Journal of Algebra 332 (2011) 500–521

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Stable degenerations of Cohen–Macaulay modules

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ARTICLE INFO

Article history:
Received 6 September 2010
Available online 7 December 2010
Communicated by Kazuhiko Kurano

MSC:
primary 13C14
secondary 13D10

Keywords:
Degeneration
Cohen–Macaulay module
Finite representation type
Stable category
Triangulated category
Isolated singularity

ABSTRACT

As a stable analogue of degenerations, we introduce the notion of stable degenerations for Cohen–Macaulay modules over a Gorenstein local algebra. We shall give several necessary and/or sufficient conditions for the stable degeneration. These conditions will be helpful to see when a Cohen–Macaulay module degenerates to another.

1. Introduction

Let k be an algebraically closed field and let R be a finite-dimensional associative k-algebra with basis $e_1 = 1, e_2, \dots, e_d$. Then the structure constants c_{ijk} are defined by $e_i \cdot e_j = \sum_k c_{ijk} e_k$. The module variety Mod_R^n of n-dimensional left R-modules consists of the d-tuple $x = (x_1, x_2, \dots, x_d)$ of $n \times n$-matrices with entries in k such that x_1 is the identity and $x_i x_j = \sum_k c_{ijk} x_k$ holds for all i, j. It is easy to see that Mod_R^n is an affine variety and the general linear group $\mathrm{GL}_n(k)$ acts on Mod_R^n by simultaneous conjugation. In such a situation each $\mathrm{GL}_n(k)$-orbit corresponds to an isomorphism class of n-dimensional modules. We denote by $\mathcal{O}(M)$ the $\mathrm{GL}_n(k)$-orbit of the point in Mod_R^n corresponding to an R-module M. Then we say that M degenerates to N if $\mathcal{O}(N)$ is contained in the Zariski closure of $\mathcal{O}(M)$. This is the definition of degeneration of modules over a finite dimensional algebra. See [1,4,10].

However, if we want to consider the degeneration of modules over a noetherian algebra R which is not necessarily finite-dimensional over k, then this definition is not applicable, since each module is not necessarily a finite-dimensional k-vector space any more even if it is assumed to be finitely

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.11.005

generated over R, and hence there is no good way to define appropriate module varieties. By this reason we have proposed a scheme-theoretical definition of degeneration in the previous paper [7]. Our definition is as follows:

Let R be an associative k-algebra where k is any field. We set $V = k[t]_{(t)}$ and $K = k(t)$ with t being a variable over R hence over k. We denote by $\mathrm{mod}(R)$ the category of all finitely generated left R-modules and R-homomorphisms. Then we have the natural functors

$$\mathrm{mod}(R) \xleftarrow{\ r\ } \mathrm{mod}(R \otimes_k V) \xrightarrow{\ \ell\ } \mathrm{mod}(R \otimes_k K),$$

where $r = -\bigotimes_V V/tV$ and $\ell = -\bigotimes_V K$. Then for modules $M, N \in \mathrm{mod}(R)$, we say that M degenerates to N if there is a module $Q \in \mathrm{mod}(R \otimes_k V)$ that is V-flat such that $\ell(Q) \cong M \otimes_k K$ and $r(Q) \cong N$. The module Q, regarded as a bimodule ${}_R Q_V$, is a flat family of R-modules with parameter in V. At the closed point in the parameter space $\mathrm{Spec} V$, the fiber of Q is N, which is a meaning of the isomorphism $r(Q) \cong N$. On the other hand, the isomorphism $\ell(Q) \cong M \otimes_k K$ means that the generic fiber of Q is essentially given by M. In [7] we have shown that the latter definition of degeneration agrees with the former if R is a finite-dimensional k-algebra.

In the present paper we want to consider the stable analogue of degenerations for Cohen–Macaulay modules over a Gorenstein local rings.

Let R be a Gorenstein commutative local k-algebra where k is any field, and set $V = k[t]_{(t)}$ and $K = k(t)$ as above. We denote by $\mathrm{CM}(R)$ the category of all (maximal) Cohen–Macaulay R-modules and all R-homomorphisms, and denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. Note that $\underline{\mathrm{CM}}(R)$ has a structure of a triangulated category. Then, similarly to the above, there are triangle functors

$$\underline{\mathrm{CM}}(R) \xleftarrow{\ \mathcal{R}\ } \underline{\mathrm{CM}}(R \otimes_k V) \xrightarrow{\ \mathcal{L}\ } \underline{\mathrm{CM}}(R \otimes_k K),$$

where $\mathcal{R}$ and $\mathcal{L}$ are induced respectively by $-\bigotimes_V V/tV$ and $-\bigotimes_V K$. Then we define that $\underline{M} \in \underline{\mathrm{CM}}(R)$ stably degenerates to $\underline{N} \in \underline{\mathrm{CM}}(R)$ if there is a Cohen–Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$. See Definition 4.1 for more detail.

The aim of this paper is to clarify the meaning of this definition by showing several implications. To explain this, let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra and assume for simplicity that k is an infinite field. For Cohen–Macaulay R-modules M and N we consider the following four conditions:

(1) $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.
(2) There is a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$ in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.
(3) $\underline{M}$ stably degenerates to $\underline{N}$.
(4) There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.

In this paper, we shall prove the following implications and equivalences of these conditions:

- In general, (1) ⇒ (2) ⇒ (3) ⇒ (4). (Theorems 5.1 and 7.1.)
- If R is an artinian ring, then (1) ⇔ (2) ⇔ (3). (Theorem 5.1.)
- If R is an isolated singularity, then (2) ⇔ (3). (Theorem 6.1.)

However there is a counter example to the implication (2) ⇒ (1) in the case when R is an isolated singularity of Krull dimension one (Example 5.2). We also give an example of a Gorenstein ring of Krull dimension zero for which the implication (4) ⇒ (3) fails (Example 7.4). Furthermore, as a consequence of the implication (3) ⇒ (4), we can show that the stable degeneration gives rise to a well-defined partial order on the set of isomorphism classes of Cohen–Macaulay modules (Theorem 8.2).

After giving some preliminary consideration for degenerations in Section 2, we make several remarks on the ring $R \otimes_k K$ in Section 3. The remaining part of the paper is devoted to giving the proofs of the results mentioned above and to constructing the examples.

2. Review of degenerations

Let us recall the precise definition of degeneration of finitely generated modules over a noetherian algebra, which is given in our previous paper [7, Definition 2.1].

Definition 2.1. Let R be a noetherian algebra over a field k, and let M and N be finitely generated R-modules. We say that M degenerates to N, or N is a degeneration of M, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[\frac{1}{t}] \cong M \otimes_k V[\frac{1}{t}]$ as a left $R \otimes_k V[\frac{1}{t}]$-module.

In the previous paper [7, Theorem 2.2] we have proved the following theorem.

Theorem 2.2. *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

By virtue of this theorem together with a theorem of Zwara [10, Theorem 1], we see that if R is a finite-dimensional algebra over k, then our definition of degeneration agrees with the original one using module varieties which is mentioned in the beginning of Introduction. We also remark from this theorem that we can always take $k[t]_{(t)}$ as V in Definition 2.1. (See [7, Corollary 2.4].)

In the rest of the paper we mainly treat the case when R is a commutative ring.

Remark 2.3. Let R be a commutative noetherian algebra over k, and suppose that a finitely generated R-module M degenerates to a finitely generated R-module N. Then the following hold.

(1) The modules M and N give the same class in the Grothendieck group, i.e. $[M] = [N]$ as elements of $K_0(\mathrm{mod}(R))$. This is actually a direct consequence of Theorem 2.2. In particular, $\operatorname{rank} M = \operatorname{rank} N$ if the ranks are defined for R-modules. Furthermore, if $(R, \mathfrak{m})$ is a local ring, then $e(I, M) = e(I, N)$ for any $\mathfrak{m}$-primary ideal I, where $e(I, M)$ denotes the multiplicity of M along I.
(2) If L is an R-module of finite length, then we have the following inequalities of lengths for any integer i:

$$\begin{cases} \operatorname{length}_R\big(\operatorname{Ext}^i_R(L, M)\big) \leqq \operatorname{length}_R\big(\operatorname{Ext}^i_R(L, N)\big), \\ \operatorname{length}_R\big(\operatorname{Ext}^i_R(M, L)\big) \leqq \operatorname{length}_R\big(\operatorname{Ext}^i_R(N, L)\big). \end{cases}$$

See [6, Lemma 4.5]. In particular, when R is a local ring, then

$$\nu(M) \leqq \nu(N), \qquad \beta_i(M) \leqq \beta_i(N) \quad \text{and} \quad \mu^i(M) \leqq \mu^i(N) \quad (i \geqq 0),$$

where ν, β_i and μ^i denote the minimal number of generators, the ith Betti number and the ith Bass number respectively.

Adding to these remarks we can prove the following result concerning Fitting ideals. Before stating the theorem, we recall the definition of the Fitting ideal of a finitely presented module. Suppose that a module M over a commutative ring A is given by a finitely free presentation

$$A^m \xrightarrow{C} A^n \longrightarrow M \longrightarrow 0,$$

where C is an $n \times m$-matrix with entries in A. Then recall that the ith Fitting ideal $\mathcal{F}_i^A(M)$ of M is defined to be the ideal $I_{n-i}(C)$ of A generated by all the $(n-i)$-minors of the matrix C. (We use the convention that $I_r(C) = A$ for $r \leqq 0$ and $I_r(C) = 0$ for $r > \min\{m, n\}$.) It is known that $\mathcal{F}_i^A(M)$ depends only on M and i, and independent of the choice of free presentation. The following lemma will be used to prove the theorem.

