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Multi Lingual Sequent Calculus and Coherent Spaces

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Abstract

We study a Gentzen style sequent calculus where the formulas on the left and right of the turnstile need not necessarily come from the same logical system. Such a sequent can be seen as a consequence between different domains of reasoning. We discuss the ingredients needed to set up the logic generalized in this fashion.

The usual cut rule does not make sense for sequents which connect different logical systems because it mixes formulas from antecedent and succedent. We propose a different cut rule which addresses this problem.

The new cut rule can be used as a basis for composition in a suitable category of logical systems. As it turns out, this category is equivalent to coherent spaces with certain relations between them.

Finally, cut elimination in this set-up can be employed to provide a new explanation of the domain constructions in Samson Abramsky's *Domain Theory in Logical Form*.

1 Introduction

This paper attempts to provide a new analysis of Samson Abramsky's *Domain Theory in Logical Form* [1,2]. The overall aim is to isolate precisely the key ingredients necessary for a *Logic of Finite Observations* [14,12].

Since we choose to emphasize the logic rather than the semantics our main objects of study are *sequents* in tradition of Proof Theory [5]:

$$\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$$

The connectives are restricted to conjunction and disjunction, that is, *positive logic*. This is in line with previous work on observational logic. We go beyond this in three respects. Firstly, we leave out the identity axiom scheme, $\phi \vdash \phi$. This is justified by the fact that *observing* a certain state of the world does not always imply that the corresponding proposition is actually true, the reason being that our instruments for observing the world are not precise enough. Measuring physical constants is an example. Secondly, we allow the formulas ϕ_i in a sequent to be from a different language than the formulas ψ_j. Technically, this seems admissible because none of the rules for disjunction or conjunction mixes formulas from different sides of sequents. However, the cut rule has to be adjusted to this new situation. As far as observational logic is concerned it seems a common situation that there is a linguistic difference between the observations one might make and the conclusions to be drawn from them. Thirdly, we allow classical sequents following the example of Gentzen's famous treatment of classical logic. At first glance there seems to be no point in this because there is no difference between intuitionistic positive logic and classical positive logic. However, this formulation will emphasize the rather pretty self-symmetry of the whole set-up.

Many aspects of this paper are a direct consequence of these three special properties of the logical system. Leaving out the identity axiom, for example, necessitates to check carefully how to retain some of its essential consequences. Doing so, we discover interpolation axioms akin to the interpolation property of the approximation (way-below) relation known from continuous domains [6,3]. Allowing sequents to 'connect' formulas from different logical languages suggests to study a category with sets of sequents (closed under the logical rules) as arrows. We call such sets of sequents *consequence relations*. The cut rule serves nicely as a composition of consequence relations but we have to work somewhat harder to find the identities. The notion of object we end up with shows close resemblance with *strong proximity lattices* which were introduced in [7,8]. While the latter were motivated by purely topological considerations (in the vain of *Stone duality*) the present paper establishes their logical proof-theoretic content. In particular, this constitutes an independent justification for the two axioms which distinguish strong proximity lattices from the structures studied in [11]. The precise connection is laid out in Sections 6 and 7 below.

Our symmetric 'classical' presentation of sequents allows us to shed new light on the open and the compact saturated interpretation of tokens in strong proximity lattices. In [7] open sets are coded as round ideals and compact saturated sets as round filters. In the present, logical reading, ideals appear naturally on the left of the turnstile, filters on the right. Besides illustrating once again the duality between open and compact subsets this now suggests to read the complement of a compact saturated set as *negative* information encoded in a token. Although we excluded negation from observational logic, we seem to get a weak form of negation for free!

There is also a semantic reading of our morphisms, that is, consequence relations. They correspond to maps from a coherent space to the Smyth power space of another one, i.e. they can be seen as certain relations between these spaces. Composition corresponds to composition in the Kleisli category associated with the Smyth power monad. Ordinary functions can be captured as well but only at the price of sacrificing symmetry.

In the last section we study cut elimination in the context of positive logic and our new form of the cut rule. The proofs are fairly simple when compared to the intricacies of Gentzens's original cut elimination theorem, but the result is nonetheless quite powerful. It allows us to describe domain constructions in purely logical terms, avoiding the translation from the logic to topological spaces which is at the heart of the proofs in [1,2]. We illustrate the technique for the construction of product.

The paper draws on a number of concepts from the existing literature. We recommend [3] as a reference for notions in domain theory and Stone duality, [7] for coherent spaces, and [13] for proof theory.

