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Almost Split Sequences for Relatively Projective Modules

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INTRODUCTION

The paper deals with almost split sequences. Introduced in [2] for the category mod Λ of finitely generated modules over an artin algebra Λ, almost split sequences were later found in the category of lattices over an order [1, 4], as well as in certain subcategories of mod Λ [6, 18, 3]. It is generally recognized that if almost split sequences exist, the subcategory has nice properties. We are concerned with the subcategory of relatively projective modules.

Let R be a field or a Dedekind domain with the field of quotients k, and let Λ and A be finite-dimensional R-algebras or R-orders, respectively, with Λ mapped into A via an R-algebra map $i\colon \Lambda \to A$. Here we understand orders and lattices in the sense of [1, p. 85, Example (b)]. Namely, A is an R-order if it is a noetherian R-algebra projective as an R-module, and $\sum = k \otimes_R A$ is a self-injective ring. A-mod denotes the category of finitely generated left A-modules if R is a field, or the category of left A-lattices if R is a Dedekind domain, where a left A-module M is a lattice if it is a finitely generated projective R-module such that $k \otimes_R M$ is a projective

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Σ-module. As pointed out in [1], the classical orders and lattices fit into this more general scheme. Denote by $\mathbf{p}(A, \Lambda)$ the full subcategory of A-mod determined by the relatively projective A-modules, i.e., by the induced modules, isomorphic to $A \otimes_\Lambda X$ with $X \in \Lambda$-mod, and their direct summands. The question is whether almost split sequences exist in $\mathbf{p}(A, \Lambda)$. To explain what it means we have to recall some notions introduced in [3].

Let $\mathscr{C}$ be a full subcategory of A-mod closed under direct sums, nonzero direct summands, and such that if a module $X \in \mathscr{C}$ is isomorphic to a module $Y \in A$-mod, then $Y \in \mathscr{C}$. An exact sequence in $\mathscr{C}$ is an exact sequence $\cdots X_{i-1} \to X_i \to X_{i+1} \cdots$ of modules in A-mod in which the nonzero X_i's are all in $\mathscr{C}$. A module $N \in \mathscr{C}$ is called Ext-projective if every exact sequence $0 \to X \to Y \to N \to 0$ in $\mathscr{C}$ splits. A module $L \in \mathscr{C}$ is called Ext-injective if every exact sequence $0 \to L \to Y \to Z \to 0$ in $\mathscr{C}$ splits. A morphism $g\colon M \to N$ in $\mathscr{C}$ is said to be right almost split in $\mathscr{C}$ if (i) g is not a splittable epimorphism and (ii) for every morphism $h\colon W \to N$, where $W \in \mathscr{C}$ and h is not a splittable epimorphism, there exists a morphism $j\colon W \to M$ satisfying $h = gj$. A morphism $f\colon L \to M$ in $\mathscr{C}$ is said to be left almost split in $\mathscr{C}$ if (i) f is not a splittable monomorphism and (ii) for every morphism $h\colon L \to W$, where $W \in \mathscr{C}$ and h is not a splittable monomorphism, there exists a morphism $j\colon M \to W$ satisfying $h = jf$. $\mathscr{C}$ is said to have right almost split morphisms if for each indecomposable $N \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism $g\colon M \to N$ which is right almost split in $\mathscr{C}$. Dually, $\mathscr{C}$ is said to have left almost split morphisms if for each indecomposable $L \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism $f\colon L \to M$ which is left almost split in $\mathscr{C}$. Finally, $\mathscr{C}$ has almost split morphisms if it has both left and right almost split morphisms.

An exact sequence $0 \to L \to^f M \to^g N \to 0$ in $\mathscr{C}$ is called almost split if f is a left almost split morphism in $\mathscr{C}$, and g is a right almost split morphism in $\mathscr{C}$. $\mathscr{C}$ is said to have almost split sequences if it satisfies the following conditions:

(a) $\mathscr{C}$ has almost split morphisms.

(b) If N is indecomposable non-Ext-projective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

(c) If L is indecomposable non-Ext-injective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

Coming back to relatively projective modules, we are interested in almost split sequences in the special case when $\mathscr{C}$ is $\mathbf{p}(A, \Lambda)$. Auslander and Smalø proved in [3] that almost split sequences exist in $\mathscr{C}$ if $\mathscr{C}$ is a dualizing R-variety closed under extensions. However, it is well known that the category $\mathbf{p}(A, \Lambda)$ generally is not closed under extensions. Accordingly, to deal with relatively projective modules, we prove the following existence theorem (Theorem 1.2).

Suppose that the direct sum of every two Ext-projective modules in $\mathscr{C}$ is Ext-projective, and the direct sum of every two Ext-injective modules in $\mathscr{C}$ is Ext-injective. Then $\mathscr{C}$ has almost split sequences if and only if it satisfies the following conditions:

(i) $\mathscr{C}$ has almost split morphisms.

(ii) If N is indecomposable non-Ext-projective in $\mathscr{C}$, then there is an exact sequence $0 \to L \to M \to^{g} N \to 0$ in $\mathscr{C}$ with g right almost split in $\mathscr{C}$.

(iii) If L is indecomposable non-Ext-injective in $\mathscr{C}$, then there is an exact sequence $0 \to L \to^{f} M \to N \to 0$ in $\mathscr{C}$ with f left almost split in $\mathscr{C}$.

This existence theorem replaces the assumption of Auslander and Smalø that $\mathscr{C}$ is closed under extensions by a weaker technical assumption that the Ext-projectives and the Ext-injectives in $\mathscr{C}$ are closed under direct sums. We use the theorem to prove the existence of almost split sequences in $\mathbf{p}(A, \Lambda)$ under certain restrictions on A and Λ. We do not know whether those restrictions imply that $\mathbf{p}(A, \Lambda)$ is closed under extensions, but we prove they imply that the Ext-projectives and the Ext-injectives are closed under direct sums so that the existence theorem works.

We now explain what the difficulties are in proving the existence of almost split sequences in $\mathscr{C} = \mathbf{p}(A, \Lambda)$. Since right almost split morphisms exist in A-mod [1, 2, 4], it is easy to prove the existence of right almost split morphisms in $\mathbf{p}(A, \Lambda)$ (Proposition 2.3(a)). (In the terminology of [4], the existence of right almost split morphisms in $\mathbf{p}(A, \Lambda)$ is a consequence of the easily verified fact that the subcategory $\mathbf{p}(A, \Lambda)$ is contravariantly finite in A-mod.) Thus we get half of the condition (i) of the existence theorem, and, for each indecomposable non-Ext-projective $N \in \mathbf{p}(A, \Lambda)$, an exact sequence $0 \to \operatorname{Ker} g \to M \to^{g} N \to 0$, where $g: M \to N$ is a right almost split morphism in $\mathbf{p}(A, \Lambda)$ with $\operatorname{Ker} g \in A$-mod. To satisfy condition (ii) of the existence theorem, it would suffice to prove $\operatorname{Ker} g \in \mathbf{p}(A, \Lambda)$. We show in Section 2 that the latter condition is satisfied when the map $i: \Lambda \to A$ is injective, and Coker i, as a Λ-bimodule, is isomorphic to $\oplus_{s=1}^{n} I_S \otimes_R P_S$, where I_S is injective in Λ-mod, and P_S is projective in mod-Λ for all s. These restrictions on the map $i: \Lambda \to A$ constitute the hypothesis of the main theorem of the present paper. Having satisfied the conditions of the existence theorem concerning right almost split morphisms, we note that the category $\mathbf{p}(A, \Lambda)$ generally is not well-behaved with respect to left almost split morphisms. So we construct an exact duality, whose domain is $\mathbf{p}(A, \Lambda)$, which in our context plays the role similar to that of the well-known duality $D = \operatorname{Hom}_R(-, R)$: A-mod $\to$ A^{op}-mod [1, 2], where A^{op} is the opposite ring of A. The construction is based on the fact that the opposite category of $\mathbf{p}(A, \Lambda)$ is equivalent to $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$, where the R-algebra A_1 has the same properties as the R-algebra A, and there exists an R-algebra map $i_1: \Lambda^{\mathrm{op}} \to A_1$ satisfying the same conditions as the map $i: \Lambda \to A$. Thus the conditions of the existence theorem concerning right almost split morphisms are satisfied for $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$, whence we conclude that the conditions of the existence theorem concerning left almost split morphisms are satisfied for the opposite category $\mathbf{p}(A, \Lambda)$. Finally, we show that in $\mathbf{p}(A, \Lambda)$ the Ext-projectives are the projectives in A-mod (Section 2), and the Ext-injectives are the direct summands of the modules of the form $A \otimes_\Lambda I$, where I is injective in Λ-mod (Section 4). Therefore the Ext-projectives and the Ext-injectives are closed under direct sums, and almost split sequences exist in $\mathbf{p}(A, \Lambda)$ under the above hypothesis on $i: \Lambda \to A$. The latter is the main theorem of the present paper. To construct A_1, we first construct the Λ^{op}-coring $C = \mathrm{Hom}_\Lambda(A, \Lambda)$, where the homomorphisms are those of right Λ-modules, with counit $\varepsilon: C \to \Lambda^{\mathrm{op}}$. Then $A_1 = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(C, \Lambda^{\mathrm{op}})$ is the set of left Λ^{op}-module homomorphisms from C into Λ^{op} with $i_1 = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\varepsilon, \Lambda^{\mathrm{op}})$ [21, 13]. We fix R, Λ, A, C, and A_1 throughout the paper, and assume that the action of R on all R-bimodules is central.

We now give examples of R-algebra maps $i: \Lambda \to A$ satisfying the hypothesis of the main theorem, claiming the existence of almost split sequences in $\mathbf{p}(A, \Lambda)$. Let G be a Frobenius group, H its proper subgroup whose intersection with gHg^{-1} is trivial whenever $g \in G - H$, and i the natural inclusion of the group algebra $\Lambda = RH$ into the group algebra $A = RG$. It is shown in Section 6 that Coker i satisfies the hypothesis. Or let A be the path algebra of the quiver (oriented graph) $1 \xleftarrow{a} 2 \xleftarrow{b} 3$ over the ring R, and Λ the R-subalgebra of A generated by a, ab, and the empty paths e_1, e_2, e_3 at the vertices 1, 2, 3, respectively. Then it is easy to check that the cokernel of the natural inclusion $i: \Lambda \to A$ satisfies the hypothesis of the main theorem.

The latter example is a very special case of the large class of R-algebra maps $i: \Lambda \to A$ which are related to BOCSes [19]. Representations of BOCSes, used in [7] to obtain important properties of tame finite-dimensional algebras, are our main motivating example; they are connected with relatively projective modules as follows. It is shown in [13] that the hypothesis on $i: \Lambda \to A$ implies the existence of a duality between $\mathbf{p}(A, \Lambda)$ and $\mathbf{i}(C, \Lambda^{\mathrm{op}})$ (where $\mathbf{i}(C, \Lambda^{\mathrm{op}})$ is the category of relatively injective C-comodules consisting of the induced C-comodules, isomorphic to $C \otimes_{\Lambda^{\mathrm{op}}} X$ with $X \in \Lambda^{\mathrm{op}}$-mod, and their direct summands) and that the category of induced C-comodules for an arbitrary C is equivalent to the category of representations of the corresponding BOCS. Using that, we show that the hypothesis of the main theorem on the map $i: \Lambda \to A$ is satisfied if and only if the counit $\varepsilon: C \to \Lambda^{\mathrm{op}}$ of the Λ^{op}-coring C is surjective, and Ker ε is isomorphic as a Λ^{op}-bimodule to $\bigoplus_{s=1}^{n} Q_s \otimes_R P_s$, where $Q_s(P_s)$ is projective in Λ^{op}-mod (mod-Λ^{op}) for all s. In other words, the hypothesis is satisfied if and only if the BOCS corresponding to C is free in the language of [7], and if the latter is the case, then the opposite category of the category $\mathbf{p}(A, \Lambda)$ of relatively projective A-modules is equivalent to the category $\mathbf{i}(C, \Lambda^{\mathrm{op}})$ of relatively injective C-comodules. Thus the main theorem relates to the existence of almost split sequences for matrix problems, originally introduced in [20, 15] as representations of differential graded categories, and later described in [19] as representations of BOCSes. More precisely, we describe in Section 5 a class of corings, called left triangular tensor corings, for which every direct summand of an induced comodule is induced; i.e., every relatively injective comodule is relatively cofree. The class contains all triangular BOCSes in the terminology of [19, 7], hence, in particular, the BOCSes occurring in the reduction of representations of finite-dimensional algebras to representations of BOCSes. For the dual rings, it means that every direct summand of an induced module is induced; i.e., every relatively projective module is relatively free. Thus in this case almost split sequences exist in the category of induced modules, hence—for representations of triangular BOCSes.

We now set the notation. Let Γ be a ring (associative with identity), Γ-Mod (Mod-Γ) the category of left (right) Γ-modules, and Γ-mod (mod-Γ) the category of finitely generated left (right) Γ-modules. Given $U, V \in \Gamma$-Mod, $\mathrm{Hom}_{\Gamma-}(U, V)$ stands for the set of homomorphisms of U into V. For $X, Y \in$ Mod-Γ, $\mathrm{Hom}_{-\Gamma}(X, Y)$ is the corresponding notation. Suppose S is a Γ-ring, i.e., a ring homomorphism $\Gamma \to S$ is given. Denote by Induc S (induc S) the full subcategory of S-mod determined by the induced modules, i.e., by the modules isomorphic to $S \otimes_\Gamma M$ with $M \in \Gamma$-Mod ($M \in \Gamma$-mod). Let $\mathbf{P}(S, \Gamma)$ ($\mathbf{p}(S, \Gamma)$) be the full subcategory of S-Mod consisting of the direct summands of all modules in Induc S (induc S). The induced modules and their direct summands are called relatively projective, or (S, Γ)-projective, modules. Likewise, if K is a Γ-coring [21], Induc K (induc K) is the full subcategory of the category K-Comod of left K-comodules which is determined by the induced comodules, i.e., by the comodules isomorphic to $K \otimes_\Gamma M$ with $M \in \Gamma$-Mod ($M \in \Gamma$-mod). $\mathbf{I}(K, \Gamma)$ ($\mathbf{i}(K, \Gamma)$) stands for the full subcategory of K-Comod consisting of the direct summands of all comodules in Induc K (induc K). The induced comodules and their direct summands are called relatively injective, or (K, Γ)-injective, comodules.

The paper utilizes categorical machinery. Namely, the above mentioned duality between $\mathbf{p}(A, \Lambda)$ and $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$ (Section 4) is obtained from an equivalence of categories $\mathbf{i}(C, \Lambda^{\mathrm{op}})$ and $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$. That equivalence is a consequence of the following very general fact we prove. If a monad $\mathbf{F} = (F, \mu, \nu)$ in a category X is a right adjoint of a comonad $\mathbf{G} = (G, \delta, \varepsilon)$

in X, then the Kleisli categories X_F and X_G are isomorphic. (The statement is a counterpart of the well-known fact [10] that the Eilenberg–Moore categories X^F and X^G are isomorphic if **F** is a left adjoint of **G**.) In particular, if a Γ-coring K is finitely generated projective as a left Γ-module, then $S = \mathrm{Hom}_{\Gamma\text{-}}(K, \Gamma)$ is a Γ-ring, and the monad in Γ-Mod determined by the endofunctor $S \otimes_\Gamma -$ is a right adjoint of the comonad determined by $K \otimes_\Gamma -$, as follows from the natural isomorphisms

$$\mathrm{Hom}_\Gamma\ (K \otimes_\Gamma M, N) \simeq \mathrm{Hom}_{\Gamma\text{-}}(M, \mathrm{Hom}_\Gamma\ (K, N)) \simeq \mathrm{Hom}_{\Gamma\text{-}}(M, S \otimes_\Gamma N)$$

with M, $N \in \Gamma$-Mod. Since the Kleisli categories here are equivalent to the categories of induced modules or induced comodules, we get an equivalence between $\mathbf{I}(K, \Gamma)$ and $\mathbf{P}(S, \Gamma)$. To come back to the original setting, we may put $\Gamma = \Lambda^{\mathrm{op}}$ and $K = C$ because C is finitely generated projective as a left Λ^{op}-module. All categorical arguments are presented in Section 3.

