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Strong stability and the incompleteness of stable models for λ-calculus

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Abstract

We prove that the class of stable models is incomplete with respect to pure λ-calculus. More precisely, we show that no stable model has the same theory as the strongly stable version of Park's model. This incompleteness proof can be adapted to the continuous case, giving an incompleteness proof for this case which is much simpler than the original proof by Honsell and Ronchi della Rocca. Moreover, we isolate a very simple finite set, $\mathscr{F}$, of equations and inequations, which has neither a stable nor a continuous model, and which is included in $Th(\mathscr{P}_{\mathrm{fs}})$ and in $T_{\overline{\Lambda_I^0}}$, the contextual theory induced by the set of essentially λI-closed terms. Finally, using an approximation theorem suitable for a large class of models (in particular stable and strongly stable non-sensible models like $\mathscr{P}_{\mathrm{fs}}$ and $\mathscr{P}_{\mathrm{s}}$), we prove that $Th(\mathscr{P}_{\mathrm{s}})$ and $Th(\mathscr{P}_{\mathrm{fs}})$ are included in $T_{\overline{\Lambda_I^0}}$, giving an operational meaning to the equality in these models. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

A model of PCF is fully abstract if the equational theory induced by the model contains the operational equivalence. It is known that the standard continuous model of PCF is not fully abstract [21], due to the fact that there are continuous functions which are not PCF-definable. In fact, PCFs terms have an essentially sequential behaviour, whereas there are continuous functions, like the *parallel or*, which have a non-sequential behaviour.

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One way of building a fully abstract model of PCF is to take a restriction of the class of continuous functions, in order to obtain sequential functions. The notion of sequentiality comes in a natural way for a function whose domain is a product of flat domains. But, when we want to extend it to higher types, none of the existing definitions of sequentiality – due to Vuillemin, Milner and Kahn–Plotkin (cf. [4]) – allow us to build a model of PCF: the categories of complete partial orders with Milner or Vuilemin-sequential functions, and concrete domains with Kahn–Plotkin-sequential functions, are not cartesian closed (cf. [9]). If we want to stay in the framework of functional models – i.e. obtained in a category in which the objects are sets and the morphisms are functions – we have to take functions that satisfy a weaker property than sequentiality.

The notion of Stability, introduced by Berry (cf. [4]), is such a property. A continuous function f is stable if it satisfies:

$$y \leqslant f(x) \Rightarrow \exists x' \leqslant x \text{ such that } y \leqslant f(x') \text{ and } x' \text{ is minimum for that property.}$$

For example the *parallel or* function is not stable. In some structures, such as dI-domains, stability is equivalent to the preservation of infima of finite subsets of compatible elements, a property more manageable than the original one.

Strong stability, introduced recently by Bucciarelli and Ehrhard [6, 8], is defined, in the framework of dI-domains, by a preservation property of more general infima, the infima of coherent sets. At ground types, a strongly stable function is sequential (for an appropriate choice of coherences).

Each of these classes leads to a cartesian closed category (c.c.c.) containing a standard model of PCF. However, these models have incomparable equational theories ([5, Chapter 9]). This can be explained by the following fact: removing functions at ground types adds new functionals.

For the untyped λ-calculus, the usual continuous models (Scott's model $\mathscr{D}_\infty$, Park's model, Engeler's model) have stable and strongly stable analogues. So one may ask if a given continuous model has the same theory as its stable or strongly stable analogue. Honsell and Ronchi della Rocca conjectured that $\mathscr{D}_\infty$ and its stable analogue have the same theory [13]. This conjecture, extended to the strongly stable case, is proved in [12]. Note that in the case of typed λ-calculus, standard models may have different theories. We obtain an analogous result in the untyped λ-calculus semantics, by considering a non-standard model (non-standard is the sense that it is not semi-sensible, i.e. equalizes solvable terms to non-solvable ones), namely the Park's model. Here we show that the continuous Park's model, its stable analogue and its strongly stable analogue have different theories. Moreover, the theory of the continuous Park's model is incomparable with the theory of the two others. This last assertion contradicts another conjecture of Honsell and Ronchi della Rocca claiming that the theory of the stable Park's model is strictly included in the theory of continuous Park's model [13].

Another natural question to ask is whether these classes of models are complete (a class of models is *complete* when it contains all the λ-theories). Honsell and Ronchi della Rocca have given a negative answer to this question in the case of the class of all continuous models (called by them *topological models*). More precisely, they have shown that there is no continuous model having $T_{\overline{\Lambda^0}}$ as its theory, where $T_{\overline{\Lambda^0}}$ is the contextual theory induced by the set of essentially closed terms (an essentially closed term is a term which is β-equivalent to a closed term).

Their proof consists in supposing the existence of a continuous model, $\mathscr{D}$, with $T_{\overline{\Lambda^0}}$ as theory, and then exhibiting two terms U and V such that:

- $U = V$ is an equation of $T_{\overline{\Lambda^0}}$, and
- U and V are distinct in $\mathscr{D}$.

The complexity of the proof comes from the fact that the first point is established syntactically: one shows by induction on the context $C[\ ]$ that $C[U] \in \overline{\Lambda^0}$ if and only if $C[V] \in \overline{\Lambda^0}$.

Actually, the terms U and V are distinct in all continuous models $\mathscr{D}$ satisfying:

1. $\mathscr{D}$ is extensional,
2. $(\Omega)\Omega =_{\mathscr{D}} \Omega$,
3. $\lambda x\ (\Omega)(\Omega)x =_{\mathscr{D}} \Omega$,
4. $\Omega \neq_{\mathscr{D}} \lambda x\ \Omega$,

where Ω is the term $(\lambda x\ (x)x)\lambda x\ (x)x$, and where $=_{\mathscr{D}}$ is the relation on λ-terms induced by $\mathscr{D}$.

It is possible to give a semantic proof of the incompleteness of continuous models by using the stable analogue $\mathscr{P}_{\mathrm{s}}$ of Park's model. Indeed, one can easily show that this model satisfies the four conditions above and equalizes U and V. That proves that no continuous model has the theory of $\mathscr{P}_{\mathrm{s}}$, and greatly simplifies the proof.

What about stable models? The main result of this paper is the incompleteness of this class of models. However we have to precise what we mean by a "class of stable models".

The various c.c.c. having stables maps as morphisms have been investigated by Amadio in [1]. In the more general structure ($CPO_{\wedge}$), stability is defined by the property of commuting with the glbs of all non-empty, finite and bounded families, which is no longer equivalent to the definition of the stability given above. This property is equivalent to the Berry's original definition of stability as soon as the domains have a finiteness property: the principal ideal generated by a compact element is finite. We are now led to consider structures such as dI-domains, Amadio's bifinite stable domains, etc., in which stable functions can be represented by stable traces. If we are interested in modelling the operational aspects of the λ-calculus, stability is interesting only in a c.c.c. where Berry's original definition is equivalent to the commutation property mentioned above, and that we denote by $(*)$ in the sequel. The class of stable models we consider here is the one of all dI-domains. Nevertheless, our proof does not use any of the specific properties of dI-domains: all we basically need is the existence of stable traces. It should be adaptable, *mutatis mutandis*, to any class of stable domains where $(*)$ is equivalent to Berry's original definition of stability.

Our proof is semantic and similar to the one described above in the continuous case. We consider the strongly stable analogue of Park's model, $\mathscr{P}_{\mathrm{fs}}$, and we prove that no stable model has the theory of $\mathscr{P}_{\mathrm{fs}}$. Moreover, we show that no continuous model has the theory of $\mathscr{P}_{\mathrm{fs}}$.

Then we improve these incompleteness results by:

- exhibiting a very simple finite set, $\mathscr{F}$, of equations and inequations, such that $\mathscr{F}$ is included in $Th(\mathscr{P}_{\mathrm{fs}})$ and has no continuous model and no stable model. The set $\mathscr{F}$ constitutes a localization of the continuous and stable incompleteness: if a model $\mathscr{D}$ or a theory $\mathscr{T}$ is such that $\mathscr{F} \subseteq Th(\mathscr{D})$ or $\mathscr{F} \subseteq \mathscr{T}$, then $Th(\mathscr{D})$ or T have neither a continuous nor a stable model;
- giving an operational meaning to these incompleteness results. Namely, using an approximation theorem valid for a large class of models, in particular continuous, stable and strongly stable non-sensible models, we prove that $Th(\mathscr{P}_{\mathrm{s}})$ and $Th(\mathscr{P}_{\mathrm{fs}})$ are included in $T_{\overline{\Lambda_I^0}}$, the contextual theory induced by the set of essentially λI-closed terms (λI-terms are λ-terms in which the parameters of an abstraction must be used in its body and an essentially λI-closed term is a term which is β-equivalent to a λI-closed term).

These results give the following corollary: there is neither a stable model nor a continuous model whose theory lies between $\mathscr{F}$ and $T_{\overline{\Lambda_I^0}}$, or between $Th(\mathscr{P}_{\mathrm{fs}})$ and $T_{\overline{\Lambda_I^0}}$. It is still unknown whether these two theories are different.

Finally, we ask whether the incompleteness of stable models could not be proved using a continuous model, rather than a strongly stable model. The proof which we have just outlined exploits the fact that the strongly stable standard model of PCF has less first-order functions than its stable analogues.

In Section 1, after fixing some terminology and notations, we give a brief exposition of the categorical interpretation of λ-calculus. Then we briefly recall Berry's stability (resp. Bucciarelli and Ehrhard's strong stability) in the framework of dI-domains (resp. of dI-domains with coherence).

The second section is devoted to the hypercoherences, a notion due to Ehrhard [10]. They give rise to a particular class of dI-domains with coherence. We adapt to this framework the model construction technique that Krivine introduced in the continuous and stable case [16].

In Section 3 we give the construction of Park's strongly stable model.

The Section 4 contains the semantical proof of the incompleteness of the class of continuous models. Actually we show that neither the theory of $\mathscr{P}_{\mathrm{s}}$ nor the one of $\mathscr{P}_{\mathrm{fs}}$ is the theory of a continuous model.

The Section 5 is devoted to proving the incompleteness of the class of stable models for λ-calculus. We show that no stable model has the same theory as $\mathscr{P}_{\mathrm{fs}}$.

In Section 6 we isolate a finite set $\mathscr{F}$ of equations and inequations, included in $Th(\mathscr{P}_{\mathrm{fs}})$ and in $T_{\overline{\Lambda_I^0}}$. We show that $\mathscr{F}$ has neither a continuous nor a stable model. The inclusion of $\mathscr{F}$ in $Th(\mathscr{P}_{\mathrm{fs}})$ then follows from previous results. Showing directly that $\mathscr{F}$ is included in $T_{\overline{\Lambda_I^0}}$ would need a very elaborate syntactic proof. To avoid it, we deduce the result from the inclusion of $Th(\mathscr{P}_{\mathrm{fs}})$ in $T_{\overline{\Lambda_I^0}}$.

This last assertion (also true for $Th(\mathscr{P}_{\mathrm{s}})$) is interesting in itself: it gives an equational viewpoint on these two models. Its proof uses a notion of approximation introduced by Honsell and Ronchi della Rocca [14], in the framework of continuous models, which can be extended to stable and strongly stable models in a natural way.

1. Preliminaries

1.1. Terminology and notations

We assume the reader familiar with basic concepts and results in λ-calculus and its semantics. The reader may consult [3] for more detail.

1.1.1. Lambda-calcul

We denote by Λ the quotient of the set of λ-terms by α-equivalence, and $\Lambda^0 \subset \Lambda$ is the set of closed terms. The terms will always be considered up to α-equivalence.

We use the applicative notation of [16]: $(u)v$ denotes the application of u to v.

We note $FV(t)$ the set of free variables of the term t and we write $t[x_1,\dots,x_n]$ to indicate that all free variables of t are among $x_1,\dots,x_n$. Sometimes we abbreviate $(\cdots(u)v_1)\cdots)v_n$ to $(u)v_1\cdots v_n$ and $\lambda x_1\cdots\lambda x_n\ t$ to $\lambda\bar{x}\ t$. A list without repetition of variables $x_1,\dots,x_n$ will be denoted by $\bar{x}$, and the length of $\bar{x}$, denoted by $l(\bar{x})$, is defined to be the number of variables occurring in $\bar{x}$. By abuse of notation, we write $FV(t)\subseteq\bar{x}$ instead of $FV(t)\subseteq\{x_1,\dots,x_n\}$. Last, a *context* is a term with a "hole", denoted by $C[\ ]$.

Starting from the set Λ built over the set of variables Var, the set Λ_I of $\lambda I-$terms is the smallest set contained in Λ such that: $Var\subseteq\Lambda_I$, if $u,v\in\Lambda_I$ then $(u)v\in\Lambda_I$ and if $u\in\Lambda_I$ and $x\in FV(u)$ then $\lambda x\ u\in\Lambda_I$.

1.1.2. Partially ordered sets

A *domain* is a partially ordered set $(\mathscr{D},\leqslant)$ with a least element written $\bot$. Two elements a,b of $\mathscr{D}$ are *compatible* if they are bounded in $\mathscr{D}$. We respectively denote by $a\vee b$, $a\wedge b$, $\bigvee A$ and $\bigwedge A$ the supremum (lub) of a and b, the infimum (glb) of a and b, the lub of the subset A of $\mathscr{D}$, and the glb of A, when they exist. A subset A of $\mathscr{D}$ is *directed* if it is not empty and such that for all $a,b\in A$ there exists $c\in A$ such that $a\leqslant c$ and $b\leqslant c$. A domain $\mathscr{D}$ is a *cpo* if every directed subset of $\mathscr{D}$ has a lub.

Now $\mathscr{D}$ and $\mathscr{E}$ are cpo's.

An element k of $\mathscr{D}$ is *compact* if for every directed subset A of $\mathscr{D}$ we have: $k\leqslant\bigvee A\Rightarrow\exists a\in A, k\leqslant a$. An element p of $\mathscr{D}$ is *prime* if it satisfies the same condition for every A which has a lub. Note that every prime element is compact and that $\bot$ is not prime (because $\bot=\bigvee\emptyset$). We denote by $\mathscr{D}_{\mathrm{c}}$ (resp. $\mathscr{D}_{\mathrm{p}}$) the set of all compact (resp. prime) elements of $\mathscr{D}$.