2 A typed propositional logic

We are considering a situation where inferences are to be drawn between different logical systems. We write

$$\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$$

as in standard proof theory and we read this as "if all ϕ_i hold then at least one ψ_j holds", as usual, but unlike in normal sequent calculus we intend to keep the formulas on the left separate from those on the right. There are many situations where such a separation might be desirable or even necessary. We discuss three of them.

Consider ordinary propositional logic. Someone could say

"It is very cold in here. I need to put on a sweater."

thus drawing an inference from an observation about the temperature to a certain action. Note that there is nothing 'logical' about this inference and, indeed, someone else might say

"It is very cold in here. I will turn on the heating."

The inference relation in this example is a subjective one and there can be many different such relations. Although it is common to combine arbitrary propositions in logic we may wish to distinguish in a situation like this between propositions about the state of the environment and propositions about actions of a certain individual.

A second example is given by Hoare Logic. When we write a triple like

$$\{x > 0\} \quad \mathtt{x := -x} \quad \{x < 0\}$$

we certainly do not mean that $x > 0$ logically implies $x < 0$, rather, we read this as

> "If $x > 0$ holds before the execution of $\mathtt{x} := -\mathtt{x}$ then $x < 0$ holds afterwards."

In this example every program fragment will give rise to a characteristic relationship between preconditions and postconditions. The logical formulas are (typically) all about the contents of program variables and there is no syntactic reason to keep pre- and postconditions separate, as in the previous example, but the separation becomes necessary because the formulas refer to the state at different times.

Our third example is from *observation logic*, [1,14,12]. In many situations, in computing in particular, there is no difference between what we observe and what we hold to be true. In more real life situations this is not so. We can observe that the thermometer reads 15° Celsius but we do not necessarily believe that this is actually the case. If the thermometer works well then we perhaps infer from this observation that the true temperature is somewhere between 14.5° and 15.5° Celsius. We arrive at a logic where ϕ ('the observation') does not necessarily imply ϕ ('the belief').

Technically, we allow formulas on the left and formulas on the right of the turnstile to come from different logical systems. These systems can be quite arbitrary; all we require is the presence of conjunction and disjunction, and the units $\perp$ (falsity) and $\top$ (truth). Each system embodies a certain 'logic' in the sense that certain formulas imply others. We capture the internal logic by referring to arbitrary $(2, 2, 0, 0)$-algebras instead of syntactically defined set of formulas. Such an algebra, for example, could have been obtained as the Lindenbaum algebra by factoring the set of formulas of the system by logical equivalence. At the other extreme, the syntactically defined set of formulas for a logical system can be regarded as a such an algebra, providing the logic contains the connectives of positive logic. Henceforth we will use the expressions "element of a $(2, 2, 0, 0)$-algebra" and "formula" interchangeably.

We require that the comma which separates formulas on the left refers to conjunction and the comma on the right refers to disjunction. The logical part of our system is given by the rules

$$\frac{}{\bot \vdash}\,(L\bot) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\,(R\bot)$$

$$\frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\,(L\top) \qquad \frac{}{\vdash \top}\,(R\top)$$

$$\frac{\phi, \psi, \Gamma \vdash \Delta}{\phi \wedge \psi, \Gamma \vdash \Delta}\,(L\wedge) \qquad \frac{\Gamma \vdash \Delta, \phi \quad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \phi \wedge \psi}\,(R\wedge)$$

$$\frac{\phi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}{\phi \vee \psi, \Gamma \vdash \Delta}\,(L\vee) \qquad \frac{\Gamma \vdash \Delta, \phi, \psi}{\Gamma \vdash \Delta, \phi \vee \psi}\,(R\vee)$$

where a double line indicates that the rule can be used in both directions. The 'backward' rules are not present in the usual sequent calculus since there they are consequences of the identity and the cut rule. The difference in character between the 'forward' and the 'backward' rules will become apparent in Section 8

Note that we cannot refer to implication or negation in the logical systems as the corresponding rules would make it necessary to transfer formulas from one side to the other. However, the logical systems themselves may still support these connectives.

On the side of structural rules we will only refer to weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\,(W)$$

and keep exchange and contraction implicit. Thus we are working with *sets* of formulas rather than sequences. The 'forward' rules $(R\bot)$ and $(L\top)$ are special cases of weakening.