Lemma 2.4. *Let $f : A \to B$ be a ring homomorphism and let M be an A-module which possesses a finitely free presentation. Then $\mathcal{F}_i^B(M \otimes_A B) = f(\mathcal{F}_i^A(M))B$ for all $i \geqq 0$.*

Proof. If M has a presentation

$$A^m \xrightarrow{(c_{ij})} A^n \longrightarrow M \longrightarrow 0,$$

then $M \otimes_A B$ has a presentation

$$B^m \xrightarrow{(f(c_{ij}))} B^n \longrightarrow M \otimes_A B \longrightarrow 0.$$

Thus $\mathcal{F}_i^B(M \otimes_A B) = I_{n-i}(f(c_{ij})) = f(I_{n-i}(c_{ij}))B = f(\mathcal{F}_i^A(M))B$. □

Theorem 2.5. *Let R be a noetherian commutative algebra over k, and M and N finitely generated R-modules. Suppose M degenerates to N. Then we have $\mathcal{F}_i^R(M) \supseteqq \mathcal{F}_i^R(N)$ for all $i \geqq 0$.*

Proof. By the assumption there is a finitely generated $R \otimes_k V$-module Q such that $Q_t \cong M \otimes_k K$ and $Q/tQ \cong N$, where $V = k[t]_{(t)}$ and $K = k(t)$. Note that $R \otimes_k V \cong S^{-1}R[t]$ where $S = k[t] \setminus (t)$. Since Q is finitely generated, we can find a finitely generated $R[t]$-module Q' such that $Q' \otimes_{R[t]} (R \otimes_k V) \cong Q$. For a fixed integer i we now consider the Fitting ideal $J := \mathcal{F}_i^{R[t]}(Q') \subseteqq R[t]$. Apply Lemma 2.4 to the ring homomorphism $R[t] \to R = R[t]/tR[t]$, and noting that $Q' \otimes_{R[t]} R \cong N$, we have

$$\mathcal{F}_i^R(N) = J + tR[t]/tR[t] \tag{2.1}$$

as an ideal of $R = R[t]/tR[t]$. On the other hand, applying Lemma 2.4 to $R[t] \to R \otimes_k K = T^{-1}R[t]$ where $T = k[t] \setminus \{0\}$, we have $\mathcal{F}_i^R(M)T^{-1}R[t] = JT^{-1}R[t]$. Therefore there is an element $f(t) \in T$ such that $f(t)J \subseteq \mathcal{F}_i^R(M)R[t]$.

Now to prove the inclusion $\mathcal{F}_i^R(N) \subseteqq \mathcal{F}_i^R(M)$, take an arbitrary element $a \in \mathcal{F}_i^R(N)$. It follows from (2.1) that there is a polynomial of the form $a + b_1t + b_2t^2 + \cdots + b_rt^r$ ($b_i \in R$) that belongs to J. Then, we have $f(t)(a + b_1t + b_2t^2 + \cdots + b_rt^r) \in \mathcal{F}_i^R(M)R[t]$. Since $f(t)$ is a non-zero polynomial whose coefficients are all in k, looking at the coefficient of the non-zero term of the least degree in the polynomial $f(t)(a + b_1t + \cdots + b_rt^r)$, we have that $a \in \mathcal{F}_i^R(M)$. □

3. Remarks for the rings $R \otimes_k K$

In this paper we are interested in the stable analogue of degenerations of Cohen–Macaulay modules over a commutative Gorenstein local ring. For this purpose, all rings considered in the rest of the paper are commutative. Furthermore $(R, \mathfrak{m}, k)$ always denotes a Gorenstein local ring which is a k-algebra, and $V = k[t]_{(t)}$ and $K = k(t)$ where t is a variable.

We note that $R \otimes_k V$ and $R \otimes_k K$ are not necessarily local rings but they are Gorenstein rings.

Remark 3.1. Let $S = \{f(t) \in k[t] \mid f(0) \neq 0\} = k[t] \setminus (t)$ and $T = k[t] \setminus \{0\}$, which are multiplicatively closed subsets of $k[t]$, hence of $R[t]$. Then, by definition, we have

$$R \otimes_k V = S^{-1}R[t], \qquad R \otimes_k K = T^{-1}R[t].$$

In particular, both rings are noetherian. Furthermore, there are natural mappings $R \to R \otimes_k V$ and $R \to R \otimes_k K$, which are flat ring homomorphisms. For any $\mathfrak{p} \in \operatorname{Spec}(R)$, the fibers of these homomorphisms are

$$\kappa(\mathfrak{p}) \otimes_R (R \otimes_k V) = S^{-1}\kappa(\mathfrak{p})[t], \qquad \kappa(\mathfrak{p}) \otimes_R (R \otimes_k K) = T^{-1}\kappa(\mathfrak{p})[t],$$

which are regular rings. Hence we see that $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein as well as R. At the same time, we have the equality of Krull dimension,

$$\dim R \otimes_k V = \dim R + 1, \qquad \dim R \otimes_k K = \dim R.$$

If $\dim R = 0$ (i.e. R is artinian), then the rings $R \otimes_k V$ and $R \otimes_k K$ are local. In fact, the ideal $\mathfrak{m}(R \otimes_k V)$ of $R \otimes_k V$ is nilpotent, and $(R \otimes_k V)/\mathfrak{m}(R \otimes_k V) \cong V$, hence $(\mathfrak{m}, t)(R \otimes_k V)$ is a unique maximal ideal of $R \otimes_k V$. By the same reason, $\mathfrak{m}(R \otimes_k K)$ is a unique maximal ideal of $R \otimes_k K$.

However we should note that $R \otimes_k V$ and $R \otimes_k K$ will never be local rings if $\dim R > 0$. Actually, if there is a prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$, then taking an $x \in R$ so that $x \notin \mathfrak{p}$, we have a maximal ideal $(\mathfrak{p}, xt - 1)R \otimes_k V$ (resp. $(\mathfrak{p}, xt - 1)R \otimes_k K$), which is distinct from the maximal ideal $(\mathfrak{m}, t)R \otimes_k V$ (resp. $\mathfrak{m}(R \otimes_k K)$).

Since $R \otimes_k K$ is non-local, there may be a lot of projective modules which are not free. The following example gives such one of them.

Example 3.2. Let $R = k[\![x, y]\!]/(x^3 - y^2)$. It is known that the maximal ideal $\mathfrak{m} = (x, y)$ is a unique non-free indecomposable Cohen–Macaulay module over R. See [5, Proposition 5.11]. In fact it is given by a matrix factorization of the polynomial $x^3 - y^2$:

$$(\varphi, \psi) = \left(\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}, \begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix} \right).$$

Therefore there is an exact sequence

$$R^2 \xrightarrow{\varphi} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Now we deform these matrices and consider the pair of matrices over $R \otimes_k K$:

$$(\Phi, \Psi) = \left(\begin{pmatrix} y - xt & x - t^2 \\ x^2 & y + xt \end{pmatrix}, \begin{pmatrix} y + xt & -x + t^2 \\ -x^2 & y - xt \end{pmatrix} \right).$$

Define the $R \otimes_k K$-module P by the following exact sequence:

$$(R \otimes_k K)^2 \xrightarrow{\Phi} (R \otimes_k K)^2 \longrightarrow P \longrightarrow 0.$$

We claim that P is a projective module of rank one over $R \otimes_k K$ but non-free. (Hence the Picard group of $R \otimes_k K$ is non-trivial.)

To prove that P is projective of rank one. let $\mathfrak{P}$ be a prime ideal of $R \otimes_k K$. If $x \notin \mathfrak{P}$, then x^2 is a unit in $(R \otimes_k K)_{\mathfrak{P}}$, therefore the matrix $\Phi_{\mathfrak{P}}$ over $(R \otimes_k K)_{\mathfrak{P}}$ is equivalent to $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ under elementary transformations, hence $P_{\mathfrak{P}} = \operatorname{Coker}(\Phi_{\mathfrak{P}}) \cong \operatorname{Coker}(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)) \cong (R \otimes_k K)_{\mathfrak{P}}$. If $x \in \mathfrak{P}$, then $x - t^2$ is a unit in $(R \otimes_k K)_{\mathfrak{P}}$, hence by the same reason we have $P_{\mathfrak{P}} \cong (R \otimes_k K)_{\mathfrak{P}}$. Therefore P is projective of rank one.

The point is to prove that P is non-free. Assume that P is a free $R \otimes_k K$-module, then $P \cong R \otimes_k K$, since it is of rank one. Let Q be an $R \otimes_k V$-module defined by the same matrix Φ:

$$(R \otimes_k V)^2 \xrightarrow{\Phi} (R \otimes_k V)^2 \longrightarrow Q \longrightarrow 0.$$

Since the pair of matrices (Φ, Ψ) gives a matrix-factorization of the polynomial $x^3 - y^2$ over the regular ring $k[\![x, y]\!] \otimes_k V$, it is easy to see that there is an exact sequence

$$\cdots \xrightarrow{\Psi} (R \otimes_k V)^2 \xrightarrow{\Phi} (R \otimes_k V)^2 \xrightarrow{\Psi} (R \otimes_k V)^2 \xrightarrow{\Phi} (R \otimes_k V)^2 \xrightarrow{\Psi} \cdots$$

In particular Q is a Cohen–Macaulay module over $R \otimes_k V$, hence it is flat over V. On the other hand, clearly we have $Q_t \cong P \cong R \otimes_k K$ and $Q/tQ \cong \operatorname{Coker}(\varphi) = \mathfrak{m}$. Hence we must have that R degenerates to $\mathfrak{m}$. This contradicts to the following proposition.

Proposition 3.3. *Let R be an integral domain and N a finitely generated torsion-free R-module. If R degenerates to N, then R is isomorphic to N.*

Proof. It follows from Theorem 2.2 that there is an exact sequence

$$0 \longrightarrow Z \xrightarrow{\binom{\phi}{\psi}} R \oplus Z \xrightarrow{(\alpha\ \beta)} N \to 0,$$

where ψ is a nilpotent endomorphism of Z. We take such a short exact sequence so that $\operatorname{rank} Z$ is minimal. (Recall that $\operatorname{rank} Z$ is the dimension of $Z \otimes_R K_R$ over K_R, where K_R is the quotient field of R.) If $Z = 0$ then clearly $R \cong N$. Therefore we assume that $Z \neq 0$, and we shall show a contradiction.

(1) If $Z \neq 0$ then $\alpha = 0$.

In fact, if $\alpha \neq 0$, then the mapping $\alpha : R \to N$ is an injection, since N is torsion-free over R. In such a case, suppose $\psi(z) = 0$ for $z \in Z$. Then, since $\alpha(\phi(z)) + \beta(\psi(z)) = 0$, we have $\alpha(\phi(z)) = 0$ hence $\phi(z) = 0$. Thus we have $\binom{\phi}{\psi}(z) = 0$. Since $\binom{\phi}{\psi}$ is an injection, we have $z = 0$. This is true for any $z \in \operatorname{Ker}(\psi)$, hence ψ must be an injection. However, since ψ is a nilpotent endomorphism of Z, we must have $Z = 0$.

(2) The restriction $\phi|_{\operatorname{Ker}(\psi)} : \operatorname{Ker}(\psi) \to R$ of ϕ is a surjection.

Let $r \in R$ be any element. Since we have shown that $\alpha = 0$, the element $\binom{r}{0}$ of $R \oplus Z$ belongs to $\operatorname{Ker}(\alpha, \beta)$. Thus there is an element $z \in Z$ such that $\binom{\phi}{\psi}(z) = \binom{r}{0}$, hence $z \in \operatorname{Ker}(\psi)$ and $\phi(z) = r$.