The results of this paper generalize those of [8, 6].

The first author was informed by W. W. Crawley-Boevey that M. C. R. Butler had worked on similar problems.

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1. Almost Split Sequences in Subcategories

The results of this section were presented at the International Conference on Representations of Algebras in Warsaw, Poland, in May 1988 [5]. Throughout the section, we assume that Δ is an artin algebra or an order over a commutative noetherian equidimensional Gorenstein ring and that Δ-mod is the category of finitely generated left Δ-modules or of Δ-lattices [4]. Throughout the section we fix $\mathscr{C}$ as a full subcategory of Δ-mod closed under direct sums, nonzero direct summands, and such that if a module $X \in \mathscr{C}$ is isomorphic to a module $Y \in \Delta$-mod, then $Y \in \mathscr{C}$. When Λ is an artin algebra, the general theory of almost split sequences in $\mathscr{C}$ was developed by Auslander and Smalø in [3]. We need to recall some notions they have introduced.

An exact sequence in $\mathscr{C}$ is an exact sequence $\cdots X_{i-1} \to X_i \to X_{i+1} \cdots$ of modules in Λ-mod in which the nonzero X_i's are all in $\mathscr{C}$. A module $N \in \mathscr{C}$ is called Ext-projective if every exact sequence $0 \to X \to Y \to N \to 0$ in $\mathscr{C}$ splits. A module $L \in \mathscr{C}$ is called Ext-injective if every exact sequence $0 \to L \to Y \to Z \to 0$ in $\mathscr{C}$ splits.

A morphism $g: M \to N$ in $\mathscr{C}$ is said to be right almost split in $\mathscr{C}$ if (i) g

is not a splittable epimorphism and (ii) for every morphism $h: W \to N$, where $W \in \mathscr{C}$ and h is not a splittable epimorphism, there exists a morphism $j: W \to M$ satisfying $h = gj$. A morphism $f: L \to M$ in $\mathscr{C}$ is said to be left almost split in $\mathscr{C}$ if (i) f is not a splittable monomorphism and (ii) for every morphism $h: L \to W$, where $W \in \mathscr{C}$ and h is not a splittable monomorphism, there exists a morphism $j: M \to W$ satisfying $h = jf$. $\mathscr{C}$ is said to have right almost split morphisms if for each indecomposable $N \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism $g: M \to N$ which is right almost split in $\mathscr{C}$. Dually, $\mathscr{C}$ is said to have left almost split morphisms if for each indecomposable $L \in \mathscr{C}$ there is an $M \in \mathscr{C}$ and a morphism $f: L \to M$ which is left almost split in $\mathscr{C}$. Finally, $\mathscr{C}$ has almost split morphisms if it has both left and right almost split morphisms.

An exact sequence $0 \to L \to^f M \to^g N \to 0$ in $\mathscr{C}$ is called almost split if f is a left almost split morphism in $\mathscr{C}$, and g is a right almost split morphism in $\mathscr{C}$. $\mathscr{C}$ is said to have almost split sequences if it satisfies the following conditions:

(a) $\mathscr{C}$ has almost split morphisms.

(b) If N is indecomposable non-Ext-projective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

(c) If L is indecomposable non-Ext-injective in $\mathscr{C}$, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.

Definition 1.1. We say that the Ext-projective (Ext-injective) modules in $\mathscr{C}$ are closed under direct sums if whenever X and Y are Ext-projective (Ext-injective) in $\mathscr{C}$, $X \oplus Y$ is Ext-projective (Ext-injective) in $\mathscr{C}$.

Theorem 1.2. *Suppose that both the* Ext-*projective and the* Ext-*injective modules in $\mathscr{C}$ are closed under direct sums. Then $\mathscr{C}$ has almost split sequences if and only if it satisfies the following conditions*:

(i) *$\mathscr{C}$ has almost split morphisms.*

(ii) *If N is indecomposable non-*Ext-*projective in $\mathscr{C}$, then there is an exact sequence $0 \to L \to M \to^g N \to 0$ in $\mathscr{C}$ with g right almost split in $\mathscr{C}$.*

(iii) *If L is indecomposable non-*Ext-*injective in $\mathscr{C}$, then there is an exact sequence $0 \to L \to^f M \to N \to 0$ in $\mathscr{C}$ with f left almost split in $\mathscr{C}$.*

Proof. The necessity is obvious. Prove the sufficiency. Let N be indecomposable non-Ext-projective in $\mathscr{C}$. Consider an exact sequence $0 \to X \to^s Y \to^t N \to 0$, where t is a minimal morphism corresponding to the right almost split morphism g given by condition (ii). Then X is a direct summand of L, so that $X \in \mathscr{C}$, and t is minimal right almost split in $\mathscr{C}$. We only have to show that s is left almost split in $\mathscr{C}$. Let $X_1, ..., X_r$ be the indecomposable direct summands of X. They are not all Ext-injective because t is not a splittable epimorphism, and the Ext-injective modules in $\mathscr{C}$ are closed under direct sums. For $j=1, ..., r$ consider an exact sequence

$$0 \longrightarrow X_j \xrightarrow{h_j} V_j \xrightarrow{k_j} W_j \longrightarrow 0 \tag{1.1}$$

in $\mathscr{C}$, where if X_j is not Ext-injective, then h_j is minimal left almost split in $\mathscr{C}$ (use condition (iii)); and if X_j is Ext-injective, then $V_j = X_j$, $W_j = 0$, $h_j = 1$. Note that h_j is a minimal morphism in both cases. Let $0 \to X \to^h V \to^k W \to 0$ be the direct sum of the exact sequences (1.1) for all j; it does not split because some of the sequences (1.1) do not split (remember, not all X_j's are Ext-injective). For each $j=1, ..., r$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_j & \xrightarrow{h_j} & V_j & \xrightarrow{k_j} & W_j & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_j} & & \downarrow{\scriptstyle v_j} & & \downarrow{\scriptstyle w_j} & & \\ 0 & \longrightarrow & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \longrightarrow & 0, \end{array}$$

where $u_j: X_j \to X$ is the natural inclusion. Really, this is obvious if X_j is Ext-injective, because $ts = 0$. If X_j is not Ext-injective, then note that su_j is not a splittable monomorphism because of the minimality of t, and use the fact that h_j is left almost split in $\mathscr{C}$. Hence we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & V & \xrightarrow{k} & W & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \longrightarrow & 0, \end{array}$$

where v is induced by the v_j's, and w by the w_j's. If w is not a splittable epimorphism, there exists a morphism $f: W \to Y$ satisfying $w = tf$, whence the top sequence splits by [16, Chap. III, Lemma 3.3, p. 74], a contradiction. Hence w is a splittable epimorphism, and $wq = 1$ for some $q: N \to W$. We now arrive at the following commutative diagram of Δ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & X & \xrightarrow{h} & V & \xrightarrow{k} & W & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \longrightarrow & 0. \end{array}$$

Since $wq = 1$, vp is an isomorphism, whence $Z \in \mathscr{C}$, $V = \operatorname{Im} p \oplus \operatorname{Ker} v$, and $\operatorname{Im} h \subseteq \operatorname{Im} p$. If v is not an isomorphism, then h is not a minimal morphism, which contradicts the minimality of all h_j's because a direct sum of minimal morphisms is a minimal morphism. Thus v is an isomorphism, w is an isomorphism, and W is indecomposable. Since t is a minimal morphism, so is k, whence none of the X_j's is Ext-injective. Therefore $r = 1$, X is indecomposable, and s is left almost split.

If L is indecomposable non-Ext-injective in $\mathscr{C}$, an exact sequence $0 \to L \to^h V \to^k W \to 0$, where h is a minimal morphism corresponding to the left almost split morphism f given by condition (iii), is almost split in $\mathscr{C}$. The proof is similar to the preceding argument. ∎

Recall that when Δ is an artin algebra, the basic existence theorem of [3] states that if $\mathscr{C}$ is a dualizing R-variety closed under extensions, then $\mathscr{C}$ has almost split sequences. Since Ext is an additive bifunctor, if a subcategory is closed under extensions, then both the Ext-projective and the Ext-injective modules in it are closed under direct sums. Thus our theorem may be viewed as an extension of the abovementioned result of Auslander and Smalø.

2. Ext-Projective Modules and Right Almost Split Morphisms in $\mathbf{p}(A, \Lambda)$

We describe the Ext-projective modules in $\mathbf{p}(A, \Lambda)$, show they are closed under direct sums, and prove $\mathbf{p}(A, \Lambda)$ satisfies conditions (i) and (ii) of Theorem 1.2 for right almost split morphisms.

For a left A-module N, consider a short relative projective resolution

$$0 \longrightarrow \Omega(N) \longrightarrow A \otimes_\Lambda N \xrightarrow{m} N \longrightarrow 0, \tag{2.1}$$

where m is the multiplication map, and $\Omega(N) = \operatorname{Ker} m$.

Lemma 2.1. *For any $N \in A$-mod, $\Omega(N)$ is injective in Λ-mod.*

Proof. We note first that the exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{i} A \longrightarrow \operatorname{Coker} i \longrightarrow 0$$

of Λ-bimodules splits as a sequence of right Λ-modules because, as a right Λ-module, Coker i is projective. Tensoring with N, we obtain the exact sequence of left Λ-modules

$$0 \longrightarrow N \xrightarrow{j} A \otimes_\Lambda N \xrightarrow{p} (\operatorname{Coker} i) \otimes_\Lambda N \longrightarrow 0, \tag{2.2}$$

where j is the map $N \simeq \Lambda \otimes_\Lambda N \to^{i \otimes 1} A \otimes_\Lambda N$. Since $mj = 1$, the sequence (2.2) splits. Denote by h: (Coker i) $\otimes_\Lambda N \to A \otimes_\Lambda N$ a unique morphism of left Λ-modules satisfying $ph = 1$ and $jm + hp = 1$. Then we get the exact sequence

$$0 \longrightarrow (\text{Coker}) \, i) \otimes_\Lambda N \xrightarrow{h} A \otimes_\Lambda N \xrightarrow{m} N \longrightarrow 0,$$

which, as a sequence of left Λ-modules, is isomorphic to (2.1). To show $\Omega(N)$ is injective as a left Λ-module, it suffices to prove so is $(\text{Coker}\, i) \otimes_\Lambda N \simeq \bigoplus_{s=1}^{n} I_s \otimes_R P_s \otimes_\Lambda N$. But $P_s \otimes_\Lambda N$ is finitely generated projective as an R-module because N can be viewed as a $\Lambda - R$ bimodule which is finitely generated projective as an R-module, and P_s is a finitely generated projective right Λ-module. Therefore $P_s \otimes_\Lambda N$ is a direct summand of a free R-module of finite rank, whence $I_s \otimes_R P_s \otimes_\Lambda N$ is injective in Λ-mod. ▮

LEMMA 2.2. *Let $0 \to L \to^f M \to^g N \to 0$ be an exact sequence in A*-mod *with $N \in \mathbf{p}(A, \Lambda)$. Then*:

(a) *$\Omega(N)$ is an* Ext-*injective module in* $\mathbf{p}(A, \Lambda)$.

(b) *There exists an exact sequence $0 \to A \otimes_\Lambda L \otimes \Omega(N) \to^h A \otimes_\Lambda M \to^{gm} N \to 0$ of A-modules, where $m: A \otimes_\Lambda M \to M$ is the multiplication map, and the restriction of h to $A \otimes_\Lambda L$ coincides with $1 \otimes f$: $A \otimes_\Lambda L \to A \otimes_\Lambda M$.*

Proof. (a) It is well known that sequence (2.1) splits as a sequence of left Λ-modules. Since N is relatively projective, it splits as a sequence of A-modules. Hence $\Omega(N) \in \mathbf{p}(A, \Lambda)$, and Lemma 2.1 implies $\Omega(N)$ is Ext-injective.

(b) Consider the following commutative diagram in A-mod.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(L) & \longrightarrow & \Omega(M) & \longrightarrow & \Omega(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \otimes_\Lambda L & \xrightarrow{1 \otimes f} & A \otimes_\Lambda M & \xrightarrow{1 \otimes g} & A \otimes_\Lambda N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Since the columns and the two bottom rows are exact (remember, A is projective as a right Λ-module), the top row is exact. It is split exact in Λ-mod by Lemma 2.1, hence it is split in A-mod because $\Omega(N) \in \mathbf{p}(A, \Lambda)$. We have $\Omega(M) = (1 \otimes f)(\Omega(L)) \oplus N_1$, where $N_1 \simeq \Omega(N)$. It is easy to see that Ker $(gm) = (1 \otimes f)(A \otimes_\Lambda L) + \Omega(M) = (1 \otimes f)(A \otimes_\Lambda L) \oplus N_1$. ∎

PROPOSITION 2.3. *Let N be an indecomposable module in $\mathbf{p}(A, \Lambda)$, and $M \to^g N$ a right almost split morphism in A-mod. Then:*

(a) *gm: $A \otimes_\Lambda M \to N$ is right almost split in $\mathbf{p}(A, \Lambda)$.*

(b) *Suppose N is not projective in A-mod, so that g is onto and we have an exact sequence $0 \to L \to^f M \to^g N \to 0$ of A-modules. Then there exists an exact sequence $0 \to X \to^s Y \to^t N \to 0$, where Y is a direct summand of $A \otimes_\Lambda M$; t is a minimal right almost split morphism in $\mathbf{p}(A, \Lambda)$; X belongs to $\mathbf{p}(A, \Lambda)$ and has no direct summands which are injective in A-mod. In fact, X is a direct summand of $A \otimes_\Lambda L$.*

(c) *A right almost split morphism $Y \to N$ in $\mathbf{p}(A, \Lambda)$ is surjective if and only if N is not* Ext-*projective.*

(d) *N is projective in A-mod if and only if it is* Ext-*projective in $\mathbf{p}(A, \Lambda)$.*

(e) *The* Ext-*projective modules in $\mathbf{p}(A, \Lambda)$ are closed under direct sums.*

Proof. (a) Since g is not a splittable epimorphism, neither is gm. We show that if $U \in \mathbf{p}(A, \Lambda)$ and h: $U \to N$ is not a splittable epimorphism, then there exists a morphism j: $U \to A \otimes_\Lambda M$ satisfying $h = (gm)\, j$. Really, since g is right almost split, $h = gk$ for some k: $U \to M$. Since m: $A \otimes_\Lambda M \to M$ has a right inverse in Λ-mod, and U is relatively projective, $k = mj$ for some j: $U \to A \otimes_\Lambda M$. Substituting, we get $h = (gm)\, j$.

(b) Let t: $Y \to N$ be a minimal right almost split morphism in $\mathbf{p}(A, \Lambda)$ corresponding to gm [2]. Then we get an exact sequence $0 \to X \to^s Y \to^t N \to 0$ with X a direct summand of Ker(gm), and Y a direct summand of $A \otimes_\Lambda M$. By Lemma 2.2(b), Ker $(gm) \simeq A \otimes_\Lambda L \oplus \Omega(N)$ so that $X \in \mathbf{p}(A, \Lambda)$. Show X has no direct summands injective in A-mod.