In what follows the letters h, k, l exclusively denote compact elements.

We say that $\mathscr{D}$ is *algebraic* if the set of the compact lower bounds of any $a\in\mathscr{D}$ is directed, and has a as lub. We say that $\mathscr{D}$ is *prime-algebraic* if any element of $\mathscr{D}$ is the lub of its prime lower bounds.

An *increasing* function f from $\mathscr{D}$ to $\mathscr{E}$ is *continuous* if, for all directed set $A \subseteq \mathscr{D}$, $f(\bigvee A) = \bigvee f(A)$.

1.1.3. Notations about sets

Let X, Y be sets, $a \subseteq X$, and let f be a function from X to Y. We denote by $f^{\bullet}(a)$ the set $\{f(\alpha);\ \alpha \in a\}$.

We denote by $\mathscr{P}_{\mathrm{f}}(X)$ the set of finite subsets of X and by $\mathscr{P}^*_{\mathrm{f}}(X)$ the set of non-empty finite subsets of X; $a \in \mathscr{P}^*_{\mathrm{f}}(X)$ will be denoted by $a \subseteq^*_{\mathrm{f}} X$. If $Y \subseteq \mathscr{P}(X)$, we denote by $Y^{>1}$ the set of elements of Y which have a cardinal greater than 1; $\sharp X$ designate the cardinal of the set X.

Let $A, B_1, \ldots, B_n$ be sets. If $A \subseteq B_1 \times \cdots \times B_n$, we denote by $(A)_i$, where $i \in \{1, \ldots, n\}$, the ith projection of A.

1.2. A categorical interpretation of the λ-calculus

For a complete presentation of the notions presented here we refer the reader to [3 Chapter 5.5; 17]. In the following, **C** is a cartesian closed category (c.c.c.) with enough points. The objects of **C** are denoted by $\mathscr{D}, \mathscr{E}, \ldots$, and $\mathscr{D} \times \mathscr{E}$ denotes the categorical cartesian product of $\mathscr{D}$ and $\mathscr{E}$. The associated projections are $p^1_{\mathscr{D}\times\mathscr{E}} : \mathscr{D} \times \mathscr{E} \to \mathscr{D}$ and $p^2_{\mathscr{D}\times\mathscr{E}} : \mathscr{D} \times \mathscr{E} \to \mathscr{E}$. In the following we take the convention that $\times$ associates to the left. From now on we will consider only locally small categories, i.e. categories **C** such that the collection of morphisms from an object to another one is a set. We denote by $\mathscr{D} \Rightarrow \mathscr{E}$ the object that internalizes the set $Hom_{\mathbf{C}}(\mathscr{D}, \mathscr{E})$ of morphisms from the object $\mathscr{D}$ to the object $\mathscr{E}$, namely $\Rightarrow$ is a bifunctor from $\mathbf{C}^{op} \times \mathbf{C}$ to **C** such that, for every object $\mathscr{A}, \mathscr{B}, \mathscr{C}$, there exists:
(i) an operation: $\Lambda_{\mathscr{A},\mathscr{B},\mathscr{C}} : Hom_{\mathbf{C}}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \to Hom_{\mathbf{C}}(\mathscr{A}, (\mathscr{B} \Rightarrow \mathscr{C}))$,
(ii) the *evaluation morphism*: $ev_{\mathscr{B},\mathscr{C}} : (\mathscr{B} \Rightarrow \mathscr{C}) \times \mathscr{B} \to \mathscr{C}$,
such that, for all morphisms $f : (\mathscr{A} \times \mathscr{B}) \to \mathscr{C}$, $h : \mathscr{A} \to (\mathscr{B} \Rightarrow \mathscr{C})$ and $g : (\mathscr{B} \Rightarrow \mathscr{C}) \times \mathscr{B} \to \mathscr{C}$,
- $ev_{\mathscr{B},\mathscr{C}} \circ (\Lambda(f) \times id_{\mathscr{B}}) = f$,
- $\Lambda(g) \circ h = \Lambda(g \circ (h \times id_{\mathscr{B}}))$,
- $\Lambda_{\mathscr{B},\mathscr{C},\mathscr{B}}(ev_{\mathscr{B},\mathscr{C}}) = id_{\mathscr{B} \Rightarrow \mathscr{C}}$.

Let d be the natural transformation making the duplication. For every object $\mathscr{D}$ of **C**, $d_{\mathscr{D}}$: $\mathscr{D} \to \mathscr{D} \times \mathscr{D}$.

An object $\mathscr{D}$ in **C** is called reflexive if there is a pair (F, G), called retraction pair, such that

$$F : \mathscr{D} \to \mathscr{D} \Rightarrow \mathscr{D}, \quad G : \mathscr{D} \Rightarrow \mathscr{D} \to \mathscr{D} \text{ and } F \circ G = id_{\mathscr{D} \Rightarrow \mathscr{D}}.$$

If moreover $G \circ F = id_{\mathscr{D}}$, $\mathscr{D}$ is called extensional.

A c.c.c. **C**, once fixed a reflexive object $\mathscr{D}$, can be denoted by $\mathbf{C}^{\mathscr{D}}$.

We now define the categorical interpretation of λ-terms in $\mathbf{C}^{\mathscr{D}}$.

Definition 1.1. The *interpretation function* $(\cdot)^*$ such that, for each term $t \in \Lambda$ with $FV(t) \subseteq \{x_1 \cdots x_n\}$,

$$(t)^*_\Gamma \in Hom_{\mathbf{C}^{\mathscr{D}}}(\mathscr{D}^n, \mathscr{D}),$$

is defined by induction on the structure of t as follows:
- $(x_i)^*_\Gamma = p^i_{\mathscr{D}^n}$,
- $(uv)^*_\Gamma = ev_{\mathscr{D},\mathscr{D}} \circ ((F \circ (u)^*_\Gamma) \times (v)^*_\Gamma) \circ d_{\mathscr{D}^n}$,
- $(\lambda x.u)^*_\Gamma = G \circ \Lambda_{\mathscr{D}^n,\mathscr{D},\mathscr{D}}((u)^*_{\Gamma,x})$,

where $\Gamma \equiv x_1, \ldots, x_n$ is a list without repetitions of variables.

In this categorical setting a λ-model is $\mathscr{M} = (D, F, G)$ where $D = Hom_{\mathbf{C}^{\mathscr{D}}}(\Pi, \mathscr{D})$, namely D is the set of points of $\mathscr{D}$ (Π is the unity of the cartesian product $\times$). Note that in the sequel we will identify $\mathscr{D}$ and D. The applicative behaviour on D is given by

$$f \bullet g = ev_{\mathscr{D},\mathscr{D}} \circ ((F \circ f) \times g),$$

for every pair of morphisms $f, g \in Hom_{\mathbf{C}^{\mathscr{D}}}(\Pi, \mathscr{D})$. We sometime denote $f \bullet g$ by $(f)g$. We denote by $(t)^*_\rho$ the morphism $(t)^*_\Gamma \circ (\rho(x_1) \times \cdots \times \rho(x_n))$, where $FV(t) = \{x_1, \ldots, x_n\}$, $\Gamma \equiv x_1, \ldots, x_n$ and ρ is a map from the set of variables to D. (We can think of a prefixed order among variables and view every $\rho(x_i)$ as a morphism from Π to $\mathscr{D}$.)

We will use the notation $t[\bar{a}/\bar{x}]^*$ as a shortland for $(t)^*_\rho$ where ρ is such that $\rho(x_i) = a_i$.

In this paper, we will be considering λ-models $(\mathscr{D}, F, G)$ from either continuous, stable or strongly stable semantics. The underlying categories are such that:
- every object is a cpo (equipped with an additional structure in the strongly stable case). The order relation of an object $\mathscr{D}$ is denoted $\leqslant_{\mathscr{D}}$ and its least element is denoted by $\perp_{\mathscr{D}}$.
- Hom-sets are partial orders having a least element, and which are complete with respect to the pointwise order on functions.
- Morphisms are continuous functions.

We denote by $\Lambda(\mathscr{D})$ the set of λ-terms with parameters in $\mathscr{D}$ and by $\Lambda^0(\mathscr{D})$ the subset of closed terms with parameters.

Lemma 1.2. *For all* $u, v \in \Lambda(\mathscr{D})$ *and* $\bar{x} \supseteq FV(u) \cup FV(v)$, *the two following assertions are equivalent*:
(i) $u^*_{\bar{x}} \leqslant v^*_{\bar{x}}$, *where* $\leqslant$ *is the order of* $\mathscr{D}^{l(\bar{x})} \Rightarrow \mathscr{D}$;
(ii) $(\lambda\bar{x}\ u)^* \leqslant (\lambda\bar{x}\ v)^*$, *where* $\leqslant$ *is the order of* $\mathscr{D}$.
And their truth does not depend on the choice of $\bar{x}$.

The same holds for equality. We denote by $\leqslant_{\mathscr{D}}$ and $=_{\mathscr{D}}$ the corresponding binary relations on $\Lambda(\mathscr{D})$. The next proposition is standard and point out in what sense the λ-calculus is modelled.

Proposition 1.3
(i) *The relation* $\leqslant_{\mathscr{D}}$ *is a partial preorder on* $\Lambda(\mathscr{D})$ *which extends the order of* $\mathscr{D}$, *and* $=_{\mathscr{D}}$ *is the associated equivalence relation.*
(ii) *These two relations are congruence with respect to application and* λ*-abstraction and they contain the* β*-equivalence* (*the* $\beta\eta$*-equivalence if* $\mathscr{D}$ *is extensional*).

Let us denote by $\mathscr{E}q$ the set of formal equations between closed terms. We call *theory* of $\mathscr{D}$, denoted by $Th(\mathscr{D})$, the subset of $\mathscr{E}q$ which corresponds to the valid equations of $\mathscr{D}$:

$$Th(\mathscr{D}) = \{u = v;\ u, v \in \Lambda^0 \text{ and } u =_{\mathscr{D}} v\}.$$

A λ-congruence (resp. $\lambda\eta$-congruence) is a congruence with respect to application and λ-abstraction which contains the β-equivalence (resp. the $\beta\eta$-equivalence) and which is not the total relation. To every λ-congruence $\sim$ we canonically associate a subset $E_{\sim}$ of $\mathscr{E}q$, by setting

$$E_{\sim} = \{u = v;\ u, v \in \Lambda^0 \text{ and } u \sim v\}.$$

A λ-theory is a set of equations of the form $E_{\sim}$. If $\sim$ is a $\lambda\eta$-congruence, we say that the associated theory is extensional. It is clear, from the previous proposition, that the theory of a model is a λ-theory, and that the theory of an extensional model is an extensional λ-theory. On the other hand, if A is a subset of Λ, with $A \neq \emptyset$, $A \neq \Lambda$, and which is closed by β-equivalence, then the following set of equations T_A:

$$T_A = \{u = v;\ u, v \in \Lambda^0 \text{ and } \forall C[\ ],\ (C[u] \in A \Leftrightarrow C[v] \in A)\},$$

is a λ-theory, that we call *contextual theory induced by A*.

1.3. Stability on dI-domains

We shall content ourselves with recalling a series of definitions and well known facts about dI-domains and stable functions. We refer the reader to [2], or to [11] (which deals with the particular case of qualitative domains) for detailed proofs.

Definition 1.4. A *dI-domain* is a domain $\mathscr{D}$ such that:
(i) every directed subset of $\mathscr{D}$ has a supremum,
(ii) every bounded subset of $\mathscr{D}$ has a supremum,
(iii) $\mathscr{D}$ is algebraic,
(iv) for each $h \in \mathscr{D}_c$, $\{a;\ a \leqslant h\}$ is finite,
(v) for all $a, b, c \in \mathscr{D}$, if a, b are $+$compatible, then $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

Remark
1. $\mathscr{D}$ is prime-algebraic.
2. If $h \in \mathscr{D}_c$ and $a \leqslant h$, then $a \in \mathscr{D}_c$.

3. Condition (iii) implies that $a \leqslant b \Leftrightarrow \forall h\ (h \leqslant a \Rightarrow h \leqslant b)$.
4. If $\mathscr{D}$ and $\mathscr{D}'$ are dI-domains, $\mathscr{D} \times \mathscr{D}'$ equipped with the product order is also a dI-domain.

Definition 1.5. A *stable* function between two dI-domains $\mathscr{D}$ and $\mathscr{E}$ is a continuous function f from $\mathscr{D}$ to $\mathscr{E}$ such that:

$$\forall a, b \in \mathscr{D}\ (a, b \text{ compatible} \Rightarrow f(a \wedge b) = f(a) \wedge f(b)).$$

A stable function f from $\mathscr{D}_1 \times \mathscr{D}_2$ to $\mathscr{E}$ is a continuous function in each argument which satisfies:

$$f(a_1 \wedge b_1, a_2 \wedge b_2) = f(a_1, a_2) \wedge f(b_1, b_2),$$

if a_1, b_1 are compatible and a_2, b_2 are compatible. To every stable function f from $\mathscr{D}$ to $\mathscr{E}$ we associate its *trace*, denoted by $Tr(f)$, and defined by

$$Tr(f) = \{(h, p) \in \mathscr{D}_{\mathrm{c}} \times \mathscr{E}_p;\ \ p \leqslant f(h),\ h \text{ minimal for that property}\}.$$

This trace is the extension to dI-domains of the notion of trace introduced by Girard in the framework of qualitative domains [11].