By claim (2) we have a homomorphism $\lambda : R \to \mathrm{Ker}(\psi)$ such that $\phi \cdot \lambda = 1_R$. Since $\psi \cdot \lambda = 0$, we have a commutative diagram

$$
\begin{array}{ccc}
R & = & R \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \binom{1}{0}} \\
Z & \xrightarrow{\binom{\phi}{\psi}} & R \oplus Z.
\end{array}
$$

It then follows that there is a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R & = & R & & & & \\
 & & {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \binom{1}{0}} & & & & \\
0 & \longrightarrow & Z & \xrightarrow{\binom{\phi}{\psi}} & R \oplus Z & \longrightarrow & N & \longrightarrow & 0 \\
 & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle (0\ 1)} & & \| & & \\
0 & \longrightarrow & Z' & \xrightarrow{f} & Z & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $Z' = Z/\lambda(R)$. Since λ is a splitting monomorphism, we have $Z \cong R \oplus Z'$ and the third row can be written as follows:

$$
0 \longrightarrow Z' \xrightarrow{\binom{\phi'}{\psi'}} R \oplus Z' \longrightarrow N \longrightarrow 0,
$$

where $\psi' = \pi \cdot f$. On the other hand it follows from the commutative diagram above that $f \cdot \pi = \psi$. Therefore $(\psi')^{n+1} = (\pi \cdot f) \cdot (\pi \cdot f) \cdots (\pi \cdot f) = \pi \cdot \psi^n \cdot f$ for any $n \in \mathbb{N}$. Thus ψ' is nilpotent as well as ψ. Since $\operatorname{rank} Z' < \operatorname{rank} Z$, this contradicts the choice of Z. $\square$

4. Definition and properties of stable degeneration

Let A be a commutative Gorenstein ring which is not necessarily local. We say that a finitely generated A-module M is Cohen–Macaulay if $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. We consider the category of all Cohen–Macaulay modules over A with all A-module homomorphisms:

$$
\mathrm{CM}(A) := \big\{ M \in \mathrm{mod}(A) \bigm| M \text{ is a Cohen–Macaulay module over } A \big\},
$$

where $\mathrm{mod}(A)$ denotes the category of all finitely generated A-modules. We can then consider the stable category of $\mathrm{CM}(A)$, which we denote by $\underline{\mathrm{CM}}(A)$. Recall that the objects of $\underline{\mathrm{CM}}(A)$ is Cohen–Macaulay modules over A, and the morphisms of $\underline{\mathrm{CM}}(A)$ are elements of $\underline{\mathrm{Hom}}_A(M, N) := \mathrm{Hom}_A(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(A)$, where $P(M, N)$ denotes the set of morphisms from M to N factoring through projective A-modules. For a Cohen–Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(A)$. Note that $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{CM}}(A)$ if and only if there are projective A-modules P_1 and P_2 such that $M \oplus P_1 \cong N \oplus P_2$ in $\mathrm{CM}(A)$.

Under such circumstances it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category. In fact, if $L \in \mathrm{CM}(A)$ then we can embed L into a projective A-module P such that the quotient P/L, which we denote by $\Omega^{-1}L$, is Cohen–Macaulay as well. We define the shift functor in $\underline{\mathrm{CM}}(A)$ by $\underline{L}[1] = \underline{\Omega^{-1}L}$. If there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{CM}(A)$, then we have the following commutative diagram by taking the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \Omega^{-1}L & \longrightarrow & 0. \end{array}$$

We define the triangles in $\underline{\mathrm{CM}}(A)$ are the sequences

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1]$$

obtained in such a way.

Let $x \in A$ be a non-zero divisor on A. Note that x is a non-zero divisor on every Cohen–Macaulay module over A. Thus the functor $- \otimes_A A/xA$ sends a Cohen–Macaulay module over A to that over A/xA. Therefore it yields a functor $\mathrm{CM}(A) \to \mathrm{CM}(A/xA)$. Since this functor maps projective A-modules to projective A/xA-modules, it induces the functor $\mathcal{R} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(A/xA)$. It is easy to verify that $\mathcal{R}$ is a triangle functor.

Now let $S \subset A$ be a multiplicative subset of A. Then, by a similar reason to the above, we have a triangle functor $\mathcal{L} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(S^{-1}A)$ which maps $\underline{M}$ to $\underline{S^{-1}M}$.

As before, let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring that is a k-algebra and let $V = k[t]_{(t)}$ and $K = k(t)$. Since $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein rings, we can apply the observation above. Actually, $t \in R \otimes_k V$ is a non-zero divisor on $R \otimes_k V$ and there are isomorphisms of k-algebras; $(R \otimes_k V)/t(R \otimes_k V) \cong R$ and $(R \otimes_k V)_t \cong R \otimes_k K$. Thus there are triangle functors $\mathcal{L} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R \otimes_k K)$ defined by the localization by t, and $\mathcal{R} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R)$ defined by taking $- \otimes_{R \otimes_k V} (R \otimes_k V)/t(R \otimes_k V) = - \otimes_V V/tV$. Now we define the stable degeneration of Cohen–Macaulay modules.

Definition 4.1. Let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ **stably degenerates to** $\underline{N}$ if there is a Cohen–Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

The following are easily verified.

Lemma 4.2. *Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$.*

Proof. Suppose M degenerates to N. Then there is $Q \in \mathrm{mod}(R \otimes_k V)$ that is V-flat and satisfies $Q/tQ \cong N$ and $Q_t \cong M \otimes_k K$. Note in this case that Q is a Cohen–Macaulay $R \otimes_k V$-module, i.e. $Q \in \mathrm{CM}(R \otimes_k V)$. In fact, if a prime ideal $\mathfrak{P}$ of $R \otimes_k V$ contains t, then, since $Q_{\mathfrak{P}}/tQ_{\mathfrak{P}} = (Q/tQ)_{\mathfrak{P}}$ is Cohen–Macaulay and since t is a non-zero divisor on Q, we see that $Q_{\mathfrak{P}}$ is a Cohen–Macaulay $R_{\mathfrak{P}}$-module. On the other hand, if $t \notin \mathfrak{P}$, then $Q_{\mathfrak{P}}$ is a Cohen–Macaulay $R_{\mathfrak{P}}$-module as well, since it is a localization of the Cohen–Macaulay $R \otimes_k K$-module Q_t. Thus we have $Q \in \mathrm{CM}(R \otimes_k V)$. Then it is clear that $\mathcal{R}(\underline{Q}) \cong \underline{N}$ and $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$. □

Proposition 4.3. *Suppose that there is a triangle*

$$\underline{L} \xrightarrow{\ \alpha\ } \underline{M} \xrightarrow{\ \beta\ } \underline{N} \xrightarrow{\ \gamma\ } \underline{L}[1],$$

in $\underline{\mathrm{CM}}(R)$. Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$.

Proof. Note that γ is an element of $\underline{\mathrm{Hom}}_R(N, \Omega^{-1}L)$, which is naturally a submodule of $\underline{\mathrm{Hom}}_R(N, \Omega^{-1}L) \otimes_k V \cong \underline{\mathrm{Hom}}_{R\otimes_k V}(N \otimes_k V, \Omega^{-1}L \otimes_k V)$. Regarding γ as an element of $\underline{\mathrm{Hom}}_{R\otimes_k V}(N \otimes_k V, \Omega^{-1}L \otimes_k V)$, we have a morphism $\gamma \otimes t : \underline{N \otimes_k V} \to \underline{L \otimes_k V}[1]$ in $\underline{\mathrm{CM}}(R \otimes_k V)$. (The morphism $\gamma \otimes t$ is a composition of $\gamma : \underline{N \otimes_k V} \to \underline{L \otimes_k V}[1]$ with the multiplication map by t.) Now embed $\gamma \otimes t$ into a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$, and we get $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ with the triangle

$$\underline{L \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{N \otimes_k V} \xrightarrow{\gamma\otimes t} \underline{L \otimes_k V}[1].$$

Notice that $\mathcal{R}(\gamma \otimes t) = 0$, since the multiplication map by t is zero in $\underline{\mathrm{CM}}(R)$. Therefore applying the triangle functor $\mathcal{R}$ to the triangle above and noting that $\mathcal{R}(\underline{X \otimes_k V}) = \underline{X}$ for any $\underline{X} \in \underline{\mathrm{CM}}(R)$, we have an isomorphism $\mathcal{R}(\underline{Q}) \cong \underline{L} \oplus \underline{N}$.

On the other hand, there is an isomorphism of triangles in $\underline{\mathrm{CM}}(R \otimes_k K)$;

$$\begin{array}{ccccccc}
\underline{L \otimes_k K} & \xrightarrow{\alpha\otimes 1} & \underline{M \otimes_k K} & \xrightarrow{\beta\otimes 1} & \underline{N \otimes_k K} & \xrightarrow{\gamma\otimes 1} & \underline{L \otimes_k K}[1] \\
{\scriptstyle t}\downarrow{\scriptstyle \cong} & & \downarrow & & \| & & {\scriptstyle t}\downarrow{\scriptstyle \cong} \\
\underline{L \otimes_k K} & \longrightarrow & \underline{Q_t} & \longrightarrow & \underline{N \otimes_k K} & \xrightarrow{\gamma\otimes t} & \underline{L \otimes_k K}[1].
\end{array}$$

Hence we have that $\mathcal{L}(\underline{Q}) = \underline{Q_t} \cong \underline{M \otimes_k K}$. As a consequence, $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$. $\square$

Proposition 4.4. *Let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$ and suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then the following hold.*

(1) *$\underline{M}[1]$ (resp. $\underline{M}[-1]$) stably degenerates to $\underline{N}[1]$ (resp. $\underline{N}[-1]$).*
(2) *$\underline{M^*}$ stably degenerates to $\underline{N^*}$, where M^* denotes the R-dual $\mathrm{Hom}_R(M, R)$.*

Proof. By the assumption, there is $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

To prove (1), consider $\underline{Q}[1]$ in $\underline{\mathrm{CM}}(R \otimes_k V)$. Then, since $\mathcal{L}$ and $\mathcal{R}$ are triangle functors, we have $\mathcal{L}(\underline{Q}[1]) \cong \underline{M \otimes_k K}[1] \cong \underline{M[1] \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}[1]) \cong \underline{N}[1]$ in $\underline{\mathrm{CM}}(R)$. This shows that $\underline{M}[1]$ stably degenerates to $\underline{N}[1]$.