Assume, to the contrary, that U is such a summand. Let $U \to^q X \to^p U$ be morphisms in A-mod satisfying $pq = 1$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{s} & Y & \xrightarrow{t} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} & & \| & & \\ 0 & \longrightarrow & U & \xrightarrow{h} & V & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

of Λ-modules, where the bottom row splits over Λ because U is injective. Since N is relatively projective, the bottom row splits over A; i.e., there exists a morphism $k: V \to U$ in A-mod satisfying $kh = 1$. Then $krsq = khpq = 1$, so that U is a direct summand of Y, contradicting the fact that t is minimal.

Since $\Omega(N)$ is injective in Λ-mod by Lemma 2.1, X has no summands of $\Omega(N)$; i.e., X is a direct summand of $A \otimes_\Lambda L$.

(c) Follows from (a) and (b).

(d) The necessity is clear. The sufficiency follows from (b).

(e) Follows from (d). ∎

3. Dualities, Adjoint Monads, and Kleisli Categories

This section contains certain facts on category theory and categories of modules which will be used in Section 4 to prove the existence of almost split sequences. Some of them, i.e., the theorem that if a monad is a right adjoint of a comonad, then their Kleisli categories are isomorphic [14], seem to be interesting on their own.

We begin with the following, apparently well-known, statement.

LEMMA 3.1. *Let $\mathfrak{A}_i$ be a preadditive category in which the idempotents split, $\mathfrak{B}_i$ a full subcategory of $\mathfrak{A}_i$, and $\mathfrak{C}_i$ the full subcategory of $\mathfrak{A}_i$ determined by the direct summands of all objects of $\mathfrak{B}_i$, $i = 1, 2$. If $\Phi: \mathfrak{B}_1 \to \mathfrak{B}_2$ is an additive equivalence of categories, then it can be extended to an additive equivalence of categories $\Psi: \mathfrak{C}_1 \to \mathfrak{C}_2$.*

Proof. For each object $C \in \mathfrak{C}_1$, choose a pair of morphisms $C \to^q B \to^p C$, where $B \in \mathfrak{B}_1$ and $pq = 1_C$. If $C \in \mathfrak{B}_1$, we put $B = C$ and $p = q = 1_C$. Then choose an object $C' \in \mathfrak{C}_1$ and morphisms $C' \to^{q'} B \to^{p'} C'$ satisfying $p'q' = 1_{C'}$ and $qp + q'p' = 1_B$. Since the idempotents split in $\mathfrak{A}_2$, $\Phi(qp) = vu$ and $\Phi(q'p') = v'u'$ for some morphisms $N \to^v \Phi B \to^u N$ and $N' \to^{v'} \Phi B \to^{u'} N'$ in $\mathfrak{A}_2$ satisfying $uv = 1_N$ and $u'v' = 1_{N'}$. If $\Phi(qp) = 1_{\Phi B}$, we choose $N = \Phi B$, $u = v = 1$, and $N' = 0$. Since Φ is an additive functor, $vu + v'u' = \Phi(qp) + \Phi(q'p') = 1_{\Phi B}$.

Define now $\Psi C = N$ for $C \in \mathfrak{C}_1$, and for each $f: C_1 \to C_2$ in $\mathfrak{C}_1$ put

$$\Psi f = u_2 \Phi(q_2 f p_1) v_1. \tag{3.1}$$

Here the morphisms p_i, q_i for C_i, as well as u_i, v_i for N_i, are chosen as described above, $i = 1, 2$. Note that $\Psi C = \Phi C$ if $C \in \mathfrak{B}_1$, and $\Psi f = \Phi f$ if f is in $\mathfrak{B}_1$.

Since

$$\Phi(q_2 f p_1) = \Phi(q_2 p_2)\, \Phi(q_2 f p_1)\, \Phi(q_1 p_1) = v_2 u_2 \Phi(q_2 f p_1)\, v_1 u_1$$

and, likewise,

$$\Phi(q'_2 f p'_1) = v'_2 u'_2 \Phi(q'_2 f p'_1)\, v'_1 u'_1,$$

we get

$$\begin{aligned} u'_2 \Phi(q_2 f p_1) &= 0, \qquad \Phi(q_2 f p_1)\, v'_1 = 0, \\ u_2 \Phi(q'_2 f p'_1) &= 0, \qquad \Phi(q'_2 f p'_1)\, v_1 = 0. \end{aligned} \tag{3.2}$$

This implies immediately that Ψ is a functor. Using (3.2) again, we see that if $\Psi f = 0$, then $\Phi(q_2 f p_1) = 0$, whence $q_2 f p_1 = 0$ because Φ is faithful. Multiplying by p_2 on the left and by q_1 on the right, we get $f = 0$; i.e., Ψ is faithful. To show Ψ is full, note that if $h: N_1 \to N_2$ is a morphism in $\mathfrak{C}_2$, then $\Psi(p_2 \Phi^{-1}(v_2 h u_1)\, q_1) = h$. The verification is straightforward. Finally, show Ψ is dense. If $M \in \mathfrak{C}_2$, then there is a pair of morphisms $M \to^t \Phi B \to^s M$, where $B \in \mathfrak{B}_1$ and $st = 1$, because Φ is dense. Since the idempotents split in $\mathfrak{A}_1$, $\Phi^{-1}(ts) = qp$ for some $C \to^q B \to^p C$, where $pq = 1$ and $C \in \mathfrak{C}_1$. Since Ψ extends Φ, we have $\Psi q \Psi p = \Psi(qp) = \Phi(qp) = ts$ and $\Psi p \Psi q = 1$; i.e., the pair of morphisms $\Psi C \to^{\Psi q} \Phi B \to^{\Psi p} \Psi C$ provides another splitting for the idempotent ts in $\mathfrak{A}_2$. Clearly, M is isomorphic to ΨC in $\mathfrak{A}_2$. Thus Ψ is an equivalence of categories.

The additivity of Ψ is obvious. ∎

Recall [2] that the contravariant additive functor $D = \operatorname{Hom}_R(-, R)$ is a duality between Λ-mod and Λ^{op}-mod, as well as between Λ-mod and C-comod. Then [13, Theorem 4.4, p. 181], together with Lemma 3.1, implies

THEOREM 3.2. *The contravariant functor D is an additive duality between* $\mathbf{p}(A, \Lambda)$ *and* $\mathbf{i}(C, \Lambda^{\text{op}})$.

We now describe the Λ-rings A for which the Λ^{op}-coring C corresponds to a free BOCS [7].

LEMMA 3.3. *Let B be a* Λ*-bimodule.*

(a) *If* $B \simeq I \otimes_R P$ *with I injective in* Λ-mod *and P projective in* mod-Λ, *then* $B^* = \operatorname{Hom}_{-\Lambda}(B, \Lambda) \simeq W \otimes_R Q$ *with W projective in* Λ^{op}-mod *and Q projective in* mod-Λ^{op}.

(b) *If* $B \simeq W \otimes_R Q$ *with W projective in* Λ-mod *and Q projective in* mod-Λ, *then*:

(i) *$B^* \simeq I \otimes_R P$ with I injective in* Λ^{op}-mod *and P projective in* mod-Λ^{op}.

(ii) $^*B = \mathrm{Hom}_\Lambda\ (B, \Lambda) \simeq I_1 \otimes_R P_1$ *with I_1 injective in* Λ-mod *and P_1 projective in* mod-Λ.

Proof. Since B is finitely generated projective as a right Λ-module in both (a) and (b), [13, Lemma 4.3, p. 181] gives an isomorphism of right Λ-modules $D(B \otimes_\Lambda M) \simeq DM \otimes_\Lambda B^*$, functorial in $M \in \Lambda$-mod. Putting $B = I \otimes_R P$ and $M = D\Lambda$, we have $B^* \simeq D(D\Lambda) \otimes_\Lambda B^* \simeq D(B \otimes_\Lambda D\Lambda) \simeq D(I \otimes_R P \otimes_\Lambda D\Lambda) \simeq DI \otimes_R D(P \otimes_\Lambda D\Lambda)$, the latter isomorphism being based on [13, Proposition 4.1, p. 179]. Here DI is projective in Λ^{op}-mod, and $D(P \otimes_\Lambda D\Lambda)$ is projective in mod-Λ^{op} because P is a direct summand of a free module, and $D\Lambda$ is injective. (a) is proved.

To prove (b), note that (i) $\Rightarrow$ (ii). The proof of (i) is similar to the proof of (a). ∎

COROLLARY 3.4. *The following are equivalent:*

(a) *The sequence*

$$0 \longrightarrow \Lambda \xrightarrow{\ i\ } A \longrightarrow \mathrm{Coker}\, i \longrightarrow 0$$

is exact (i.e., the map i is injective), and Coker $i \simeq \bigoplus_{s=1}^n I_s \otimes_R P_s$ *with I_s injective in* Λ-mod *and P_s projective in* mod-Λ.

(b) *The sequence*

$$0 \longrightarrow \mathrm{Ker}\, \varepsilon \longrightarrow C \xrightarrow{\ \varepsilon\ } \Lambda^{\mathrm{op}} \longrightarrow 0$$

is exact (i.e., the map ε is surjective), and Ker $\varepsilon \simeq \bigoplus_{s=1}^n W_s \otimes_R Q_s$ *with W_s projective in* Λ^{op}-mod *and Q_s projective in* mod-Λ^{op}.

Proof. If the first or second sequence is exact, it is split as a sequence of right Λ- or Λ^{op}-modules, respectively. Therefore, applying to the sequences the contravariant additive functors $\mathrm{Hom}_{-\Lambda}(-, \Lambda)$ or $\mathrm{Hom}_{-\Lambda^{\mathrm{op}}}(-, \Lambda^{\mathrm{op}})$, we obtain exact sequences of Λ^{op}- or Λ-bimodules, respectively. It remains to use Lemma 3.3, [13, Corollary 4.2, pp. 179–180], and the fact that $\varepsilon = \mathrm{Hom}_{-\Lambda}(i, \Lambda)$. ∎

Prove now that $\mathbf{i}(C, \Lambda^{\mathrm{op}})$ is equivalent to $\mathbf{p}(\Lambda_1, \Lambda^{\mathrm{op}})$ if condition (b) of Corollary 3.4 is satisfied. We have to turn to general category theory; the notions used here are defined, for example, in Chapters IV and VI of [17].

Let $\mathfrak{A}$ be a category, $\mathbf{F} = (F, \mu, \eta)$ a monad in $\mathfrak{A}$ with multiplication $\mu: F^2 \to F$ and unit $\eta: I \to F$, $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad in $\mathfrak{A}$ with comultiplication $\delta: G \to G^2$ and counit $\varepsilon: G \to I$. Suppose that the monad $\mathbf{F}$ is a right adjoint of the comonad $\mathbf{G}$; i.e., the functor F is a right adjoint of the functor G under an adjunction $\alpha\colon \mathfrak{A}(GX, Y) \to \mathfrak{A}(X, FY)$, and the following two diagrams commute for all X, $Y \in \mathfrak{A}$.

$$\begin{array}{ccc} \mathfrak{A}(GX, Y) & \xrightarrow{\alpha} & \mathfrak{A}(X, FY) \\ {\scriptstyle \mathfrak{A}(\varepsilon X, Y)}\uparrow & & \uparrow{\scriptstyle \mathfrak{A}(X, \eta Y)} \\ \mathfrak{A}(X, Y) & = & \mathfrak{A}(X, Y) \end{array} \tag{3.3}$$

$$\begin{array}{ccc} \mathfrak{A}(GX, Y) & \xrightarrow{\alpha} & \mathfrak{A}(X, FY) \\ {\scriptstyle \mathfrak{A}(\delta X, Y)}\uparrow & & \uparrow{\scriptstyle \mathfrak{A}(X, \mu Y)} \\ \mathfrak{A}(G^2X, Y) & \xrightarrow{\alpha^2} & \mathfrak{A}(X, F^2Y). \end{array} \tag{3.4}$$

We prove that under these circumstances the Kleisli categories $\mathfrak{A}_F$ and $\mathfrak{A}_G$ are isomorphic [14]. For the convenience of the reader, we give a full treatment. Note that the diagrams (3.3) and (3.4) are obtained by reversing the arrows in the diagrams from [10, p. 390, top], defining a monad which is a left adjoint of a comonad. If **F** is a left adjoint of **G**, then [10, Proposition 3.3, p. 389] shows that the Eilenberg–Moore categories $\mathfrak{A}^F$ and $\mathfrak{A}^G$ are isomorphic.

In the remaining part of this section, we fix the monad **F** and its left adjoint comonad **G** which make the diagrams (3.3) and (3.4) commute.

Define a map $\Theta\colon \mathfrak{A}_G \to \mathfrak{A}_F$ as follows. For each $X \in \mathfrak{A}$ put $\Theta X = X$. For each morphism $f\colon GX \to Y$ in $\mathfrak{A}_G$ put

$$\Theta f = \alpha(f)\colon X \to FY. \tag{3.5}$$

THEOREM 3.5. *$\Theta\colon \mathfrak{A}_G \to \mathfrak{A}_F$ is an isomorphism of categories.*

Proof. Show first that Θ is a functor. For each identity morphism $\varepsilon X\colon GX \to X$ in $\mathfrak{A}_G$ we have $\Theta(\varepsilon X) = \alpha(\varepsilon X) = \eta X$, using the diagram (3.3). Hence Θ preserves identity morphisms. Given $g\colon GY \to Z$ and $f\colon GX \to Y$, the composite morphism in $\mathfrak{A}_G$ is $gGf\delta X\colon GX \to Z$. Then

$$\begin{aligned} \Theta(gGf\delta X) &= \alpha(gGf\delta X) = \mu Z\alpha^2(gGf) = \mu Z\alpha(\alpha(gGf)) \\ &= \mu Z\alpha(\alpha(g)f) = \mu ZF(\alpha(g))\,\alpha(f), \end{aligned}$$

using the diagram (3.4). But $\mu ZF(\alpha(g))\,\alpha(f)$ is the composite of $\alpha(g)$ and $\alpha(f)$ in $\mathfrak{A}_F$.

