6 we prove 2.3(f).

§3. HOMOLOGY

Let k be a field, and let $A = k[d]/(d^q)$. If M is a left module over A, define, for

$$1 \leq i \leq q-1,$$

$$H_n(M, d^i) = \mathrm{Ker}(d^i\colon M_n \to M_{n+i|d|})/\mathrm{Im}(d^{q-i}\colon M_{n-(q-i)|d|} \to M_n).$$

The following four propositions, which show that these homology groups are similar to the usual ones, are all elementary and their proofs are left to the reader.

PROPOSITION 3.1. *Let M be a left A-module which is bounded below. The following four conditions are equivalent*:

(a) *M is a free A-module,*
(b) *$H_n(M, d) = 0$ for all n,*
(c) *$H_n(M, d^i) = 0$ for some i and all n,*
(d) *$H_n(M, d^i) = 0$ for all i and all n.*

PROPOSITION 3.2. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of left A-modules. Then the following is a long exact sequence*:

$$\cdots \to H_n(M', d^i) \to H_n(M, d^i) \to H_n(M'', d^i) \to H_{n+i|d|}(M', d^{q-1}) \to H_{n+i|d|}(M, d^{q-i}) \to \cdots$$

PROPOSITION 3.3. *Assume* char $k|q$. *Let $f: M \to N$ be a map of left A-modules. Then $f_*: H_n(M, d) \to H_n(N, d)$ is an isomorphism for all n if and only if $f_*: H_n(M, d^i) \to H_n(N, d^i)$ is an isomorphism for all n and all i, $1 \le i \le q-1$.*

§4. PROOF OF THEOREM 2.4

We first do the case $p = 2$. The proof for p odd is a generalization of this but is somewhat complicated due to certain technicalities, but the basic ideas already appear in the proof for $p = 2$.

We first prove a combinatorial lemma and a corollary.

LEMMA 4.1. *Let $\{A_\alpha\}$ be a collection of non-empty subsets of $\{0, 1, \ldots, t-1\}$. $A_\alpha = i_{\alpha_1} < \cdots < i_{\alpha_s}$. Then there is a non-empty subset $I = i_1 < \cdots < i_s$ of $\{0, 1, \ldots, t-1\}$ such that $I \cap A_\alpha \neq \varnothing$ all α and there is a β such that $I \cap A_\beta = i_{\beta_1} = i_1$.*

PROPOSITION 3.4. *Let ${}^iA = A/d^iA$, $0 < i < q$. Then*

$$\mathrm{Tor}^A_{2s,t}({}^iA, M) = H_{t-ps|d|}(M, d^{p-i}) \quad s > 0$$

and

$$\mathrm{Tor}^A_{2s+1,t}({}^iA, M) = H_{t-(ps+i)|d|}(M, d^i), \quad s \ge 0.$$

Proof. Put all A_α with $|A_\alpha| = 1$ into I_1. Assume we have constructed I_n such that $I_n \cap A_\alpha \neq \varnothing$ for all α with $|A_\alpha| \le n$ and that $I_n \cap A_{\beta_n} = i_{1_n} = i_{\beta_{n1}}$ for some β_n with $|A_{\beta_n}| \le n$. I_1 satisfies this for $n = 1$. To show that we can construct I_{n+1}. Consider all A_α such that $|A_\alpha| = n+1$ and $I_n \cap A_\alpha = \varnothing$. If $i_{\alpha_1} > i_{1_n}$, then find $i_{\alpha_j} \notin A_{\beta_n}$ and add to I_n to form I_n'. Do this for all such A_α. Now consider all A_α such that $I_n' \cap A_\alpha = \varnothing$ and $|A_\alpha| = n+1$. Let $\bar{A}_\alpha = A_\alpha \cap \{0, 1, \ldots, i_{n_1} - 1\} \subset \{0, 1, \ldots, i_{n_1} - 1\}$. By induction on t, we can find $\bar{I}$ and $\bar{A}_\beta$ satisfying the lemma. Let $I_{n+1} = I_n' \cup \bar{I}$ and let $\beta_{n+1} = \beta$. This induction works if $i_{n_1} < t$, i.e. if $I_n \neq \varnothing$. If $I_n = \varnothing$ and there exist A_α with $|A_\alpha| = n+1$, let I_{n+1} be the smallest i_{α_1} in these A_α and some element from each of the other A_α which is not in this A_β. This can be done because of the equal cardinality.