To prove (2) we set $\tilde{Q} = \mathrm{Hom}_{R\otimes_k V}(Q, R \otimes_k V)$, and consider $\underline{\tilde{Q}} \in \underline{\mathrm{CM}}(R \otimes_k V)$. Then it is easy to see that there are isomorphisms; $\tilde{Q}_t \cong \mathrm{Hom}_{R\otimes_k K}(Q_t, R \otimes_k K)$ and $\tilde{Q}/t\tilde{Q} \cong \mathrm{Hom}_R(Q/tQ, R)$. Thus $\mathcal{L}(\underline{\tilde{Q}}) \cong \underline{\mathrm{Hom}_{R\otimes_k K}(M \otimes_k K, R \otimes_k K)} \cong \underline{\mathrm{Hom}_R(M, R) \otimes_k K}$ and $\mathcal{R}(\underline{\tilde{Q}}) \cong \underline{\mathrm{Hom}_R(N, R)}$. Therefore $\underline{M^*}$ stably degenerates to $\underline{N^*}$. $\square$

Proposition 4.5. *Let $\underline{M}, \underline{N}, \underline{X} \in \underline{\mathrm{CM}}(R)$. If $\underline{M} \oplus \underline{X}$ stably degenerates to $\underline{N}$, then $\underline{M}$ stably degenerates to $\underline{N} \oplus \underline{X}[1]$.*

Proof. Take $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ satisfying that $\mathcal{L}(\underline{Q}) \cong \underline{(M \oplus X) \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

First of all we note that there is a natural isomorphism

$$\underline{\mathrm{Hom}}_{R\otimes_k V}\big((M \oplus X) \otimes_k V, Q\big)_t \cong \underline{\mathrm{Hom}}_{R\otimes_k K}\big((M \oplus X) \otimes_k K, Q_t\big).$$

Thus there is a morphism $\alpha : \underline{(M \oplus X) \otimes_k V} \to \underline{Q}$ in $\underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\alpha) = \alpha_t$ is an isomorphism in $\underline{\mathrm{CM}}(R \otimes_k K)$. Here, replacing α with $t\alpha$ if necessary, we may assume that $\mathcal{R}(\alpha) = 0$. Now let $\beta : \underline{X \otimes_k V} \to \underline{(M \otimes_k V) \oplus (X \otimes_k V)} \cong \underline{(M \oplus X) \otimes_k V}$ be a natural splitting monomorphism and we set $\gamma = \alpha \cdot \beta$. Then we can embed the morphism γ into a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$:

$$\underline{X \otimes_k V} \xrightarrow{\gamma} \underline{Q} \longrightarrow \underline{Q'} \longrightarrow \underline{X \otimes_k V}[1].$$

Since $\mathcal{R}(\gamma) = \mathcal{R}(\alpha) \cdot \mathcal{R}(\beta) = 0$, we have that

$$\mathcal{R}(\underline{Q'}) \cong \mathcal{R}(\underline{Q}) \oplus \mathcal{R}(\underline{X \otimes_k V}[1]) \cong \underline{N} \oplus \underline{X}[1].$$

On the other hand, since there is a triangle in $\underline{\mathrm{CM}}(R \otimes_k K)$;

$$\underline{X \otimes_k K} \xrightarrow{\mathcal{L}(\beta)} \underline{(M \oplus X) \otimes_k K} \longrightarrow \underline{M \otimes_k K} \longrightarrow \underline{X \otimes_k K}[1],$$

noting that $\mathcal{L}(\alpha) : \underline{(M \oplus X) \otimes_k K} \to \underline{Q_t}$ is an isomorphism, we have that $\mathcal{L}(\underline{Q'}) \cong \underline{M \otimes_k K}$. This shows that $\underline{M}$ stably degenerates to $\underline{N} \oplus \underline{X}[1]$. □

Remark 4.6. The zero object in $\underline{\mathrm{CM}}(R)$ can stably degenerate to a non-zero object. For example, there is a triangle

$$\underline{X} \longrightarrow \underline{0} \longrightarrow \underline{X}[1] \xrightarrow{1} \underline{X}[1].$$

for any $\underline{X} \in \underline{\mathrm{CM}}(R)$. Hence $\underline{0}$ stably degenerates to $\underline{X} \oplus \underline{X}[1]$ by Proposition 4.3.

5. Conditions for stable degeneration

Let $(R, \mathfrak{m}, k)$ be a Gorenstein local k-algebra as before. The main purpose of this section is to prove the following theorem.

Theorem 5.1. *Consider the following three conditions for Cohen–Macaulay R-modules M and N:*

(1) *$R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.*
(2) *There is a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$ in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.*
(3) *$\underline{M}$ stably degenerates to $\underline{N}$.*

Then, in general, the implications (1) ⇒ (2) ⇒ (3) hold.

Furthermore, if R is artinian, then the conditions (1), (2) and (3) are all equivalent.

In the next section we shall prove that the implication (3) ⇒ (2) holds if R is a Gorenstein complete local ring with only an isolated singularity (Theorem 6.1).

Proof. (1) ⇒ (2): Suppose that $R^m \oplus M$ degenerates to $R^n \oplus N$. Then by Theorem 2.2, we have a short exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} (R^m \oplus M) \oplus Z \to (R^n \oplus N) \to 0,$$

where ψ is nilpotent. In such a case Z is a Cohen–Macaulay module as well. See [7, Remark 4.3.1] or [8, Proof of Theorem 3.2]. Then converting this short exact sequence into a triangle in $\underline{\mathrm{CM}}(R)$, we have $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$, where, since $\psi \in \mathrm{End}_R(Z)$ is nilpotent, $\underline{\psi} \in \underline{\mathrm{End}}_R(Z)$ is nilpotent as well.

(2) ⇒ (3): Through the natural injective homomorphism

$$\underline{\mathrm{Hom}}_R(Z, M) \hookrightarrow \underline{\mathrm{Hom}}_{R\otimes_k V}(Z \otimes_k V, M \otimes_k V),$$
$$\underline{\phi} \mapsto \underline{\phi \otimes 1_V}$$

we regard $\underline{\phi}$ as a morphism in $\underline{\mathrm{CM}}(R \otimes_k V)$. Likewise $\underline{\psi}$ is regarded as a morphism in $\underline{\mathrm{CM}}(R \otimes_k V)$ which is nilpotent as well. Note that there is a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$;

$$\underline{Z \otimes_k V} \xrightarrow{\binom{\phi}{\psi}} \underline{M \otimes_k V} \oplus \underline{Z \otimes_k V} \longrightarrow \underline{N \otimes_k V} \longrightarrow \underline{Z \otimes_k V}[1].$$

Now consider a morphism $t + \underline{\psi} : \underline{Z \otimes_k V} \to \underline{Z \otimes_k V}$, and we have a triangle of the form;

$$\underline{Z \otimes_k V} \xrightarrow{\binom{\phi}{t+\psi}} \underline{M \otimes_k V} \oplus \underline{Z \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{Z \otimes_k V}[1],$$

for some $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$. Note that $\mathcal{L}(t + \underline{\psi})$ is an isomorphism in $\underline{\mathrm{CM}}(R \otimes_k K)$, since $t \in R \otimes_k K$ is a unit and $\mathcal{L}(\underline{\psi})$ is a nilpotent morphism. Thus, applying the functor $\mathcal{L}$ to the triangle above, we have that $\mathcal{L}(\underline{Q}) \cong \mathcal{L}(\underline{M \otimes_k V}) = \underline{M \otimes_k K}$. On the other hand, since $\mathcal{R}(t + \underline{\psi}) = \underline{\psi}$, we see that $\mathcal{R}(\underline{Q}) \cong N$. Thus $\underline{M}$ stably degenerates to $\underline{N}$.

(3) ⇒ (1): In this proof we assume that $\dim R = 0$. Since $\underline{M}$ stably degenerates to $\underline{N}$, there is $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$. By definition, we have isomorphisms $Q_t \oplus P_1 \cong (M \otimes_k K) \oplus P_2$ in $\mathrm{CM}(R \otimes_k K)$ for some projective $R \otimes_k K$-modules P_1, P_2, and $Q/tQ \oplus R^a \cong N \oplus R^b$ in $\mathrm{CM}(R)$ for some $a, b \in \mathbb{N}$. As we have remarked in Remark 3.1, $R \otimes_k K$ is a local ring, hence P_1 and P_2 are free. Thus $Q_t \oplus (R \otimes_k K)^c \cong (M \otimes_k K) \oplus (R \otimes_k K)^d$ for some $c, d \in \mathbb{N}$.

Now setting $\tilde{Q} = Q \oplus (R \otimes_k V)^{a+c}$, we have isomorphisms

$$\tilde{Q}_t \cong \left(M \oplus R^{a+d}\right) \otimes_k K, \qquad \tilde{Q}/t\tilde{Q} \cong N \oplus R^{b+c}.$$

Noting that Q, hence $\tilde{Q}$, is V-flat, since it is a Cohen–Macaulay module over $R \otimes_k V$, we conclude that $M \oplus R^{a+d}$ degenerates to $N \oplus R^{b+c}$. □

In Theorem 5.1 the implication (2) ⇒ (1) does not holds even in the case when $\dim R = 1$.

Example 5.2. As in Example 3.2, let $R = k[\![x, y]\!]/(x^3 - y^2)$ and let $\mathfrak{m} = (x, y)R$. The ring R can be identified with the subring $k[\![s^2, s^3]\!]$ of the formal power series ring $S = k[\![s]\!]$ by mapping x, y to s^2, s^3 respectively. Note in this case that $\mathrm{End}_R(\mathfrak{m}) = S$. Actually $\mathfrak{m}$ is the conductor of the ring extension $R \subset S$ and hence $\mathfrak{m}$ is identical with an ideal s^2S of S.

In this case, there is an exact sequence

$$0 \longrightarrow \mathfrak{m} \xrightarrow{\binom{j}{s}} R \oplus \mathfrak{m} \xrightarrow{(x\ -s)} \mathfrak{m} \longrightarrow 0,$$

where j is a natural inclusion $\mathfrak{m} \subset R$. This is in fact a unique AR sequence in $\mathrm{CM}(R)$. (See [5, Proposition 5.11].) Note that $\underline{\mathrm{End}}_R(\mathfrak{m}) = S/s^2S$ and there is a triangle

$$\underline{\mathfrak{m}} \xrightarrow{\underline{s}} \underline{\mathfrak{m}} \xrightarrow{\underline{-s}} \underline{\mathfrak{m}} \longrightarrow \underline{\mathfrak{m}}[1],$$

in $\underline{\mathrm{CM}}(R)$. Since $\underline{s} \in \underline{\mathrm{End}}_R(\mathfrak{m}) = S/s^2S$ is nilpotent, the condition (2) in Theorem 5.1 holds for $M = 0$ and $N = \mathfrak{m}$. Hence $\underline{0}$ stably degenerates to $\underline{\mathfrak{m}}$ in this case. However we can prove the following proposition, and hence the condition (1) in Theorem 5.1 does not hold.