To finish the proof, we note that the functor $\mathfrak{A}_F \to \mathfrak{A}_G$ given by $X \mapsto X$ for all $X \in \mathfrak{A}$, and $g \mapsto \alpha^{-1}(g)$ for each $g\colon X \to FY$, is an inverse of Θ. ∎

Let $\mathfrak{A}_0^F$ be the full subcategory of $\mathfrak{A}^F$ consisting of the free F-algebras, i.e., of the algebras of the form $(FX, \mu X)$ with $X \in \mathfrak{A}$. Likewise, denote by $\mathfrak{A}_0^G$ the full subcategory of $\mathfrak{A}^G$ consisting of the cofree G-coalgebras, i.e., of the coalgebras of the form $(GX, \delta X)$ with $X \in \mathfrak{A}$. Consider the following commutative diagram of categories and functors whose rows are canonical adjunctions for the Eilenberg–Moore and Kleisli categories involved, with S_i a left adjoint of T_i, $i = 1, 2, 3, 4$.

$$\begin{array}{ccccccccccc}
\mathfrak{A} & \xleftarrow{S_1} & \mathfrak{A}^F & \xrightarrow{T_1} & \mathfrak{A} & \qquad & \mathfrak{A}^G & \xleftarrow{S_3} & \mathfrak{A} & \xrightarrow{T_3} & \mathfrak{A}^G \\
\| & & \uparrow{\scriptstyle U} & & \| & & \uparrow{\scriptstyle V} & & \| & & \uparrow{\scriptstyle V} \\
\mathfrak{A} & \xleftarrow{S_2} & \mathfrak{A}_F & \xrightarrow{T_2} & \mathfrak{A} & & \mathfrak{A}_G & \xleftarrow{S_4} & \mathfrak{A} & \xrightarrow{T_4} & \mathfrak{A}_G.
\end{array}$$

Here U and V are the unique functors making the diagrams commute. It is well known [17, Exercises 1, 2, p. 144] that U, V are fully faithful, and $U(\mathfrak{A}_F) = \mathfrak{A}_0^F$, $V(\mathfrak{A}_G) = \mathfrak{A}_0^G$; in other words, the restriction of U (which we still denote by U) $\mathfrak{A}_F \to \mathfrak{A}_0^F$ is an equivalence, and so is $V: \mathfrak{A}_G \to \mathfrak{A}_0^G$. Explicitly,

$$\begin{aligned} UX &= (FX, \mu X), & Uf &= \mu Y Ff, \\ U^{-1}(FX, \mu X) &= X, & U^{-1}g &= g\eta X, \end{aligned} \tag{3.6}$$

for all $X, Y \in \mathfrak{A}$, $f: X \to FY$ in $\mathfrak{A}$, and all morphisms $g: (FX, \mu X) \to (FY, \mu Y)$ of free F-algebras. V and V^{-1} are defined dually. We obtain

COROLLARY 3.6. *In the setting of Theorem* 3.5, $\Phi = V\Theta^{-1}U^{-1}$: $\mathfrak{A}_0^F \to \mathfrak{A}_0^G$ *is an equivalence of categories.*

PROPOSITION 3.7. *Suppose* $\mathfrak{A}$ *is an abelian category, F preserves cokernels, and G preserves kernels. Then*

(a) *F and G are additive functors.*

(b) $\mathfrak{A}^F$ *and* $\mathfrak{A}^G$ *are abelian categories.*

(c) *The equivalence* $\Phi = V\Theta^{-1}U^{-1}$: $\mathfrak{A}_0^F \to \mathfrak{A}_0^G$, *described in Corollary* 3.6, *is additive.*

(d) *Denote by* add $\mathfrak{A}_0^F$ (add $\mathfrak{A}_0^G$) *the full subcategory of* $\mathfrak{A}^F(\mathfrak{A}^G)$ *determined by the direct summands of objects in* $\mathfrak{A}_0^F$ $(\mathfrak{A}_0^G)$. *Let* Ψ: add $\mathfrak{A}_0^F \to$ add $\mathfrak{A}_0^G$ *be the equivalence of categories constructed from* Φ *according to Lemma* 3.1. *If* $C_1 \to^f C_2 \to^g C_3 \to 0$ *is an exact sequence in* $\mathfrak{A}^F$ *with* $C_i \in$ add $\mathfrak{A}_0^F$, $i = 1, 2, 3$, *then* $\Psi C_1 \to^{\Psi f} \Psi C_2 \to^{\Psi g} \Psi C_3 \to 0$ *is an exact sequence in* $\mathfrak{A}^G$.

(e) *If* $f: X \to Y$ *is monic in* $\mathfrak{A}$, *then* $\Phi(Ff): GX \to GY$ *is monic in* $\mathfrak{A}^G$.

Proof. (a) Follows from [9, Proposition 1.4, p. 14].

(b) This is just [10, Proposition 5.3, p. 395].

(c) It follows from (3.5), using [17, Chap. IV, Theorem 3, p. 83], that Θ is an additive isomorphism; then so is Θ^{-1}. V and U^{-1} are also additive, as follows from (3.6).

(d) Here, as in the proof of Lemma 3.1, we have $C_i \to^{q_i} B_i \to^{p_i} C_i$, where $B_i = FX_i$ for some $X_i \in \mathfrak{A}$ and $p_i q_i = 1$. It follows from the formulae (3.1), (3.5), and (3.6) that

$$\begin{aligned} &\Phi(q_2 f p_1) = G(\alpha^{-1}(q_2 f p_1 \eta X_1))\, \delta X_1, && \Psi f = u_2 \Phi(q_2 f p_1)\, v_1, \\ &\Phi(q_3 g p_2) = G(\alpha^{-1}(q_3 g p_2 \eta X_2))\delta X_2, && \Psi g = u_3 \Phi(q_3 g p_2)\, v_2, \end{aligned} \tag{3.7}$$

where $\Psi C_i \to^{v_i} \Phi B_i \to^{u_i} \Psi C_i$, $\Phi B_i = GX_i$, and $u_i v_i = 1$, $i = 1, 2, 3$. Remember that we also have morphisms p_i', q_i', u_i', v_i' satisfying $p_i' q_i' = 1$, $q_i p_i + q_i' p_i' = 1$, $u_i' v_i' = 1$, $v_i u_i + v_i' u_i' = 1$ for all i.

Assume now that g is a cokernel of f in $\mathfrak{A}^F$, and show Ψg is a cokernel of Ψf in $\mathfrak{A}^G$. As follows from [10, Proposition 5.2, pp. 394–395], it suffices to prove that Ψg is a cokernel of Ψf in $\mathfrak{A}$. Let $h\colon \Psi C_2 \to Y$ be a morphism in $\mathfrak{A}$ satisfying $h\Psi f = hu_2 \Phi(q_2 f p_1)\, v_1 = 0$. Multiplying by u_1 on the right and using (3.2), we get $0 = hu_2 \Phi(q_2 f p_1) = hu_2 G(\alpha^{-1}(q_2 f p_1 \eta X_1))\, \delta X_1$. The latter expression is the composite of hu_2 and $\alpha^{-1}(q_2 f p_1 \eta X_1)$ in $\mathfrak{A}_G$. By Theorem 3.5, the composite of $\alpha(hu_2)$ and $q_2 f p_1 \eta X_1$ in $\mathfrak{A}_F$ is zero, i.e., $\mu YF(\alpha(hu_2))\, q_2 f p_1 \eta X_1 = 0$. Using the equivalence $U\colon \mathfrak{A}_F \to \mathfrak{A}_0^F$ given by (3.6), we get $\mu YF(\alpha(hu_2))\, q_2 f p_1 = 0$, which yields $\mu YF(\alpha(hu_2))\, q_2 f = 0$, after the multiplication by q_1 on the right. Since g is a cokernel of f in $\mathfrak{A}^F$,

$$\mu YF(\alpha(hu_2))\, q_2 = kg \tag{3.8}$$

for a unique $k\colon C_3 \to FY$ in $\mathfrak{A}^F$. We claim that

$$\mu YF(\alpha(hu_2))\, q_2' p_2' = 0. \tag{3.9}$$

To prove this, it suffices to show that $\mu YF(\alpha(hu_2))\, q_2' p_2' \eta X_2 = 0$, using U. Theorem 3.5 implies that the latter equality is equivalent to

$$hu_2 G(a^{-1}(q_2' p_2' \eta X_2))\, \delta X_2 = hu_2 \Phi(q_2' p_2') = hu_2 \Phi(q_2' 1_{B_2} p_2') = 0,$$

which holds because of (3.7) and (3.2). Multiplying (3.8) by p_2 on the right and adding to (3.9), we get

$$\mu YF(\alpha(hu_2)) = kgp_2. \tag{3.10}$$

Multiplying by ηX_2 on the right and using properties of the natural transformations η and μ, we obtain $\alpha(hu_2) = kgp_2\eta X_2$, so that $hu_2 = \alpha^{-1}(kp_3q_3gp_2\eta X_2)$.

Since kp_3 is a morphism in $\mathfrak{A}_0^F$, (3.6) implies $kp_3 = \mu YF(kp_3\eta X_3)$. Using the diagram (3.4), we obtain

$$\begin{aligned} hu_2 &= \alpha^{-1}(\mu YF(kp_3\eta X_3)\, q_3\, gp_2\eta X_2) \\ &= \alpha^{-1}[\alpha^{-1}(F(kp_3\eta X_3)\, q_3\, gp_2\eta X_2)]\, \delta X_2 \\ &= \alpha^{-1}[kp_3\eta X_3\alpha^{-1}(q_3\, gp_2\eta X_2)]\, \delta X_2 \\ &= \alpha^{-1}(kp_3\eta X_3)\, G(\alpha^{-1}(q_3\, gp_2\eta X_2))\, \delta X_2 \\ &= \alpha^{-1}(kp_3\eta X_3)\, v_3u_3G(\alpha^{-1}(q_3\, gp_2\eta X_2))\, \delta X_2, \end{aligned}$$

the last step being based on (3.2) and (3.7). Multiplying by v_2 on the right, we obtain $h = \alpha^{-1}(kp_3\eta X_3)\, v_3\, \Psi g$, using (3.1) and (3.7). To finish the proof, it remains to show that $h = j\Psi g$, for some morphism $j\colon \Psi C_3 \to Y$ in $\mathfrak{A}$, implies

$$j = \alpha^{-1}(kp_3\eta X_3)\, v_3. \tag{3.11}$$

Multiplying by u_2 on the right, we get

$$\begin{aligned} hu_2 &= j\Psi gu_2 = ju_3\Phi(g_3\, gp_2)\, v_2u_2 = ju_3\Phi(q_3\, gp_2) \\ &= ju_3G(\alpha^{-1}(q_3gp_2\eta X_2))\, \delta X_2, \end{aligned}$$

using (3.2) and (3.7). Then, using (3.4), we get

$$\begin{aligned} \alpha(hu_2) &= \alpha(ju_3G(\alpha^{-1}(q_3gp_2\eta X_2))\, \delta X_2) = \mu Y\alpha^2(ju_3G(\alpha^{-1}(q_3\, gp_2\eta X_2))\, \delta X_2) \\ &= \mu Y\alpha[\alpha(ju_3G(\alpha^{-1}(q_3\, gp_2\eta X_2))\, \delta X_2)] \\ &= \mu Y\alpha[\alpha(ju_3)\, \alpha^{-1}(q_3\, gp_2\eta X_2)] \\ &= \mu YF(\alpha(ju_3))\, q_3\, gp_2\eta X_2. \end{aligned}$$

Passing to $\mathfrak{A}^F$ via U, we obtain $\mu YF(\alpha(hu_2)) = \mu YF(\alpha(ju_3))\, q_3\, gp_2$. Comparing with (3.10), we get $\kappa = \mu YF(\alpha(ju_3))\, q_3$ because g and p_2 are epi in $\mathfrak{A}^F$. Multiplying by p_3 on the right, we obtain $kp_3 = \mu YF(\alpha(ju_3))\, q_3\, p_3$. But $\mu YF(\alpha(ju_3))\, q'_3\, p'_3 = 0$ by (3.9), whence $kp_3 = \mu YF(\alpha(ju_3))$. To obtain (3.11), it suffices to multiply the latter equality by ηX_3 on the right, then use properties of the natural transformations η, μ, and $u_3v_3 = 1$.

(e) We have

$$\begin{aligned} \Phi(Ff) &= G(\alpha^{-1}(Ff\eta X))\, \delta X = G(a^{-1}(\eta Yf))\, \delta X \\ &= G(f\varepsilon X)\, \delta X = GfG\varepsilon X\delta X = Gf, \end{aligned}$$

using (3.3). Since f is monic in $\mathfrak{A}$, f is a kernel because $\mathfrak{A}$ is abelian. Hence Gf is a kernel, by assumption, therefore monic in $\mathfrak{A}$. Then Gf is monic in $\mathfrak{A}^G$. ∎

PROPOSITION 3.8. *Let K be a Γ-coring which, as a left Γ-module, is finitely generated projective. Then*:

(a) $^*K = \mathrm{Hom}_{\Gamma-}(K, \Gamma)$ *is a Γ-ring.*

(b) *There exists an additive equivalence of categories Φ*: Induc $^*K \to$ Induc K *whose restriction $\Phi|$*induc $^*K$: induc $^*K \to$ induc K *is an equivalence. The equivalence Φ can be extended to an additive equivalence Ψ: $\mathbf{P}(^*K, \Gamma) \to \mathbf{I}(K, \Gamma)$, whose restriction $\Psi|\mathbf{p}(^*K, \Gamma)$: $\mathbf{p}(^*K, \Gamma) \to \mathbf{i}(K, \Gamma)$ is an a equivalence of categories.*

(c) *Let K be flat as a right Γ-module. Then*:

(i) *If $L \to M \to N \to 0$ is an exact sequence of left $^*K$-modules with $L, M, N \in \mathbf{P}(^*K, \Gamma)$, then $\Psi L \to \Psi M \to \Psi N \to 0$ is an exact sequence of left K-comodules.*

(ii) *For each injective module I in Γ*-Mod, $^*K \otimes_\Gamma I$ *is injective as a left Γ-module, hence* Ext-*injective in* $\mathbf{P}(^*K, \Gamma)$.

(iii) *An induced module $^*K \otimes_\Gamma M$ is* Ext-*injective in* $\mathbf{P}(^*K, \Gamma)$ *if and only if it is a direct summand of $^*K \otimes_\Gamma I$, where I is injective in Γ*-Mod.

Proof. (a) Let $\mu: K \to K \otimes_\Gamma K$ and $\varepsilon: K \to \Gamma$ be the comultiplication and counit, respectively, of K. For $f, g \in {}^*K$, define the product gf as the composite of the maps $K \to^\mu K \otimes_\Gamma K \to^{1 \otimes g} K \otimes_\Gamma \Gamma = K \to^f \Gamma$, and the structure map $\Gamma \to {}^*K$ as the map sending each $\gamma \in \Gamma$ to the morphism $\gamma\varepsilon$ (remember, $^*K$ is a Γ-bimodule). $^*K$ is the opposite ring of the one defined in [21, 3.2. Proposition (a)(c), p. 398].

(b) For $M, N \in \Gamma$-Mod we have the following natural isomorphisms of abelian groups

$$\begin{aligned}\mathrm{Hom}_\Gamma(K \otimes_\Gamma M, N) &\simeq \mathrm{Hom}_\Gamma(M, \mathrm{Hom}_{\Gamma-}(K, N))\\ &\simeq \mathrm{Hom}_\Gamma(M, {}^*K \otimes_\Gamma N) \qquad (3.12)\end{aligned}$$

because K is finitely generated projective as a left Γ-module. Hence the monad in Γ-Mod induced by the functor $^*K \otimes_\Gamma -$ is a right adjoint of the comonad in Γ-Mod induced by the functor $K \otimes_\Gamma -$. Really, the commutativity of the diagrams (3.3) and (3.4) is obtained by applying the isomorphism (3.12), which is functorial in the Γ-bimodule K, to the morphisms $\varepsilon: K \to \Gamma$ and $\mu: K \to K \otimes_\Gamma K$ of Γ-bimodules. $\Phi = V\Theta^{-1}U^{-1}$ is an additive equivalence Induc $^*K \to$ Induc K by Proposition 3.7(c), and the properties of Θ, as defined by (3.5), ensure that $\Phi|\text{induc } {*K}$ is an equivalence. The rest follows from Lemma 3.1.

(c) The endofunctor $K \otimes_\Gamma \ldots$ of Γ-Mod preserves kernels, hence Proposition 3.7(d) implies (i).

(ii) Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in Γ-Mod. Since K is flat and I injective, the following two sequences are exact in Γ-Mod:

$$0 \to K \otimes_\Gamma X \to K \otimes_\Gamma Y \to K \otimes_\Gamma Z \to 0,$$

$$0 \to \mathrm{Hom}_{\Gamma\text{-}}(K \otimes_\Gamma Z, I) \to \mathrm{Hom}_{\Gamma\text{-}}(K \otimes_\Gamma Y, I) \to \mathrm{Hom}_{\Gamma\text{-}}(K \otimes_\Gamma X, I) \to 0.$$

Using the natural isomorphism (3.12), we conclude that the latter exact sequence is isomorphic to the sequence

$$0 \to \mathrm{Hom}_{\Gamma\text{-}}(Z, {*K} \otimes_\Gamma I) \to \mathrm{Hom}_{\Gamma\text{-}}(Y, {*K} \otimes_\Gamma I)$$
$$\to \mathrm{Hom}_{\Gamma\text{-}}(X, {*K} \otimes_\Gamma I) \to 0.$$

Therefore ${*K} \otimes_\Gamma I$ is injective in Γ-Mod. The rest is clear.

