ELSEVIER
Linear Algebra and its Applications 341 (2002) 291–298
LINEAR ALGEBRA AND ITS APPLICATIONS
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n-Tuples of operators satisfying $\sigma_T(AB) = \sigma_T(BA)$

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Received 26 December 2000; accepted 31 May 2001

Submitted by K. Okubo

Dedicated to Professor T. Ando on the occasion of his 70th birthday

Abstract

For "criss-cross commuting" tuples $\mathbf{A}$ and $\mathbf{B}$ of Banach space operators we give two sufficient conditions for the spectral equality $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Criss-cross commuting; Spectral equality; Taylor spectrum

1. Introduction

If A and B are operators on a Banach space, it is well known ([1] and [2, Proposition 6, p. 16]) that the spectra of the two products AB and BA are very nearly the same:

$$\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}. \tag{1.1}$$

Necessary and sufficient for full equality is that either of the following two conditions hold:

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[1] The author was partially supported by a research grant from NSF.

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$$0 \in \sigma(AB) \cap \sigma(BA), \tag{1.2}$$

$$0 \notin \sigma(AB) \cup \sigma(BA). \tag{1.3}$$

If more generally $\mathbf{A}$ and $\mathbf{B}$ are "criss-cross commuting" systems of operators ([6]; see Definition 1.1 below), then in particular each of the systems $\mathbf{AB} := (A_1B_1, \ldots, A_nB_n)$ and $\mathbf{BA} := (B_1A_1, \ldots, B_nA_n)$ of products is commutative, and the analog of (1.1) is true for the Taylor spectrum. Thus it is clear that for full equality we will need the analog either of (1.2) or of (1.3). In this paper, we find one condition sufficient for the analog of (1.2), and another sufficient for the analog of (1.3).

Definition 1.1. $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are said to *criss-cross commute* if

$$A_iB_kA_j = A_jB_kA_i \quad \text{and} \quad B_iA_kB_j = B_jA_kB_i \quad (\text{all } i, j, k).$$

As we mentioned before if $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ criss-cross commute, then each of $\mathbf{AB}$ and $\mathbf{BA}$ is a commuting n-tuple. Several authors have obtained analogs of (1.1) for various joint spectra, under special conditions. In [6], Li proved that if $\mathbf{A}$ and $\mathbf{B}$ are criss-cross commuting, then $\sigma_T(\mathbf{AB})\backslash\{0\} = \sigma_T(\mathbf{BA})\backslash\{0\}$, and in [7], he showed that $\mathrm{ind}(\mathbf{AB} - \mathbf{z}) = \mathrm{ind}(\mathbf{BA} - \mathbf{z})$ for $\mathbf{z} \neq 0$. In [12], Wrobel proved the first equality under the stronger condition $A_1 = \cdots = A_n = A$ and $AB_j = B_jA$ (all $j = 1, \ldots, n$). In [5], Harte extended this result to different kinds of joint spectra for criss-cross commuting pairs of n-tuples.

For a commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ acting on a Banach space $\mathscr{X}$, let $K(\mathbf{T} - \mathbf{z})$ denote the Koszul complex associated with $\mathbf{T} - \mathbf{z}$ (cf. Section 3 below, [4,9–11]). We define the Taylor spectrum $\sigma_T(\mathbf{T})$ and approximate point spectrum $\sigma_\pi(\mathbf{T})$ as follows:

$$\sigma_T(\mathbf{T}) := \left\{\mathbf{z} \in \mathbb{C}^n : K(\mathbf{T} - \mathbf{z}) \text{ is not exact}\right\}$$

and

$$\sigma_\pi(\mathbf{T}) := \left\{\mathbf{z} \in \mathbb{C}^n : \inf_{\|x\|=1} \sum_{i=1}^{n} \|(T_i - z_i)x\| = 0\right\}.$$

Given a bounded linear operator T on a Banach space $\mathscr{X}$, we let T^* denote the adjoint of T, acting on $\mathscr{X}^*$, the dual space of $\mathscr{X}$. Also, $\sigma(T)$ and $\sigma_\pi(T)$ denote the spectrum and the approximate point spectrum of T, respectively. In this paper, we show that for a pair $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ of criss-cross commuting n-tuples on a Banach space, if (i) $\mathbf{0} \in \sigma_\pi(\mathbf{A}) \cap \sigma_\pi(\mathbf{A}^*)$ or (ii) there exists an invertible operator which is a linear combination of $\{A_1, \ldots, A_n\}$, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$, where $\mathbf{A}^* \equiv (A_1^*, \ldots, A_n^*)$.