Theorem 1.6. *Let f be a stable function from $\mathscr{D}$ to $\mathscr{E}$. Then*
1. *f is characterized by its trace and we have*

$$f(a) = \bigvee \{p; \exists h \leqslant a,\ (h, p) \in Tr(f)\}.$$

2. *A subset T of $\mathscr{D}_{\mathrm{c}} \times \mathscr{E}_p$ is the trace of a stable function f^T iff it satisfies*
   (i) *if $h_1, \ldots, h_n$ are compatible and $(h_1, p_1), \ldots, (h_n, p_n) \in T$ then $p_1, \ldots, p_n$ are compatible*;
   (ii) *$(h, p) \in T$ and $p' \leqslant p$ implies $(h', p') \in T$ for a certain $h' \leqslant h$*;
   (iii) *(h, p), $(h', p) \in T$ and h, h' compatible implies $h = h'$.*

Example. 1. $Tr(id) = \{(p, p);\ \ p \in \mathscr{D}_{\mathrm{p}}\}$.

2. Let $\varepsilon_{h,k}$ be the function from $\mathscr{D}$ to $\mathscr{E}$ defined by

$$\varepsilon_{h,k}(a) = \begin{cases} k & \text{if a } \geqslant h, \\ \perp & \text{otherwise.} \end{cases} \tag{1}$$

Then $\varepsilon_{h,k}$ is stable and its trace is $\{(h, p'); p' \in \mathscr{E}_{\mathrm{p}} \text{ and } p' \leqslant k\}$.

We denote by $[\mathscr{D}, \mathscr{E}]$ the set of stable functions from $\mathscr{D}$ to $\mathscr{E}$, and by $\mathscr{D} \Rightarrow \mathscr{E}$ the set of traces of these functions.

Theorem 1.7. *The domain $(\mathscr{D} \Rightarrow \mathscr{E}, \subseteq)$ is isomorphic, as a poset, to $([\mathscr{D}, \mathscr{E}], \leqslant)$, where $\leqslant$ denotes Berry's order defined by*

$$f \leqslant g \textit{ iff } \forall a, b \in \mathscr{D}, (a \leqslant b \Rightarrow f(a) = f(b) \wedge g(a)).$$

Theorem 1.8. *The category of dI-domains and stable functions is cartesian closed.*

1.4. dI-domains with coherence

Strong stability, a notion due to Bucciarelli and Ehrhard, can be expressed in the framework of dI-domains. Roughly speaking, a function is strongly stable if it preserves the glb of certain (not necessary bounded) subsets of its domain, which are said to be *coherent*. Here we shall content ourselves with stating the definitions and the main results without proof (cf. [8] or [6] which deals with the particular case of qualitative domains).

A *coherence* on a dI-domain $\mathscr{D}$ is a subset $\mathscr{C}$ of $\mathscr{P}^*_{\mathrm{f}}(\mathscr{D})$, that is a set of finite and non-empty subsets of $\mathscr{D}$. The *canonical coherence* on $\mathscr{D}$ is $\mathscr{C}_{\mathscr{D}} = \{A \subseteq^*_{\mathrm{f}} \mathscr{D};\ A \text{ bounded}\}$. A *dI-domain with coherence* is a pair $(\mathscr{D},\mathscr{C})$, where $\mathscr{D}$ is a dI-domain and where $\mathscr{C}$ is a coherence on $\mathscr{D}$.

Definition 1.9. Let $(\mathscr{D}_1,\mathscr{C}_1)$ and $(\mathscr{D}_2,\mathscr{C}_2)$ be two dI-domains with coherence.

A *continuous* function f, from $\mathscr{D}_1$ to $\mathscr{D}_2$, is *strongly stable* with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$ if it satisfies:
(i) $\forall A \in \mathscr{C}_1\ \ f^\bullet(A) \in \mathscr{C}_2$;
(ii) $\forall A \in \mathscr{C}_1\ \ f(\bigwedge A) = \bigwedge f^\bullet(A)$.

Fact 1.10. *If $\mathscr{C}_{\mathscr{D}_1} \subseteq \mathscr{C}_1$ and if f from $\mathscr{D}_1$ to $\mathscr{D}_2$ is strongly stable with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$, then f is stable.*

All the coherences considered latter contain the canonical coherence, and so all the strongly stable functions met will be stable.

Definition 1.11. Let $(E, \leqslant)$ be a poset and $A, B \subseteq E$. We say that A is *Egli-Milner smaller* than B, we write $A \sqsubseteq B$, if
1. $\forall a \in A\ \exists b \in B.\ \ a \leqslant b$,
2. $\forall b \in B\ \exists a \in A.\ \ a \leqslant b$.

Definition 1.12. A coherence $\mathscr{C}$ on a dI-domain $\mathscr{D}$ is *acceptable* if it satisfies
(i) $a \in \mathscr{D}$ implies $\{a\} \in \mathscr{C}$,
(ii) if $A \in \mathscr{C}$ and $B \sqsubseteq A$, then $B \in \mathscr{C}$,
(iii) if $X_1, \ldots, X_n$ are directed subsets of $\mathscr{D}$ such that for any family $a_1 \in X_1, \ldots, a_n \in X_n$ we have $\{a_1, \ldots, a_n\} \in \mathscr{C}$, then $\{\bigvee X_1, \ldots, \bigvee X_n\} \in \mathscr{C}$.

One can easily show that the canonical coherence is the smallest acceptable coherence.

Theorem 1.13 (Bucciarelli and Ehrhard [8]). *The category of dI-domains with acceptable coherence and strongly stable functions is cartesian closed.*

In this category the cartesian product of $(\mathscr{D}_1, \mathscr{C}(\mathscr{D}_1))$ and $(\mathscr{D}_2, \mathscr{C}(\mathscr{D}_2))$, denoted by $(\mathscr{D}_1 \times \mathscr{D}_2, \mathscr{C}(\mathscr{D}_1 \times \mathscr{D}_2))$, is defined by
– $\mathscr{D}_1 \times \mathscr{D}_2$ is the usual product,
– $\mathscr{C}(\mathscr{D}_1 \times \mathscr{D}_2) = \{C \subseteq \mathscr{D}_1 \times \mathscr{D}_2;\ (C)_1 \in \mathscr{C}(\mathscr{D}_1) \text{ and } (C)_2 \in \mathscr{C}(\mathscr{D}_2)\}$.
In particular, when $\mathscr{C}(\mathscr{D}_1)$ and $\mathscr{C}(\mathscr{D}_2)$ are standard, so is $\mathscr{C}(\mathscr{D}_1 \times \mathscr{D}_2)$.

Examples. (1) Let $(\mathscr{D}_1, \mathscr{C}_1)$ and $(\mathscr{D}_2, \mathscr{C}_2)$ be two dI-domains with acceptable coherence. The function $\varepsilon_{h,k}$ from $\mathscr{D}_1$ to $\mathscr{D}_2$ defined above is strongly stable with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$. Indeed, this function is continuous, since it is stable. Let us check that if we have $A = \{a_1, \ldots, a_n\} \in \mathscr{C}_1$, then $\varepsilon_{h,k}{}^\bullet(A) \in \mathscr{C}_2$ and $\bigwedge \varepsilon_{h,k}{}^\bullet(A) = \varepsilon_{h,k}(\bigwedge A)$. There are three possible values for $\varepsilon_{h,k}{}^\bullet(A): \{\perp\}$, $\{\perp, k\}$ or $\{k\}$. These three sets are Egli-Milner lower than $\{k\}$. Since $\mathscr{C}_2$ is acceptable, they belong to $\mathscr{C}_2$. If $\bigwedge \varepsilon_{h,k}{}^\bullet(A) = k$, then for every $i \leqslant n$, $a_i \geqslant h$, and so $a_1 \wedge \cdots \wedge a_n \geqslant h$, which implies that $\varepsilon_{h,k}(\bigwedge A) = k$. If $\bigwedge \varepsilon_{h,k}{}^\bullet(A) = \perp$, then there is $j \leqslant n$ such that $a_j \ngeqslant h$, and so $a_1 \wedge \cdots \wedge a_n \ngeqslant h$, which implies that $\varepsilon_{h,k}(\bigwedge A) = \perp$.

(2) Let $\mathscr{B} = \{\perp, T, F\}$ be the flat domain of booleans. This dI-domain admits two acceptable coherences: the canonical coherence and the full coherence $\mathscr{C}(\mathscr{B}) = \mathscr{P}_{\mathrm{f}}^*(\mathscr{B})$. Let us now consider Berry's function, g_{b}, from $\mathscr{B}^3$ to $\mathscr{B}$ defined by

$$g_{\mathrm{b}}(a,b,c) = \begin{cases} T & \text{if } (a,b,c) \geqslant (\perp, T, F) \text{ or } (F, \perp, T) \text{ or } (T, F, \perp), \\ \perp & \text{otherwise.} \end{cases}$$

It is well known that g_{b} is stable, and it is easy to check that it is also strongly stable with respect to the canonical coherence. On the other hand, g_{b} is not strongly stable from $(\mathscr{B}^3, \mathscr{C}(\mathscr{B}^3))$ to $(\mathscr{B}, \mathscr{C}(\mathscr{B}))$: the set $A = \{(\perp, T, F), (F, \perp, T), (T, F, \perp)\}$ belongs to $\mathscr{C}(\mathscr{B}^3)$ since for every i, $1 \leqslant i \leqslant 3$, $(A)_i = \{\perp, T, F\} \in \mathscr{C}(\mathscr{B})$. Now $g_{\mathrm{b}}(\bigwedge A) = g_{\mathrm{b}}(\perp, \perp, \perp) = \perp \neq \bigwedge g_{\mathrm{b}}^\bullet(A) = T$.

(3) Let f be the function from $\mathscr{B}$ to $\mathscr{B}$ defined by $f(a) = V$ iff $a = V$ or $a = F$. If we take $\mathscr{P}_{\mathrm{f}}^*(\mathscr{B})$ as coherence, then f is not strongly stable (for it does not preserve the glb of $\{V, F\}$). On the other hand, one can easily show that it is strongly stable with respect to the canonical coherence.

2. Hypercoherences

In [10], Ehrhard introduced the notion of hypercoherence, a class of dI-domains with acceptable coherence which is stable under products and exponentials and can be described in a very simple way.

A hypercoherence H is a pair (D, Γ) where D is a set and Γ is a set of finite and non-empty subsets of D which contains all singletons. A hypercoherence H induces $(\mathscr{D}(H), \mathscr{C}(H))$ a dI-domain with an acceptable coherence. One of the advantages of hypercoherences is that, with the exclusive help of simple webs, they generate the stack of structures which constitutes dI-domains with acceptable coherence.

The category of hypercoherences is a sub-category of dI-domains with acceptable coherence, in which
– domains and coherences are closely related,
– at first order, strong stability and sequentiality coincide.
The cpo's associated to hypercoherences are dI-domains of particular kind: they are *qualitative domains*.

A *qualitative domain* [11] is a set $\mathscr{D}$ such that
- $\emptyset \in \mathscr{D}$,
- $a \in \mathscr{D}$ iff all its finite subsets belong to $\mathscr{D}$.

The *web* of $\mathscr{D}$ is the set $|\mathscr{D}| = \bigcup \mathscr{D}$ $(=\{\alpha; \{\alpha\} \in \mathscr{D}\})$.
In what follows we always denote qualitative domains by calligraphic letters $\mathscr{D}$, $\mathscr{D}_1, \ldots$.

A qualitative domain is a dI-domain when ordered by inclusion. Compact elements of $\mathscr{D}$ are finite elements, and prime elements are singletons. So we can identify the web of $\mathscr{D}$ and the set of prime element of $\mathscr{D}$, if we consider $\mathscr{D}$ as a dI-domain. Up to this identification the set of traces of all stable functions from $\mathscr{D}$ to $\mathscr{E}$ is a qualitative domain, the web of which is $\mathscr{D}_{\mathrm{c}} \times |\mathscr{E}|$. This qualitative domain is isomorphic, as a poset, to the set of stable functions ordered by Berry's order. Qualitative domains and stable functions form a c.c.c. [11].

If we equip $\mathscr{D}$ and $\mathscr{E}$ with acceptable coherences and if we use the same identification, then the set of traces of all strongly stable functions from $\mathscr{D}$ to $\mathscr{E}$ is a qualitative domain with acceptable coherence. Moreover, this set is isomorphic, as a poset, to the set of strongly stable functions from $\mathscr{D}$ to $\mathscr{E}$ ordered by Berry's order. Qualitative domains with acceptable coherence and strongly stable functions form a c.c.c. [6].

Definition 2.1. A *hypercoherence* is a pair $H = (D, \Gamma)$, where D is a set and where $\Gamma \subseteq \mathscr{P}^*_{\mathrm{f}}(D)$ is such that $\{\alpha\} \in \Gamma$, for every $\alpha \in D$.

D is called the *web* of the hypercoherence H, and Γ the *precoherence* of H (the atomic coherence in [10]).

Definition 2.2. Let A be a set. We say that u is a *multisection* of A, written $u \triangleleft A$, if
1. $u \subseteq \bigcup A$,
2. $\forall a \in A.\ \ u \cap a \neq \emptyset$

Note that a multisection cannot be empty. The notation $u \triangleleft^*_{\mathrm{f}} A$ means that u is a finite multisection of A.

Definition 2.3. Let $H = (D, \Gamma)$ be a hypercoherence. Define

$$\mathscr{D}(H) = \{a \subseteq D;\ \forall u \subseteq D,\ u \subseteq^*_{\mathrm{f}} a \Rightarrow u \in \Gamma\},$$

$$\mathscr{C}(H) = \{A \subseteq^*_{\mathrm{f}} \mathscr{D}(H);\ \forall u \subseteq D,\ u \triangleleft^*_{\mathrm{f}} A \Rightarrow u \in \Gamma\},$$

where $u \triangleleft^*_{\mathrm{f}} A$ means that u is a finite multisection of A.

Proposition 2.4. $(\mathscr{D}(H), \mathscr{C}(H))$ *is a qualitative domain with acceptable coherence.*

We can remark that $|\mathscr{D}(H)| = D$.