(iii) The sufficiency follows from (ii). For the necessity, consider an exact sequence $0 \to M \to I \to N \to 0$ in Γ-Mod with I injective. Tensoring with ${*K}$, which is projective as a right Γ-module, we obtain the exact sequence

$$0 \to {*K} \otimes_\Gamma M \to {*K} \otimes_\Gamma I \to {*K} \otimes_\Gamma N \to 0$$

of ${*K}$-modules. It splits because ${*K} \otimes_\Gamma M$ is Ext-injective in $P({*K}, \Gamma)$. ∎

COROLLARY 3.9. (a) *Let* $\Psi: \mathbf{p}(A_1, \Lambda^{\mathrm{op}}) \to \mathbf{i}(C, \Lambda^{\mathrm{op}})$ *be the additive equivalence of categories given by Proposition* 3.8(b). *Then* $D\Psi: \mathbf{p}(A_1, \Lambda^{\mathrm{op}}) \to \mathbf{p}(A, \Lambda)$ *is an additive duality.*

(b) *If* $X \to Y \to Z \to 0$ *is an exact sequence of left* A_1*-modules with* X, Y, $Z \in \mathbf{p}(A_1, \Lambda^{\mathrm{op}})$, *then* $0 \to D\Psi Z \to D\Psi Y \to D\Psi X$ *is an exact sequence of left* A*-modules.*

(c) *The structure map* $i_1: \Lambda^{\mathrm{op}} \to A_1$ *of the* Λ^{op}*-ring is injective, and* Coker i_1, *as a* Λ^{op}*-bimodule, is isomorphic to* $\bigoplus_{s=1}^{n} I_s \otimes_R P_s$, *where* I_s *is injective in* Λ^{op}-mod *and* P_s *is projective in* mod-Λ^{op}.

Proof. (a) follows from Theorem 3.2. (b) follows from Proposition 3.8(c)(i) and the fact that D is exact. The proof of (c) is almost the same as that of (b) ⇒ (a) in Corollary 3.4: one only has to use Lemma 3.3(b)(ii) instead of Lemma 3.3(b)(i). ∎

4. Ext-Injective Modules and Left Almost Split Morphisms in $\mathbf{p}(A, \Lambda)$; Existence of Almost Split Sequences

Proposition 4.1. *Let L be an indecomposable module in $\mathbf{p}(A, \Lambda)$, $D\Psi: \mathbf{p}(A_1, \Lambda^{\mathrm{op}}) \to \mathbf{p}(A, \Lambda)$ the duality given by Corollary 3.9(b), and $(D\Psi)^{-1}$ an inverse duality.*

(a) *Let $f: M \to (D\Psi)^{-1} L$ be a right almost split morphism in $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$, then $D\Psi f: L \to D\Psi M$ is a left almost split morphism in $\mathbf{p}(A, \Lambda)$. Thus left almost split morphisms exist in $\mathbf{p}(A, \Lambda)$.*

(b) *If $(D\Psi)^{-1} L$ is not projective in A_1*-mod, *then there exists an exact sequence $0 \to L \to^h V \to^k W \to 0$ in $\mathbf{p}(A, \Lambda)$ with h left almost split in $\mathbf{p}(A, \Lambda)$.*

(c) *The following are equivalent:*

(i) *L is* Ext-*injective.*

(ii) *$(D\Psi)^{-1} L$ is projective in A_1*-mod.

(iii) *L is a direct summand of $A \otimes_\Lambda I$, where I is injective in* Λ-mod.

(iv) *L is injective in* Λ-mod.

(d) *A left almost split morphism $L \to X$ in $\mathbf{p}(A, \Lambda)$ is injective if and only if L is not* Ext-*injective.*

(e) *The* Ext-*injective modules in $\mathbf{p}(A, \Lambda)$ are closed under direct sums.*

Proof. (a) Note that $(D\Psi)^{-1} L$ is indecomposable and use Proposition 2.3(a).

(b) By Proposition 2.3(b), there exists an exact sequence of A_1-modules $0 \to X \to^s V \to^t (D\Psi)^{-1} L \to 0$ with $X, Y \in \mathbf{p}(A_1, \Lambda^{\mathrm{op}})$ and t right almost split in $\mathbf{p}(A_1, \Lambda^{\mathrm{op}})$. Then the sequence $0 \to L \to^h V \to^k W$ of A-modules, where $V = D\Psi Y$, $W = D\Psi X$, $h = D\Psi t$, and $k = D\Psi s$, is exact by Corollary 3.9(b), and h is left almost split by part (a). It remains to show k is onto. Using Lemma 2.2 and Proposition 2.3(a)(b), it suffices to show that if $q: U_1 \to U_2$ is a monomorphism in Λ^{op}-mod, and $1 \otimes q: A_1 \otimes_{\Lambda^{\mathrm{op}}} U_1 \to A_1 \otimes_{\Lambda^{\mathrm{op}}} U_2$ is the induced monomorphism in A_1-mod, then $\Psi(1 \otimes q)$ is injective. This relation follows from Proposition 3.7(e) (remember, both A_1 and C are projective as right Λ^{op}-modules).

(c) (i) $\Rightarrow$ (ii) follows from (b).

To prove (ii) $\Rightarrow$ (iii), note that $L = D\Psi(D\Psi)^{-1} L$, and $D\Psi(A_1) \simeq D\Psi(A_1 \otimes_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{op}}) = D(C \otimes_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{op}}) \simeq A \otimes_\Lambda D\Lambda$ according to [13, Lemma 4.3, p. 181]. Since $D\Lambda$ is injective in Λ-mod, and D, Ψ are additive functors, the statement follows.

(iii) $\Rightarrow$ (iv). Since A is finitely generated projective as a right Λ-module, we have $A \otimes_\Lambda I \simeq \mathrm{Hom}_{\Lambda-}(C, I)$, where $C = \mathrm{Hom}_{-\Lambda}(A, \Lambda)$ is a Λ-bimodule projective (hence flat) as a right Λ-module by Corollary 3.4. It remains to use the well-known consequence of the adjoint associativity stating that if J is an injective left S-module, and B is an $S-T$-bimodule flat as a T-module, then $\mathrm{Hom}_S\ (B, J)$ is an injective left T-module.

(iv) $\Rightarrow$ (i). If $0 \to L \to Y \to Z \to 0$ is an exact sequence in $\mathbf{p}(A, \Lambda)$, then it splits over Λ because L is Λ-injective, hence splits over A because Z is relatively projective.

(d) Follows from (b) and (c).

(e) Follows from (c). ∎

THEOREM 4.2. (a) *Let N be an indecomposable non-*Ext*-projective module in* $\mathbf{p}(A, \Lambda)$, *and* $0 \to L \to M \to N \to 0$ *an almost split sequence in* A-mod. *There exists an almost split sequence* $0 \to X \to Y \to N \to 0$ *in* $\mathbf{p}(A, \Lambda)$, *where X (Y) is a direct summand of* $A \otimes_\Lambda L$ ($A \otimes_\Lambda M$).

(b) *Let L be an indecomposable non-*Ext*-injective module in* $\mathbf{p}(A, \Lambda)$. *There exists an almost split sequence* $0 \to L \to V \to W \to 0$ *in* $\mathbf{p}(A, \Lambda)$.

Proof. Follows from Proposition 2.3, Proposition 4.1, and Theorem 1.2. ∎

5. RELATIVELY PROJECTIVE AND INDUCED MODULES

In view of the recent applications of representations of BOCSes to representations of finite-dimensional algebras [7], the question of whether almost split sequences exist for representations of BOCSes is important. As shown in [6], they do exist in the special case of representations of partially ordered sets. Since the problem of classifying the representations of a BOCS can be viewed as the problem of finding a canonical form for a certain collection of matrices [20, 15, 19], the indicated question is also important for linear algebra. If K is a Γ-coring, then, in the algebraic language of [13], the category of representations of a BOCS is just the Kleisli category of the comonad in Γ-mod induced by the endofunctor $K \otimes_\Gamma -$; the category is equivalent to the category induc K of induced K-comodules. However, the existence theorem of Section 4 does not apply immediately because relatively injective comodules, which are *direct summands* of induced comodules, are not, in general, representations of BOCSes. In this connection, it is natural to ask whether a Γ-coring K has the property that every direct summand of an induced comodule is induced; i.e., every relatively injective comodule is relatively cofree. The question is equivalent to asking whether idempotents split in Induc K. If the answer is "yes," almost split sequences exist in induc K. The question seems also interesting on its own because one wants to have a class of corings well-behaved with respect to induced comodules. Although it was known that idempotents split in the category of representations of a partially ordered set, as well as in some other cases, the general answer was missing. In this section we describe a large class of corings with the desired property; the class contains all known special cases. In particular, it contains all triangular BOCSes in the terminology of [19, 7]. The dualities of Section 3 yield a class of Γ^{op}-rings for which every direct summand of an induced module is induced, i.e., every relatively projective module is relatively free. For these rings, almost split sequences exist in the category of induced modules.

Let Γ be an R-algebra (for the moment, R can be replaced by any commutative ring), and K a Γ-coring with comultiplication $\mu\colon K \to K \otimes_\Gamma K$, counit $\varepsilon\colon K \to \Gamma$, and a grouplike g. Recall that $\mu(g) = g \otimes g$ and $\varepsilon(g) = 1$. Following [19], for $U = \operatorname{Ker} \varepsilon$ consider two R-linear maps $\Gamma \to U$ and $U \to U \otimes_\Gamma U$ which are denoted by the same letter D and given by the formulae

$$D\gamma = \gamma g - g\gamma \tag{5.1}$$

and

$$Du = \mu(u) - u \otimes g - g \otimes u \tag{5.2}$$

for all $\gamma \in \Gamma$, $u \in U$.

Then D extends uniquely to an R-differential D of degree 1 on the graded tensor ring $T(U)$ of the Γ-bimodule U. Here the elements of Γ are assigned degree 0, and the elements of U are assigned degree 1; D satisfies the Leibniz formula

$$D(vw) = (Dv)\, w + (-1)^{\deg v}\, v(Dw) \tag{5.3}$$

for all homogeneous v, $w \in T(U)$, and

$$D^2 = 0. \tag{5.4}$$

Thus $T(U)$ is a differential graded algebra (DG-algebra) [16, p. 190], except that the differential D is of degree 1 rather than -1.

We now describe in the language of [13] a category which, in fact, coincides with the category of representations of the differential graded category (DGC) corresponding to the coring K [20, 15, 19].

For every M, $N \in \Gamma$-Mod, define an R-linear map

$$F\colon \operatorname{Hom}_{\Gamma-}(K \otimes_\Gamma M, N) \to \operatorname{Hom}_R(M, N) \oplus \operatorname{Hom}_{\Gamma}(U \otimes_\Gamma M, N)$$

as follows. For each $\phi: K \otimes_\Gamma M \to N$ put $F\phi = (\phi_0, \hat{\phi})$, where $\phi_0: M \to N$ and $\hat{\phi}: U \otimes_\Gamma M \to N$ are given by $\phi_0(x) = \phi(g \otimes x)$ for all $x \in M$ and $\hat{\phi} = \phi \mid U \otimes_\Gamma M$.

PROPOSITION 5.1. (a) *The map F is well-defined and injective.*

(b) Im *F consists of all pairs* $(\psi_0, \hat{\psi})$ *satisfying*

$$\gamma\psi_0(x) = \psi_0(\gamma x) + \hat{\psi}(D\gamma \otimes x) \tag{5.5}$$

for all $x \in M$, $\gamma \in \Gamma$, *where* $\psi_0: M \to N$ *is an R-linear map, and* $\hat{\psi}: U \otimes_\Gamma M \to N$ *is a Γ-linear map.*

(c) *If* $\phi \in \mathrm{Hom}_\Gamma\ (K \otimes_\Gamma M, N)$, $\psi \in \mathrm{Hom}_{\Gamma-}(K \otimes_\Gamma L, M)$, *and* $\phi \circ \psi$ *is the composite of* ϕ *and* ψ *in the Kleisli category* $\Gamma\text{-Mod}_{K \otimes \Gamma}$ *of the comonad in* Γ-Mod *determined by the endofunctor* $K \otimes_\Gamma -$, *then* $F(\phi \circ \psi) = (\chi_0, \hat{\chi})$, *where*

$$\chi_0 = \phi_0 \psi_0 \tag{5.6}$$

and

$$\hat{\chi}(u \otimes x) = \phi_0 \hat{\psi}(u \otimes x) + \hat{\phi}(u \otimes \psi_0(x)) + \sum_i \hat{\phi}(a_i \otimes \hat{\psi}(b_i \otimes x)) \tag{5.7}$$

for all $x \in L$, $u \in U$, $Du = \sum_i a_i \otimes b_i$.

(d) *The image of the identity morphism* $\varepsilon \otimes 1_M$ *of M in* $\Gamma\text{-Mod}_{K \otimes \Gamma}$ *is*

$$F(\varepsilon \otimes 1_M) = (1_M, 0). \tag{5.8}$$

(e) *F is an isomorphism between* $\Gamma\text{-Mod}_{K \otimes \Gamma -}$ *and the category* $\mathfrak{B}(K, \Gamma)$ *whose objects are those of* Γ-Mod, *and whose morphisms are all pairs* $(\psi_0, \hat{\psi})$ *satisfying* (5.5), *with the composition defined by* (5.6), (5.7).

(f) *If an R-linear map* ψ_0 *and a Γ-linear map* $\hat{\psi}$ *satisfy* (5.5) *for* γ *equal to* γ_1 *and* γ_2 *in Γ and all* $x \in M$, *then they satisfy* (5.5) *for* $\gamma = \gamma_1\gamma_2$ *and all x. Thus* $(\psi_0, \hat{\psi}) \in \mathrm{Im}\ F$ *if* (5.5) *holds for any set of generators of Γ as an R-algebra.*

Proof. (a) Since $\varepsilon(g) = 1$, then $K = g\Gamma \oplus U$ is a direct sum of right Γ-modules, and the restriction of ε to $g\Gamma$ is an isomorphism $g\Gamma \xrightarrow{\sim} \Gamma$ in Mod-Γ. Then $K \otimes_\Gamma M \simeq g \otimes M \oplus U \otimes_\Gamma M$ is a direct sum of R-modules for each M in Γ-Mod, and the map $g \otimes M \to M$ sending $g \otimes x$ to x for each $x \in M$ is an isomorphism of R-modules. Therefore F is well-defined and injective.

(b) If $(\psi_0, \hat{\psi})$ is of the form $F\psi$ for some Γ-linear map

$\psi: K \otimes_\Gamma L \to M$, then (5.5) holds; the verification is straightforward. Suppose now that the pair $(\psi_0, \hat{\psi})$ satisfies (5.5), and show that the R-linear map $\psi: K \otimes_\Gamma L \to M$ defined by $\psi(g\gamma \otimes x) = \psi_0(\gamma x)$ and $\psi(u \otimes x) = \hat{\psi}(u \otimes x)$ for all $x \in M$, $\gamma \in \Gamma$, $u \in U$, is Γ-linear. Since $\hat{\psi}$ is Γ-linear, we only have to show that $\psi(\gamma g \otimes x) = \gamma\psi(g \otimes x)$. Using (5.5), we have $\psi(\gamma g \otimes x) = \psi((g\gamma + D\gamma) \otimes x) = \psi(g\gamma \otimes x) + \psi(D\gamma \otimes x) = \psi_0(\gamma x) + \hat{\psi}(D\gamma \otimes x) = \gamma\psi_0(x) = \gamma\psi(g \otimes x)$.