2. The singular case

First we prove:

Theorem 2.1. *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *and* $\mathbf{B} = (B_1, \ldots, B_n)$ *be criss-cross commuting. If* $\mathbf{0} \in \sigma_\pi(\mathbf{A}) \cap \sigma_\pi(\mathbf{A}^*)$, *then* $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.

For the proof of Theorem 2.1, we need the following result.

Theorem 2.2 [10, Theorem 3.6]. *Let* $\mathbf{A} \equiv (A_1, \ldots, A_n)$ *be a commuting n-tuple of operators. Then* $\sigma_T(\mathbf{A}) = \sigma_T(\mathbf{A}^*)$.

Proof of Theorem 2.1. By Li's Theorem [6], it suffices to show that $\mathbf{0} \in \sigma_T(\mathbf{AB}) \cap \sigma_T(\mathbf{BA})$. By assumption, there exists a sequece $\{x_n\}$ of unit vectors in $\mathscr{X}$ such that $A_j x_n \to 0$ as $n \to \infty$ $(j = 1, \ldots, n)$. Then $B_j A_j x_n \to 0$ as $n \to \infty$ (all $j = 1, \ldots, n$). Thus, we have $\mathbf{0} \in \sigma_T(\mathbf{BA})$. Now, since $\mathbf{0} \in \sigma_\pi(\mathbf{A}^*)$, there exists a sequence $\{f_n\}$ of unit vectors in $\mathscr{X}^*$ such that $A_j^* f_n \to 0$ as $n \to \infty$. Then $B_j^* A_j^* f_n \to 0$ as $n \to \infty$ (all $j = 1, \ldots, n$). Thus, we have $\mathbf{0} \in \sigma_T(\mathbf{B}^*\mathbf{A}^*)$. Since $\sigma_T(\mathbf{B}^*\mathbf{A}^*) = \sigma_T((\mathbf{AB})^*)$, we must have $\mathbf{0} \in \sigma_T(\mathbf{AB})$ by Theorem 2.2. Hence $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$, as desired. □

Next we study the condition

$$\mathbf{0} \in \sigma_T(\mathbf{A}) \quad \Rightarrow \quad \mathbf{0} \in \sigma_\pi(\mathbf{A}) \cap \sigma_\pi(\mathbf{A}^*). \tag{$*$}$$

Definition 2.3. An n-tuple $\mathbf{A} = (A_1, \ldots, A_n)$ is called *strongly commuting* if, for each $1 \leqslant j \leqslant n$, there exist operators H_j and K_j, each with real spectrum, such that $A_j = H_j + \mathrm{i}K_j$ and $\mathbf{S} = (H_1, K_1, \ldots, H_n, K_n)$ is a commuting $2n$-tuple (cf. [8]).

Theorem 2.4. *If* $\mathbf{A} = (A_1, \ldots, A_n)$ *is a strongly commuting n-tuple*, *then* $\mathbf{A}$ *has condition* $(*)$.

For the proof of Theorem 2.4 we need the following result.

Theorem 2.5 [3, Theorem 2.1]. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a strongly commuting n-tuple of operators. Then* $\sigma_T(\mathbf{T}) = \sigma_\pi(\mathbf{T})$.