COROLLARY 4.2. *Let M' be a free Λ_t'-module with all elements of degree ≥ 0. Assume $M' \subset \overline{M}$, a Λ_t-module. Let m_1' be a non-zero decomposable in M' such that $|m_1'| < |P_{2t}{}^0|$. Then there is a $\lambda \in \Lambda_t'$ such that $\lambda P^t m_1' \neq 0 \in M'$.*

Proof. $m'_1 = \Sigma \lambda_\alpha m_\alpha$, with $|\lambda_\alpha| > 0$. Assume λ_α are nominals $\lambda_\alpha = P_t^{i_{\alpha_1}} \cdots P_t^{i_{\alpha_s}}$ and let $A_\alpha = \{i_{\alpha_1}, \ldots, i_{\alpha_s}\}$. Note $P_{2t}{}^0$ does not appear as $|m_1'| < |P_{2t}{}^0|$. Here m_α are a Λ_t-base for M'. Apply the lemma and let $\lambda = P_t^{i_1} \cdots P_t^{i_s}$. Then $\lambda P_t{}^t m_1' = P_t^{i_1} \cdots P_t^{i_s} P_t{}^t(\Sigma \lambda_\alpha m_\alpha) = P_t(2^{i_1} - 1)P_t^{i_2} \cdots P^{i_s} P_{2t}{}^0 \lambda_\beta m_\beta + P_t^{i_2} \cdots P_t^{i_s} P_t{}^t (P_t^{i_1})^2 \cdots m_\beta +$ other terms $\neq 0$ in M' as all terms with $P_t{}^t$ in them contain $(P_t{}^j)^2$ for some j.

Proof of 2.4 *for $p = 2$.* Let M be a Λ_t-module which is free as a Λ_t'-module. We must show M is Λ_t-free. Embed M in its injective envelope (see [3]). That is, $M \subset \overline{M}$ with $\overline{M}$ a Λ_t-module which is injective and the extension is essential; i.e. if $N \subset \overline{M}$ and $N \neq 0$, then $N \cap M \neq 0$. We wish to show that $M = \overline{M}$ as injectives over Λ_t are free [5]. Since $\overline{M}$ is free, M is bounded below, and Λ_t is finite dimensional, we see that $\overline{M}$ is also bounded below. As Λ_t'-modules, M is free hence injective, hence $\overline{M} = M \oplus M'$ as Λ_t'-modules with all three modules free over Λ_t'. Let $m' \neq 0$ be an element of minimal dimension in M'. $P_t{}^t(m') = m_1 \oplus m_1'$. Note $m_1' \neq 0$ or we would have $(P_t{}^t)^2 m' = P_t{}^t m_1 \in M$ and $(P_t{}^t)^2 m' = P_t(2^t - 1)P_{2t}{}^0 m' \neq 0 \in M'$ as m' must generate a free Λ_t'-module in M'. Further,

$$P_t(2^t - 1)P_{2t}{}^0 P_t{}^t(m') = P_t(2^t - 1)P_{2t}{}^0 m_1 \oplus P_t(2^t - 1)P_{2t}{}^0 m_1'$$

(because $P_t(2^t - 1)P_{2t}{}^0 \in \Lambda'$). This must be in M as $P_t(2^t - 1)P_{2t}{}^0 P_t{}^t$ is the top class of Λ_t and (some multiple of m') $\in M$ by the essential extension property. Hence, m_1' is Λ_t'-decomposable in M'. By Corollary 4.2, there is an element $\lambda \in \Lambda_t'$, $|\lambda| > 0$, such that $\lambda P_t{}^t m_1' \in M'$ and is non-zero. Then $\lambda P_t{}^t P_t{}^t m' = \lambda P_t{}^t m_1 \oplus \lambda P_t{}^t m_1' = \lambda P_t(2^t - 1)P_{2t}{}^0 m' = 0$, a contradiction. Thus m' did not exist and $M = \overline{M}$.