Proposition 5.3. *Let $R = k[\![s^2, s^3]\!] \subset S = k[\![s]\!]$ and let $\mathfrak{m} = (s^2, s^3)R$ as above. Then R^m never degenerates to $R^n \oplus \mathfrak{m}$ for $m, n \in \mathbb{N}$.*

Proof. Suppose that R^m degenerates to $R^n \oplus \mathfrak{m}$ for some $m, n \in \mathbb{N}$, and we shall seek a contradiction. Comparing the ranks, we have $m = n + 1$ by Remark 2.3(1). Then it follows from Theorem 2.2 that there is an exact sequence

$$0 \longrightarrow Z \xrightarrow{\binom{\phi}{\psi}} R^{n+1} \oplus Z \longrightarrow R^n \oplus \mathfrak{m} \longrightarrow 0,$$

such that the endomorphism ψ of Z is nilpotent. Note that Z is also a Cohen–Macaulay module over R. Since R and $\mathfrak{m}$ are unique indecomposable Cohen–Macaulay R-modules over R, Z can be described as $R^a \oplus \mathfrak{m}^b$ for some $a, b \in \mathbb{N}$. Therefore the above sequence is described as

$$0 \longrightarrow R^a \oplus \mathfrak{m}^b \xrightarrow{\begin{pmatrix} \phi_1 & \phi_2 \\ \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}} R^{n+1} \oplus R^a \oplus \mathfrak{m}^b \longrightarrow R^n \oplus \mathfrak{m} \longrightarrow 0,$$

where $\phi = (\phi_1\ \phi_2)$ and $\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$. Note that $S = \mathrm{End}_R(\mathfrak{m}) = \mathrm{Hom}_R(\mathfrak{m}, R)$ and $\mathrm{Hom}_R(R, \mathfrak{m}) = \mathfrak{m}$. Thus,

$$\begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \in \begin{pmatrix} R^{a\times a} & S^{a\times b} \\ \mathfrak{m}^{b\times a} & S^{b\times b} \end{pmatrix}.$$

As the first step of the proof we claim that

$$\det(\psi_{22}) \equiv cs \pmod{s^2S}, \tag{5.1}$$

for some $c \in k \setminus \{0\}$.

In fact, converting the short exact sequence above into the stable category, we have the triangle in $\underline{\mathrm{CM}}(R)$;

$$\underline{\mathfrak{m}}^b \xrightarrow{\underline{\psi}} \underline{\mathfrak{m}}^b \longrightarrow \underline{\mathfrak{m}} \longrightarrow \underline{\mathfrak{m}}^b[1],$$

where we should note that $\underline{\psi} = \underline{\psi_{22}}$. Note that $\underline{\mathrm{End}}_R(\mathfrak{m}) = S/s^2S$ and $\underline{\mathfrak{m}}[1] \cong \underline{\mathfrak{m}}$ in $\underline{\mathrm{CM}}(R)$. It is easy to see that there are triangles in $\underline{\mathrm{CM}}(R)$;

$$\begin{cases} \underline{\mathfrak{m}} \xrightarrow{1} \underline{\mathfrak{m}} \longrightarrow \underline{0} \longrightarrow \underline{\mathfrak{m}}, \\ \underline{\mathfrak{m}} \xrightarrow{\underline{s}} \underline{\mathfrak{m}} \longrightarrow \underline{\mathfrak{m}} \longrightarrow \underline{\mathfrak{m}}, \\ \underline{\mathfrak{m}} \xrightarrow{0} \underline{\mathfrak{m}} \longrightarrow \underline{\mathfrak{m}}^{\oplus 2} \longrightarrow \underline{\mathfrak{m}}. \end{cases}$$

Since $\underline{\psi_{22}}$ is an element $(S/s^2S)^{b\times b}$, noticing that the cone of $\underline{\psi_{22}}$ is $\underline{\mathfrak{m}}$, we can make $\underline{\psi_{22}}$ into the following form after elementary transformations of matrices over S/s^2S:

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & s \end{pmatrix}.$$

In particular, $\det(\underline{\psi_{22}})$ is equal to s up to a unit in S/s^2S. Since the natural projection $S \to S/s^2S$ sends $\det(\psi_{22})$ to $\overline{\det}(\underline{\psi_{22}})$, this shows the equality (5.1).

Now let us denote by $\chi_{\psi_{22}}(T)$ the characteristic polynomial of the matrix $\psi_{22} \in S^{b\times b}$. We have from (5.1) that

$$\chi_{\psi_{22}}(T) = \det(TE - \psi_{22}) \equiv T^b + \cdots + (-1)^b cs \quad \bigl(\operatorname{mod} s^2S\bigr). \tag{5.2}$$

Now we consider the matrix $\psi = \left(\begin{smallmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{smallmatrix}\right)$ as an element of $\left(\begin{smallmatrix} S^{a\times a} & S^{a\times b} \\ S^{b\times a} & S^{b\times b} \end{smallmatrix}\right)$, which is a nilpotent matrix as well. Therefore, as for the characteristic polynomial $\chi_\psi(T)$ of ψ we have the equality

$$\chi_\psi(T) = T^{a+b}. \tag{5.3}$$

On the other hand, since every entry of ψ_{21} is in $\mathfrak{m} = s^2S$, we have

$$\psi \equiv \begin{pmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{pmatrix} \quad \bigl(\operatorname{mod} s^2S\bigr),$$

and thus

$$\chi_\psi(T) \equiv \chi_{\psi_{11}}(T) \cdot \chi_{\psi_{22}}(T) \quad \bigl(\operatorname{mod} s^2S\bigr). \tag{5.4}$$

Since every entry of ψ_{11} is in $R = k[\![s^2, s^3]\!]$, we see that ψ_{11} mod s^2S is just a matrix with entries in k. Thus $\chi_{\psi_{11}}(T)$ mod $s^2S \in k[T]$. Combining this observation with (5.2), (5.3) and (5.4), we have the equality of elements in $(S/s^2S)[T]$;

$$T^{a+b} = \bigl(T^a + d_1T^{a-1} + \cdots + d_a\bigr)\bigl(T^b + \cdots + (-1)^b cs\bigr),$$

with $d_i \in k$. Setting $d_0 = 1$ and $\ell = \max\{i \mid 0 \leqslant i \leqslant a,\ d_i \neq 0\}$, we see that the nontrivial term $(-1)^b d_\ell cs T^{a-\ell}$ appears in the right-hand side. This is a contradiction. □

6. The case of isolated singularity

In this section we shall prove the equivalence (2) ⇔ (3) for the conditions in Theorem 5.1 if the Gorenstein local ring is an isolated singularity. The goal of this section is to prove the following theorem.

Theorem 6.1. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local ring that is a k-algebra, and let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$. Assume that R has only an isolated singularity, and k is an infinite field. If $\underline{M}$ stably degenerates to $\underline{N}$, then there is a triangle in $\underline{\mathrm{CM}}(R)$;*

$$\underline{Z} \xrightarrow{\binom{\underline{\phi}}{\underline{\psi}}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1],$$

where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.

To prove this theorem we need some auxiliary lemmas. For the proof of the following lemma the reader should refer to [3, Swan's Lemma 5.1].

Lemma 6.2 *(Swan's Lemma). Let R be a noetherian ring and t a variable. Assume that an $R[t]$-module L is a submodule of $W \otimes_R R[t]$ with W being a finitely generated R-module. Then, there is an exact sequence of $R[t]$-modules;*

$$0 \longrightarrow X \otimes_R R[t] \longrightarrow Y \otimes_R R[t] \longrightarrow L \longrightarrow 0,$$

where X and Y are finitely generated R-modules.

Lemma 6.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian local k-algebra, where k is an infinite field, and let $V = k[t]_{(t)}$ and $K = k(t)$ as before. Suppose that a finitely generated $R \otimes_k V$-module P' satisfies the following conditions:*

(1) P' is a submodule of a finitely generated free $R \otimes_k V$-module,
(2) the localization $P = P'_t$ by t is a projective $R \otimes_k K$-module.

Then there is a short exact sequence of $R \otimes_k V$-modules;

$$0 \longrightarrow X \otimes_k V \longrightarrow (X \otimes_k V) \oplus (R^n \otimes_k V) \longrightarrow P' \longrightarrow 0,$$

where X is a finitely generated R-module and n is a non-negative integer.

Proof. Recall that $R \otimes_k V = S^{-1}R[t]$ and $R \otimes_k K = T^{-1}R[t]$ where $S = k[t] \setminus (t)$ and $T = k[t] \setminus \{0\}$. Since P' is a finitely generated $S^{-1}R[t]$-module, we find a finitely generated $R[t]$-submodule L of P' which satisfies $S^{-1}L = P'$. By the assumption, we may assume $P' \subseteqq S^{-1}R[t]^r$ for some $r \geqq 0$. Replacing L with $L \cap R[t]^r$ if necessary, we may take a submodule of a free $R[t]$-module as such an L. Thus we can apply Swan's Lemma 6.2 to L and we have an exact sequence of $R[t]$-modules

$$0 \longrightarrow X \otimes_R R[t] \longrightarrow Y \otimes_R R[t] \longrightarrow L \longrightarrow 0, \tag{6.1}$$

where X, Y are finitely generated R-modules.

We note that there is a polynomial $f(t) \in T$ such that $L_{f(t)}$ is projective over $R[t]_{f(t)}$. In fact, taking an epimorphism from a free module to L, we have an exact sequence $0 \to L_1 \to R[t]^m \to L \to 0$. This sequence will split if it is localized by T, since $P = T^{-1}L$ is projective. Therefore it splits after localizing it by some element $f(t) \in T$.