Proof of Theorem 2.4. By Theorem 2.5 we have $\mathbf{0} \in \sigma_\pi(\mathbf{A})$. Since by Theorem 2.2, $\sigma_T(\mathbf{A}^*) = \sigma_T(\mathbf{A})$ and since $\mathbf{A}^* = (A_1^*, \ldots, A_n^*)$ is also strongly commuting, by Theorem 2.5 we have $\mathbf{0} \in \sigma_\pi(\mathbf{A}^*)$. □

For the rest of this section, we consider the single operator case.

Corollary 2.6. *If an operator A satisfies condition* $(*)$, *then* $\sigma(AB) = \sigma(BA)$ *for every operator* $B \in B(\mathscr{X})$.

Proof. If $0 \notin \sigma(A)$, then it is clear that $\sigma(AB) = \sigma(BA)$. If $0 \in \sigma(A)$, then by Theorem 2.1 we have $\sigma(AB) = \sigma(BA)$. $\square$

We now let

$$\Pi := \{(x, f) \in \mathscr{X} \times \mathscr{X}^* : \|f\| = f(x) = \|x\| = 1\}.$$

Definition 2.7. For an operator $T \in B(\mathscr{X})$, *the numerical range* $V(T)$ of T is defined by

$$V(T) := \{f(Tx): \ (x, f) \in \Pi\}.$$

An operator T is said to be *Hermitian* if $V(T) \subseteq \mathbf{R}$; T is said to be *normal* if there exist Hermitian operators H and K such that $HK = KH$ and $T = H + \mathrm{i}K$.

It is well known that $\sigma(T) \subseteq \overline{V(T)}$, where $\overline{V(T)}$ is the closure of $V(T)$. Hence normal operators satisfy condition $(*)$. We thus have:

Corollary 2.8. *If A is normal, then $\sigma(AB) = \sigma(BA)$ for every operator $B \in B(\mathscr{X})$.*

3. The nonsingular case

For a *commuting* n-tuple of operators $\mathbf{A} = (A_1, \ldots, A_n)$, we consider the following properties (P_1) and (P_2):

$$\exists \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n \ \text{ and } \ \mathbf{a} \circ \mathbf{A} := a_1 A_1 + \cdots + a_n A_n \text{ is invertible}, \quad (\mathrm{P}_1)$$

$$\mathbf{0} = (0, \ldots, 0) \notin \sigma_T(\mathbf{A}). \quad (\mathrm{P}_2)$$

Proposition 3.1. *For a commuting n-tuple of operators $\mathbf{A} = (A_1, \ldots, A_n)$, (P_1) implies (P_2).*

Proof. By the spectral mapping theorem we have

$$\sigma(\mathbf{a} \circ \mathbf{A}) = \mathbf{a} \circ \sigma_T(\mathbf{A}).$$

Hence

$$\begin{aligned} 0 \notin \sigma(\mathbf{a} \circ \mathbf{A}) &\iff 0 \notin \mathbf{a} \circ \sigma_T(\mathbf{A}) \\ &\iff \mathbf{a} \circ \mathbf{z} := a_1 z_1 + \cdots + a_n z_n \neq 0 \\ &\qquad (\text{all } \mathbf{z} = (z_1, \ldots, z_n) \in \sigma_T(\mathbf{A})). \end{aligned}$$

Therefore, we have $\mathbf{0} \notin \sigma_T(\mathbf{A})$, so (P_2) holds. $\square$

Remark 3.2. In general, $(P_2) \not\Rightarrow (P_1)$. To see this, we shall need the following result [10, Theorem 4.1]. There exists a 5-tuple $\mathbf{A} = (A_1, \dots, A_5)$ such that $\mathbf{A}$ is non-singular but the equation $A_1B_1 + \cdots + A_5B_5 = I$ cannot be solved for $B_1, \dots, B_5 \in (\mathbf{A})'$, where $(\mathbf{A})' := \{T : A_iT = TA_i \text{ for all } A_i\ (i = 1, \dots, 5)\}$. Assume now that $(P_2) \Rightarrow (P_1)$. Then, for $\mathbf{A} = (A_1, \dots, A_5)$ as above, there exists $\mathbf{a} = (a_1, \dots, a_5)$ such that $\mathbf{a} \circ \mathbf{A}$ is invertible. Since it is clear that $\sum_{i=1}^{5} a_i(\mathbf{a} \circ \mathbf{A})^{-1}A_i = I$ and $a_i(\mathbf{a} \circ \mathbf{A})^{-1} \in (\mathbf{A})'$ $(i = 1, \dots, 5)$, we get a contradiction.