As the examples above suggest, this calculus is not about finding tautologies but rather, each relation $\vdash$ between formulas of two logical systems embodies a particular, possibly subjective, inference. Whatever the reasons are for holding such an inference as true, there are other inferences which should in such a situation also be held as true. The rules above formalize precisely this reasoning: If ϕ, ψ, Γ entails Δ then $\phi \wedge \psi, \Gamma$ should also entail Δ, and so on. Our objects of study are therefore relations between sets of formulas which are closed under the rules from above. We fix this in a definition:

Definition 2.1 *For two algebras* $\langle L; \wedge, \vee, \top, \bot\rangle$ *and* $\langle M; \wedge', \vee', \top', \bot'\rangle$ *of type* $(2, 2, 0, 0)$*, a* consequence relation from L to M *is a binary relation between finite sets from L and M closed under* $(L\bot)$*,* $(R\bot)$*,* $(L\top)$*,* $(R\top)$*,* $(L\wedge)$*,* $(R\wedge)$*,* $(L\vee)$*,* $(R\vee)$*, and* (W)*.*

If, according to a consequence relation $\vdash$, the formula ϕ implies ψ, and if, according to a second relation $\vdash'$, ψ implies σ, then it makes sense to combine these two inferences and to say that ϕ implies σ according to the composition

$\vdash \circ \vdash'$ of the two given consequence relations. This composition corresponds to the composition of relations and functions in an obvious way. However, consequence relations relate *sets* of formulas rather than single formulas and the meaning of a set as an antecedent is different from the meaning of the same set as a succedent. A logically correct composition is given by the following rule [1]:

$$\frac{\begin{array}{cc} \Gamma \vdash \Delta_1 & \Theta_1 \vdash' \Lambda \\ \vdots & \vdots \\ \Gamma \vdash \Delta_n & \Theta_m \vdash' \Lambda \end{array}}{\Gamma(\vdash \circ \vdash')\Lambda}(\textsf{Cut-Comp})$$

subject to the condition that for every choice function $f \in \prod_i \Delta_i$ there exists an index j so that $\Theta_j \subseteq \{f_1, \ldots, f_n\}$.

We call it *cut composition* in reference to Gentzen's classical cut rule.

The intuition behind this cut rule and its side-condition is that if Γ entails all the Δ_i's then at least one formula in each Δ_i is true. If for every possibility, coded by a choice function, these formulas cover one of the Θ_j's then Γ also entails Λ.

The only problem with the cut rule is that it looks rather asymmetric whereas the other rules are perfectly symmetric. That is to say if we take a rule and interchange left and right as well as the connectives $\vee$ and $\perp$ with their duals $\wedge$ and $\top$ then we again get a rule. Fortunately, we can do the same with the cut rule as the following lemma shows:

Lemma 2.2 *The side-condition of* Cut-Comp *is equivalent to the following self-dual condition:*

$$(\forall f \in \prod_i \Delta_i)(\forall g \in \prod_j \Theta_j)\ \{f_1, \ldots, f_n\} \cap \{g_1, \ldots, g_m\} \neq \emptyset$$

3 A proof-theoretic analysis of identities

We are interested in constructing a category of consequence relations. As a first step we observe that Cut-Comp preserves consequence relations and that it is associative.

Lemma 3.1 *Given consequence relations $\vdash$ from L to M and $\vdash'$ from M to N the sequents $\Gamma(\vdash \circ \vdash')\Lambda$ that arise from the rule* Cut-Comp *form a consequence relation.*

Lemma 3.2 *The composition of consequence relations induced by* Cut-Comp *is associative.*

[1] The premise of this rule is meant to be read as two families of sequents, not as proof trees.

It remains to find identities. One might be tempted to employ ordinary logical implication between formulas of one world for this. However, this is somewhat against the spirit of this paper where we want to suppress purely logical equivalences in order to exhibit the properties of inferences which are, in some sense, subjective or observational. As we have argued, for such inferences it is not necessarily the case that a formula ϕ implies itself. That is, we refuse the identity axioms

$$\frac{}{\phi \Vdash \phi}$$

We reserve the symbol $\Vdash$ to represent a consequence relation that has identical source and target algebra L.

On the other hand, Gentzen's original cut rule

$$\frac{\Gamma \Vdash \Delta, \phi \quad \phi, \Theta \Vdash \Lambda}{\Gamma, \Theta \Vdash \Delta, \Lambda}(\mathsf{Cut})$$

makes sense even in an observational interpretation, and we would therefore expect it to hold for an identity consequence relation. More precisely, if $\Vdash \circ \Vdash = \Vdash$ for a consequence relation on some algebra L then $\Vdash$ should be closed under Gentzen's cut. As it turns out, this can be shown if consequence relations are assumed to be *interpolative* in the following sense:

Definition 3.3 *We say that* $\Vdash$ has interpolants *if the following are satisfied:*

(L-Int) *If* $\phi, \Gamma \Vdash \Delta$ *then there exists* $\phi' \in L$ *so that* $\phi \Vdash \phi'$ *and* $\phi', \Gamma \Vdash \Delta$.