Proof of 2.4 *for p odd.* Let $M \subset \overline{M}$ be the Λ_t-injective envelope of M. As Λ_t'-modules, $\overline{M} = M \oplus M'$ and we can write $(P_t{}^t)^{p-1} m' = m_1 \oplus m_1'$, where m' is a lowest dimensional element of M'. $m_1' \neq 0$ as before and

$$P_t(p^t - 1)(P_{2t}{}^0)^{p-1}(P_t{}^t)^{p-1} m' = P_t(p^t - 1)(P_{2t}{}^0)^{p-1} m_1 \oplus P_t(p^t - 1)(P_{2t}{}^0)^{p-1}(m_1') \in M$$

by the essential extension property. Hence m_1' is decomposable and non-zero. The following combinatorial lemma is now needed.

LEMMA 4.3. *Let $\{A_\alpha\}$ be a collection of weighted nonempty, subsets of $\{0, 1, \ldots, t - 1\}$, with each i_{α_j} having a weighting $\leq p - 1$. $A_\alpha = i_{\alpha_1}{}^{u_{\alpha_1}}, \ldots, i_{\alpha_s}{}^{u_{\alpha_s}}$. Then, there is a non-empty-weighted subset $I = i_1{}^{u_1}, \ldots, i_s{}^{u_s}$ of $\{0, 1, \ldots, t - 1\}$ such that $I \cup A_\alpha$ has an element of weight $\geq p$ for all α and there is a β such that $I \cup A_\beta$ has only $i_{\beta_1} = i_1$ of weight $\geq p$ and that has weight p.*

Proof. The proof of Lemma 4.1 generalizes almost directly. After forming I_n', consider those A_α with $|A_\alpha| = n + 1$, $I_n' \cup A_\alpha$ having weights $\leq p - 1$, and $i_{\alpha_1} = i_{1_n}$. If such exist, increase the weight of i_{1_n} in I_n and change A_{β_n}. Then continue the proof as before.

We now use the same argument as in the mod 2 case to see that there is a $\lambda \in \Lambda'$ such that $\lambda P_t{}^t m_1' \neq 0$ in M'. However, since $P_t(p^t - (p - 1)) = a P_t{}^0 (P_t{}^1 \cdots P_t{}^{t-1} P_{2t}{}^0)^{p-1}$ with

$a \neq 0$, it may not be true that $\lambda(P_t{}^t)^p m' = 0$. Thus, we must discuss the case when $\lambda = (P_t{}^0)^i$, $1 \leq i \leq p-2$. Recall that $m_1' = \Sigma \lambda_\alpha m_\alpha$. If λ_β has $P_t{}^1, \ldots, P_t{}^{t-1}$ with weight $<p-1$ or $P_{2t}{}^0$ with weight $<p-2$, then we can multiply λ by an appropriate $P_t{}^i$ or $P_{2t}{}^0$ so that $P_t{}^i \lambda P_t{}^t m' = 0$ but $P_t{}^i \lambda P_t{}^t m_1' \neq 0$. Hence, this modification finishes the proof unless $\lambda_\beta = (P_t{}^0)^{p-i}(P_t{}^1 \cdots P_t{}^{t-1})^{p-1}(P_{2t}{}^0)^{p-2}$. However, then $|\lambda_\beta| > |(P_t{}^t)^{p-1}|$ unless $i = p-2$. Furthermore, if this λ_β appears in m_1', we can use the argument on β' to reach our contradiction unless this λ_β is the only term in the sum.

Thus, we need only show a contradiction if

$$(P_t{}^t)^{p-1} m' = m_1 \oplus (P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} (P_{2t}{}^0)^{p-2} m_\beta .$$

Apply $(P_t{}^0)^{p-2} P_t{}^t$ to both sides of this equation to see that

$$(P_t{}^0)^{p-2} P_t(p^t - (p-1))(P_{2t}{}^0)^{p-1} m' = 2(P_t{}^0 P_t{}^1 \cdots P_t{}^{t-1} P_{2t}{}^0)^{p-1} m_\beta .$$

Since $P_t(p^t - (p-1)) = a P_t{}^0 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1}$, we see that $am' = 2m_\beta$, $a \neq 0$ as M' is a free Λ_t'-module and m' has minimal dimension.