Remark

1. If $\{\alpha_1, \ldots, \alpha_n\} \in \Gamma$, then $\{\{\alpha_1\}, \ldots, \{\alpha_n\}\} \in \mathscr{C}(H)$ (for the only multisection of this set is $\{\alpha_1, \ldots, \alpha_n\}$).
2. If $A \subseteq^*_{\mathrm{f}} \mathscr{D}(H)$ and $\emptyset \in A$, then $A \in \mathscr{C}(H)$ (for there is no multisection of A).

Examples. (1) Let n be a natural number ($n \geqslant 1$), D be a set, and $\Gamma = \{u \subseteq^*_{\mathrm{f}} D;\ \sharp u \leqslant n\}$. Then, $H = (D, \Gamma)$ is a hypercoherence with

$$\mathscr{D}(H) = \Gamma \cup \{\emptyset\} \quad \text{and} \quad \mathscr{C}(H) = \{A \subseteq^*_{\mathrm{f}} \mathscr{D};\ \sharp \cup A \leqslant n\} \cup \{A \subseteq^*_{\mathrm{f}} \mathscr{D};\ \emptyset \in A\}.$$

(2) Let H be the hypercoherence $(D, \mathscr{P}^*_{\mathrm{f}}(D))$. Then

$$\mathscr{D}(H) = \mathscr{P}(D) \quad \text{and} \quad \mathscr{C}(H) = \mathscr{P}^*_{\mathrm{f}}(\mathscr{D}(H)).$$

(3) The qualitative domain $\mathscr{B}$ of booleans, the web of which is $|\mathscr{B}| = \{t, f\}$, is generated by a unique hypercoherence: $|\mathscr{B}|$ admits only two precoherences, $\Gamma_1 = \{\{t\}, \{f\}\}$ and $\Gamma_2 = \{\{t\}, \{f\}, \{t, f\}\}$. Set $H_1 = (|\mathscr{B}|, \Gamma_1)$ and $H_2 = (|\mathscr{B}|, \Gamma_2)$. It is easy to check that the domain generated by H_2 has a top (and so is not $\mathscr{B}$), whereas the domain generated by H_1 is $\mathscr{B}$. Furthermore the coherence associated to H_1 is the coherence $\mathscr{C}(\mathscr{B})$ defined in Example following Theorem 1.13 (set $T = \{t\}$ and $F = \{f\}$).

We denote by $A_1 + \cdots + A_n$ the disjoint union of the sets $A_1, \ldots, A_n$, that is the set $(A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup \cdots \cup (A_n \times \{n\})$. In that context, we denote by $p_i(C)$, where $i \in \{1, \ldots, n\}$, the ith projection of the subset C of $A_1 + \cdots + A_n$: $p_i(C) = \{\alpha \in A_i;\ (\alpha, i) \in C\}$.

If $H_1 = (D_1, \Gamma_1), \ldots, H_n = (D_n, \Gamma_n)$ are hypercoherences, the product $(\mathscr{D}(H_1), \mathscr{C}(H_1)) \times \cdots \times (\mathscr{D}(H_n), \mathscr{C}(H_n))$ is generated by a hypercoherence H, induced by $H_1, \ldots, H_n$:

$$H = (D_1 + \cdots + D_n, \Gamma),$$

where Γ is the set of non-empty and finite subsets u of $D_1 + \cdots + D_n$ satisfying: if there exists $k \leqslant n$ such that $p_k(u) \neq \emptyset$ and $p_j(u) = \emptyset$ for all $1 \leqslant j \leqslant n$ such that $j \neq k$, then $p_k(u) \in \Gamma_k$.

In particular, Γ contains all the sets $u \subseteq^*_{\mathrm{f}} D_1 + \cdots + D_n$ such that there are $i, j \leqslant n$, $i \neq j$, with $p_i(u) \neq \emptyset$ and $p_j(u) \neq \emptyset$.

Actually $(\mathscr{D}(H), \mathscr{C}(H))$ is

$$\mathscr{D}(H) = \{a \subseteq D_1 + \cdots + D_n;\ \forall i \leqslant n.\ p_i(a) \in \mathscr{D}(H_i)\},$$

$$\mathscr{C}(H) = \{A \subseteq^*_{\mathrm{f}} \mathscr{D}(H);\ \forall i \leqslant n.\ p_i^{\bullet}(A) \in \mathscr{C}(H_i)\}.$$

Note that there is a natural identification between the points of $\mathscr{D}(H)$, which are subsets of $D_1 + \cdots + D_n$, and the n-tuples of the product $\mathscr{D}(H_1) \times \cdots \times \mathscr{D}(H_n)$: to $a \in \mathscr{D}(H)$ corresponds $(p_1(a), \ldots, p_n(a))$.

A strongly stable function from H_1 to H_2 is by definition a function from $\mathscr{D}(H_1)$ to $\mathscr{D}(H_2)$ which is strongly stable with respect to $\mathscr{C}(H_1)$ and $\mathscr{C}(H_2)$.

We denote by $[H_1, H_2]$ the set of strongly stable functions from H_1 to H_2, and by $H_1 \Rightarrow H_2$ the set of their traces. If $H_1 = (D_1, \Gamma_1)$ and $H_2 = (D_2, \Gamma_2)$ are two hypercoherences, the qualitative domain $H_1 \Rightarrow H_2$ is generated by a hypercoherence $H_1 \rightarrow H_2$, induced by H_1 and H_2:

$$H_1 \rightarrow H_2 = ((\mathscr{D}(H_1))_{\mathrm{c}} \times D_2, \Gamma(H_1 \rightarrow H_2)), \quad \text{where} \quad \Gamma(H_1 \rightarrow H_2) \quad \text{is}$$

$$\{u \subseteq (\mathscr{D}(H_1))_{\mathrm{c}} \times D_2;\ (u)_1 \in \mathscr{C}(H_1) \Rightarrow (u)_2 \in \Gamma_2 \text{ and } (u)_1 \in \mathscr{C}(H_1)^{>1} \Rightarrow (u)_2 \in {\Gamma_2}^{>1}\},$$

and where $Y^{>1}$ denotes the set of elements of Y which have a cardinal strictly greater than 1.

Proposition 2.5. *The ordered sets* $([H_1, H_2], \leqslant)$ *and* $(\mathscr{D}(H_1 \rightarrow H_2), \subseteq)$ *are isomorphic* ($\leqslant$ *denotes Berry's order*).

Theorem 2.6 (Ehrhard [10]). *Hypercoherences and strongly stable functions form a cartesian closed category.*

We show now how the technique developed by Krivine in the continuous and the stable cases [16] can be adapted in the strongly stable framework.

Definition 2.7. Let $H_1 = (D_1, \Gamma_1)$, $H_2 = (D_2, \Gamma_2)$ be two hypercoherences. We say that H_1 is a sub-structure of H_2, and we write $H_1 \preccurlyeq H_2$, if $D_1 \subseteq D_2$ and $\Gamma_1 = \Gamma_2 \cap \mathscr{P}(D_1)$.

Proposition 2.8
- (i) *Let H_1 and H_2 be two hypercoherences such that $H_1 \preccurlyeq H_2$. Then $\mathscr{D}(H_1) \subseteq \mathscr{D}(H_2)$ and $\mathscr{C}(H_1) = \mathscr{C}(H_2) \cap \mathscr{P}(\mathscr{D}(H_1))$.*
- (ii) *If $H_1 \preccurlyeq H_1'$ and $H_2 \preccurlyeq H_2'$, then $H_1 \rightarrow H_2 \preccurlyeq H_1' \rightarrow H_2'$.*
- (iii) *Let $(H_n)_{n \geqslant 0}$ be a sequence of hypercoherences, which is increasing for the relation $\preccurlyeq$, and let $H = \bigcup_{n \geqslant 0} H_n$ be the hypercoherence defined as follows: $D = \bigcup_{n \geqslant 0} D_n$ and $\Gamma = \bigcup_{n \geqslant 0} \Gamma_n$. Then $H \rightarrow H = \bigcup_{n \geqslant 0} (H_n \rightarrow H_n)$.*

The proof is immediate.

Definition 2.9. Let H_1 and H_2 be two hypercoherences. A *rigid embedding of hypercoherence* from H_1 to H_2 is an *injective* function i from D_1 to D_2 such that

$$u \in \Gamma_1 \;\Leftrightarrow\; i^{\bullet}(u) \in \Gamma_2.$$

An embedding projection pair from a poset $\mathscr{D}$ to a poset $\mathscr{E}$ is a pair of function (f, g) such that $f \circ g \leqslant id_{\mathscr{E}}$ and $g \circ f = id_{\mathscr{D}}$.

Proposition 2.10. *Let i be a rigid embedding of hypercoherence from H_1 to H_2. Then i generates (i^+, i^-) a strongly stable embedding projection pair from $(\mathscr{D}(H_1), \mathscr{C}(H_1))$ to $(\mathscr{D}(H_2), \mathscr{C}(H_2))$, where, for every a in $\mathscr{D}(H_1)$ and for every b in $\mathscr{D}(H_2)$:*

$$i^+(a) = \{i(\alpha);\ \alpha \in a\} \quad \textit{and} \quad i^-(a) = \{\alpha;\ i(\alpha) \in b\}.$$

Proof. Let i be a rigid embedding of hypercoherence from H_1 to H_2. It is immediate to check that (i^+, i^-) is an embedding projection pair. It is easy to see that $i^+(a) \in \mathscr{D}(H_2)$ for every $a \in \mathscr{D}(H_1)$, that i^+ is continuous and that $i^+(\bigcap A) = \cap\, i^{+\bullet}(A)$ for every $A \in \mathscr{C}(H_1)$. Let $A \in \mathscr{C}(H_1)$ and let $u \subseteq^*_{\mathrm{f}} D_2$ be such that $u \lhd i^{+\bullet}(A)$. There is a unique $v \subseteq^*_{\mathrm{f}} D_1$ such that $u = i^+(v)$, and clearly $v \lhd A$. Hence we have $v \in \Gamma_1$ and, i being a rigid embedding of hypercoherence, $u \in \Gamma_2$. Thus $i^{+\bullet}(A) \in \mathscr{C}(H_2)$.

It is also routine to check that i^- is strongly stable. $\square$

Observe moreover that $i^+ \circ i^- = id_{\mathscr{D}_2}$ iff i is a surjective.

Proposition 2.11. *Let $H = (D, \Gamma)$ be a hypercoherence, and let i be a rigid embedding from $H \to H$ to H. Then $(\mathscr{D}(H), i^+, i^-)$ is a λ-model. In this model, for $a, b \in \mathscr{D}(H)$, one has*

$$(a)b = \{\alpha \in D; \exists h \subseteq b.\ i(h, \alpha) \in a\}.$$

Furthermore $\mathscr{D}(H)$ is extensional iff i is surjective.

Proof. Let Φ_a be the strongly stable function whose trace is $i^-(a)$. We have

$$\begin{aligned}(a)b = \Phi_a(b) &= \{\alpha \in D;\ \exists h \subseteq b,\ (h, \alpha) \in Tr(\Phi_a)\}\\ &= \{\alpha \in D;\ \exists h \subseteq b,\ i(h, \alpha) \in a\}. \quad \square\end{aligned}$$

3. Park's strongly stable model

The Park model [20] was first defined in the framework of continuous semantics. It is a variant of the Scott model $\mathscr{D}_\infty$[22], but with a very different equational theory (the model is not semi-sensible). This model has a stable analogue (which was defined by Honsell and Ronchi della Rocca [13]), and a strongly stable analogue that we construct now.

We start from a non-empty, countable set A of atoms, which does not contain any pairs. We simultaneously define the set of formulas Δ and a bijection i from $\mathscr{P}_{\mathrm{f}}(\Delta) \times \Delta$ to Δ as follows:
- each $\alpha \in A$ is a formula;
- if $h \in \mathscr{P}_{\mathrm{f}}(\Delta)$ and $\alpha \in \Delta$, and if $(h, \alpha) \neq (\{\alpha\}, \alpha)$ with $\alpha \in A$, then $(h, \alpha) \in \Delta$ and $h \to \alpha = (h, \alpha)$ (from now on, $h \to \alpha$ stands for $i(h, \alpha)$);
- finally we set $\{\alpha\} \to \alpha = \alpha$, for every $\alpha \in A$.

We define a notion of rank on Δ: $rk(\alpha)=0$ if $\alpha\in A$; $rk(h,\alpha)=\max\{\max\{rk(\beta);\ \beta\in h\}, rk(\alpha)\}+1$.

We denote by Δ_n the set of formulas of rank $\leqslant n$. We build a sequence of hypercoherences $(H_n=(D_n,\Gamma_n))_{n\geqslant 0}$, increasing for $\preccurlyeq$, such that, for every n, $D_n\subseteq\Delta_n$ and the restriction i_n of i to $(\mathscr{D}(H_n))_{\mathrm{c}}\times D_n$ is an isomorphism of hypercoherence from $H_n\longrightarrow H_n$ to H_{n+1}.

- H_0 is the hypercoherence $(D_0,\Gamma_0)=(A,\mathscr{P}^*_{\mathrm{f}}(A))$.
- $H_{n+1}=(D_{n+1},\Gamma_{n+1})=i(H_n)$, i.e.:
  $D_{n+1}=i^{\bullet}((\mathscr{D}(H_n))_{\mathrm{c}}\times D_n)$ and
  $\Gamma_{n+1}=\{i^{\bullet}(h);\ h\in\Gamma(H_n\longrightarrow H_n)\}$.

Lemma 3.1. $H_n\preccurlyeq H_{n+1}$, *for every integer n.*

Proof. Let us show first that $H_0\preccurlyeq H_1$. It is clear that $D_0\subseteq D_1$. Let us see that $\Gamma_0=\Gamma_1\cap\mathscr{P}(D_0)$ $(=\Gamma_1\cap\mathscr{P}(A))$. Since $\Gamma_0\subseteq\mathscr{P}(A)$, it suffices to show that for every $h=\{\alpha_1,\ldots,\alpha_n\}\in\mathscr{P}(A)$, we have $h\in\Gamma_0\Leftrightarrow h\in\Gamma_1$. Since A does not contain any pairs and i is injective, $\{(\{\alpha_1\},\alpha_1),\ldots,(\{\alpha_n\},\alpha_n)\}$ is the unique $k\subseteq\mathscr{P}_{\mathrm{f}}(\Delta)\times\Delta$ such that $i^{\bullet}(k)=h$, and $k\in\Gamma(H_0\rightarrow H_0)$ (this does not depend on the choice of Γ_0). So the equivalence holds because $h\in\Gamma_0=\mathscr{P}^*_{\mathrm{f}}(A)$ (observe here that there is no choice for Γ_0).