(c) Since $\mu(g) = g \otimes g$, we have

$$\chi_0(x) = \chi(g \otimes x) = \phi(g \otimes \psi(g \otimes x)) = \phi(g \otimes \psi_0(x)) = \phi_0(\psi_0(x))$$

for all $x \in L$. Hence (5.6) holds. Also, using (5.2), we get $\hat{\chi}(u \otimes x) = \chi(u \otimes x) = \phi(g \otimes \psi(u \otimes x)) + \phi(u \otimes \psi(g \otimes x)) + \sum_i \phi(a_i \otimes \psi(b_i \otimes x))$ for all $u \in U$, $x \in L$. This is precisely (5.7).

(d) Obvious.

(e) Follows from (a), (b), (c), and (d).

(f) Suppose $\gamma_1\psi_0(x) = \psi_0(\gamma_1 x) + \hat{\psi}(D\gamma_1 \otimes x)$ and $\gamma_2\psi_0(x) = \psi_0(\gamma_2 x) + \hat{\psi}(D\gamma_2 \otimes x)$ for all $x \in M$. Then $\gamma_1\gamma_2\psi_0(x) = \gamma_1[\psi_0(\gamma_2 x) + \hat{\psi}(D\gamma_2 \otimes x)] = \gamma_1\psi_0(\gamma_2 x) + \hat{\psi}(\gamma_1 D\gamma_2 \otimes x) = \psi_0(\gamma_1\gamma_2 x) + \hat{\psi}(D\gamma_1 \otimes \gamma_2 x) + \hat{\psi}(\gamma_1 D\gamma_2 \otimes x) = \psi_0(\gamma_1\gamma_2 x) + \hat{\psi}\{[(D\gamma_1)\gamma_2 + \gamma_1 D\gamma_2] \otimes x\} = \psi_0(\gamma_1\gamma_2 x) + \hat{\psi}(D(\gamma_1\gamma_2) \otimes x)$. ∎

For the rest of this section suppose that the identity $1 \in \Gamma$ can be written as a sum of pairwise othogonal idempotents $1 = e_1 + \cdots + e_t$ in such a way that, as a Γ-bimodule, U is isomorphic to $\bigoplus_{s=1}^n \Gamma e_{i_s} \otimes_R e_{j_s}\Gamma$. (This is the case, for instance, when the Γ-coring K satisfies condition (b) of Corollary 3.4.) If u_s corresponds to $e_{i_s} \otimes e_{j_s}$ under the abovementioned isomorphism, then

$$U = \bigoplus_{s=1}^{n} \Gamma u_s \Gamma \tag{5.9}$$

and

$$U \otimes_\Gamma U \simeq \bigoplus_{s,t=1}^{n} \Gamma u_s \Gamma \otimes_\Gamma \Gamma u_t \Gamma. \tag{5.10}$$

We often consider the isomorphism $U \simeq \bigoplus_{s=1}^n \Gamma e_{i_s} \otimes_R e_{j_s}\Gamma$ as identification.

Remark 5.2. As follows from Proposition 5.1(b), a pair $(\psi_0, 0)$, where ψ_0 is an R-linear map, is a morphism in $\mathfrak{B}(K, \Gamma)$ if and only if ψ_0 is a homomorphism of Γ-modules. Then Proposition 5.1(c)(d)(e) implies that there exists an embedding of the category Γ-Mod into $\mathfrak{B}(K, \Gamma)$ sending every Γ-module M to itself, and every homomorphism $\theta: M \to N$ in Γ-Mod to the morphism $(\theta, 0)$: $M \to N$ in $\mathfrak{B}(K, \Gamma)$. Generally, the embedding is not full.

LEMMA 5.3. *Let Γ and Λ be R-algebras, $1 = e_1 + \cdots + e_t$ and $1 = f_1 + \cdots + f_v$ representations of the identities of Γ and Δ, respectively, as sums of pairwise orthogonal idempotents. Let $M \in \Gamma$-Mod, $N \in \Delta$-Mod, and $U = \bigoplus_{s=1}^n \Delta f_{i_s} \otimes_R e_{j_s}\Gamma$.*

(a) *For each Λ-linear map $\hat{\sigma}: U \otimes_\Gamma M \to N$ and $s = 1, ..., n$, consider the R-linear map $\sigma_s: e_{j_s}M \to f_{i_s}N$ given by*

$$\sigma_s(x) = \hat{\sigma}(f_{i_s} \otimes e_{j_s} \otimes x)$$

for all $x \in e_{j_s}M$. The map $\hat{\sigma} \mapsto (\sigma_1, ..., \sigma_n)$ is an R-isomorphism

$$\operatorname{Hom}_{\Delta-}(U \otimes_\Gamma M, N) \simeq \bigoplus_{s=1}^n \operatorname{Hom}_R(e_{j_s}M, f_{i_s}N).$$

In particular, if $\Delta = \Gamma$, $v = t$, and $f_i = e_i$ for $i = 1, ..., t$, then the map $\hat{\sigma} \mapsto (\sigma_1, ..., \sigma_n)$, where

$$\sigma_s(x) = \hat{\sigma}(u_s \otimes x), \tag{5.11}$$

is an R-isomorphism

$$\operatorname{Hom}_{\Gamma-}(U \otimes_\Gamma M, N) \simeq \bigoplus_{s=1}^n \operatorname{Hom}_R(e_{j_s}M, e_{i_s}N).$$

(b) *For each Δ-bimodule V, the evaluation at $f_i \otimes f_j$ map $\operatorname{Hom}_{\Delta\ \Delta}(\Delta f_i \otimes_R f_j\Delta, V) \to V$, sending every homomorphism $\phi \in \operatorname{Hom}_{\Delta\ \Delta}(\Delta f_i \otimes_R f_j\Delta, V)$ to $\phi(f_i \otimes f_j)$, is an isomorphism $\operatorname{Hom}_{\Delta\ \Delta}(\Delta f_i \otimes_R f_j\Delta, V) \simeq f_i V f_j$ natural in V.*

Proof. (a) We have the following natural isomorphisms for $P_s = \Delta f_{i_s}$ and $Q_s = e_{j_s}\Gamma$:

$$\begin{aligned}
\operatorname{Hom}_{\Delta-}(U \otimes_\Gamma M, N) &\simeq \operatorname{Hom}_{\Delta-}\left(\bigoplus_{s=1}^n P_s \otimes_R Q_s \otimes_\Gamma M, N\right) \\
&\simeq \bigoplus_{s=1}^n \operatorname{Hom}_\Delta\ (P_s \otimes_R Q_s \otimes_\Gamma M, N) \\
&\simeq \bigoplus_{s=1}^n \operatorname{Hom}_R(Q_s \otimes_\Gamma M, \operatorname{Hom}_{\Delta-}(P_s, N)) \\
&= \bigoplus_{s=1}^n \operatorname{Hom}_R(e_{j_s}\Gamma \otimes_\Gamma M, \operatorname{Hom}_{\Delta-}(\Delta f_{i_s}, N)) \\
&\simeq \bigoplus_{s=1}^n \operatorname{Hom}_R(e_{j_s}M, f_{i_s}N).
\end{aligned}$$

(b) In a similar way, we get

$$\begin{aligned}\operatorname{Hom}_{\Lambda-\Lambda}(\Lambda f_i \otimes_R f_j \Lambda, V) &\simeq \operatorname{Hom}_{\Lambda-\Lambda}(\Lambda f_i \otimes_R R \otimes_R f_j \Lambda, V)\\ &\simeq \operatorname{Hom}_{R-\Lambda}(R \otimes_R f_j \Lambda, \operatorname{Hom}_{\Lambda-}(\Lambda f_i, V))\\ &\simeq \operatorname{Hom}_{R-\Lambda}(R \otimes_R f_j \Lambda, f_i V)\\ &\simeq \operatorname{Hom}_{R-R}(R, \operatorname{Hom}_{-\Lambda}(f_j \Lambda, f_i V))\\ &\simeq \operatorname{Hom}_{R-R}(R, f_i V f_j) \simeq f_i V f_j\end{aligned}$$

because we assume that R acts centrally on all Λ-bimodules. ∎

Remark 5.4. Since $\hat{\sigma}(u_s \otimes x) = 0$ if $x \in e_p M$ with $p \neq j_s$, (5.11) allows us to view σ_s as an R-linear map $M \to N$ whose image is contained in $e_{i_s} N$ and whose kernel contains $e_p M$ whenever $p \neq j_s$. At our convenience, we will treat σ_s either as a map from $e_{j_s} M$ into $e_{i_s} N$, or as a map from M into N.

In the remaining part of this section we assume that

$$De_p = 0 \text{ for } p = 1, ..., t, \tag{5.12}$$

i.e., $e_p g = g e_p$ for all p.

LEMMA 5.5. (a) *If* $\gamma \in e_p \Gamma e_q$, *then* $D\gamma = \sum_f a_f u_{s(f)} b_f$, *where* $a_f \in e_p \Gamma e_{i_{s(f)}}$, $b_f \in e_{j_{s(f)}} \Gamma e_q$, *and* p, $q = 1, ..., t$.

(b) $Du_s = \sum_h a_h u_{p(h)} \otimes b_h u_{q(h)} c_h$, *where* $a_h \in e_{i_s} \Gamma e_{i_{p(h)}}$, $b_h \in e_{j_{p(h)}} \Gamma e_{i_{q(h)}}$, $c_h \in e_{j_{q(h)}} \Gamma e_{j_s}$, *and* $s = 1, ..., n$.

(c) *If* $(\phi_0, \hat{\phi})$: $M \to N$ *is a morphism in* $\mathfrak{B}(K, \Gamma)$, *then* $\phi_0(e_p M) \subset e_p N$ *for all* $p = 1, ..., t$.

Proof. (a) and (b) Since $\gamma = e_p \gamma e_q$ and $u_s = e_{i_s} u_s e_{j_s}$, (5.3) and (5.12) imply $D\gamma = e_p(D\gamma) e_q$ and $Du_s = e_{i_s}(Du_s) e_{j_s}$.

(c) For any $x \in M$, we have $\phi_0(e_p x) = e_p \phi_0(x) + \hat{\phi}(De_p \otimes x) = e_p \phi_0(x)$, using (5.5) and (5.12). ∎

For every morphism $(\phi_0, \hat{\phi})$: $M \to N$ in $\mathfrak{B}(K, \Gamma)$ consider a $(t+n)$-tuple of R-linear maps $G(\phi_0, \hat{\phi}) = (\phi_{01}, ..., \phi_{0t}, \phi_1, ..., \phi_n)$, where $\phi_1, ..., \phi_n$ are obtained from $\hat{\phi}$ according to (5.11), and $\phi_{0p} = \phi_0 | e_p M$: $e_p M \to e_p N$. The latter makes sense because of Lemma 5.5(c).

PROPOSITION 5.6. (a) *The map G is injective.*

(b) *Im G consists of all* $(t+n)$-*tuples* $(\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n)$ *of R-linear maps satisfying*

$$\gamma \psi_{0q}(x) = \psi_{0p}(\gamma x) + \sum_f a_f \psi_{s(f)}(b_f x) \tag{5.13}$$

for all $x \in e_q M$ *and* $\gamma \in e_p \Gamma e_q$, *where* $D\gamma = \sum_f a_f u_{s(f)} b_f$, *as explained in Lemma* 5.5(a), $p, q = 1, ..., t$.

(c) *If* $(\phi_0, \hat{\phi})$: $M \to N$, $(\psi_0, \hat{\psi})$: $L \to M$ *are morphisms in* $\mathfrak{B}(K, \Gamma)$, *and* $G((\phi_0, \hat{\phi}) \circ (\psi_0, \hat{\psi})) = (\chi_{01}, ..., \chi_{0t}, \chi_1, ..., \chi_n)$, *then*

$$\chi_{0p} = \phi_{0p} \psi_{0p} \tag{5.14}$$

for all p and

$$\chi_s(x) = \phi_{0i_s} \psi_s(x) + \phi_s \psi_{0j_s}(x) + \sum_h a_h \phi_{p(h)}(b_h \psi_{q(h)}(c_h x)) \tag{5.15}$$

for all $x \in e_{j_s} L$, *where* $s = 1, ..., n$ *and* $Du_s = \sum_h a_h u_{p(h)} \otimes b_h u_{q(h)} c_h$ *as explained in Lemma* 5.5(b).

(d) *The image of the identity morphism is*

$$G(1_M, 0) = (1_{e_1 M}, ..., 1_{e_t M}, 0, ..., 0).$$

(e) *G is an isomorphism between* $\mathfrak{B}(K, \Gamma)$ *and the category* $\mathfrak{D}(K, \Gamma)$ *whose objects are those of* Γ-Mod, *and whose morphisms are all* $(t+n)$-*tuples* $(\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n)$ *satisfying* (5.13), *with the composition defined by* (5.14), (5.15).

(f) *If a* $(t+n)$-*tuple* $(\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n)$ *satisfies* (5.13) *for all* $x \in e_r M$ *and a fixed* $\gamma_2 \in e_q \Gamma e_r$, *as well as for all* $x \in e_q M$ *and a fixed* $\gamma_1 \in e_p \Gamma e_q$, *then it satisfies* (5.13) *for all* $x \in e_r M$ *and* $\gamma = \gamma_1 \gamma_2$. *Thus if S is a set of generators of the form* $e_p \gamma e_q$ *for the R-algebra* Γ, *then* $(\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n) \in \operatorname{Im} G$ *if* (5.13) *holds for every pair* γ *and x, where* $\gamma = e_p \gamma e_q \in S$ *and* $x \in e_q M$.

Proof. (a) Follows from Lemma 5.5(c) and Lemma 5.3(a).

(b) If

$$G(\psi_0, \hat{\psi}) = (\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n), \tag{5.16}$$

then $(\psi_0, \hat{\psi})$ satisfies (5.5), so that, using Lemma 5.5(c), we have $\gamma\psi_{0q}(x) = \gamma\psi_0(x) = \psi_0(\gamma x) + \hat{\psi}(D\gamma \otimes x) = \psi_0(\gamma x) + \sum_f a_f \hat{\psi}(u_{s(f)} \otimes b_f x) = \psi_{0p}(\gamma x) + \sum_f a_f \psi_{s(f)}(b_f x)$. Thus (5.13) holds.

Suppose now that a $(t+n)$-tuple $(\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n)$ of R-linear maps satisfies (5.13). Construct a pair $(\psi_0, \hat{\psi})$, where ψ_0: $M \to N$ is an R-linear map, and $\hat{\psi}$: $U \otimes_\Gamma M \to N$ is a Γ-linear map satisfying (5.16). We put $\psi_0 = \bigoplus_{p=1}^t \psi_{0p}$ and choose $\hat{\psi}$ to be a unique Γ-linear map which corresponds to the n-tuple $(\psi_1, ..., \psi_n)$ according to Lemma 5.3(a). Check

$(\psi_0, \hat{\psi})$ satisfies (5.5). Using Proposition 5.1(f), we may assume $\gamma \in e_p \Gamma e_q$. Since (5.13) holds, we get

$$\gamma\psi_0(x) = \gamma\psi_0(e_1 x + \cdots + e_t x) = \gamma \sum_{r=1}^{t} \psi_{0r}(e_r x) = \gamma\psi_{0q}(e_q x)$$
$$= \psi_{0p}(\gamma e_q x) + \sum_f a_f \psi_{s(f)}(b_f e_q x) = \psi_0(\gamma x)$$
$$+ \sum_f a_f \psi_{s(f)}(b_f x) = \psi_0(\gamma x) + \hat{\psi}(D\gamma \otimes x).$$

Here we have used the orthogonality of the idempotents $e_1, ..., e_t$, Lemma 5.5(a), the Γ-linearity of $\hat{\psi}$, and (5.11). Thus (5.5) holds, whence $(\psi_0, \hat{\psi})$: $M \to N$ is a morphism in $\mathfrak{B}(K, \Gamma)$. It is obvious that $(\psi_0, \hat{\psi})$ satisfies (5.16).