Write $P_t{}^t m' = m_2 \oplus m_2'$. Then

$$P_t(p^t - (p-1))(P_{2t}{}^0)^{p-1} m' = P_t{}^t (P_t{}^t)^{p-1} m' =$$

$$P_t{}^t m_1 + \frac{a}{2} [(P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} (P_{2t}{}^0)^{p-2} m_2 +$$

$$(P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} (P_{2t}{}^0)^{p-2} m_2' + 2 P_t{}^0 (P_t{}^1 \cdots P_t{}^{t-1} P_{2t}{}^0)^{p-1} m'].$$

Looking at the term in the summand M', we see that

$$(P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} (P_{2t}{}^0)^{p-2} m_2' = 0.$$

Furthermore, $m_2' \neq 0$ or otherwise we would have $m_1' = 0$. Since $|P_{2t}{}^0| > |m_2'| - |m'|$, we have $(P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} m_2' = 0$. Under these conditions we prove below that there is a $\lambda \in \Lambda'$ such that $\lambda (P^t)^{p-2} m_2' \neq 0 \in M'$ and that

$$\lambda (P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{t-1})^{p-1} (P_{2t}{}^0)^{p-2} = 0.$$

Applying $\lambda (P_t{}^t)^{p-2}$ to the equation $P_t{}^t m' = m_2 \oplus m_2'$, and looking at the component in M', we obtain $\lambda (P_t{}^t)^{p-1} m' = 0 = \lambda (P_t{}^t)^{p-2} m_2' \neq 0$; a contradiction.

To construct such a λ, let $m_2' = \Sigma \lambda_\alpha m_\alpha$, where λ_α are monomials in Λ' and m_α are a Λ'-basis of M'. We need the following formulae which are easily proved. Let

$$\lambda_\alpha = (P_t{}^{i_1})^{u_1} \cdots (P_t{}^{i_s})^{u_s},\ i_1 < \cdots < i_s .$$

Case 1. If $i_1 > 0$, then

$$(P_t{}^t)^{p-2} \lambda_\alpha \equiv b (P_t{}^0)^2 (P_t{}^1 \cdots P_t{}^{i_1 - 1})^{p-1} (P_t{}^{i_1})^{u_1 - 1} (P_t{}^{i_2})^{u_2} \cdots (P_t{}^{i_s})^{u_s} (P_{2t}{}^0)^{p-2} \bmod (P_t{}^0)^3,$$

where $b \not\equiv 0(p)$.

Case 2. If $i_1 = 0$, $s > 1$, and $u_1 = p-1$, then

$$(P_t{}^t)^{p-2} \lambda_\alpha \equiv b P_t{}^0 (P_t{}^{i_2})^{u_2} \cdots (P_t{}^{i_s})^{u_s} (P_{2t}{}^0)^{p-2} \bmod (P_t{}^0)^2.$$

Case 3. If $i_1 = 0$, $s > 1$, and $0 < u_1 < p-1$, then

$$(P_t{}^t)^{p-2} \lambda_\alpha \equiv b (P_t{}^0)^{p-2-u_1} (P_t{}^1 \cdots P_t{}^{i_2 - 1})^{p-1} (P_t{}^{i_2})^{u_2 - 1} (P^{i_3})^{u_3} \cdots$$
$$(P_t{}^{i_s})^{u_s} (P_{2t}{}^0)^{p-2} \bmod (P_t{}^0)^{p-1-u_1}.$$

Case 4. If $i_1 = 0$, $s = 0$, and $u_1 = p - 1$, then

$$(P_t{}^t)^{p-2}\lambda_\alpha \equiv bP_t{}^0(P_{2t}{}^0)^{p-2} \bmod (P_t{}^0)^2.$$

Case 5. If $i_1 = 0$, $s = 0$, and $u_1 = p - 2$, then $(P_t{}^t)^{p-2}\lambda_\alpha \equiv b(P_{2t}{}^0)^{p-2} \bmod P_t{}^0$. If case 5 occurs in the sum $m_2' = \Sigma\lambda_\alpha m_\alpha$, choose $\lambda = (P_t{}^0)^{p-1}$. If case 5 does not occur, and case 3 with $u_1 = p - 2$ does, choose $\lambda = (P_t{}^0)^{p-1}P_t{}^{i_2}$. If case 5 and case 3 with $u_1 = p - 2$ do not occur, and case 2 or 4 do, choose $\lambda = (P_t{}^0)^{p-2}$. If cases 2, 4, 5, and case 3 with $u_1 = p - 2$ do not occur, and case 3 with $u_1 = p - 3$ does, choose $\lambda = (P_t{}^0)^{p-2}P_t{}^{i_2}$. If cases 2, 4, 5, and case 3 with $u_1 \geq p - 3$ do not occur, and case 1 does, choose $\lambda = (P_t{}^0)^{p-3}P_t{}^{i_1}$. Finally, if only case 3 with $u_1 < p - 3$ occurs, choose $\lambda = (P_t{}^0)^{u_1+1}P^{i_2}$, with the largest u_1 for the different case 3's. This λ has the appropriate property and proves the theorem.