We suppose now that $H_{n-1}\preccurlyeq H_n$ and we show that $H_n\preccurlyeq H_{n+1}$. We have $H_{n-1}\rightarrow H_{n-1}\preccurlyeq H_n\rightarrow H_n$, by Proposition 2.8. Since i_n is an isomorphism from $H_n\rightarrow H_n$ to H_{n+1} and i_{n-1} is the restriction of i_n to $(\mathscr{D}(H_{n-1}))_{\mathrm{c}}\times D_{n-1}$, we have $i_{n-1}(H_{n-1}\rightarrow H_{n-1})\preccurlyeq i_n(H_n\rightarrow H_n)$ that is $H_n\preccurlyeq H_{n+1}$. $\square$

Let $H=\bigcup_{n\geqslant 0}H_n$ (cf. Proposition 2.8). One can easily check that $j=\bigcup_{n\geqslant 0}i_n$ is a surjective rigid embedding from $H\rightarrow H$ to H. By Proposition 2.11 $(\mathscr{D}(H),j^+,j^-)$ is an extensional λ-model. It is the strongly stable version of Park's model.

Remarks

1. Park's stable model is constructed in the framework of Girard's coherent spaces, making the same identifications (cf. [13]).
2. The strongly stable version of Scott's $\mathscr{D}_\infty$ model can be similarly constructed: the only difference is that we identify each atom α with $\emptyset\rightarrow\alpha$. Note that for $\mathscr{D}_\infty$ one can choose an arbitrary precoherence on A, while for Park's model it is necessary to choose the maximal precoherence on A.

From now on, we restrict ourselves to the case where A is reduced to one point. Note however that all the following results hold for a model built on an arbitrary set of atoms.

Let $\mathscr{P}_{\mathrm{s}}$ (resp. $\mathscr{P}_{\mathrm{fs}}$) be the Park stable (resp. strongly stable) model built on a set of atoms reduced to one element. Set $p_0=A=\{\alpha_0\}$ $(=D_0)$.
In the sequel, $\mathscr{P}$ *denotes indifferently one of the two models,* $\mathscr{P}_{\mathrm{s}}$ *and* $\mathscr{P}_{\mathrm{fs}}$.

The three following lemmas set out the few properties of $\mathscr{P}$ (that are common to the two models) that we will need for proving our incompleteness results.

Note that Proposition 2.11 — which specifies how to compute the interpretation of a term in a Krivine strongly stable model — and the similar theorem which holds for the stable case (cf. [16]) make the proofs of these lemmas very easy.

Lemma 3.2. *Let $\gamma, \gamma', \delta \in D$ and d be a compact element of $\mathscr{P}$. If $\gamma \in D \backslash A$ and $\gamma' = d \to \delta$ with $\gamma \in d \cup \{\delta\}$, then $rk(\gamma') > rk(\gamma)$.*

Proof. If $\gamma \in d$ then $\max\{rk(\beta);\ \beta \in d\} \geqslant rk(\gamma)$ and $rk(\gamma') > rk(\gamma)$. Let us suppose that $\gamma = \delta$ then $\delta \notin A$ and $rk(\gamma') > rk(\gamma)$. □

So the only element in the web of $\mathscr{P}$ that satisfies $h \to \beta \in h$ is $\{\alpha_0\} \to \alpha_0$ $(\in \{\alpha_0\})$.

Lemma 3.3. *Let $a_1, \ldots, a_n \in \mathscr{P}$. Then*

$$(p_0)a_1 \cdots a_n = p_0 \cap a_1 \cap \cdots \cap a_n = \begin{cases} p_0 & \textit{if } \forall i\ a_i \supseteq p_0 \\ \emptyset & \textit{otherwise.} \end{cases}$$

Proof. The assertion follows from $(p_0)a = \{\beta; \exists h \subseteq a, h \to \beta \in p_0\}$, and from the fact that we have $h \to \beta \in p_0$ iff $h = \{\alpha_0\}$ and $\beta = \alpha_0$. □

Lemma 3.4. *The model $\mathscr{P}$ satisfies*

(i) $\Omega^* = p_0 = \{\alpha_0\}$,

(ii) $(\lambda x\ \Omega)^* = \{\emptyset \to \alpha_0\}$,

(iii) Ω^* *and* $(\lambda x\ \Omega)^*$ *are incompatible*,

(iv) $(\Omega^*)\Omega^* = \Omega^*$, $(\Omega^*)(\lambda x\ \Omega)^* = \emptyset$ *and* $(\lambda x\ (\Omega)(\Omega)x)^* = \Omega^*$,

where Ω is the term $(\lambda x\ (x)x)\lambda x\ (x)x$.

Proof. (i) let $\delta = \lambda x\ (x)x$. We have

$$\Omega^* = \{\beta;\ \exists h \subseteq \delta^*,\ h \to \beta \in \delta^*\}$$

and

$$\delta^* = \{k \to \gamma;\ \gamma \in (k)k \text{ and } k \text{ minimal for that property}\}.$$

Let $\beta \in \Omega^*$. We have $\beta \in (h)h$, with h minimal for that property and $h \subseteq \delta^*$. Hence there is $h' \subseteq h$ such that $h' \to \beta \in h$. So we have $h' \to \beta \in \delta^*$, and the minimality of h forces $h' = h$, that is $h \to \beta \in h$. According to Lemma 3.2 the only element in the web of $\mathscr{P}$ that satisfies this property is $\{\alpha_0\} \to \alpha_0$. This proves that $\Omega^* \subseteq \{\alpha_0\}$.

Towards the converse, it suffices to show that $\{\alpha_0\} \to \alpha_0 \in \delta^*$ and that $\{\alpha_0\} \subseteq \delta^*$. The second statement is a consequence of the first one, since $\{\alpha_0\} \to \alpha_0 = \alpha_0$; the last statement follows from $\alpha_0 \in (\{\alpha_0\})\{\alpha_0\}$ with $\{\alpha_0\}$ minimal for that property. And this results from Lemma 3.3.

(ii) We have $(\lambda x\ \Omega)^* = \{h \to \beta; \beta \in \Omega^*$ and h minimal for that property$\}$. Hence $(\lambda x\ \Omega)^* = \{\emptyset \to \beta; \beta \in \Omega^*\}$, and the result follows from (i).

(iii) We have $\alpha_0 = \{\alpha_0\} \to \alpha_0$, therefore $F(\Omega^*) = \{(\{\alpha_0\}, \alpha_0)\}$; on the other hand $F((\lambda x\ \Omega)^*) = \{(\emptyset, \alpha_0)\}$. But $\{(\{\alpha_0\}, \alpha_0), (\emptyset, \alpha_0)\}$ is not a trace, by Theorem 1.6. So, by extensionality, we have that Ω^* and $(\lambda x\ \Omega)^*$ are incompatible.

(iv) The first two equalities follow immediately from the points (i) and (ii) and from Lemma 3.3. As for the third one, it follows from

$$(\lambda x\ (\Omega)(\Omega)x)^* = \{h \to \alpha;\ \alpha \in (p_0)(p_0)h \text{ and } h \text{ minimal}\}$$

and from Lemma 3.3. □

4. The continuous incompleteness theorem

As mentioned in the introduction, the incompleteness of continuous models is a theorem of Honsell and Ronchi della Rocca [14]. These authors proved, with the help of a non-trivial syntactic argument, that no continuous model has the theory $T_{\overline{\Lambda^0}}$ (the contextual theory induced by the set of essentially closed terms).

We give now a semantic proof of this incompleteness result, by showing:

Theorem 4.1. *There is no continuous model with the same theory as $\mathscr{P}$.*

Our proof uses the same terms U and V than those considered by Honsell and Ronchi della Rocca [14]:

$$U = \lambda x \lambda z\ (\Omega)\ U_1\ U_2,$$

where $U_1 = (x)\ (\Omega)z\ \Omega$ and $U_2 = (x)\ \Omega\ (\Omega)z$. And

$$V = \lambda x \lambda z\ (\Omega)\ V_0,$$

where $V_0 = (x)\ (\Omega)z\ (\Omega)z$.

Lemma 4.2. *Let $\mathscr{D}$ be a continuous model such that*
1. *$\mathscr{D}$ is extensional,*
2. *$(\Omega)\Omega =_{\mathscr{D}} \Omega$,*
3. *$\lambda x\ (\Omega)(\Omega)x =_{\mathscr{D}} \Omega$,*
4. *$\Omega \neq_{\mathscr{D}} \lambda x\ \Omega$.*

Then, the terms U and V have distinct interpretations in $\mathscr{D}$.

Proof. Remember first that, in a continuous model, the order on functions is the extensional order. Let $\mathscr{D}$ be a continuous model satisfying the four hypothesis. Condition (3) implies (3'): $(\Omega)(\Omega)c =_{\mathscr{D}} (\Omega)c$, for every $c \in \mathscr{D}$. We have to find two points a and f of $\mathscr{D}$ such that $(U^*)fa \neq (V^*)fa$. We use again the counterexamples built by

Honsell and Ronchi della Rocca, to show that U and V are distinct in a continuous model having $T_{\overline{\Lambda^0}}$ as theory [14]. Two cases arise:

1. $(\Omega^*)c \leqslant \Omega^*$ for every $c \in \mathscr{D}$. Then $\Omega^* < (\lambda x\ \Omega)^*$ and $(\Omega^*)\bot < \Omega^*$ (by (1) and (4)). Take $a = \bot$ and $f = G(\hat{f})$ where $\hat{f}$ is the function defined by

$$\hat{f}(z) = \begin{cases} (\lambda x\ \Omega)^* & \text{if } z \nleqslant (\Omega^*)\bot, \\ \Omega^* & \text{otherwise.} \end{cases}$$

   It is easy to check that $\hat{f}$ is continuous, and that $(U^*)f\bot = \Omega^*$ while $(V^*)f\bot = (\Omega^*)\bot$ (by (2) and (3′)).
2. There is $a_0 \in \mathscr{D}$ such that $(\Omega^*)a_0 \nleqslant \Omega^*$. Then $(\Omega^*)\bot < (\Omega^*)a_0$ (otherwise we would have $(\Omega^*)a_0 \leqslant \Omega^*$, by (2)). Take $a = a_0$ and $f = G(\hat{f})$ where $\hat{f}$ is the function defined by

$$\hat{f}(z) = \begin{cases} \Omega^* & \text{if } z \nleqslant \Omega^*, \\ \bot & \text{otherwise.} \end{cases}$$

   It is easy to check that $\hat{f}$ is continuous, and that $(U^*)fa_0 = (\Omega^*)\bot$ while $(V^*)fa_0 = (\Omega^*)a_0$ (by (3′)). □

Lemma 4.3
(i) *The model $\mathscr{P}$ satisfies the four conditions of the previous lemma.*
(ii) *The terms U and V are equalized in $\mathscr{P}$.*

Proof. (i) Follows immediately from Lemma 3.4.

(ii) The terms U and V are equalized in $\mathscr{P}$ iff $(U^*)fa = (V^*)fa$, for all $a, f \in \mathscr{P}$. Let $\hat{f} = F(f)$, we have $\hat{f}(u,v) = (f)uv$, for every $u, v \in \mathscr{P}$. Lemma 3.3 gives

$$(U^*)fa = p_0 \cap U_1[f/x, a/z]^* \cap U_2[f/x, a/z]^* \quad \text{and} \quad (V^*)fa = p_0 \cap V_0[f/x, a/z]^*.$$

Now $U_1[f/x, a/z]^* = \hat{f}((\Omega^*)a, \Omega^*)$ and $U_2[f/x, a/z]^* = \hat{f}(\Omega^*, (\Omega^*)a)$. Since $(\Omega^*)a \subseteq \Omega^*$ for every $a \in \mathscr{P}$ by Lemma 3.3, we can use the stability of $\hat{f}$

$$\hat{f}((\Omega^*)a, (\Omega^*) \cap \hat{f}(\Omega^*, (\Omega^*)a) = \hat{f}((\Omega^*)a, (\Omega^*)a) = V_0[f/x, a/z]^*.$$

Therefore $(U^*)fa = (V^*)fa$, for all $a, f \in \mathscr{P}$. □

This establishes Theorem 4.1, and finishes our semantic proof of the incompleteness of continuous models.

Proposition 4.4. *The theory of Park's continuous model is incomparable with the theory of its stable analogue stable and with the theory of its strongly stable analogue.*

Proof. Denote by $\mathscr{P}_c$ the Park continuous model. We have $Th(\mathscr{P}) \nsubseteq Th(\mathscr{P}_c)$ by Lemma 4.3 (just consider U and V). The converse follows from the fact that $\mathscr{P}_c$

equalizes all non-solvable closed terms of order zero [14], while $\mathscr{P}$ distinguishes, for instance, Ω and $(\Omega)\lambda x\ \Omega$ (Lemma 3.4). □

5. The stable incompleteness theorem

Now we prove that stable models are incomplete with respect to pure λ-calculus. More precisely, we prove:

Theorem 5.1. *There is no stable model with the same theory as* $\mathscr{P}_{\mathrm{fs}}$.

The proof of this theorem consists:
- in exhibiting two terms X and Z that are equalized in $\mathscr{P}_{\mathrm{fs}}$, and
- in showing that there is no *stable* model $\mathscr{D}$ which satisfies
  (1) $\mathscr{D}$ is extensional,
  (2) $(\Omega)\Omega =_{\mathscr{D}} \Omega$,
  (3) $(\Omega)\lambda x\ \Omega \neq_{\mathscr{D}} \Omega$,
  and which equalizes X and Z.

