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COMMUNICATION

A 1-TOUGH NONHAMILTONIAN MAXIMAL PLANAR GRAPH

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We construct a maximal planar graph which is 1-tough but nonhamiltonian. The graph is an answer to Chvátal's question on the existence of such a graph.

1. Introduction

In this note we consider only finite undirected graph without loops or multiple edges. Our terminology and notation will be standard except as indicated. For a graph $G=(V,E)$ we denote by $k(G)$ the number of connected components of G. A graph $G=(V,E)$ is 1-*tough* if $k(G-S)\leq|S|$ for every nonempty property subset S of the vertex set V. Clearly 1-toughness is a necessary condition for a graph to be hamiltonian [1]. Since the graph shown in Fig. 1 is 1-tough but nonhamiltonian, the 1-toughness is not a sufficient condition in general.

The question when a maximal planar graph is hamiltonian is of considerable interest. However no nontrivial necessary and sufficient condition for a maximal planar graph to be hamiltonian is known. When V. Chvátal visited our Department 1979, he raised the following question: is the 1-toughness a sufficient condition for a maximal planar graph to be hamiltonian? We give an answer to the question by constructing a 1–tough nonhamiltonian maximal planar graph.

2. Construction

First we give an easy lemma.

Lemma. *Let $G=(V,E)$ and $S\subset V$. If $G-v$ is* 1-*tough for a vertex $v\in V$, and if $k(G-S)>|S|$, then v does not belong to S but all of its neighbours do.*

Proof. Immediate.

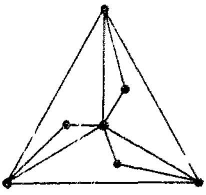

Fig. 1. A 1-tough nonhamiltonian planar graph.

We can construct a 1-tough nonhamiltonian maximal planar graph as follows.

Theorem. *Form a maximal planar graph G as follows: begin with K_4 the complete graph with four vertices, and in each inner triangular face of K_4 place the graph G_1 of Fig. 2 so that K_4 and G_1 have precisely the triangle in common. (See Fig. 3.) Then G is 1-tough but nonhamiltonian.*

Proof. First we show that G is 1-tough. Let T denote the set of vertices of degree three in G, then T has the properties that

(i) for every v in T, the graph $G - v$ is hamiltonian, so $G - v$ is 1-tough, and
(ii) every vertex not in T has a neighbour in T.

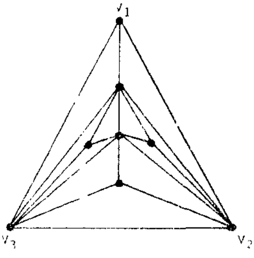

Fig. 2. A maximal planar graph G_1.

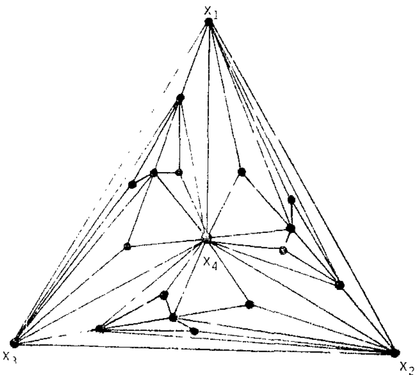

Fig. 3. A 1-tough nonhamiltonian maximal planar graph G.

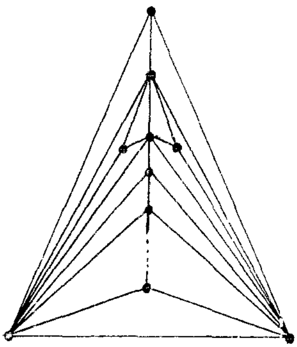

Fig. 4. A maximal planar graph.

These properties together with the above Lemma show that the only S that can possibly satisfy $k(G-S)>|S|$ is the complement of T. However this set S has $k(G-S)=|S|=9$.

Next we show that G is nonhamiltonian. Denote the set of vertices inside the triangle $x_ix_jx_k$ by S_{ij}, and suppose that G has a hamiltonian cycle C. One of the three set S_{ij} contains neither of the two vertices that are neighbours of x_4 in C. This set S_{ij} contains three vertices u, v, w of degree three such that u and v are neighbours of x_4. Since neither u nor v is adjacent to x_4 in C, the portion of C passing through u and v has the form $\ldots, a, u, b, v, x_j, \ldots$ such that a, b and x_j are the only neighbours of w in G. Now w cannot be adjacent to b in C and it can be adjacent to at most one of a, x_j in C. Hence w cannot have two neighbours in C, a contradiction.

We can similarly construct infinitely many maximal planar graphs which are 1-tough but nonhamiltonian as follows: place the graph in Fig. 4, instead of G_1, in the interior of one of three inner faces of K_4, and place G_1 in each interior of other inner faces. The proof is left to the reader.

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Reference

[1] V. Chvátal, Tough graph and Hamiltonian circuits, Discrete Math. 5 (1973) 215–228.