§5. PROOFS OF THE MAIN THEOREMS

Before proving 2.1 and 2.2, we first prove Theorem 2.5 which is the main step in deducing 2.2 from 2.1.

Proof of Theorem 2.5. Recall we have $K' \subset K$, with $K'(n-1)$-connected and $z \in K$ with $z^q = 0$ and $|z^{q-1}| < n$. Let M be a K-module. Following the style of the proof of Theorem 2.4, let $M \subset \overline{M}$, the K-injective envelope. As K modules, $\overline{M} = M \oplus M'$ with all three modules free over K′ and bounded below. Let $m' \in M'$ be an element of minimal dimension. $z^{q-1}(m') = m_1 \oplus m_1'$. Let $\kappa' \in K'$ be the top dimentional element. Then $\kappa' z^{q-1}(m') \in M$ and $\kappa'(m_1') = 0$. Since M' is free over K′, m_1' is K′-decomposable in M'. Since $|z^{q-1}| < n$, $m_1' = 0$. Hence $z^{q-1}(m') = m_1$ and $z(m_1) = 0$. Since $H(M, z) = 0$, $m_1 = z^{q-1}(\tilde{m})$, $\tilde{m} \in M$. Thus $z^{q-1}(m' - \tilde{m}) = 0$ is $\overline{M}$ free over K, hence $H(\overline{M}, z) = 0$ and $m' - \tilde{m} = z(\tilde{m}_1)$, $\tilde{m}_1 \in M$ by dimensional reasons. Thus $m' \in M$, a contradiction.

Proof of Theorem 2.1. This theorem was proved for $p = 2$ by Adams and Margolis [1]. We follow their proof with the following modifications for p odd. The results of their §2 carry over as is; in particular a Λ_t'-module M is free if and only if $H(M, P_t{}^s) = 0$ for all s and for $P_{2t}{}^0$ (this follows by induction from our Theorem 2.5). Most of their §3 goes over without change except for their Lemma 3.6 and its use in the proof of their Lemma 3.5. Let B be a Hopf subalgebra of $'\mathscr{A}$ such that $P_t{}^t \in B$. Then $\Lambda_t \subset B$. Assume M is a B-module such that M is a free Λ_t'-module. Then we must show that $H(M/P_{2t}{}^0M, P_t{}^t) = 0$ in order to complete the proof of their Lemma 3.5. By Theorem 2.4, M is a free Λ_t-module. Let $E = Z_p[P_{2t}{}^0]/(P_{2t}{}^0)^p$. Then Λ_t is a free E-module and hence so is M. Hence $M/P_{2t}{}^0M$ is a free module over $\Lambda_t/\Lambda_t\bar{E} = Z_p[P_t{}^0, \ldots, P_t{}^{t-1}, \tilde{P}_t{}^t]/\{(P_t{}^0)^p, \ldots, (P_t{}^{t-1})^p, (\tilde{P}_t{}^t)^p\}$, where $\tilde{P}_t{}^t$ is the image of $P_t{}^t$. Thus $H(M/P_{2t}{}^0M, P_t{}^t) = 0$.

Let ${}_1\Lambda_t$ be the subalgebra of $\mathscr{A}$ generated by $P_t{}^0, \ldots, P_t{}^{t-1}$, and $E = E(Q_0, Q_1, \ldots)$. The structure of ${}_1\Lambda_t$ is determined by Proposition 2.3 and the following relations (see [4]): $[P_t{}^j, Q_1] = 0$ if $i > j$ and $[P_t{}^j, Q_i] = Q_{i+t}P_t(p^j - p^i)$ if $i \leq j$. Note that

$$|Q_i| = (2p-2)(p^{i-1} + p^{i-2} + \cdots + p + 1) + 1,$$

and $|P_t{}^j| = p^j(2p-2)(p^{t-1} + p^{t-2} + \cdots + p + 1)$.