(c) The formulae (5.14) and (5.15) are immediate consequences of (5.6) and (5.7) in view of Lemma 5.5(c) and (5.11).

(d) Obvious.

(e) Follows from (a), (b), (c), and (d).

(f) Similar to the proof of Proposition 5.1(f). ∎

Remark 5.7. If R is a field, Γ is a finitely generated R-algebra, and we consider only finite-dimensional over R modules in Γ-Mod, the problem of classifying the objects of $\mathfrak{D}(K, \Gamma)$ is the problem of finding a canonical form for a finite collection of matrices under a given set of admissible transformations, as follows from (5.13), (5.14), (5.15). $\mathfrak{D}(K, \Gamma)$ is the category introduced in [20, 15].

DEFINITION 5.8. Suppose Γ, as an R-algebra, has a set of generators Z which admits a filtration $Z_1 \subset Z_2 \subset \cdots \subset Z_m = Z$ such that $e_p \in Z_1$ for $p = 1, ..., t$. Let $\Gamma_h = R[Z_h]$ be the R-subalgebra of Γ generated by Z_h. Suppose also that the set $E = \{u_1, ..., u_n\}$ admits a filtration $E_1 \subset E_2 \subset \cdots E_l = E$ such that:

(i) $DZ_1 = 0$ and $DZ_h \subset \Gamma_{h-1} E\Gamma$ for all $h > 1$.

(ii) $DE_1 = 0$ and $DE_h \subset \mathrm{Im}(\Gamma E_{h-1} \Gamma \otimes_\Gamma U \to U \otimes_\Gamma U)$ for all $h > 1$.

Then the Γ-coring K is called *left triangular*.

The Γ-coring K is called a *left triangular tensor coring* if the set of generators Z of the R-algebra Γ satisfies the following conditions for $h = 2, ..., m$:

(iii) For all $z \in Z - Z_1$, we have $z = e_p z e_q$ for some p, $q = 1, ..., t$.

(iv) A unique map $\Gamma_{h-1} e_p \otimes_R e_q \Gamma_{h-1} \to \Gamma_{h-1} z \Gamma_{h-1}$ of Γ_{h-1}-bimodules sending $e_p \otimes e_q$ to z is an isomorphism (see Lemma 5.3(b)).

(v) $W_h = \sum_{z \in Z_h - Z_{h-1}} \Gamma_{h-1} z \Gamma_{h-1}$ is a direct sum of the Γ_{h-1}-subbimodules $\Gamma_{h-1} z \Gamma_{h-1}$ of Γ.

(vi) Let $T(W_h)$ be the tensor ring of the Γ_{h-1}-bimodule W_h. A unique map of Γ_{h-1}-rings $T(W_h) \to \Gamma_h$ sending z to z for all $z \in Z_h - Z_{h-1}$ is an isomorphism.

Examples of left triangular tensor corings are given in [7].

PROPOSITION 5.9. *Let K be a left triangular Γ-coring, ψ: $K \otimes_\Gamma L \to M$ a Γ-linear map, and $GF\psi = G(\psi_0, \hat{\psi}) = (\psi_{01}, ..., \psi_{0t}, \psi_1, ..., \psi_n)$. Then the following are equivalent*:

(a) *ψ is an isomorphism in Γ-$\mathrm{Mod}_{K \otimes_\Gamma}$.*

(b) *ψ_0 is an isomorphism in R-Mod.*

(c) *ψ_{0p} is an isomorphism in R-Mod for $p = 1, ..., t$.*

Proof. (b)$\Leftrightarrow$(c) follows from Lemma 5.5(c). (a)$\Rightarrow$(b) is immediate from the formulae (5.6) and (5.8). For (b)$\Rightarrow$(a), assume that ψ_0 is an isomorphism in R-Mod and construct a left inverse ϕ: $K \otimes_\Gamma M \to L$ of ψ in Γ-$\mathrm{Mod}_{K \otimes_\Gamma}$ such that ϕ_0 is an isomorphism in R-Mod, where $F\phi = (\phi_0, \hat{\phi})$. It is easy to see that such a ϕ is a two-sided inverse of ψ.

It is obvious from the formulae (5.6) and (5.8) that we must put $\phi_0 = \psi_0^{-1}$. As follows from Proposition 5.1(b)(c)(d), we must find such a Γ-linear map $\hat{\phi}$: $U \otimes_\Gamma M \to L$ that

$$\gamma\psi_0^{-1}(y) = \psi_0^{-1}(\gamma y) + \hat{\phi}(D\gamma \otimes y) \tag{5.17}$$

for all $y \in M$, $\gamma \in Z$, and

$$0 = \psi_0^{-1}\hat{\psi}(u \otimes x) + \hat{\phi}(u \otimes \psi_0(x)) + \sum_i \hat{\phi}(a_i \otimes \hat{\psi}(b_i \otimes x)) \tag{5.18}$$

for all $x \in L$, $u \in U$, $Du = \sum_i a_i \otimes b_i$ with a_i, $b_i \in U$.

By Lemma 5.3(a), any collection $\{\phi_1, ..., \phi_n\}$ of R-linear maps with $\phi_s: e_{j_s}M \to e_{i_s}L$ uniquely determines a Γ-linear map $\hat{\phi}$: $U \otimes_\Gamma M \to L$ such that $\phi_s(y) = \hat{\phi}(u_s \otimes y)$ for all y. Using the collection of maps $\{\psi_1, ..., \psi_n\}$ which corresponds to the map $\hat{\psi}$, determine the values of the $\phi_1, ..., \phi_n$ for the desired map $\hat{\phi}$ by induction on $h = h(s)$—the least positive integer such that $u_s \in E_h$ (see Definition 5.8). If $h = 1$, then $Du_s = 0$, and we put

$$\phi_s(y) = -\psi_0^{-1}\psi_s\psi_0^{-1}(y) \tag{5.19}$$

for all $y \in e_{j_s}M$. We note that $\psi_0^{-1}(y) \in e_{j_s}L$, as follows from Lemma 5.5(c), so that (5.19) makes sense. If $h = h(s) > 1$, fix a decomposition

$$Du_s = \sum_f a_f u_f \otimes v_f, \tag{5.20}$$

where $a_f \in \Gamma$, $u_f \in E_{h-1}$, $v_f \in U$. Assuming by induction that ϕ_s has already been defined for all s with $h(s) < k$, pick an s with $h(s) = k$ and define

$$\phi_s(y) = -\psi_0^{-1}\psi_s\psi_0^{-1}(y) - \sum_f a_f\phi_f\hat{\psi}(v_f \otimes \psi_0^{-1}(y)), \tag{5.21}$$

as suggested by (5.18). Here we treat ϕ_f as a map $M \to L$, according to Remark 5.4. Note that since the Γ-coring K is left triangular, $h(f) < k$ for all the summation indices f in (5.21). Therefore ϕ_f has already been defined by induction, and (5.21) makes sense.

We now show that the Γ-linear map $\hat{\phi}$ which corresponds to the collection $\{\phi_1, ..., \phi_n\}$, defined according to (5.19) and (5.21), satisfies (5.17) and (5.18).

LEMMA 5.10. *If $\lambda \in \Gamma$, and $\hat{\phi}$ satisfies* (5.18) *for $u = D\lambda$, then $\hat{\phi}$ satisfies* (5.17) *for $\gamma = \lambda$.*

Proof of Lemma 5.10. Since $Du = D(D\lambda) = 0$, (5.18) yields $\hat{\phi}(D\lambda \otimes \psi_0(x)) = -\psi_0^{-1}\hat{\psi}(D\lambda \otimes x)$. Then, using (5.5), we have

$$\begin{aligned}\psi_0^{-1}(\lambda y) + \hat{\phi}(D\lambda \otimes y) &= \psi_0^{-1}(\lambda\psi_0(x)) + \hat{\phi}(D\lambda \otimes \psi_0(x))\\ &= \psi_0^{-1}(\lambda\psi_0(x)) - \psi_0^{-1}\hat{\psi}(D\lambda \otimes x) = \lambda x = \lambda\psi_0^{-1}(y).\end{aligned}$$

Thus (5.17) holds for $\gamma = \lambda$. ∎

LEMMA 5.11. *If $\hat{\phi}$ satisfies* (5.17) *for $\gamma = c$, then it satisfies* (5.18) *for $u = cu_s d$, where d is an arbitrary element of Γ, and $s = 1, ..., n$.*

Proof of Lemma 5.11. We have, using (5.20), that

$$Du = Dc \otimes u_s d + \sum_f ca_f u_f \otimes v_f d - cu_s \otimes Dd,$$

where $a_f = 0$ if $Du_s = 0$, and

$$\begin{aligned}&\psi_0^{-1}\hat{\psi}(cu_s d \otimes x) + \hat{\phi}(cu_s d \otimes \psi_0(x)) + \hat{\phi}(Dc \otimes \hat{\psi}(u_s d \otimes x))\\ &\quad + \sum_f \hat{\phi}(ca_f u_f \otimes \hat{\psi}(v_f d \otimes x)) - \hat{\phi}(cu_s \otimes \hat{\psi}(Dd \otimes x))\\ &= \psi_0^{-1}(c\hat{\psi}(u_s \otimes dx)) + c\hat{\phi}(u_s \otimes d\psi_0(x)) + \hat{\phi}(Dc \otimes \hat{\psi}(u_s \otimes dx))\\ &\quad + c\sum_f a_f\hat{\phi}(u_f \otimes \hat{\psi}(v_f \otimes dx)) - c\hat{\phi}(u_s \otimes \hat{\psi}(Dd \otimes x))\\ &= c\psi_0^{-1}\hat{\psi}(u_s \otimes dx) + c\hat{\phi}(u_s \otimes \psi_0(dx)) + c\sum_f a_f\hat{\phi}(u_f \otimes \hat{\psi}(v_f \otimes dx))\\ &= c\{\psi_0^{-1}\psi_s\psi_0^{-1}[\psi_0(dx)] + \phi_s[\psi_0(dx)]\\ &\quad + \sum_f a_f\phi_f\hat{\psi}[v_f \otimes \psi_0^{-1}[\psi_0(dx)]]\} = 0,\end{aligned}$$

using the assumption that (5.17) holds for c, the fact that formula (5.5) holds for $\gamma = d$, and formula (5.21). Hence (5.18) holds for $u = cu_s d$. ▮

We now show how to use Lemmas 5.10 and 5.11 to finish the proof. It is an easy consequence of (5.5) that (5.17) holds for all $\gamma \in Z_1$; hence it holds for all $\gamma \in \Gamma_1$, according to Proposition 5.1(f). Since the Γ-coring K is left triangular, it follows from Lemma 5.10 that (5.18) holds for all $u \in \Gamma_1 E\Gamma$; then Lemma 5.11 implies that (5.15) holds for all $\gamma \in Z_2$, and so on. Continuing this argument, we obtain that (5.17) holds for all $\gamma \in \Gamma_m = \Gamma$, whence (5.18) holds for all $u \in \Gamma_m E\Gamma = U$. ▮

For the rest of this section, we assume that the Γ-coring K is a left triangular tensor coring, and restrict ourselves to those modules in Γ-Mod which are projective over R.

PROPOSITION 5.12. *Let M be a Γ-module, and $N = \bigoplus_{p=1}^{t} N_p$ a direct sum of the R-modules N_p. Given R-isomorphisms $\psi_{0p}: e_p M \to N_p$ for $p = 1, ..., t$, and arbitrary R-linear maps $\psi_s: e_{j_s} M \to N_{i_s}$ for $s = 1, ..., n$, there exists a structure of a left Γ-module on N for which*:

(a) *$N_p = e_p N$ and*

(b) *$(\psi_0, \hat{\psi})$: $M \to N$ is an isomorphism in $\mathfrak{B}(K, \Gamma)$, where $\psi_0 = \bigoplus_{p=1}^{t} \psi_{0_p}$ and $\hat{\psi}$: $U \otimes_\Gamma M \to N$ is a unique Γ-linear map determined by the collection $\{\psi_1, ..., \psi_n\}$, according to Lemma* 5.3(a).

Proof. The argument is similar to the proof of [7, 4.2. Proposition, p. 465]. We begin with the observation that the natural map $\Gamma_h E\Gamma \otimes_\Gamma M \to \Gamma_{h+1} E\Gamma \otimes_\Gamma M$ is a monomorphism of left Γ_h-modules (we view it as the inclusion map of $\Gamma_h E\Gamma \otimes_\Gamma M$ into $\Gamma_{h+1} E\Gamma \otimes_\Gamma M$). To prove this, note first that $e_{j_s}\Gamma \otimes_\Gamma M \simeq e_{j_s} M$ is R-projective because M is R-projective by assumption. Then, since each u_s is identified with $e_{i_s} \otimes e_{j_s}$, tensoring the inclusion map $\Gamma_h e_{i_s} \to \Gamma_{h+1} e_{i_s}$ with $e_{j_s} M$ and passing to the direct sums produce the desired monomorphism.

For $h = 1, ..., m$, we will define on N a structure of a left Γ_h-module with the property that the Γ_{h+1}-structure extends the Γ_h-structure if $h \leqslant m - 1$. Assuming temporarily that such Γ_h-structures have already been defined, denote by $\hat{\psi}_h: \Gamma_h E\Gamma \otimes_\Gamma M \to N$ a unique Γ_h-linear map corresponding to the collection $\{\psi_1, ..., \psi_n\}$, according to Lemma 5.3(a) (recall that $e_p \in \Gamma_h$ for $p = 1, ..., t$ and all h, by Definition 5.8). Then $\hat{\psi}_{h+1}$ extends $\hat{\psi}_h$. Really, since $\Gamma_h E\Gamma \otimes_\Gamma M$ is a Γ_h-submodule of $\Gamma_{h+1} E\Gamma \otimes_\Gamma M$, and the Γ_{h-1}-module structure on N extends the Γ_h-module structure, $\hat{\psi}_{h+1} | \Gamma_h E\Gamma \otimes_\Gamma M$ is a Γ_h-linear map $\Gamma_h E\Gamma \otimes_\Gamma M \to N$ corresponding to the collection $\{\psi_1, ..., \psi_n\}$, according to Lemma 5.3(a). By the uniqueness, $\hat{\psi}_{h+1} | \Gamma_h E\Gamma \otimes_\Gamma M = \hat{\psi}_h$.

Proceed by induction on $h = 1, \ldots, m$. For each h, we define a Γ_h-module structure on N, and show that the pair $(\psi_0, \hat{\psi}_h)$ satisfies (5.5) for all $\gamma = z \in Z_h$ and $\hat{\psi} = \hat{\psi}_h$. We denote by γx the image of $\gamma \otimes x$ under the structure map $\Gamma \otimes_R M \to M$, and by $\gamma \circ y$ the image of $\gamma \otimes y$ under the structure map $\Gamma_h \otimes_R N \to N$.