Now we localize the exact sequence (6.1) by $f(t) \in T$, and then we have an isomorphism of $R[t]_{f(t)}$-modules

$$Y \otimes_R R[t]_{f(t)} \cong L_{f(t)} \oplus \big(X \otimes_R R[t]_{f(t)}\big). \tag{6.2}$$

Since we assume that k is an infinite field, there is an element $c \in k$ such that $f(c) \neq 0$. Consider the R-algebra homomorphism $\sigma : R[t]_{f(t)} \to R$ which sends t to c, and we regard R as an $R[t]_{f(t)}$-algebra through σ. We note that $(X \otimes_R R[t]_{f(t)}) \otimes_{R[t]_{f(t)}} R \cong X$ for any R-module X. Hence, taking $- \otimes_{R[t]_{f(t)}} R$ with (6.2), we have

$$Y \cong (L_{f(t)} \otimes_{R[t]_{f(t)}} R) \oplus X.$$

Noting that $L_{f(t)}$ is a projective $R[t]_{f(t)}$-module, we see that $L_{f(t)} \otimes_{R[t]_{f(t)}} R$ is a projective R-module as well. Hence $L_{f(t)} \otimes_{R[t]_{f(t)}} R \cong R^n$ as R-modules for some $n \geqslant 0$, since R is local. Therefore we have $Y \cong R^n \oplus X$. Substituting this into (6.1) and taking the localization by S, we have a short exact sequence

$$0 \longrightarrow X \otimes_k V \longrightarrow (X \otimes_k V) \oplus (R^n \otimes_k V) \longrightarrow P' \longrightarrow 0. \qquad \square$$

Proposition 6.4. *Let R be a Gorenstein local k-algebra, where k is an infinite field. Suppose we are given a Cohen–Macaulay $R \otimes_k V$-module P' satisfying that the localization $P = P'_t$ by t is a projective $R \otimes_k K$-module. Then there is a Cohen–Macaulay R-module X with a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$ of the following form:*

$$\underline{X \otimes_k V} \longrightarrow \underline{X \otimes_k V} \longrightarrow \underline{P'} \longrightarrow \underline{X \otimes_k V}[1]. \tag{6.3}$$

Proof. Let $\ell = \dim R + 1 = \dim R \otimes_k V$, and we note that P' is an ℓth syzygy module of a Cohen–Macaulay $R \otimes_k V$-module P''. ($R \otimes_k V$ is a Gorenstein ring and every Cohen–Macaulay module over a Gorenstein ring is a syzygy module of a Cohen–Macaulay module.) In such a situation, P''_t is a projective $R \otimes_k K$-module, since so is P'_t. Therefore we can apply Lemma 6.3 to P'' to get a short exact sequence

$$0 \longrightarrow X \otimes_k V \longrightarrow (X \otimes_k V) \oplus (R^n \otimes_k V) \longrightarrow P'' \longrightarrow 0,$$

where X is just a finitely generated R-module. Now take the ℓth syzygy of this sequence. We notice that the ℓth syzygy module $\Omega^{\ell}_{R \otimes_k V}(X \otimes_k V)$ is isomorphic, as an object of $\underline{\mathrm{CM}}(R \otimes_k V)$, to $\Omega^{\ell}_R(X) \otimes_k V$ which is a Cohen–Macaulay $R \otimes_k V$-module. In this way we obtain a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$;

$$\underline{\Omega^{\ell}_R X \otimes_k V} \longrightarrow \underline{\Omega^{\ell}_R X \otimes_k V} \longrightarrow \underline{\Omega^{\ell}_{R \otimes_k V} P''} \longrightarrow \underline{\Omega^{\ell}_R X \otimes_k V}[1].$$

Setting $P' = \Omega^{\ell}_{R \otimes_k K} P''$ and replacing $\Omega^{\ell}_R X$ with X, we have a desired triangle. $\square$

The following lemma is an analogue, or one might say a higher-dimensional version, of the Fitting lemma.

Lemma 6.5. *Let R be a Gorenstein complete local ring which has only an isolated singularity, and let $\underline{X} \in \underline{\mathrm{CM}}(R)$. Given an endomorphism $\underline{\psi} \in \underline{\mathrm{End}}_R(X)$, we have a direct decomposition $\underline{X} = \underline{X_1} \oplus \underline{X_2}$ and automorphisms $\underline{\alpha}$, $\underline{\beta}$ of $\underline{X}$ such that*

$$\underline{\alpha} \cdot \underline{\psi} \cdot \underline{\beta} = \begin{pmatrix} \underline{\psi_1} & 0 \\ 0 & \underline{\psi_2} \end{pmatrix},$$

where $\underline{\psi_1} : \underline{X_1} \to \underline{X_1}$ is an automorphism and $\underline{\psi_2} : \underline{X_2} \to \underline{X_2}$ is a nilpotent endomorphism.

Proof. Recall that $\mathrm{CM}(R)$ is a Krull-Schmidt category and so is $\underline{\mathrm{CM}}(R)$. Therefore $\underline{X}$ is uniquely decomposed into a direct sum of indecomposable objects; $\underline{X} \cong \underline{Y_1} \oplus \cdots \oplus \underline{Y_n}$. According to this decomposition, $\underline{\psi}$ is described as an $n \times n$-matrix $(\underline{\psi_{ij}})$, where $\underline{\psi_{ij}} \in \underline{\mathrm{Hom}}_R(Y_j, Y_i)$. If there is an isomorphism $\underline{\psi_{ij}}$, then $\underline{\psi}$ is arranged into the form $\left(\begin{smallmatrix} \underline{\psi_{ij}} & 0 \\ 0 & * \end{smallmatrix}\right)$, more precisely, there are automorphisms $\underline{\alpha_1}$ and $\underline{\beta_1}$ of $\underline{X}$ such that $\underline{\alpha_1} \cdot \underline{\psi} \cdot \underline{\beta_1} = \left(\begin{smallmatrix} \underline{\psi_{ij}} & 0 \\ 0 & * \end{smallmatrix}\right)$. Hence, by induction on n, it is enough to prove that $\underline{\psi}$ is nilpotent if all $\underline{\psi_{ij}}$ are non-isomorphic. For an integer N, each entry of the matrix $(\underline{\psi_{ij}})^N$ is a composition of morphisms; $\underline{Y_j} = \underline{Z_0} \to \underline{Z_1} \to \cdots \to \underline{Z_N} = \underline{Y_i}$, where each $\underline{Z_k}$ is one of $\underline{Y_1}, \ldots, \underline{Y_n}$. Take an integer ℓ satisfying $(\mathrm{rad}\, \underline{\mathrm{End}}_R(Y_i))^{\ell} = 0$ for $i = 1, 2, \ldots, n$. Note that this is possible, since each $\underline{\mathrm{End}}_R(Y_i)$ is an artinian local ring by the assumption. If $N > \ell n$, then some $\underline{Y_k}$ appears at least $(\ell+1)$-times among $\underline{Z_0}, \underline{Z_1}, \dots, \underline{Z_N}$, therefore the composition $\underline{Z_0} \to \underline{Z_1} \to \cdots \to \underline{Z_N}$ is a zero morphism. Thus $(\underline{\psi_{ij}})^N = 0$ if $N > \ell n$. □

Now we proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. To prove the theorem, let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra, and assume that R has only an isolated singularity and that k is an infinite field.

For $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$, we assume that $\underline{M}$ stably degenerates to $\underline{N}$. Then, by definition we have $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\underline{Q_t} \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\underline{Q/tQ} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$. Note that there is a natural isomorphism

$$\underline{\mathrm{Hom}}_{R\otimes_k V}(M \otimes_k V, Q)_t \cong \underline{\mathrm{Hom}}_{R\otimes_k K}(M \otimes_k K, Q_t).$$

Therefore there is a morphism $\rho : \underline{M \otimes_k V} \to \underline{Q}$ in $\underline{\mathrm{CM}}(R \otimes_k V)$ with $\rho_t : \underline{M \otimes_k K} \to \underline{Q_t}$ is an isomorphism. Now take a cone of ρ, and we get a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$;

$$\underline{M \otimes_k V} \xrightarrow{\ \rho\ } \underline{Q} \longrightarrow \underline{P'} \longrightarrow \underline{M \otimes_k V}[1]. \tag{6.4}$$

By the choice of ρ, we have that $\underline{P'_t} \cong \underline{0}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$, i.e. P'_t is a projective $R \otimes_k K$-module. By virtue of Lemma 6.4 we have a Cohen–Macaulay R-module X and a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$;

$$\underline{X \otimes_k V} \xrightarrow{\ \mu\ } \underline{X \otimes_k V} \longrightarrow \underline{P'} \longrightarrow \underline{X \otimes_k V}[1].$$

Utilizing the octahedron axiom, it follows from this triangle together with (6.4) that there is a commutative diagram in which all rows and columns are triangles in $\underline{\mathrm{CM}}(R \otimes_k V)$.

$$\begin{array}{ccccccc}
 & & \underline{X \otimes_k V} & = & \underline{X \otimes_k V} & & \\
 & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\
\underline{M \otimes_k V} & \longrightarrow & \underline{W} & \xrightarrow{\ \nu\ } & \underline{X \otimes_k V} & \xrightarrow{\ \lambda\ } & \underline{M \otimes_k V}[1] \\
\| & & \downarrow & & \downarrow & & \| \\
\underline{M \otimes_k V} & \xrightarrow{\ \rho\ } & \underline{Q} & \longrightarrow & \underline{P'} & \longrightarrow & \underline{M \otimes_k V}[1] \\
 & & \downarrow & & \downarrow & & \\
 & & \underline{X \otimes_k V}[1] & = & \underline{X \otimes_k V}[1] & &
\end{array} \tag{6.5}$$

Taking the localization by t, and noting that $\underline{P'_t} \cong \underline{0}$, we have the following commutative diagram in which all rows and columns are triangles in $\underline{\mathrm{CM}}(R \otimes_k K)$.

$$
\begin{array}{ccccccc}
 & & \underline{X \otimes_k K} & = & \underline{X \otimes_k K} & & \\
 & & \downarrow & & \downarrow {\scriptstyle \mu} \, {\scriptstyle \cong} & & \\
\underline{M \otimes_k K} & \longrightarrow & \underline{W_t} & \xrightarrow{\nu_t} & \underline{X \otimes_k K} & \xrightarrow{\lambda_t} & \underline{M \otimes_k K}[1] \\
\| & & \downarrow & & \downarrow & & \| \\
\underline{M \otimes_k K} & \underset{\cong}{\xrightarrow{\rho_t}} & \underline{Q_t} & \longrightarrow & \underline{0} & \longrightarrow & \underline{M \otimes_k K}[1] \\
 & & \downarrow & & \downarrow & & \\
 & & \underline{X \otimes_k K}[1] & = & \underline{X \otimes_k K}[1]. & &
\end{array}
$$

From this diagram we see that ν_t is a splitting epimorphism, and hence $\lambda_t = 0$ in $\underline{\mathrm{CM}}(R \otimes_k K)$. Notice that λ is an element of $\mathrm{Ext}^1_{R \otimes_k V}(X \otimes_k V, M \otimes_k V)$, and that there is a natural isomorphism

$$\mathrm{Ext}^1_{R \otimes_k V}(X \otimes_k V, M \otimes_k V) \cong \mathrm{Ext}^1_R(X, M) \otimes_k V.$$

Thus that $\lambda_t = 0$ forces that $t^n \lambda = 0$ in $\mathrm{Ext}^1_R(X, M) \otimes_k V$ for some $n > 0$. However, since t is a non-zero divisor on $\mathrm{Ext}^1_R(X, M) \otimes_k V$, this implies that $\lambda = 0$ as an element $\mathrm{Ext}^1_{R \otimes_k V}(X \otimes_k V, M \otimes_k V)$.