In order to prove Theorem 2.2, we note that $\mathscr{A}//E = '\mathscr{A}$. Let M be an $\mathscr{A}$-module which is free over E and free over $'\mathscr{A}$. By Proposition 2.2 of [1], it is enough to show that $Z_p \otimes_E M$ is free over $'\mathscr{A}$. By Theorem 2.1, it is enough to show that $Z_p \otimes_E M$ is free over ${}_1\Lambda_t//E = Z_p[P_t^0{}_E, \ldots, P_t^{t-1}]/\{(P_t^0)^p, \ldots, (P_t^{t-1})^p\}$. This follows immediately from the following proposition.

PROPOSITION 5.1. *Let M be a ${}_1\Lambda_t$-module which is bounded below. If M is free over E and $H(M, P_t^j) = 0$, $j < t$, then M is free over ${}_1\Lambda_t$.*

Before proving 5.1, we prove two lemmas, the first of which is analogous to Lemma 2.3 of [1].

LEMMA 5.2. *Let M be a $Z_p[P]/P^p \otimes E(Q)$-module which is bounded below where $p\,|P| \neq 2\,|Q|$. If $H(M, P) = 0$ and $H(M, Q) = 0$, then $H(M/QM, P) = 0$.*

Proof. Since $H(M, Q) = 0$, $Q\colon M/QM \to QM$ is an isomorphism of degree $|Q|$. Thus, $H_n(M/QM, P^i) \approx H_{n+|Q|}(QM, P^i)$. Also, $0 \to QM \to M \to M/QM \to 0$ is exact. By Proposition 3.2 we get a long exact sequence $\cdots \to H_n(QM, P^i) \to H_n(M, P^i) \to H_n(M/QM, P^i) \to H_{n+i|P|}(QM, P^{p-i}) \to H_{n+i|P|}(M, P^{p-i}) \to \cdots$. Since $H_*(M, P) = 0$, we obtain

$$H_n(M/QM, P^i) \approx H_{n+i|P|}(QM, P^{p-i}).$$

Combining isomorphisms, we get

$$\begin{aligned} H_n(M/QM, P) &\approx H_{n+|P|}(QM, P^{p-1}) \approx H_{n+|P|-|Q|}(M/QM, P^{p-1}) \\ &\approx H_{n+|P|-|Q|+(p-1)|P|}(QM, P) \approx H_{n+p|P|-2|Q|}(M/QM, P). \end{aligned}$$

Since $p\,|P| - 2\,|Q| \neq 0$ and M/QM is bounded below, $H_*(M/QM, P) = 0$.

LEMMA 5.3. *Let M be a ${}_1\Lambda_t//E \otimes E(Q_t, Q_{t+1}, \ldots, Q_{t+r})$-module which is bounded below. If $H(M, P_t^j) = 0$ for $j < t$ and $H(M, Q_i) = 0$ for $t \leq i \leq t+r$, then M is a free module.*

Proof. Note that $p\,|P_t^j| \neq 2\,|Q_i|$, if $j < t$ and $i \geq t$. This lemma follows from Lemma 5.2, induction, Theorem 2.1 of [1] and Proposition 2.2 of [1] (see the proof of Theorem 2.1 of [1]).

Proof of 5.1. Let ${}_1\Lambda_t^r \subset {}_1\Lambda_t$ be the subalgebra generated by $P_t^0, \ldots, P_t^{t-1}$, and $E^r = E(Q_0, Q_1, \ldots, Q_{t+r})$ with $r \geq t$. In order to prove 5.1, it is enough to prove Proposition 5.1^r which is the same as 5.1 with E replaced by E^r and ${}_1\Lambda_t$ replaced by ${}_1\Lambda_t^r$ (for example, see the "Proof of Theorem 3.1 from Theorem 3.2" in [1]). Proposition 5.1^r now follows easily from Lemma 5.3 and theorem 2.5 by downward induction, because $|Q_i| < |P_t^0| = 1 +$ the connectivity of ${}_1\Lambda_t^r//E^r$, if $i < t$.

§6. PROOF OF 2.3(f)

The algebra Λ_1 was studied by Liulevicius [2] and 2.3(f) for $t = 1$ was proved by him. Instead of calculating with Milnor matrices as it is easy to do to prove the other parts of Proposition 2.3, we work directly with $\mathscr{A}^*$, the dual of the Steenrod algebra.