Indeed, by Lemma 3.4, $\mathscr{P}_{\mathrm{fs}}$ satisfies these the conditions (1)–(3).

Berry [4] showed that the standard model of PCF contains a non-sequential stable function, g_{b} (defined in the second part of the Example following Theorem 1.13). Therefore, no term of PCF is interpreted by this function. It allows for the construction of two terms X', Z' of PCF, which are distinguished by g_{b} in the standard model of PCF, while they cannot be distinguished by any term of PCF. This implies that the stable standard model of PCF is not fully-abstract. Let us present an *untyped variant* g of g_{b}, from which we define our terms X and Z which are distinguished in any stable model satisfying conditions (1)–(3).

Let $\mathscr{D}$ be dI-domain, and let a, $True$, $False$ be points of $\mathscr{D}$ satisfying:
- $True$ and $False$ are incompatible,
- $a < True$.

Then $a \not\geqslant False$ and

Proposition 5.2. *There is a stable function g from $\mathscr{D}^3$ to $\mathscr{D}$ such that*
(i) $g(a, True, False) = g(False, a, True) = g(True, False, a) = True$;
(ii) $g(a, False, True) = g(True, a, False) = g(False, True, a) = \bot$.

Proof. $\mathscr{D}$ being algebraic, there exist two *incompatible* compact elements t and f such that $t \leqslant True$ and $f \leqslant False$ (otherwise the set $A = \{h \vee k;\ h \leqslant True,\ k \leqslant False\}$ would be directed and we would have $\bigvee A \geqslant True,\ False$). Let g be the function defined by the following trace:

$$Tr(g) = \bigcup_{\substack{\bot < p \leqslant True \\ p \text{ prime}}} \{((\bot, t, f), p),\ ((f, \bot, t), p),\ ((t, f, \bot), p)\}.$$

It is easy to check that the set above satisfies the conditions of Theorem 1.6 for being a trace and that the associated function satisfies equalities (i) and (ii). □

Consider the terms:

$$X = \lambda x\lambda z\ (\Omega)W_1W_2W_3 \quad \text{and} \quad Z = \lambda x\lambda z\ (\Omega)W_4W_5W_6,$$

where

$$\begin{array}{llll} W_1 = (x)\ (\Omega)z \quad \Omega \quad \lambda y\ \Omega & & W_4 = (x)\ (\Omega)z\ \ \lambda y\ \Omega \quad \Omega \\ W_2 = (x)\ \lambda y\ \Omega\ \ (\Omega)z \quad \Omega & & W_5 = (x) \quad \Omega \quad (\Omega)z\ \ \lambda y\ \Omega \\ W_3 = (x) \quad \Omega \quad \lambda y\ \Omega\ \ (\Omega)z & & W_6 = (x)\ \lambda y\ \Omega \quad \Omega \quad (\Omega)z \end{array}$$

Intuitively, Ω corresponds to *True*, $\lambda y\ \Omega$ corresponds to *False*, and $(\Omega)z$ corresponds to a, when $\perp$ is substituted for z. If we substitute g for x and $\perp$ for z, we see that the triple W_1, W_2, W_3 corresponds to the cycle where g takes the value *True*, while W_4, W_5, W_6 corresponds to the cycle where g takes the value $\perp$.

Lemma 5.3. *Let $(\mathscr{D}, F, G)$ be a stable model satisfying the conditions* (1)–(3). *Then in $\mathscr{D}$*:
- *Ω^* and $(\lambda x\ \Omega)^*$ are incompatible.*
- *$(\Omega^*)\perp < \Omega^*$.*
- *$X = Z \notin Th(\mathscr{D})$.*

Proof.
- Ω^* and $(\lambda x\ \Omega)^*$ are incompatible in $\mathscr{D}$: Firstly we show that

$$F((\lambda x\ \Omega)^*) \not\subseteq F(\Omega^*): \tag{$*$}$$

Indeed, assume that $F((\lambda x\ \Omega)^*) \subseteq F(\Omega^*)$. By (1) we have $G \circ F = id$, and so $(\lambda x\ \Omega)^* \leqslant \Omega^*$. Hence

$$\Omega^* = (\lambda x\ \Omega)^*(\lambda x\ \Omega)^* \leqslant (\Omega^*)(\lambda x\ \Omega)^* \leqslant (\Omega^*)\Omega^* = \Omega^* \text{ by (2)},$$

and $(\Omega)\lambda x\ \Omega =_{\mathscr{D}} \Omega$, which contradicts (3). Now, if we suppose that Ω^* and $(\lambda x\ \Omega)^*$ are compatible in $\mathscr{D}$ then $F(\Omega^*)$ and $F((\lambda x\ \Omega)^*)$ are compatible (for F is increasing); thus there is a trace which contains simultaneously $F(\Omega^*)$ and $F((\lambda x\ \Omega)^*)$. We have $(\lambda x\ \Omega)^* = G(\{(\perp, k);\ k \leqslant \Omega^*\})$, and so $F((\lambda x\ \Omega)^*) = \{(\perp, k);\ k \leqslant \Omega^*\}$, since $F \circ G = id$. On the other hand, we have $(\Omega^*)\Omega^* = \bigvee\{k;\ \exists h \leqslant \Omega^*,\ (h,k) \in F(\Omega^*)\}$, and by (2):

$$k \leqslant \Omega^* \Rightarrow \exists h \leqslant \Omega^*; \quad (h,k) \in F(\Omega^*).$$

Then, Theorem 1.6 implies $h = \perp$ for every $(h,k) \in F(\Omega^*)$, by compatibility of $F(\Omega^*)$ and $F((\lambda x\ \Omega)^*)$. Hence $F((\lambda x\ \Omega)^*) \subseteq F(\Omega^*)$, in contradiction with ($\bigstar$).

- $(\Omega^*)\bot < \Omega^*$:
  We already know that $(\Omega^*)\bot \leqslant \Omega^*$ (by (2)). Assume that these two points are equal. Then

  $$(\Omega^*)\bot = \bigvee\{k;\ (\bot, k) \in F(\Omega^*)\} = \Omega^*.$$

  Assume $k \leqslant \Omega^*$. Then there is $k' \leqslant (\Omega^*)\bot$ such that $(\bot, k') \in F(\Omega^*)$ and $k \leqslant k'$, and by Theorem 1.6 we have $(\bot, k) \in F(\Omega^*)$. Hence $F((\lambda x\ \Omega)^*) \subseteq F(\Omega^*)$, in contradiction with (★).
- $X = Z \notin Th(\mathscr{D})$.
  Set $True = \Omega^*$, $False = (\lambda x\ \Omega)^*$ and $a = ((\Omega^*)\bot)$, and consider a stable function g from $\mathscr{D}^3$ to $\mathscr{D}$ satisfying the conditions of Proposition 5.2. We denote by g the element $G(g)$ of $\mathscr{D}$. We have $W_i[g/x, \bot/z] =_{\mathscr{D}} \Omega$, for $1 \leqslant i \leqslant 3$, and $W_i[g/x, \bot/z] =_{\mathscr{D}} \bot$, for $4 \leqslant i \leqslant 6$. Hence $(X)\ g\ \bot =_{\mathscr{D}} (\Omega)\ \Omega\ \Omega\ \Omega =_{\mathscr{D}} \Omega$, and $(Z)\ g\ \bot =_{\mathscr{D}} (\Omega)\ \bot\ \bot\ \bot <_{\mathscr{D}} \Omega$. Therefore $(X)\ g\ \bot \neq_{\mathscr{D}} (Z)\ g\ \bot$. □

To establish Theorem 5.1, it suffices to show:

Proposition 5.4. *$X = Z$ is an equation of $\mathscr{P}_{\mathrm{fs}}$.*

Proof. In this proof, we abbreviate $(\lambda y\ \Omega)^*$ by $\lambda y\ \Omega^*$. The terms X and Z are equalized in $\mathscr{P}_{\mathrm{fs}}$ iff $(X^*)fa = (Z^*)fa$ for all $a, f \in \mathscr{P}_{\mathrm{fs}}$. (By extensionality of $\mathscr{P}_{\mathrm{fs}}$, or also because both terms X and Z begin with a λ.)

By Lemma 3.3 we have

$$(X^*)fa = \begin{cases} \{\alpha_0\} & \text{if for each } i \in \{1,2,3\},\ \alpha_0 \in W_i[f/x, a/z]^*, \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$(Z^*)fa = \begin{cases} \{\alpha_0\} & \text{if for each } i \in \{4,5,6\},\ \alpha_0 \in W_i[f/x, a/z]^*, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\hat{f} = F(f)$, we have $\hat{f}(u,v,w) = (f)uvw$, for each $(u,v,w) \in (\mathscr{P}_{\mathrm{fs}})^3$. Assume that $(X^*)fa = \{\alpha_0\}$. Then $\alpha_0 \in \hat{f}(\Omega^*, \lambda y\ \Omega^*, (\Omega^*)a) \cap \hat{f}((\Omega^*)a, \Omega^*, \lambda y\ \Omega^*) \cap \hat{f}(\lambda y\ \Omega^*, (\Omega^*)a, \Omega^*)$. Two cases arise:

1. $\alpha_0 \in a$. Then $(\Omega^*)a = \Omega^* = \{\alpha_0\}$. Thus:

   $$\alpha_0 \in W_1[f/x, a/z]^* = \hat{f}(\Omega^*, \Omega^*, \lambda y\ \Omega^*) = W_5[f/x, a/z]^*.$$

   $$\alpha_0 \in W_2[f/x, a/z]^* = \hat{f}(\lambda y\ \Omega^*, \Omega^*, \Omega^*) = W_6[f/x, a/z]^*.$$

   $$\alpha_0 \in W_3[f/x, a/z]^* = \hat{f}(\Omega^*, \lambda y\ \Omega^*, \Omega^*) = W_4[f/x, a/z]^*.$$

   Therefore $\alpha_0 \in (Z^*)fa$, and $(X^*)fa = (Z^*)fa$.
2. $\alpha_0 \notin a$. Then $(\Omega^*)a = \emptyset$. We have $\alpha_0 \in \hat{f}(\Omega^*, \lambda y\ \Omega^*, \emptyset) \cap \hat{f}(\emptyset, \Omega^*, \lambda y\ \Omega^*) \cap \hat{f}(\lambda y\ \Omega^*, \emptyset, \Omega^*)$. Now $A = \{(\Omega^*, \lambda y\ \Omega^*, \emptyset), (\emptyset, \Omega^*, \lambda y\ \Omega^*), (\lambda y\ \Omega^*, \emptyset, \Omega^*)\}$ belongs to

$\mathscr{C}((\mathscr{P}_{\mathrm{fs}})^3)$, because for each $i \in \{1,2,3\}$, $(A)_i = \{\emptyset, \Omega^*, \lambda y\ \Omega^*\} \in \mathscr{C}(\mathscr{P}_{\mathrm{fs}})$. Since $\hat{f}$ is strongly stable, we have:

$$\alpha_0 \in \hat{f}((\Omega^*, \lambda y\ \Omega^*, \emptyset) \cap (\emptyset, \Omega^*, \lambda y\ \Omega^*) \cap (\lambda y\ \Omega^*, \emptyset, \Omega^*)) = \hat{f}(\emptyset, \emptyset, \emptyset).$$

And since $\hat{f}$ is increasing:

$$\alpha_0 \in W_4[f/x, a/z]^* \cap W_5[f/x, a/z]^* \cap W_6[f/x, a/z]^*, \text{that is } \alpha_0 \in (Z^*)fa.$$

Thus $(X^*)fa = (Z^*)fa$.
An similar reasoning shows that if $(Z^*)fa = \{\alpha_0\}$, then $(X^*)fa = \{\alpha_0\}$. □

Note that the argument showing that A is a coherent subset is the one used in second part of the Example given after Theorem 1.13 for proving that Berry's function is not strongly stable.

Moreover we have:

Corollary 5.5. *There is neither continuous model, nor stable model which has the theory of Park's strongly stable model.*

6. Improvement of the incompleteness results

In this section, we give an operational view of the incompleteness results by exhibiting a particular contextual theory which has neither continuous model nor stable model. For this purpose, we use a notion of approximation which is valid in a very large class of models, including $\mathscr{P}_{\mathrm{s}}$ and $\mathscr{P}_{\mathrm{fs}}$.

For starting with, let us show that the incompleteness of stable models and continuous models can be expressed in terms of a very simple finite set of equations and inequations between terms of pure λ-calculus.

If $\mathscr{F}$ is a given finite set of equations and inequations, we say that a λ-theory T contains $\mathscr{F}$ if:
1. all equations of $\mathscr{F}$ belong to T,
2. $u = v$ does not belong to T if $u \neq v$ belongs to $\mathscr{F}$.

We say that $\mathscr{F}$ is equivalent to $\mathscr{F}'$ if $\mathscr{F}$ and $\mathscr{F}'$ are contained in the same theories. Note that if T is a λ-theory containing $\mathscr{F}$ and if $\mathscr{F}' \subseteq \mathscr{F}$, then T contains $\mathscr{F}'$.

Write $\varepsilon = \lambda y \lambda x\ (y)x$, $I = \lambda x\ x$ and

$$\mathscr{F}_1 = \{\varepsilon = I,\ (\Omega)\Omega = \Omega,\ X = Z,\ (\Omega)\lambda x\ \Omega \neq \Omega\},$$

$$\mathscr{F}_2 = \{\varepsilon = I,\ (\Omega)\Omega = \Omega,\ \lambda x\ (\Omega)(\Omega)x = \Omega,\ U = V,\ \lambda x\ \Omega \neq \Omega\},$$

$$\mathscr{F} = \{\varepsilon = I,\ (\Omega)\Omega = \Omega,\ \lambda x\ (\Omega)(\Omega)x = \Omega,\ X = Z,\ U = V,\ (\Omega)\lambda x\ \Omega \neq \Omega\}.$$

Remark. $\mathscr{F}$ is equivalent to $\mathscr{F}_1 \cup \mathscr{F}_2$ ($\lambda x\ \Omega \neq \Omega$ is a consequence of $(\Omega)\lambda x\ \Omega \neq \Omega$).