Now getting back to the diagram (6.5), we conclude from $\lambda = 0$ that the second row splits and that $\underline{W}$ is isomorphic to $\underline{M \otimes_k V} \oplus \underline{X \otimes_k V}$. Thus we have a triangle in $\underline{\mathrm{CM}}(R \otimes_k V)$:

$$\underline{X \otimes_k V} \longrightarrow \underline{M \otimes_k V} \oplus \underline{X \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{X \otimes_k V}[1].$$

Send this triangle by the functor $\mathcal{R} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R)$, and we get a triangle in $\underline{\mathrm{CM}}(R)$ of the following form:

$$\underline{X} \xrightarrow{\binom{\underline{\phi}}{\underline{\psi}}} \underline{M} \oplus \underline{X} \longrightarrow \underline{N} \longrightarrow \underline{X}[1].$$

We should note that we did not use so far the assumption that R is an isolated singularity.

It remains to prove that we can take a nilpotent endomorphism as $\underline{\psi}$. For this, we apply Lemma 6.5 to the $\underline{\psi}$ above. As in the lemma, we have a decomposition $\underline{X} = \underline{X_1} \oplus \underline{X_2}$ and automorphisms $\underline{\alpha}, \underline{\beta}$ of $\underline{X}$ such that

$$\underline{\alpha} \cdot \underline{\psi} \cdot \underline{\beta} = \begin{pmatrix} \underline{\psi_1} & 0 \\ 0 & \underline{\psi_2} \end{pmatrix},$$

where $\underline{\psi_1} : \underline{X_1} \to \underline{X_1}$ is an automorphism and $\underline{\psi_2} : \underline{X_2} \to \underline{X_2}$ is a nilpotent endomorphism. Then we have an isomorphism of triangles in $\underline{\mathrm{CM}}(R)$;

$$\begin{array}{ccccccc}
\underline{X_1} \oplus \underline{X_2} & \xrightarrow{\binom{\underline{\phi}}{\underline{\psi}}} & \underline{M} \oplus \underline{X_1} \oplus \underline{X_2} & \longrightarrow & \underline{N} & \longrightarrow & (\underline{X_1} \oplus \underline{X_2})[1] \\
{\scriptstyle \underline{\beta}^{-1}} \downarrow \cong & & {\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & \underline{\alpha} \end{pmatrix}} \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\underline{X_1} \oplus \underline{X_2} & \xrightarrow[\begin{pmatrix} \underline{\phi\beta} \\ \cdots \\ \underline{\psi_1} & 0 \\ 0 & \underline{\psi_2} \end{pmatrix}]{} & \underline{M} \oplus \underline{X_1} \oplus \underline{X_2} & \longrightarrow & \underline{N} & \longrightarrow & (\underline{X_1} \oplus \underline{X_2})[1].
\end{array}$$

Since $\underline{\psi_1}$ is an isomorphism, we can split $\underline{X_1}$ off from the triangle in the second row above, and we get the triangle of the form;

$$\underline{X_2} \xrightarrow{\binom{\underline{\phi}'}{\underline{\psi_2}}} \underline{M} \oplus \underline{X_2} \longrightarrow \underline{N} \longrightarrow \underline{X_2}[1].$$

Since $\underline{\psi_2}$ is nilpotent, this is the triangle we wanted. □

As a direct consequence of Theorem 6.1, we have the following corollary.

Corollary 6.6. *Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras. Assume that the both R_1 and R_2 are isolated singularities, and that k is an infinite field. Suppose there is a k-linear equivalence $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ of triangulated categories. Then, for $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ stably degenerates to $\underline{N}$ if and only if $F(\underline{M})$ stably degenerates to $F(\underline{N})$.*

Proof. Assume that $\underline{M}$ stably degenerates to $\underline{N}$ for $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R_1)$. By Theorem 6.1 there is a triangle

$$\underline{Z} \xrightarrow{\binom{\underline{\phi}}{\underline{\psi}}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1],$$

where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$. Applying the functor F to this, we have a triangle in $\underline{\mathrm{CM}}(R_2)$;

$$F(\underline{Z}) \xrightarrow{\binom{F(\underline{\phi})}{F(\underline{\psi})}} F(\underline{M}) \oplus F(\underline{Z}) \longrightarrow F(\underline{N}) \longrightarrow F(\underline{Z})[1].$$

Since $F(\underline{\psi})$ is nilpotent as well, Theorem 5.1 forces that $F(\underline{M})$ stably degenerates to $F(\underline{N})$. □

Remark 6.7. Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras as above. Then it hardly occurs that there is a k-linear equivalence of categories between $\mathrm{CM}(R_1)$ and $\mathrm{CM}(R_2)$. In fact, if it occurs, then R_1 is isomorphic to R_2 as a k-algebra. (See [2, Proposition 5.1].)

On the other hand, an equivalence between $\underline{\mathrm{CM}}(R_1)$ and $\underline{\mathrm{CM}}(R_2)$ may happen for non-isomorphic k-algebras. For example, let $R_1 = k[\![x, y, z]\!]/(x^n + y^2 + z^2)$ and $R_2 = k[\![x]\!]/(x^n)$ with characteristic of k being odd and $n \in \mathbb{N}$. Then, by Knoerrer's periodicity [5, Theorem 12.10], we have an equivalence $\underline{\mathrm{CM}}(k[\![x, y, z]\!]/(x^n + y^2 + z^2)) \cong \underline{\mathrm{CM}}(k[\![x]\!]/(x^n))$. Since $k[\![x]\!]/(x^n)$ is an artinian Gorenstein ring, the stable degeneration of modules over $k[\![x]\!]/(x^n)$ is equivalent to a degeneration up to free summands by Theorem 5.1. Moreover the degeneration problem for modules over $k[\![x]\!]/(x^n)$ is known to be equivalent to the degeneration problem for Jordan canonical forms of square matrices of size n. Thus by virtue of Corollary 6.6, it is easy to describe the stable degenerations of Cohen–Macaulay modules over $k[\![x, y, z]\!]/(x^n + y^2 + z^2)$.

7. Degeneration and stable degeneration

In this section we shall show that a stable degeneration implies a degeneration after adding some identical Cohen–Macaulay module. More precisely the main result of this section is the following.

Theorem 7.1. *Let R be a Gorenstein complete local k-algebra, where we assume that k is an infinite field. Let $M, N \in \mathrm{CM}(R)$, and suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then, there exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.*

Before giving a proof of this theorem we need a lemma and a proposition which are basically proved by using Swan's Lemma 6.2.

Lemma 7.2. *Let R be a noetherian local k-algebra, where k is an infinite field, and let $K = k(t)$ as before. Given a finitely generated projective $R \otimes_k K$-module P, there are a finitely generated R-module X and a non-negative integer n such that*

$$P \oplus (X \otimes_k K) \cong \left(R^n \otimes_k K\right) \oplus (X \otimes_k K),$$

as $R \otimes_k K$-modules.

Proof. Recall that $R \otimes_k K = (R \otimes_k V)_t$. Since P is a finitely generated $(R \otimes_k V)_t$-module, we find a finitely generated $R \otimes_k V$-submodule P' of P satisfying $P'_t \cong P$. Since P is projective, we may assume $P \subseteqq (R \otimes_k K)^r$ for some $r \geqq 0$. Replacing P' with $P' \cap (R \otimes_k V)^r$ if necessary, we may take a submodule of a free $R \otimes_k V$-module as such a P'. Thus we can apply Lemma 6.3 to P' and we have an exact sequence of $R \otimes_k V$-modules;

$$0 \longrightarrow X \otimes_k V \longrightarrow (X \otimes_k V) \oplus (R^n \otimes_k V) \longrightarrow P' \longrightarrow 0,$$

where X is a finitely generated R-module and n is a non-negative integer. Taking the localization by t and noting that $P = P'_t$ is projective, we have an isomorphism

$$P \oplus (X \otimes_k K) \cong \left(R^n \otimes_k K\right) \oplus (X \otimes_k K). \qquad \square$$

Proposition 7.3. *Let R be a Cohen–Macaulay local k-algebra, where k is an infinite field, and let $K = k(t)$ as before. Given a finitely generated projective $R \otimes_k K$-module P, there are a Cohen–Macaulay R-module X and a non-negative integer n such that*

$$P \oplus (X \otimes_k K) \cong \left(R^n \otimes_k K\right) \oplus (X \otimes_k K), \tag{7.1}$$

as $R \otimes_k K$-modules.

Proof. We have already shown that there is a finitely generated R-module X which satisfies the isomorphism (7.1). Take an integer ℓ so that $2\ell > \dim R$ and we consider the 2ℓth syzygy module of X, i.e. there is an exact sequence

$$0 \longrightarrow \Omega^{2\ell} X \longrightarrow F_{2\ell-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow X \longrightarrow 0,$$

where each F_i is a free R-module. Since P is a projective $R \otimes_k K$-module, it is a direct summand of a free $R \otimes_k K$-module $R^m \otimes_k K$. Therefore there is an idempotent $\epsilon \in \mathrm{End}_{R\otimes_k K}(R^m \otimes_k K)$ such that $P = \mathrm{Coker}(\epsilon) = \mathrm{Im}(1-\epsilon) = \mathrm{Ker}(\epsilon)$. Thus there is an exact sequence with $2\ell + 2$ terms

$$0 \to P \to R^m \otimes_k K \xrightarrow{\epsilon} R^m \otimes_k K \xrightarrow{1-\epsilon} \cdots \xrightarrow{1-\epsilon} R^m \otimes_k K \xrightarrow{\epsilon} R^m \otimes_k K \to P \to 0.$$

Tensoring the first exact sequence with K over k and taking the direct sum of these exact sequences, we obtain an exact sequence of the form

$$0 \to P \oplus \left(\Omega^{2\ell} X \otimes_k K\right) \to G_{2\ell-1} \to \cdots \to G_0 \to P \oplus (X \otimes_k K) \to 0,$$

where $G_i = R^m \otimes_k K \oplus F_i \otimes_k K$ that is a free $R \otimes_k K$-module. On the other hand there is an exact sequence of the form

$$0 \to \left(\Omega^{2\ell} X \otimes_k K\right) \to H_{2\ell-1} \to \cdots \to H_0 \to \left(R^n \otimes_k K\right) \oplus (X \otimes_k K) \to 0,$$

where each H_i is a free $R \otimes_k K$-module. Since $(R^n \otimes_k K) \oplus (X \otimes_k K) \cong P \oplus (X \otimes_k K)$, it follows from Schanuel's lemma that there is an isomorphism of the form

$$P \oplus \left(\Omega^{2\ell} X \otimes_k K\right) \oplus \left(R^a \otimes_k K\right) \cong \left(\Omega^{2\ell} X \otimes_k K\right) \oplus \left(R^b \otimes_k K\right),$$

for some $a, b \in \mathbb{N}$. Note that in such a situation we must have $a < b$ whenever $P \neq 0$. Thus setting $X' = \Omega^{2\ell} X \oplus R^a$, we have

$$P \oplus \left(X' \otimes_k K\right) \cong \left(R^{b-a} \otimes_k K\right) \oplus \left(X' \otimes_k K\right)$$

and clearly X' is a Cohen–Macaulay module over R. □

Now we can prove Theorem 7.1.