Let $h = 1$. For all $\gamma \in \Gamma_1$, put $\gamma \circ y = \psi_0(\gamma \psi_0^{-1}(y))$. Then $1 \circ y = \psi_0(1\psi_0^{-1}(y)) = y$, and $\gamma_1\gamma_2 \circ y = \psi_0(\gamma_1\gamma_2\psi_0^{-1}(y)) = \psi_0\{\gamma_1\psi_0^{-1}[\psi_0(\gamma_2\psi_0^{-1}(y))]\} = \gamma_1 \circ (\gamma_2 \circ y)$. Hence N is a left Γ_1-module, and, clearly, $N_p = e_p N$ because $e_p \in \Gamma_1$ for all p. Let $\hat{\psi}_1 : \Gamma_1 E\Gamma \otimes_\Gamma M \to N$ be a unique Γ_1-linear map determined by $\{\psi_1, \ldots, \psi_n\}$, according to Lemma 5.3(a). We have $\gamma \circ \psi_0(x) = \psi_0[\gamma\psi_0^{-1}(\psi_0(x))] = \psi_0(\gamma x)$ for all $x \in M$. This is precisely (5.5) for $\hat{\psi} = \hat{\psi}_1$ because $D\gamma = 0$ for all $\gamma \in \Gamma_1$. In particular, (5.5) is satisfied for all $\gamma = z \in Z_1$.

Suppose now that a structure of a left Γ_h-module on N is defined for some $h \geqslant 1$, and the pair $(\psi_0, \hat{\psi}_h)$ satisfies (5.5) for all $\gamma = z \in Z_h$ and $\hat{\psi} = \hat{\psi}_h$. Extend the Γ_h-module structure to a Γ_{h+1}-module structure on N in such a way that (5.5) is satisfied for all $\gamma = z \in Z_{h+1}$ and $\hat{\psi} = \hat{\psi}_{h+1}$. For each $z \in Z_{h+1} - Z_h$, define a Γ_h-bimodule map $\Gamma_h z \Gamma_h \to \mathrm{End}_R(N)$. Taking into account Definition 5.8(iii)(iv) and Lemma 5.3(b), it suffices to construct an R-endomorphism σ of N satisfying $\sigma = e_p \sigma e_q$. Choose σ to be the R-endomorphism of N given, for all $y \in N$, by the right-hand side of the formula

$$z \circ y = \psi_0(z\psi_0^{-1}(y)) + \hat{\psi}_h(Dz \otimes \psi_0^{-1}(y)). \tag{5.22}$$

The formula makes sense because $Dz \in \Gamma_h E\Gamma$, and $\sigma \in e_p \mathrm{End}_R(N) e_q$ because $e_p, e_q \in \Gamma_1$ and the pair $(\psi_0, \hat{\psi}_h)$ satisfies (5.5) for all $\gamma \in \Gamma_1$. Taking into account Definition 5.8(v)(vi), we obtain a Γ_h-bimodule map $W_{h+1} \to \mathrm{End}_R(N)$ and, hence, a Γ_h-ring map $\Gamma_{h+1} = T(W_{h+1}) \to \mathrm{End}_R(N)$. Thus N has acquired a Γ_{h+1}-module structure which extends its Γ_h-module structure, and which on the set Z_{h+1} is given by (5.22) (recall that (5.5), and hence (5.22), is satisfied by $(\psi_0, \hat{\psi}_h)$ for all $\gamma = z \in Z_h$). Since $\hat{\psi}_{h+1}$ extends $\hat{\psi}_h$, as explained above, (5.22) shows that $(\psi_0, \hat{\psi}_{h+1})$ satisfies (5.5) for all $\gamma = z \in Z_{h+1}$.

By induction, we have for all h that N is a Γ_h-module, $\hat{\psi}_h$ is a Γ_h-linear map, and the pair $(\psi_0, \hat{\psi}_h)$ satisfies (5.5) for all $\gamma = z \in Z_h$ and $\hat{\psi} = \hat{\psi}_h$. Putting $h = m$, we get that N is a $\Gamma_m = \Gamma$-module, $\hat{\psi}_m$ is a Γ-linear map, and the pair $(\psi_0, \hat{\psi}_m)$ satisfies (5.5) for all $\gamma = z \in Z_m = Z$ and $\hat{\psi} = \hat{\psi}_m$. Using Proposition 5.1(f)(b), we see that $(\psi_0, \hat{\psi}_m)$: $M \to N$ is a morphism in $\mathfrak{B}(K, \Gamma)$ which is an isomorphism by Proposition 5.9. ∎

We are now ready to prove the main result of this section.

THEOREM 5.13. *Idempotents split in* $\mathfrak{B}(K, \Gamma)$.

Proof. Let $\chi=(\chi_0, \hat{\chi})$: $M \to M$ be an idempotent morphism in $\mathfrak{B}(K, \Gamma)$, then $\psi \circ \chi \circ \psi^{-1}$ is an idempotent for each isomorphism ψ in $\mathfrak{B}(K, \Gamma)$. Construct a Γ-module N and an isomorphism $\psi=(\psi_0, \hat{\psi})$: $M \to N$ such that $\psi \circ \chi \circ \psi^{-1}=(\phi_0, 0)$. Then Remark 5.2 would imply that $\phi_0 \in \operatorname{End}_{\Gamma}(N)$ and $\phi_0^2=\phi_0$. Since idempotents split in Γ-Mod, there exist morphisms σ_0 and τ_0 in Γ-Mod satisfying $\phi_0=\sigma_0\tau_0$ and $\tau_0\sigma_0=1_L$ for some $L \in \Gamma$-Mod. Putting $\sigma=(\sigma_0, 0)$ and $\tau=(\tau_0, 0)$, we obtain $\sigma \circ \tau=\psi \circ \chi \circ \psi^{-1}$ and $\tau \circ \sigma=1_L$ in $\mathfrak{B}(K, \Gamma)$, whence $\chi=(\psi^{-1} \circ \sigma \circ \psi) \circ (\psi^{-1} \circ \tau \circ \psi)$ and $1_L=(\psi^{-1} \circ \tau \circ \psi) \circ (\psi^{-1} \circ \sigma \circ \psi)$. Thus the existence of the desired isomorphism ψ would imply the theorem.

We first record some consequences of the fact that χ is an idempotent. According to formulae (5.6) and (5.7), this is equivalent to $\chi=(\chi_0, \hat{\chi})$ satisfying the conditions

$$\chi_0=\chi_0^2 \tag{5.23}$$

and

$$\hat{\chi}(u \otimes x)=\chi_0\hat{\chi}(u \otimes x)+\hat{\chi}(u \otimes \chi_0(x))+\sum_i \hat{\chi}(a_i \otimes \hat{\chi}(b_i \otimes x)) \tag{5.24}$$

for all $x \in M$, $u \in U$, $Du=\sum_i a_i \otimes b_i$. Substituting $\chi_0(x)$ for x in (5.24), we obtain that

$$0=\chi_0\hat{\chi}(u \otimes \chi_0(x))+\sum_i \hat{\chi}(a_i \otimes \hat{\chi}(b_i \otimes \chi_0(x))) \tag{5.25}$$

for all $x \in M$, $u \in U$, $Du=\sum_i a_i \otimes b_i$.

Introduce a binary relation on the set $E=\{u_1, ..., u_n\}$ by putting $u_i \to u_j$ if $Du_j \in U \otimes_\Gamma U$ has a nonzero component in the direct summand $\Gamma u_i \Gamma \otimes_\Gamma U$ of $U \otimes_\Gamma U$ (see (5.10)). Since the Γ-coring K is left triangular, the transitive closure of the binary relation $\to$ is a partial ordering, which we denote by $\leqslant$. By induction on the partial ordering in E, show that for each $u \in E$ and every idempotent χ in $\mathfrak{B}(K, \Gamma)$, there exists an isomorphism $\psi=(\psi_0, \hat{\psi})$: $M \to N$ in $\mathfrak{B}(K, \Gamma)$ such that

$$\phi=(\phi_0, \hat{\phi})=\psi \circ \chi \circ \psi^{-1}$$

and $\hat{\phi}(v \otimes x)=0$ for all $x \in M$ and all $v \leqslant u$ in E. Then Lemma 5.3(a) implies the existence of ψ for which $\phi=(\phi_0, 0)$, and the theorem follows, as explained above.

Let u be a minimal element of the partially ordered set E. Then $Du=0$. Let

$$N_p=e_pM, \qquad p=1, ..., t, \tag{5.26}$$

be a collection of R-modules, and

$$\psi_{0p} = 1_{e_p M} \tag{5.27}$$

a collection of R-linear isomorphisms $e_p M \to N_p$. To define a collection of R-linear maps $\psi_s : e_{j_s} M \to N_{i_s}$, put

$$\psi_s = 0 \qquad \text{if } u \neq u_s \tag{5.28}$$

and

$$\psi_s(x) = \hat{\chi}(u \otimes \chi_0(x)) - \chi_0 \hat{\chi}(u \otimes x) \qquad \text{for all } x \in e_{j_s} M \text{ if } u = u_s. \tag{5.29}$$

By Proposition 5.12, there exists a Γ-module structure on $N = \bigoplus'_{p=1} N_p = M$ and an isomorphism $\psi = (\psi_0, \hat{\psi}) : M \to N$ in $\mathfrak{B}(K, \Gamma)$ satisfying $\psi_0 = \bigoplus'_{p=1} \psi_{0p} = 1_M$, $\hat{\psi}(u_s \otimes x) = \psi_s(x)$ for all s and x. Then $\psi^{-1} = (1_M, -\hat{\psi})$, as follows from formulae (5.6) and (5.21). For $\phi = (\phi_0, \hat{\phi}) = \psi \circ \chi \circ \psi^{-1}$, we want to show that $\phi_s = 0$ if $u_s = u$. Using formulae (5.6) and (5.7), we have

$$(\psi \circ \chi)_0 = \chi_0, \qquad (\psi \mathbin{\hat{\circ}} \chi)(u \otimes x) = \hat{\chi}(u \otimes x) + \hat{\psi}(u \otimes \chi_0(x)) \tag{5.30}$$

because $\psi_0 = 1_M$ and $Du = 0$. Further,

$$\hat{\phi}(u \otimes x) = -\chi_0 \hat{\psi}(u \otimes x) + \hat{\chi}(u \otimes x) + \hat{\psi}(u \otimes \chi_0(x)). \tag{5.31}$$

But

$$\begin{aligned} &-\chi_0 \hat{\psi}(u \otimes x) + \hat{\psi}(u \otimes \chi_0(x)) \\ &\quad = -\chi_0[\hat{\chi}(u \otimes \chi_0(x)) - \chi_0 \hat{\chi}(u \otimes x)] + \hat{\chi}(u \otimes \chi_0^2(x)) - \chi_0 \hat{\chi}(u \otimes \chi_0(x)) \\ &\quad = -\chi_0 \hat{\chi}(u \otimes \chi_0(x)) + \chi_0 \hat{\chi}(u \otimes x) + \hat{\chi}(u \otimes \chi_0(x)) - \chi_0 \hat{\chi}(u \otimes \chi_0(x)) \\ &\quad = \hat{\chi}(u \otimes x), \end{aligned} \tag{5.32}$$

using (5.23), (5.25), and (5.24), together with $Du = 0$. This implies $\hat{\phi}(u \otimes x) = 0$, and the base of induction has been established.

Suppose now that ξ is an isomorphism in $\mathfrak{B}(K, \Gamma)$ such that $\phi = (\phi_0, \hat{\psi}) = \xi \circ \chi \circ \xi^{-1}$ is an idempotent satisfying $\hat{\phi}(v \otimes x) = 0$ for all x and all $v < u$ in E. It suffices to find an isomorphism ψ in $\mathfrak{B}(K, \Gamma)$ such that $\zeta = (\zeta_0, \hat{\zeta}) = \psi \circ \phi \circ \psi^{-1}$ has the property $\hat{\zeta}(v \otimes x) = 0$ for all x and all $v \leqslant u$ in E. Without loss of generality, we may assume that $\phi = \chi$, i.e., that the idempotent $\chi = (\chi_0, \hat{\chi})$ has the property $\hat{\chi}(v \otimes x) = 0$ for all $x \in M$ and all $v < u$ in E. The argument here is similar to the case when u is a minimal element in E. We consider a collection of R-modules and R-linear maps defined by formulae (5.26), (5.27), (5.28), and (5.29). Using Proposition 5.12, we obtain an isomorphism $\psi = (1_M, \hat{\psi})$ in $\mathfrak{B}(K, \Gamma)$ with inverse $\psi^{-1} = (1_M, \hat{\theta})$. We have

$$\hat{\chi}(v \otimes x) = \hat{\psi}(v \otimes x) = \hat{\theta}(v \otimes x) = 0 \qquad \text{for all } x \in M \text{ and all } v < u. \tag{5.33}$$

Really, this is the assumption about $\hat{\chi}$; $\hat{\psi}$ satisfies the condition by Lemma 5.3(a), in view of (5.28), (5.29); and for $\hat{\theta}$ we have

$$\hat{\theta}(v \otimes x) = -\hat{\psi}(v \otimes x) \qquad \text{for all } x \in M \text{ and all } v \leqslant u, \tag{5.34}$$

according to formulae (5.6), (5.21), and the assumption that the Γ-coring K is left triangular. Coming back to $\phi = (\phi_0, \hat{\phi}) = \psi \circ \chi \circ \psi^{-1}$, note that (5.33) implies

$$\hat{\phi}(v \otimes x) = 0 \qquad \text{for all } x \in M \text{ and all } v < u, \tag{5.35}$$

as follows from (5.7). We note that, as before, formulae (5.6) and (5.7) imply (5.30) because of $\psi_0 = 1_M$,

$$Du = \sum_i a_i \otimes b_i \qquad \text{with } a_i \in \bigoplus_{v < u} \Gamma v \Gamma, \tag{5.36}$$

and (5.33). Likewise, (5.33) and (5.34) imply (5.31). Finally, computation (5.32) goes through similarly to the previous case, but, instead of $Du = 0$, we rely on (5.36) and (5.33). Since (5.35) holds, and (5.31), (5.32) imply $\hat{\phi}(u \otimes x) = 0$ for all x, the theorem is proved. ∎

COROLLARY 5.14. *Idempotents split in* induc K.

6. RELATIVELY PROJECTIVE MODULES OVER FROBENIUS GROUPS

The content of this section was communicated to the second author by Jacques Lewin.

Recall that a finite group G is a Frobenius group if it contains a proper subgroup H with $H \cap H^g = 1$ for all $g \in G - H$, where $x^g = gxg^{-1}$. It is well known [12, p. 317] that G is a split extension $G = KH$ for a normal subgroup $K = \{1, k_1, ..., k_u\}$ of G, and $k_i^a = k_i^b$ implies $a = b$ for a, $b \in H$ and any i with $1 \leqslant i \leqslant u$. Hence $Hk_iH = \bigcup_{a \in H} ak_iH = \bigsqcup_{a \in H} k_i^a H$, where $\bigsqcup$ stands for disjoint union. If h is the number of elements of H, then Hk_iH consists of h^2 elements.

For an arbitrary commutative ring R, denote by RG the group algebra of G over R, and by RH the R-subalgebra of G generated by H. Then a unique RH-bimodule map $RH \otimes_R RH \to RHk_iRH$ sending $1 \otimes 1$ to k_i is an isomorphism. Since G is a disjoint union of double cosets of H, we get the following direct sum of RH-bimodules: $RG = RH \oplus (\oplus_i RHk_iRH) \simeq RH \oplus (\oplus_i RH \otimes_R RH)$ for some values of i between 1 and u. Since the R-algebra RH is self-injective, condition (a) of Corollary 3.4 is satisfied. Thus, assuming again that R is a field or a Dedekind domain, and applying Theorem 4.2, we obtain the following statement.

THEOREM 6.1. *Category* $\mathbf{p}(RG, RH)$ *has almost split sequences.*

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