Theorem 3.3. *Let $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{B} = (B_1, \dots, B_n)$ be criss-cross commuting n-tuples. If there exists an invertible operator T which is a linear combination of $\{A_1, \dots, A_n\}$, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

Proof. We need to recall the construction of the Koszul complex. Let E be the exterior algebra on n generators, that is, E is the complex algebra with identity e_0 generated by indeterminates $e_1, \dots, e_n$ such that $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j, where $\wedge$ denotes multiplication. The elements $e_{j_1} \wedge \cdots \wedge e_{j_k}$, $1 \leqslant j_1 < \cdots < j_k \leqslant n$ form a basis for the subspace of k-forms, E_k $(k = 1, \dots, n)$, while $E_0 = \mathbb{C}e$. Thus, E is a graded algebra, with $E = \bigoplus_{k=0}^{n} E_k$. For $\mathscr{X}$ a Banach space, let $E_k(\mathscr{X}) := E_k \otimes_{\mathbb{C}} \mathscr{X}$. For $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define D_k and $D^k : E_k(\mathscr{X}) \to E_{k-1}(\mathscr{X})$ by

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1}(A_{j_i}B_{j_i} - z_{j_i})x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k}$$

and

$$D^k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1}(B_{j_i}A_{j_i} - z_{j_i})x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

respectively (here $\check{e}_{j_i}$ means deletion). We thus have two chain complexes (Koszul complexes)

$$K(\mathbf{AB} - \mathbf{z}) : 0 \to E_n(\mathscr{X}) \overset{D_n}{\to} E_{n-1}(\mathscr{X}) \overset{D_{n-1}}{\to} \cdots \overset{D_2}{\to} E_1(\mathscr{X}) \overset{D_1}{\to} E_0(\mathscr{X}) \to 0$$

and

$$K(\mathbf{BA} - \mathbf{z}) : 0 \to E_n(\mathscr{X}) \overset{D^n}{\to} E_{n-1}(\mathscr{X}) \overset{D^{n-1}}{\to} \cdots \overset{D^2}{\to} E_1(\mathscr{X}) \overset{D^1}{\to} E_0(\mathscr{X}) \to 0.$$

By hypothesis, there exist complex numbers a_i $(i = 1, \dots, n)$ such that $T := a_1A_1 + \cdots + a_nA_n$ is invertible. For $i = 1, \dots n$, $x \in \mathscr{X}$, and a complex number z,

$$\begin{aligned}(A_iB_i - z)x &= (A_iB_i - z)TT^{-1}x \\ &= (A_iB_i - z)(a_1A_1 + \cdots + a_nA_n)T^{-1}x \\ &= T(B_iA_i - z)T^{-1}x,\end{aligned}$$

because $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ criss-cross commute. It follows that

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = TD^k\big((T^{-1}x) \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}\big) \quad (k = 0, \ldots, n).$$

This identity readily implies that $K(\mathbf{AB} - \mathbf{z})$ is exact if and only if so is $K(\mathbf{BA} - \mathbf{z})$. Therefore, $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$. □

We proved in Section 2 that a normal operator on a Banach space satisfies condition (∗). Applying this result and Theorems 2.1 and 3.3, we obtain the following:

Corollary 3.4. *Let $\mathbf{A} = (A, \ldots, A)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ be criss-cross commuting. If A is normal, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

Proof. If $0 \in \sigma(A)$, then Theorem 2.1 applies; if $0 \notin \sigma(A)$, then use Theorem 3.3. □

We conclude this section with an application of Theorem 3.3 to commuting n-tuples of operators in a somewhat restricted form.