Recall that in the dual basis to the monomial basis for $\mathscr{A}^*$, $\mathscr{P}^{(r_1, r_2, \ldots)}$ is dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots$. Hence P_t^t is dual to $\xi_t^{p^t}$, $P_t(p^t - (p-1))$ is dual to $\xi_t^{p^t-(p-1)}$, and P_{2t}^0 is dual to

ξ_{2t}. By 2.3(a), $(P_{2t}{}^0)^{p-1} = (p-1)!P_{2t}(p-1) = -P_{2t}(p-1)$ and so $(P_{2t}{}^0)^{p-1}$ is dual to $-\xi_{2t}{}^{p-1}$.

$$P_t(p^t-(p-1))(P_{2t}{}^0)^{p-1} = -\mathscr{P}^{(0,\ldots,p^t-(p-1),0,\ldots,p-1,0,\ldots)}$$

so $P_t(p^t-(p-1))(P_{2t}{}^0)^{p-1}$ is dual to $-\xi_t{}^{p^t-(p-1)}\xi_{2t}{}^{p-1}$. Recall also that

$$\psi(\xi_t) = \sum_{i=0}^t \xi_{t-i}{}^{p^i} \otimes \xi_i .$$

Let $\Psi = (1 \otimes \cdots \otimes 1 \otimes \psi)(1 \otimes \cdots \otimes \psi) \cdots (1 \otimes \psi)\psi$ where there are $(p-1)$ morphisms in the composition. $\Psi\colon \mathscr{a}^* \to (\mathscr{a}^*)^p$. To prove 2.3(f) we must prove the following proposition.

PROPOSITION 6.1.

(a) $\Psi(-\xi_t{}^{p^t-(p-1)}\xi^{p-1}) = \xi_t{}^{p^t} \otimes \xi_t{}^{p^t} \otimes \cdots \otimes \xi_t{}^{p^t} +$ *other terms.*

(b) Ψ (*other monomials*) = *other terms*. The following lemma is straightforward.

LEMMA 6.2

$$\Psi(\xi_t) = \sum_{i_1+\ldots+i_p=t} \xi_{i_1}{}^{p^{j_1}} \otimes \xi_{i_2}{}^{p^{j_2}} \otimes \cdots \otimes \xi_{i_p}{}^{p^{j_p}}$$

where

$$j_1 = t - i_1 = i_2 + \cdots + i_p,\ j_2 = t - i_1 - i_2 = i_3 + \cdots + i_p,\ \ldots,\ j_{p-1} = i_p,\ j_p = 0.$$

COROLLARY 6.3.

(a) $\Psi(\xi_t) = \Sigma\, 1 \otimes \cdots \otimes \xi_t \otimes 1 \otimes \cdots \otimes 1 +$ *terms involving* ξ_i, $i \neq t$.

(b) $\Psi(\xi_{2t}) = \Sigma\, 1 \otimes \cdots \otimes \xi_t{}^{p^t} \otimes 1 \otimes \cdots \otimes \xi_t \otimes \cdots \otimes 1 +$ *terms involving* ξ_i, $i \neq t$.

(c) *if* $s \neq t$ *or* $2t$, $\Psi(\xi_s) =$ *terms involving* ξ_i, $i \neq t$ *or* $\xi_t{}^{p^q}$ *with* $q > t$.

Hence, to prove Proposition 6.1, we need only consider monomials in ξ_t and ξ_{2t} of the correct dimension, that is, $\xi_t{}^{(s+1)p^t-p+s+1}\xi_{2t}{}^{p-1-s}$ with $0 \leq s \leq p-1$.

The following lemma is well-known.

LEMMA 6.4. *Let* $v_p(n) =$ *the highest power of* p *which divides* n. *Let*

$$n = n_1 + \cdots + n_k = \Sigma\, a_i p^i,\ n_j = \Sigma\, a_{ji} p^i.$$

Then $v_p((n_1, \ldots, n_k)) = (\Sigma a_{ji} - \Sigma a_i)/p - 1$, *where* $(n_1, \ldots, n_k) = n!/n_1! \cdots n_k!$.