Proposition 6.1. *$\mathscr{F}$ is contained in $Th(\mathscr{P}_{\mathrm{fs}})$ and in $T_{\overline{\Lambda_I^0}}$, where $T_{\overline{\Lambda_I^0}}$ is the contextual theory induced by the set of essentially λI-closed terms.*

Proof. By Lemmas 3.4 and 4.3 and by Proposition 5.4, $Th(\mathscr{P}_{\mathrm{fs}})$ contains $\mathscr{F}$. Moreover it is immediate that $\Omega = (\Omega)\lambda x\ \Omega$ does not belong to $T_{\overline{\Lambda_I^0}}$. In order to prove that $\mathscr{F}$ is contained in $T_{\overline{\Lambda_I^0}}$ we have to show that the equational part of $\mathscr{F}$ is contained in $T_{\overline{\Lambda_I^0}}$; this results from the next proposition. □

Proposition 6.2. $Th(\mathscr{P}) \subseteq T_{\overline{\Lambda_I^0}}$.

Since the proof of this proposition is rather long and technical, we devote the remainder of this section to it.

Corollary 6.3. *$\mathscr{F}$ has neither continuous model, nor stable model.*

Indeed, by the results obtained in Section 4, $\mathscr{F}_2$ has no continuous model, and by the results obtained in Section 5, $\mathscr{F}_1$ has no stable model.

6.1. The approximation property

In order to prove Proposition 6.2 we use a notion of approximation introduced by Honsell and Ronchi della Rocca [14] in the framework of continuous semantics, which can be naturally extended to a very large class of models, in particular to *non-sensible* models (like Park's model), for which the standard approximation theorem of Wadsworth and Hyland [23, 24, 15] is not available.

From now on $\mathscr{D}$ denotes either a continuous, stable or strongly stable model.

Let us add a constant c_0 to the λ-calculus, and consider the calculus induced by the β-reduction on $\Lambda(c_0)$. We call *approximants* the β-normal terms of $\Lambda(c_0)$; we denote by $\mathscr{A}$ the set of approximants.

The set $\mathscr{A}$ is inductively defined by

- $c_0 \in \mathscr{A}$,
- $x \in \mathscr{A}$ for every variable x,
- if $u_1, \ldots, u_k \in \mathscr{A}$ then $\lambda x_1 \cdots \lambda x_n\ (x)u_1 \cdots u_k \in \mathscr{A}$,
  and $\lambda x_1 \cdots \lambda x_n\ (c_0)u_1 \cdots u_k \in \mathscr{A}$.

Consider $t \in \Lambda$. The *direct approximant* of t is the term t_0 of $\mathscr{A}$ obtained from t by replacing in it each redex $(\lambda x\ q)r$ by $((c_0)\lambda x\ q)r$. It allows to see canonically t as a normal term.

The set of *approximate normal forms* of t, denoted by $AF(t)$, is the set of direct approximants of t and of the terms to which t can be β-reduced (in one or several steps):

$$AF(t) = \{t_0';\ t\ \beta\ t'\}.$$

The interpretation of the approximants in $\mathscr{D}$ is determined as soon as the interpretation p_0 of c_0 in $\mathscr{D}$ is fixed. We set then $AF_{p_0}(t)^*_{\bar{a}/\bar{x}} = \{u^*_{\bar{a}/\bar{x}};\ u \in AF(t)\}$, for $t \in \Lambda$, $\bar{x} \supseteq FV(t)$ and for every $\bar{a} \in \mathscr{D}^{l(\bar{x})}$. In particular if $t \in \Lambda^0$, $AF_{p_0}(t)^* = \{u^*;\ u \in AF(t)\}$.

Definition 6.4. A model $\mathscr{D}$ satisfies the *approximation property* if one can interpret in it c_0 by $p_0 < (\lambda x\ x)^*$ such that $AF_{p_0}(t)^*_{\bar{a}/\bar{x}}$ is directed and $t[\bar{a}/\bar{x}]^* = \bigvee AF_{p_0}(t)^*_{\bar{a}/\bar{x}}$, for every term $t[\bar{x}]$ and every $\bar{a} \in \mathscr{D}^{l(\bar{x})}$.

Now we present a class of models in which the approximation property is satisfied: the stratified models. This notion is similar to the notion of approximable application of Longo [18], although it is weaker.

Definition 6.5. (i) A projection from $\mathscr{D}$ to $\mathscr{D}$ is an element π of $\mathscr{D} \Rightarrow \mathscr{D}$ such that $\pi \circ \pi = \pi$ and $\pi \leqslant id$; in particular $\pi(\bot) = \bot$.

(ii) A projection of $\mathscr{D}$ is an element $p \in \mathscr{D}$ such that $p \leqslant_{\mathscr{D}} \lambda x\ x$ and $p \circ p = p$, where $a \circ b$ is by definition $\lambda z\ (a)(b)z$.

It is easy to check that projections of $\mathscr{D}$ are exactly the elements $G(\pi)$ in $\mathscr{D}$ where π is a projection from $\mathscr{D}$ on $\mathscr{D}$ (if p is given, π is the application $x \rightsquigarrow (p)x$). Observe that $p =_{\mathscr{D}} \lambda x\ (p)x$ and $(p)\bot =_{\mathscr{D}} \bot$.

Definition 6.6. A model $(\mathscr{D}, F, G)$ is stratified if there exists a sequence of projections $(p_n)_{n \in \omega}$ of $\mathscr{D}$, satisfying:
- $p_n \leqslant_{\mathscr{D}} p_{n+1}$, for every n,
- $\bigvee p_n =_{\mathscr{D}} \lambda x\ x$,
- $((p_{n+1})a)b = (p_n)(a)(p_n)b$.

In order to prove that every stratified model satisfies the approximation property, we use an extended λ-calculus, the labelled λ-calculus. This calculus is very similar to the indexed λ-calculus used by Hyland and Wadsworth [23, 24, 15]. Here we use the syntax introduced by Parigot in [19].

Let $C = \{c_n\}_{n \in \omega}$ be a set of constants that we call *labels*. The set Λ_e of labelled terms is the subset of $\Lambda(C)$ inductively defined by
- $x \in \Lambda_e$, for every variable x,
- if $u, v \in \Lambda_e$ then $(c_n)u$, $(c_n)\lambda x\ u$ and $(u)v$ belong to Λ_e.

Remark
1. Λ_e contains no labels, and no term starting by a λ.
2. Every labeled term is β-normal.
3. Λ_e is closed under substitution.

Definition 6.7
(i) A γ-redex is a term of the form $((c_{n+1})\lambda x\ u)v$, its reduct is $(c_n)u[(c_n)v/x]$.
(ii) A ε-redex is a term of the form $(c_n)(c_m)u$, it reduces to $(c_p)u$, where $p = min(n, m)$.

It is easy to check that Λ_e is closed by γ- and ε-reductions. We call labelled λ-calculus the calculus on Λ_e generated by the γ- and ε-reductions. We denote by $=_{\gamma\varepsilon}$ the equivalence relation induced by the γ- and ε-reductions.

Theorem 6.8. *The labelled λ-calculus is strongly normalizing and Church–Rosser.*

An analogous theorem is proved in [3] for an extension of this calculus and a straightforward verification shows that this result also applies to our calculus.

To every term t of Λ_e, we can associate a term $o(t)$ of Λ, by "forgetting" the labels; more precisely, we define by induction on the labelled terms, a surjection o from Λ_e to Λ by
$o(x)=x$ for every variable x,
$o((c_n)u)=o(u)$,
$o((u)v)=(o(u))o(v)$,
$o((c_n)\lambda x\ u)=\lambda x\ o(u)$.
Conversely, to every term t of Λ we can associate the set $E(t)$ of terms $u\in\Lambda_e$ which are ε-normal and such that $o(u)=t$, and it is easy to check that $E(x)=\{x\}\cup\{(c_n)x; n\in\omega\}$, $E((u)v)=\{(w)s; w\in E(u)$ and $s\in E(v)\}\cup\{(c_n)(w)s; w\in E(u), s\in E(v)$ and $n\in\omega\}$, $E(\lambda x\ u)=\{(c_n)\lambda x\ w; w\in E(u)$ and $n\in\omega\}$. In order to give an interpretation of labelled terms in a stratified model $(\mathscr{D},F,G))$ we must introduce an extension of the interpretation $(\cdot)^*$ to Λ_e. To this aim, the canonical interpretation of c_n in $\mathscr{D}$ consists in taking $c_n^*=p_n$. In what follows, we always suppose that the c_n are interpreted in that way. And we add the following rule: $((c_n)t)^*_{\bar a/\bar x}=(p_n)(t)^*_{\bar a/\bar x}$, for every $t\in\Lambda$, $\bar x\supseteq FV(t)$ and $\bar a\in\mathscr{D}^{l(\bar x)}$. Then we abbreviate $AF_{p_0}(t)^*_{\bar a/\bar x}$ by $AF(t)^*_{\bar a/\bar x}$, and we set $E(t)^*_{\bar a/\bar x}=\{u^*_{\bar a/\bar x};\ u\in E(t)\}$.

Proposition 6.9. *Let $(\mathscr{D},F,G)$ be a stratified model. Then*
(i) *Let $u,v\in\Lambda_e$ be such that $\bar x\supseteq FV(u)\cup FV(v)$. If $u=_{\gamma\varepsilon}v$ then $(u)^*_{\bar a/\bar x}=(v)^*_{\bar a/\bar x}$.*
(ii) *Let $t\in\Lambda$ be such that $\bar x\supseteq FV(t)$. $E(t)^*_{\bar a/\bar x}$ is directed for $\leqslant_{\mathscr{D}}$ and $t^*_{\bar a/\bar x}=\bigvee E(t)^*_{\bar a/\bar x}$.*

Proof. (i) It is an immediate consequence of stratification.

(ii) The fact that the sequence (p_n) is increasing and that the relation $\leqslant_{\mathscr{D}}$ is contextually closed, ensures that, for every $t\in\Lambda$, the set $E(t)^*_{\bar a/\bar x}$ is directed for $\leqslant_{\mathscr{D}}$. Let us show by induction on $t[\bar x]$ that, for every $\bar a\in\mathscr{D}^{l(\bar x)}$,

$$t[\bar a/\bar x]^*=\bigvee\{u[\bar a/\bar x]^*;\ u\in E(t)\}.$$

Sometimes we abbreviate $t[\bar a/\bar x]^*$ by $t[\ ]^*$. Let us recall that $(a)b$ is an abbreviation for $(F(a))(b)$.

1. $t\equiv x_i$ We have $E(t)=\{(c_n)x_i\}$ and $E(t)^*_{\bar a/\bar x}=\{(p_n)a_i\}$. So $\bigvee E(t)^*_{\bar a/\bar x}=\bigvee(p_n)a_i=(\bigvee p_n)a_i=a_i$ (since F is continuous and $\bigvee p_n=id$).
2. $t\equiv(u)v$ then $E(t)=\{(w)s,(c_n)(w)s; w\in E(u), s\in E(v)\}$ and $E(t)^*_{\bar a/\bar x}=\{(w[\ ]^*)s[\ ]^*, (p_n)(w[\ ]^*)s[\ ]^*;\ w\in E(u), s\in E(v)\}$. Moreover $\bigvee E(t)^*_{\bar a/\bar x}=(\bigvee E(u)^*_{\bar a/\bar x})\bigvee E(v)^*_{\bar a/\bar x}$

(since F is continuous and $\bigvee p_n = id$); since we have $u[\ ]^* = \bigvee E(u)^*_{\bar{a}/\bar{x}}$ and $v[\ ]^* = \bigvee E(v)^*_{\bar{a}/\bar{x}}$ by inductive hypothesis, we deduce $t[\ ]^* = \bigvee E(t)^*_{\bar{a}/\bar{x}}$.

3. $t \equiv \lambda y\ u$. Then $E(t) = \{(c_n)\lambda y\ w; w \in E(u)\}$ and $E(t)^*_{\bar{a}/\bar{x}} = \{(p_n)G(b \rightsquigarrow w[b/y, \bar{a}/\bar{x}]^*; w \in E(u)\}$. Since F, G are continuous, and $\bigvee p_n = id$, we have $\bigvee E(t)^*_{\bar{a}/\bar{x}} = G(\bigvee A_{\bar{a}})$, where $A_{\bar{a}}$ is $\{b \rightsquigarrow w[b/y, \bar{a}/\bar{x}]^*; w \in E(u)\}$. The set $E(u)^*_{\bar{a}/\bar{x}}$ is directed for $\leqslant_{\mathscr{D}}$, so $A_{\bar{a}}$ is directed in $\mathscr{D} \Rightarrow \mathscr{D}$. Since the lub of $A_{\bar{a}}$ is the extensional lub, $\bigvee A_{\bar{a}}$ is $b \rightsquigarrow \bigvee\{w[b/y, \bar{a}/\bar{x}]^*;\ w \in E(u)\}$. And, by inductive hypothesis, $\bigvee\{w[b/y, \bar{a}/\bar{x}]^*; w \in E(u)\} = u[b/y, \bar{a}/\bar{x}]^*$; so $t[\ ]^* = \bigvee E(t)^*_{\bar{a}/\bar{x}}$. $\square$

Lemma 6.10

(i) $((c_0)\lambda x\ u)v \leqslant_{\mathscr{D}} (c_0)u[(c_0)v/x] \leqslant_{\mathscr{D}} u[(c_0)v/x]$, *for every* u, v *in* $\Lambda(\mathscr{D} \cup \{c_0\})$.

(ii) *Let* $t, s \in \Lambda$ *be such that* t *β-reduces to* s *in one step. Then* $t_0 \leqslant_{\mathscr{D}} s_0$, *where* t_0 *and* s_0 *are the direct approximants of* t *and* s.