Corollary 3.5. *Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a commuting n-tuple operators which is non-singular, that is, $\mathbf{0} = (0, \ldots, 0) \notin \sigma(\mathbf{A})$. Suppose that for $i = 1, \ldots, n$, if $0 \in \sigma(A_i)$, then 0 is an isolated point of $\sigma(A_i)$. Let $\mathbf{B} = (B_1, \ldots, B_n)$ be an n-tuple of operators such that $\mathbf{A}$ and $\mathbf{B}$ criss-cross commute. Then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

For the proof of Corollary 3.5, we need the following lemma. For $i = 1, \ldots, n$, let $P_i : \mathbb{C}^n \to \mathbb{C}$ denote the orthogonal projection onto the ith coordinate.

Lemma 3.6. *Let $M \subseteq \mathbb{C}^n$ be a compact subset and assume that $\mathbf{0} \notin M$. Assume that $0 \in P_i(M)$ and that 0 is an isolated point of $P_i(M)$ (all $i = 1, \ldots, n$). Then there exist numbers $a_1, \ldots, a_n$ such that $a_1x_1 + \cdots + a_nx_n \neq 0$ for every $(x_1, \ldots, x_n) \in M$.*

Proof. Since 0 is an isolated point of $P_i(M)$, there exist positive numbers b_i, c_i such that if $0 \neq z \in P_i(M)$, then $b_i \leqslant |z| \leqslant c_i$ $(i = 1, \ldots, n)$. Let $a_1 := 1$ and select $a_2 > 0$ such that $a_2b_2 > c_1$. Next select $a_3 > 0$ such that $a_3b_3 > c_1 + a_2c_2$. Inductively, select $a_i > 0$ such that $a_ib_i > c_1 + a_2c_2 + \cdots + a_{i-1}c_{i-1}$ $(i = 3, \ldots n)$. Then for all $(x_1, \ldots, x_n) \in M$ we must have

$$x_1 + a_2x_2 + \cdots + a_nx_n \neq 0.$$

For, assume that $x_1 + a_2x_2 + \cdots + a_nx_n = 0$. If $x_n \neq 0$, then

$$\begin{aligned} a_nb_n &\leqslant a_n|x_n| \\ &= |x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1}| \\ &\leqslant c_1 + a_2c_2 + \cdots + a_{n-1}c_{n-1} \\ &< a_nb_n, \end{aligned}$$

a contradiction. Thus, we must have $x_n = 0$. Therefore, $x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} = 0$, and a repeated application of the above argument shows that $x_1 = x_2 = \cdots = x_n = 0$, contradicting the hypothesis on M. □

Proof of Corollary 3.5. If there exists i such that $0 \notin \sigma(A_i)$, then let $T := 0 \cdot A_1 + \cdots + 0 \cdot A_{i-1} + A_i + 0 \cdot A_{i+1} + \cdots + 0 \cdot A_n = A_i$. Since T is invertible, by Theorem 3.3 we have $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$. Thus, we may assume $0 \in \sigma(A_i)$ (all $i = 1, \ldots, n$). By the hypothesis and Lemma 3.6 (applied to $M := \sigma_T(\mathbf{A})$), there exist numbers $a_1, \ldots, a_n$ such that $a_1x_1 + \cdots + a_nx_n \neq 0$ for all $(x_1, \ldots, x_n) \in \sigma_T(\mathbf{A})$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be given by $f(z_1, \ldots, z_n) := a_1z_1 + \cdots + a_nz_n$, and let $T := f(A_1, \ldots, A_n)$. By the Spectral Mapping Theorem for the Taylor spectrum we have $\sigma(T) = \{a_1x_1 + \cdots + a_nx_n : (x_1, \ldots, x_n) \in \sigma_T(\mathbf{A})\}$, and thus $0 \notin \sigma(T)$. Therefore, T is invertible, so by Theorem 3.3 we have $\sigma_T(\mathbf{AB} = \sigma_T(\mathbf{BA})$. □

Acknowledgement

The authors would like to thank the referee for several useful suggestions.

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