Proof of 6.1(a). By 6.3(b), $\Psi(\xi_{2t}{}^{p-1}) = (1, \ldots, 1)\xi_t{}^{p^t} \otimes \cdots \otimes \xi_t{}^{p^t} \otimes \xi_t{}^{p-1} +$ terms involving ξ_i, $i \neq t$ or $\xi_t{}^{p_q}$ with $q > t$, where $(1, \ldots, 1) = (p-1)!/1! \cdots 1! = (p-1)! - 1$. By 6.3(a), $\Psi(\xi_t{}^{p_t-(p-1)}) = 1 \otimes \cdots \otimes 1 \otimes \xi_t{}^{p^t-(p-1)} +$ other terms. Hence

$$\Psi(\xi_t{}^{p^t-(p-1)}\xi_{2t}{}^{p-1}) = -\xi_t{}^{p^t} \otimes \cdots \otimes \xi_t{}^{p^t} + \text{other terms}.$$

We now study $\Psi(\xi_t{}^{(s+1)p^t-p-s+1}\xi_{2t}{}^{p-1-s})$ with $1 \leq s \leq p-1$.

LEMMA 6.5. *If* $\Psi(\xi_{2t}{}^{p-1-s})$ *has more than s ones in a term and only* ξ_t*'s in that term, then some exponent of* ξ_t *is* $> p^t$.

Proof. This follows by dimensional reasons. If all exponents are $\leq p^t$, then the dimension of the term is

$$\leq (p-s-1)p^t(p^{t-1} + \cdots + p + 1) < (p-1-s)(p^{2t-1} + \cdots + p + 1) = |\xi_{2t}{}^{p-1-s}|.$$

LEMMA 6.6. *Given an ordered $(p-s)$-tuple, $(a_1, \ldots, a_{p-s})$, with all $a_i \leq p^t$, then the number of times terms of the form*

$$1 \otimes \cdots \otimes \xi_t^{a_1} \otimes \cdots \otimes \xi_t^{a_2} \otimes \cdots \otimes \xi_t^{a_{p-s}} \otimes \cdots \otimes 1$$

appear in the expansion of $\Psi(\xi_{2t})^{p-1-s}$ *is* $\equiv 0(p)$ *if* $1 \leq s \leq p-1$.

Proof. By 6.3(b), the only possible tuple that can appear is $(p^t, p^t, \ldots, p^t, p-s-1)$ and this appears $\binom{p}{s}$-times, but $\binom{p}{s} \equiv 0(p)$ if $1 \leq s \leq p-1$.

LEMMA 6.7. *Given an ordered p-tuple,* $(n_1, \ldots, n_p)$, *with all* $n_j \leq p^t$ *and fewer than s* p^t*'s, then the number of times the term* $\xi_t^{n_1} \otimes \cdots \otimes \xi_t^{n_p}$ *appears in the expansion of*

$$\Psi(\xi_t^{(s+1)p_t-p+s+1})$$

is $\equiv 0(p)$ if $1 \leq s \leq p-1$.

Proof. We must show $v_p((n_1, \ldots, n_p)) \geq 1$, where

$$n_1 + \cdots + n_p = (s+1)p^t - p + s + 1.$$

By Lemma 6.4, $v_p((n_1, \ldots, n_p)) \geq 1$, unless $\Sigma_j a_{ji} = a_i$ for all i. Assume $\Sigma_j a_{ji} = a_i$. We are given that $a_{ji} = 0$ if $i > t$, hence $a_i = 0$ if $i > t$, and that $\Sigma_j a_{jt} = a_t < s$. Then

$$n \leq (s-1)p^t + (p^t - 1) = sp^t - 1 < (s+1)p^t - p + s + 1 = n,$$

contradiction.

Proof of 6.1(b). Consider

$$\Psi(\xi_t^{(s+1)p^t-p+s+1}\xi_{2t}^{p-1-s}) = \Psi(\xi_t^{(s+1)p^t-p+s+1})\Psi(\xi_{2t}^{p-1-s})$$

with $1 \leq s \leq p-1$. In order to obtain a term $\xi_t^{p^t} \otimes \cdots \otimes \xi_t^{p^t}$ in the expansion, the number of ones in expansion of $\Psi(\xi_{2t}^{p-1-s})$ must be $\leq s$ by Lemma 6.5. If the number of ones in this expansion is s, then the $\xi_t^{p^t} \otimes \cdots \otimes \xi_t^{p^t}$ which appears would appear in a mutltiple of p times by Lemma 6.6 because $\Psi(\xi_t^{(s+1)p^t-p+s+1})$ is commutative. If the number of ones is less than s, then the corresponding term in $\Psi(\xi_t^{(s+1)p^t-p+s+1})$ is zero by Lemma 6.7. This proves 6.1 and hence 2.3(f).

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