(R-Int) *If* $\Gamma \Vdash \Delta, \phi$ *then there exists* $\phi' \in L$ *so that* $\Gamma \Vdash \Delta, \phi'$ *and* $\phi' \Vdash \phi$.

Of course, if the identity axioms of sequent calculus are adopted then interpolation is trivial. Looking at this from the other end, we can say that interpolation will provide us with some of the consequences of the identity axiom scheme.

Lemma 3.4 *If* $\Vdash$ *has interpolants, then* $\Vdash \subseteq \Vdash \circ \Vdash$.

Theorem 3.5 *A consequence relation* $\Vdash$ *with interpolants is closed under* Cut*, if and only if* $\Vdash \circ \Vdash \subseteq \Vdash$.

Corollary 3.6 *If* $\Vdash$ *has interpolants, then it is an idempotent with respect to* Cut-Comp *if and only if it is closed under Gentzen's cut rule.*

Another way of casting this is to observe that for interpolative consequence relations Cut and Cut-Comp are inter-definable.

From this, we take our cue to define the objects of a category.

Definition 3.7 $\langle L; \wedge, \vee, \top, \bot; \Vdash \rangle$ *is a* coherent sequent calculus *if* $\Vdash$ *is a consequence relation from* L *to* L *such that* $\Vdash$ *has interpolants and is closed under* Cut.

The relations $\Vdash$ are in fact idempotents but not identities for all consequence relations. This is not surprising because, as yet, we do not have any axioms that make sure that identities and other consequence relations interact in a sensible way. Say that a consequence relation $\vdash$ from L to M is *compatible* *with* $\Vdash_L$ *and* $\Vdash_M$ if

$$\Vdash_L \circ \vdash \, = \, \vdash \, = \, \vdash \circ \Vdash_M$$

Definition 3.8 *The category* MLS *(for Multi Lingual Sequents) has coherent sequent calculi as objects and compatible consequence relations between them as arrows.*

The facts that $\Vdash_L$ is self-compatible on both sides, and that composition of compatible consequent relations preserves compatibility are both evident from the definition. From Lemma 2.2 and the preceding discussion it is clear that MLS is self-dual.

The properties of idempotent consequence relations $\Vdash$ are inherited by compatible consequence relations as follows:

Lemma 3.9 *A consequence relation* $\vdash$ *from* L *to* M *is compatible if and only if*

(L-Int′) $\phi, \Gamma \vdash \Delta$ *implies that there exists* $\phi' \in L$ *so that* $\phi \Vdash_L \phi'$ *and* $\phi', \Gamma \vdash \Delta$*;*

(R-Int′) $\Gamma \vdash \Delta, \phi$ *implies that there exists* $\phi' \in L$ *so that* $\phi' \Vdash_L \phi$ *and* $\Gamma \vdash \Delta, \phi'$*;*

(L-Cut) *if* $\Gamma \Vdash_L \phi$ *and* $\phi, \Theta \vdash \Lambda$*, then* $\Gamma, \Theta \vdash \Lambda$*; and*

(R-Cut) *if* $\Gamma \vdash \Delta, \phi$ *and* $\phi \Vdash_M \Lambda$*, then* $\Gamma \vdash \Delta, \Lambda$*.*

A different perspective on the definition of MLS is given by the following. One can restrict the logic to situations where a proposition ϕ does imply itself and thus adopt the identity rule for all consequence relations $\Vdash$ from a logical system to itself. As the identity morphism on a $(2,2,0,0)$-algebra one can then take the smallest consequence relation generated by the identity rules, which will yield precisely the logically valid sequents of the system. Compatibility is not an issue and one obtains immediately a (self-dual) category RMLS (Reflexive Multi Lingual Sequents). Now, the category MLS, that we are interested in, is precisely the category one obtains from RMLS by *splitting the idempotents*, a technique which is well-known from category theory, [4, 1.28].

It is worthwhile to note that all definitions and theorems up to this point would still make sense if one did not allow the application of the logical rules from the lower sequent to the upper sequent. One would then get consequence relations in which, for example, $\phi \wedge \psi, \Gamma \vdash \Delta$ was true but not $\phi, \psi, \Gamma \vdash \Delta$. At present, we cannot see any applications for such a logic.