Proof of Theorem 7.1. We assume that $\underline{M}$ stably degenerates to $\underline{N}$ for $M, N \in \mathrm{CM}(R)$. By definition there is a Cohen–Macaulay $R \otimes_k V$-module Q such that $\underline{Q_t} \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\underline{Q/tQ} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$. Thus $Q_t \oplus P_1 \cong (M \otimes_k K) \oplus P_2$ in $\mathrm{CM}(R \otimes_k K)$ for projective $R \otimes_k K$-modules P_1 and P_2, and $Q/tQ \oplus R^a \cong N \oplus R^b$ in $\mathrm{CM}(R)$ for some $a, b \in \mathbb{N}$. It then follows from Proposition 7.3 there are Cohen–Macaulay R-modules X_1, X_2 and integers n_1, n_2 satisfying

$$P_i \oplus (X_i \otimes_k K) \cong \left(R^{n_i} \otimes_k K\right) \oplus (X_i \otimes_k K),$$

for $i = 1, 2$. Now setting $X = X_1 \oplus X_2$ we have an isomorphism in $\mathrm{CM}(R \otimes_k K)$;

$$Q_t \oplus \left(R^{n_1} \otimes_k K\right) \oplus (X \otimes_k K) \cong (M \otimes_k K) \oplus \left(R^{n_2} \otimes_k K\right) \oplus (X \otimes_k K).$$

We denote by $\tilde{Q}$ the Cohen–Macaulay module $Q \oplus (R^{a+n_1} \otimes_k V) \oplus (X \otimes_k V)$ over $R \otimes_k V$. Then it follows

$$\tilde{Q}_t \cong (M \otimes_k K) \oplus \left(R^{a+n_2} \otimes_k K\right) \oplus (X \otimes_k K),$$

in $\mathrm{CM}(R \otimes_k K)$ and $\tilde{Q}/t\tilde{Q} \cong N \oplus R^{b+n_1} \oplus X$ in $\mathrm{CM}(R)$. Therefore $M \oplus R^{a+n_2} \oplus X$ degenerates to $N \oplus R^{b+n_1} \oplus X$. □

The converse of Theorem 7.1 does not hold even in the case when R is artinian. The following example is taken from the paper [4] of Riedtmann.

Example 7.4. Let $R = k[\![x, y]\!]/(x^2, y^2)$. Note that R is an artinian Gorenstein local ring. Now consider the modules $M_\lambda = R/(x - \lambda y)R$ for all $\lambda \in k$. We denote by k the unique simple module $R/(x, y)R$ over R. In this case, it is known by [4, Example 3.1] that $R \oplus k^2$ degenerates to $M_\lambda \oplus M_\mu \oplus k^2$ for any choice of $\lambda, \mu \in k$.

We claim that $\underline{R}$ never stably degenerates to $\underline{M_\lambda \oplus M_\mu}$ if $\lambda + \mu \neq 0$.

In fact, if there is such a stable degeneration, then it follows from Theorem 5.1 that R^m degenerates to $M_\lambda \oplus M_\mu \oplus R^n$ for some $m, n \in \mathbb{N}$. Since $[R^m] = [M_\lambda \oplus M_\mu \oplus R^n]$ in the Grothendieck group, we have $m > n \geqq 0$. Now we apply Theorem 2.5 to obtain an inclusion of Fitting ideals; $\mathcal{F}_n^R(M_\lambda \oplus M_\mu \oplus R^n) \subseteq \mathcal{F}_n^R(R^m)$. We note that $\mathcal{F}_n^R(R^m) = 0$ since $n < m$, and an easy computation shows that

$$\mathcal{F}_n^R\left(M_\lambda \oplus M_\mu \oplus R^n\right) = \mathcal{F}_0^R(M_\lambda)\mathcal{F}_0^R(M_\mu)\mathcal{F}_n^R\left(R^n\right) = (x - \lambda y)(x - \mu y)R = (\lambda + \mu)xyR.$$

Hence we must have $\lambda + \mu = 0$.

8. Stable degeneration order

As an application of Theorem 7.1 we can define the stable degeneration order for Cohen–Macaulay modules.

Definition 8.1. Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra as before, and let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$. If there is a sequence of objects $\underline{L_0}, \underline{L_1}, \underline{L_2}, \dots, \underline{L_n}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{L_0} = \underline{M}$, $\underline{L_n} = \underline{N}$ and $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $i = 0, 1, \dots, n-1$, then we write $\underline{M} \leqslant_{st} \underline{N}$.

Theorem 7.1 shows that the relation $\leqslant_{st}$ gives a partial order on the set of isomorphism classes of objects in $\underline{\mathrm{CM}}(R)$. In fact we can prove the antisymmetric law for $\leqslant_{st}$.

Theorem 8.2. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local algebra over an infinite field k, and let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$. If $\underline{M} \leqslant_{st} \underline{N}$ and $\underline{N} \leqslant_{st} \underline{M}$, then $\underline{M} \cong \underline{N}$.*

Proof. Since $\underline{M} \leqslant_{st} \underline{N}$, there are $\underline{L_0}, \underline{L_1}, \underline{L_2}, \dots, \underline{L_n}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{L_0} = \underline{M}$, $\underline{L_n} = \underline{N}$ and $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $i = 0, 1, \dots, n-1$. It follows from Theorem 7.1 that $L_i \oplus R^{a_i} \oplus X_i$ degenerates to $L_{i+1} \oplus R^{b_i} \oplus X_i$, where $X_i \in \mathrm{CM}(R)$ and $a_i, b_i \in \mathbb{N}$ for each $i = 0, 1, \dots, n-1$. Then, set

$$L_i' = L_i \oplus R^{b_0 + \dots + b_{i-1} + a_i + \dots + a_{n-1}} \oplus X_0 \oplus \dots \oplus X_{n-1},$$

and we can see that L_i' degenerates to L_{i+1}' for $i = 0, 1, \dots, n-1$. Therefore, we have $L_0' \leqslant_{deg} L_n'$ under the ordinary degeneration order, thus setting $a = a_0 + \dots + a_{n-1}$, $b = b_0 + \dots + b_{n-1}$ and $X = X_1 \oplus \dots \oplus X_{n-1}$, we have

$$M \oplus R^a \oplus X \leqslant_{deg} N \oplus R^b \oplus X.$$

(See [6, Definition 4.11] for the detail of degeneration order $\leqslant_{deg}$.) Similarly, using the assumption that $\underline{N} \leqslant_{st} \underline{M}$, we get

$$N \oplus R^c \oplus Y \leqslant_{deg} M \oplus R^d \oplus Y,$$

for some $c, d \in \mathbb{N}$ and $Y \in \mathrm{CM}(R)$. Thus we have the inequality

$$M \oplus R^{a+c} \oplus X \oplus Y \leqslant_{deg} N \oplus R^{b+c} \oplus X \oplus Y \leqslant_{deg} M \oplus R^{b+d} \oplus X \oplus Y.$$

Since there is a degeneration, we have the equality $[M \oplus R^{a+c} \oplus X \oplus Y] = [M \oplus R^{b+d} \oplus X \oplus Y]$ in the Grothendieck group $K_0(\mathrm{mod}(R))$. See Remark 2.3. Thus it follows that $[R^{a+c}] = [R^{b+d}]$, hence $a + c = b + d$. Since $\leqslant_{deg}$ satisfies the antisymmetric law [6, Theorem 2.2, Proposition 4.4], we have the isomorphism

$$M \oplus R^{a+c} \oplus X \oplus Y \cong N \oplus R^{b+c} \oplus X \oplus Y.$$

Note that $\mathrm{CM}(R)$ is a Krull–Schmidt category, and thus this isomorphism forces $M \oplus R^a \cong N \oplus R^b$. Therefore $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$. $\square$

If R is an isolated singularity, then we can prove that the stable degeneration for Cohen–Macaulay modules is transitive. More precisely we can prove the following proposition.

Proposition 8.3. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local algebra over an infinite field k, and assume that R has only an isolated singularity.*

For $\underline{L}, \underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$, if $\underline{L}$ stably degenerates to $\underline{M}$ and if $\underline{M}$ stably degenerates to $\underline{N}$, then $\underline{L}$ stably degenerates to $\underline{N}$. In particular, the partial order $\underline{M} \leqslant_{st} \underline{N}$ is equivalent to saying that $\underline{M}$ stably degenerates to $\underline{N}$.

Proof. The proof proceeds as in the same way as the proof of [9, Theorem 2.1]. In fact, if there are triangles

$$\underline{Z_1} \xrightarrow{\binom{\underline{\phi_1}}{\underline{\psi_1}}} \underline{L} \oplus \underline{Z_1} \longrightarrow \underline{M} \longrightarrow \underline{Z_1}[1],$$

$$\underline{Z_2} \xrightarrow{\binom{\underline{\phi_2}}{\underline{\psi_2}}} \underline{M} \oplus \underline{Z_2} \longrightarrow \underline{N} \longrightarrow \underline{Z_2}[1],$$

with $\underline{\psi_1}$ and $\underline{\psi_2}$ being nilpotent, then we can construct a new triangle of the form;

$$\underline{Z_3} \xrightarrow{\binom{\underline{\phi_3}}{\underline{\psi_3}}} \underline{L} \oplus \underline{Z_3} \longrightarrow \underline{N} \longrightarrow \underline{Z_3}[1],$$

where $\underline{\psi_3}$ may not be nilpotent. (One can prove this completely in a similar way to the proof of [9, Theorem 2.1] but by replacing short exact sequences there by triangles.) As in the same manner of the last half of the proof of Theorem 6.1, we can replace $\underline{\psi_3}$ by a nilpotent endomorphism by utilizing Lemma 6.5. $\square$

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