Proof. (i) We have $((c_0)\lambda x\ u)v \leqslant_{\mathscr{D}} ((c_1)\lambda x\ u)v =_{\mathscr{D}} (c_0)u[(c_0)v/x] \leqslant_{\mathscr{D}} u[(c_0)v/x]$, by stratification.

(ii) There exists a context $C[\ ]$ and terms q, r such that

$$t = C[(\lambda x\ q)r] \quad \text{and} \quad s = C[q[r/x]].$$

moreover $t_0 = C_0[((c_0)\lambda x\ q_0)r_0]$. Let us denote q'_0 the direct approximant of $q[r/x]$.

- If r is not an abstraction we have $q'_0 = q_0[r_0/x]$.
- Otherwise, $q'_0 = q_0[(c_0)r_0/x^f, r_0/x^a]$, where x^f represents the functional occurrences of x in q (the ones preceded by "("), and x^a the others.

In the both cases we have $q_0[(c_0)r_0/x] \leqslant_{\mathscr{D}} q'_0$. Moreover we have $C_0[q'_0] = s_0$, if $q[r/x]$ does not create redex in s, and $C_0[(c_0)q'_0] = s_0$ otherwise. Since $t_0 = C_0[((c_0)\lambda x\ q_0)r_0] \leqslant_{\mathscr{D}} C_0[(c_0)q_0[(c_0)r_0/x]] \leqslant_{\mathscr{D}} C_0[q_0[(c_0)r_0/x]]$ (with (i)), we obtain $t_0 \leqslant_{\mathscr{D}} s_0$. $\square$

The Church–Rosser property implies that the set $AF(t)^*_{\bar{a}/\bar{x}}$ is directed, for every $t \in \Lambda$ and $\bar{a} \in \mathscr{D}^{l(\bar{x})}$. Moreover, since $t_0 \leqslant_{\mathscr{D}} t$ for every t, we have $\bigvee AF(t)^*_{\bar{a}/\bar{x}} \leqslant_{\mathscr{D}} t[\bar{a}/\bar{x}]^*$.

Proposition 6.11. *Every stratified model satisfies the approximation property.*

Proof. We have $t[\bar{a}/\bar{x}]^* = \bigvee E(t)^*_{\bar{a}/\bar{x}}$; by Lemma 6.10 it is sufficient to prove that $\bigvee E(t)^*_{\bar{a}/\bar{x}} \leqslant_{\mathscr{D}} \bigvee AF(t)^*_{\bar{a}/\bar{x}}$.

Let $u \in E(t)$ and v be its $\gamma\varepsilon$-normal form. If $((c_n)\lambda x\ q)r$ is a subterm of v, then $n = 0$ since v is normal. Let v' be the term obtained from v by deleting all labels different from c_0. We have $u =_{\mathscr{D}} v \leqslant_{\mathscr{D}} v'$ and $v' \in AF(t)$ (we can translate the γ-reduction from u to v in Λ_e by a β-reduction from t to t' in Λ, so $(t')_0 = v'$). $\square$

Let us now come back to $\mathscr{P}$ (we recall that $\mathscr{P}$ simultaneously designates $\mathscr{P}_s$ and $\mathscr{P}_{fs}$), and set $p_0 = \{\alpha_0\}$. We have $p_0 \subset (\lambda x\ x)^*$ since $\alpha_0 = \{\alpha_0\} \to \alpha_0$.

Proposition 6.12. *If we interpret the constant* c_0 *by* p_0, *then* $\mathscr{P}$ *satisfies the approximation property.*

We can now focus on the proof of Proposition 6.2.

6.2. *The theory of $\mathscr{P}_{\mathrm{s}}$ and the theory of $\mathscr{P}_{\mathrm{fs}}$ are included in $T_{\overline{\Lambda_I^0}}$*

The notion of λI-(closed) term can be extended to $\Lambda(c_0)$ by considering c_0 as a λI-closed term. We denote by $\overline{\Lambda_I^0}$ (resp. $\overline{\Lambda_I^0(c_0)}$) the set of essentially λI-closed terms of Λ (resp. of $\Lambda(c_0)$), that is the set of terms that can be β-reduced to a λI-closed term. Let us start by showing that in $\mathscr{P}$, the interpretation of a λI-closed term always contains p_0 $(=\{\alpha_0\})$.

Lemma 6.13. *Let $u \equiv u[x, x_1, \ldots, x_n] \in \mathscr{A}$ be such that $x \in FV(u)$. Then: $p_0 \not\subseteq u[\emptyset/x, p_0/x_1, \ldots, p_0/x_n]^*$.*

Proof. By induction on u. We abbreviate $[\emptyset/x, p_0/x_1, \ldots, p_0/x_n]$ by $[\ ]$.
- If $u \equiv x$, we have $u[\ ]^* = \emptyset$.
- If $u \equiv \lambda y\ v$, we have $v[c_0/y]$ β-equivalent to $(\lambda y\ v)c_0 = (u)c_0$. By induction we have $p_0 \not\subseteq v[c_0/y][\ ]^*$, since $v[c_0/y]$ is a β-normal term and $x \in FV(v[c_0/y])$. But $v[c_0/y][\ ]^* = ((u)c_0)[\ ]^* = (u[\ ]^*)p_0$. If we had $p_0 \subseteq u[\ ]^*$, we would have $v[c_0/y]$ $[\ ]^* \supseteq (p_0)p_0 = p_0$, contradiction.
- If $u \equiv (\xi)u_1 \cdots u_m$, where ξ denotes a variable or c_0:
  - $\xi \equiv x$ gives $u[\ ]^* = (\emptyset)u_1[\ ]^* \cdots u_m[\ ]^* = \emptyset$.
  - $\xi \equiv y \neq x$ or $\xi = c_0$ implies $u[\ ]^* = (p_0)u_1[\ ]^* \cdots u_m[\ ]^*$. But $x \in FV(u)$; so there exists j such that $x \in FV(u_j)$. By induction we have $p_0 \not\subseteq u_j[\ ]^*$; hence $u[\ ]^* = \emptyset$, by Lemma 3.3. □

Proposition 6.14. *Let $u \in \Lambda_I^0(c_0)$ is a β-normal term. Then $u^* \supseteq p_0$. If moreover u is of the form $(c_0)u_1 \cdots u_k$, then $u^* = p_0$.*

Proof. We show the first part by induction on u:
- It is obvious for $u \equiv c_0$.
- If $u \equiv (v)w$, then v and w satisfy the hypothesis of the proposition, and we have $p_0 \subseteq v^*$ and $p_0 \subseteq w^*$ by induction. Hence $(p_0)p_0 = p_0 \subseteq u^*$.
- If $u \equiv \lambda y\ v$, then $FV(v) = \{y\}$, since u is a closed term, and $v[c_0/y]$ satisfies the hypothesis of the proposition. Hence, by induction, $p_0 \subseteq (v[p_0/y])^* = ((\lambda y\ v)^*)p_0 = (u^*)p_0$. Now $u^* = \{k \to \beta;\ \beta \in v[k/y]^*,\ k$ minimal for that property$\}$. Therefore, either $\emptyset \to \alpha_0 \in u^*$ or $\{\alpha_0\} \to \alpha_0 = \alpha_0 \in u^*$. Applying the previous lemma to v, we obtain $\alpha_0 \notin v[\emptyset/y]^*$ (for $p_0 = \{\alpha_0\}$). So we have $p_0 \subseteq u^*$.

For the second assertion, it suffices to remark that we have $u_i^* \supseteq p_0$ for every i, since the u_i are β-normal terms and belong to $\Lambda_I^0(c_0)$; hence $u^* = p_0$, by Lemma 3.3. □

Let us recall that a term t has order 0 if t is not β-convertible to any abstraction. Such a term is not solvable.

Corollary 6.15. *Let us consider* $t \in \overline{\Lambda_I^0}$. *Then*
(i) $t^* \supseteq p_0$;
(ii) *if* t *is a term of order* 0, *we have* $t^* = p_0$.

Proof. (i) If t β-reduces to a λI-closed term t', it is clear that the direct approximant t'_0 of t' is a λI-closed term of $\Lambda(c_0)$. But $t'_0 \in \mathscr{A}$, hence $(t')^* \supseteq (t'_0)^* \supseteq p_0$ by Proposition 6.14.

(ii) If t is a λI-closed term of order 0, then, for every t' such that $t\beta t'$, the term t' is a λI-closed of order 0. So every element u of $AF(t')$ is of the form $(c_0)u_1 \cdots u_m$, where the u_i are also λI-closed terms. Proposition 6.14 gives $u^* = p_0$; hence $t^* = (t')^* = p_0$ by the approximation property. $\square$

Lemma 6.16. *Let* $w \in \mathscr{A}$ *be a closed term which is not essentially a* λI*-term. Then* $p_0 = \{\alpha_0\} \not\subseteq w^*$.

Proof. By induction on the length of w. Note that the hypothesis is equivalent to $w \notin \Lambda_I$, since w is a β-normal term.

- If w does not begin with a λ, it is of the form $(c_0)w_1 \cdots w_p$, where $w_1, \ldots, w_p$ are closed terms and where $w_j \notin \Lambda_I$ for a certain j. We have $p_0 \not\subseteq w_j^*$ by induction, and so $w^* = \emptyset$ by Lemma 3.3.
- If $w \equiv \lambda x\ s$, two cases arise
  - $x \notin FV(s)$. Then s is a closed term and we have $w^* = \{\emptyset \to \beta;\ \beta \in s^*\}$; hence $\alpha_0 \notin w^*$.
  - $x \in FV(s)$. In this case $s[c_0/x]$ is a closed term and does not belong to Λ_I. So we have $\alpha_0 \notin s[\{\alpha_0\}/x]^*$ by induction. On the other hand $\alpha_0 \in w^*$ implies $\alpha_0 \in s[\{\alpha_0\}/x]^*$ (since $\alpha_0 = \{\alpha_0\} \to \alpha_0$) and this not the case. $\square$

To establish that $Th(\mathscr{P}) \subseteq T_{\overline{\Lambda_I^0}}$, it suffices to show that $\mathscr{P}$ never equalizes an essentially λI-closed term to a term which is not. Indeed, let u and v be such that $u =_{\mathscr{P}} v$. We have $C[u] =_{\mathscr{P}} C[v]$ for every context $C[\ ]$, and so $C[u] \in \overline{\Lambda_I^0}$ iff $C[v] \in \overline{\Lambda_I^0}$.

Towards a contradiction, assume that $u \in \overline{\Lambda_I^0}$ and $v \notin \overline{\Lambda_I^0}$. We have $u^* \supseteq p_0$ by Corollary 6.15.

If v is not β-equivalent to a closed term, there is a free variable x of v which occurs free in all the elements of $AF(v)$. Let $[\ ]$ be the interpretation of the free variables of v in which x is interpreted by $\emptyset$ and the other variables are interpreted by p_0. We have $p_0 \not\subseteq w[\ ]^*$ for every $w \in AF(v)$ (Lemma 6.13); therefore $p_0 \not\subseteq v[\ ]^*$ by the approximation property, and $u \neq_{\mathscr{P}} v$.

If v is β-equivalent to a closed term but is not β-equivalent to a λI-term, then v β-reduces to a closed term v' which is not a λI-term. Let us show that $p_0 \not\subseteq (v')^* = v^*$. Every element of $AF(v')$ is a closed term and is not a λI-term; by the approximation property, we just have to establish that $p_0 \not\subseteq w^*$ for every $w \in AF(v')$. This is assured by Lemma 6.16.

Thus, Proposition 6.2 is established and it is proved that $T_{\overline{\Lambda_I^0}}$ contains $\mathscr{F}$.

7. Conclusion

This paper explored the representativeness of classes of λ-models with respect to their theories. Natural questions concerned with this problematic are still open:

- Is the class of strongly stable models incomplete? We conjecture that it is, but we did not manage to prove it, neither syntactically (considering a particular contextual theory) nor semantically (using a particular continuous or stable model).
- Are the theories of $\mathscr{P}_{\mathrm{s}}$ and $\mathscr{P}_{\mathrm{fs}}$ incomparable? We conjecture a positive answer to this question. We have proved that the theory of the continuous Park's model is incomparable with $Th(\mathscr{P}_{\mathrm{s}})$ and $Th(\mathscr{P}_{\mathrm{fs}})$ (Proposition 4.4). Moreover Lemma 5.3 and Proposition 5.4 say us that $Th(\mathscr{P}_{\mathrm{fs}}) \not\subseteq Th(\mathscr{P}_{\mathrm{s}})$. In order to state that $Th(\mathscr{P}_{\mathrm{s}}) \not\subseteq \mathrm{Th}(\mathscr{P}_{\mathrm{fs}})$, one should provide a more subtle analysis of the structural differences existing between $\mathscr{P}_{\mathrm{fs}}$ and $\mathscr{P}_{\mathrm{s}}$.

	$Th(\mathscr{P}_{\mathrm{c}})$	$Th(\mathscr{P}_{\mathrm{s}})$	$Th(\mathscr{P}_{\mathrm{fs}})$	$T_{\overline{\Lambda^0}}$	$T_{\overline{\Lambda^0_I}}$
$\mathscr{F}$	$\not\subseteq$	$\not\subseteq$	$\subsetneq$	$\not\subseteq$	$\subsetneq$
$Th(\mathscr{P}_{\mathrm{c}})$		$\asymp$	$\asymp$	$\subsetneq$	$\asymp$
$Th(\mathscr{P}_{\mathrm{s}})$			$\not\supseteq$ [? $\asymp$]	$\asymp$	$\subsetneq$
$Th(\mathscr{P}_{\mathrm{fs}})$				$\asymp$	$\subseteq$ [? $\neq$]
$T_{\overline{\Lambda^0}}$					$\asymp$

The array summarizes results and conjectures concerning the theories of Park's models. Note that the symbol $\asymp$ signifies that the corresponding theories are incomparable, and that $\mathscr{P}_{\mathrm{c}}$ denotes the continuous Park's model. The array must be read as follows: $\mathscr{F} \not\subseteq Th(\mathscr{P}_{\mathrm{c}})$.

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