4 Some proof theory

Consider the following definitions for a compatible consequence relation $\vdash$ from L to M. For $X \subseteq L$ and $Y \subseteq M$, define

$$X[\vdash] = \{\phi \in M \mid \exists \Gamma \subseteq_{\mathsf{fin}} X. \Gamma \vdash \phi\}$$

$[\vdash]Y = \{\phi \in L \mid \exists\Delta \subseteq_{\mathsf{fin}} Y. \phi \vdash \Delta\}$

As usual, for singletons we write $\phi[\vdash]$ for $\{\phi\}[\vdash]$, and $[\vdash]\phi$ for $[\vdash]\{\phi\}$.

A *filter of* L is a set $F \subseteq L$ such that $F = F[\Vdash_L]$; an *ideal of* L is a set $I \subseteq L$ such that $I = [\Vdash_L]I$. Let $\mathsf{Filt}(L)$ and $\mathsf{Idl}(L)$ denote the partial orders of filters and ideals, respectively, both ordered by inclusion.

Proposition 4.1 *A set X is a filter if and only if the following hold:*

(i) $\top \in X$*;*

(ii) $\phi, \psi \in X$ *if and only if* $\phi \wedge \psi \in X$*; and*

(iii) $\phi \in X$ *if and only if for some* $\psi \in X$*,* $\psi \Vdash \phi$*.*

Furthermore, in any filter X, if $\phi \in F$ (or $\psi \in F$), then $\phi \vee \psi \in F$.

Lemma 4.2 *$X[\vdash]$ is always a filter and $[\vdash]Y$ is always an ideal.*

Lemma 4.3 *The assignments $X \mapsto X[\Vdash_L]$ and $X \mapsto [\Vdash_L]X$ are Scott-continuous retractions on the powerset of the coherent sequent calculus L.*

Corollary 4.4 *$\mathsf{Filt}(L)$ and $\mathsf{Idl}(L)$ are continuous lattices.*

By looking at the internal structure more closely we discover the following:

Lemma 4.5 *In $\mathsf{Filt}(L)$ and $\mathsf{Idl}(L)$ directed suprema are unions and finite meets are intersections.*

Later, we will exhibit further properties of the lattices $\mathsf{Filt}(L)$ and $\mathsf{Idl}(L)$. We have singled out directed suprema and finite meets because of the following:

Lemma 4.6 *The assignments*

$$L \longmapsto \mathsf{Filt}(L)$$

$$\vdash \longmapsto (F \mapsto F[\vdash])$$

define a functor Filt from MLS to the category SL of directed complete meet semilattices and Scott-continuous semilattice homomorphisms.

Dually, the assignments

$$L \longmapsto \mathsf{Idl}(L)$$

$$\vdash \longmapsto (I \mapsto [\vdash]I)$$

define a contravariant functor Idl from MLS to SL.

As $\mathsf{Filt}(L)$ and $\mathsf{Idl}(L)$ are meet semilattices, it is worthwhile to characterize the meet-prime elements.

Lemma 4.7 *For sets $X \subseteq L$, the following are equivalent.*

(i) *X is a meet-prime element of $\mathsf{Filt}(L)$;*

(ii) *X is a filter, $\perp \notin X$ and $\phi \vee \psi \in X$ if and only if $\phi \in X$ or $\psi \in X$;*

(iii) *$X \cap \Delta \neq \emptyset$, for a finite Δ, if and only if $\Gamma \Vdash_L \Delta$ for some $\Gamma \subseteq_{\mathsf{fin}} X$.*

The dual conditions for meet-prime elements of $\mathsf{Idl}(L)$ *also are equivalent.*

A set satisfying any of these equivalent conditions is called a *prime filter*. A set satisfying the dual conditions is called a *prime ideal*. Note that in (3) we are neither assuming X to be a filter nor do we refer to the operations of the algebra.

5 Consistency

Consider the role that filters and ideals play in logic. Roughly, a filter corresponds to a theory. One typically says that a filter (or theory) is *consistent* if it is not the entire language. Then one formulation of soundness and completeness has it that a filter is consistent if and only if it has a model. The latter means essentially, that it is contained in a prime filter. But closer inspection of the proofs of completeness theorems, say for Gentzen's system **K** [5], shows that more is proved. In particular, we have nearly complete freedom to choose, apart from the formulas in F, what formulas are *not* to be satisfied in a particular model.

Say that a pair of sets (X, Y) for $X \subseteq L$ and $Y \subseteq M$ is $\vdash$-*consistent* provided that for all $\Gamma \subseteq_{\mathsf{fin}} X$ and $\Delta \subseteq_{\mathsf{fin}} Y$, it is the case that $\Gamma \not\vdash \Delta$. The idea here is to understand X as a set of formulas that 'hold' in L and Y as a set of formulas that do not hold in M. So the least we should expect is that $\vdash$ does not contradict this understanding.

Consistency has to do essentially with filters and ideals as the following lemma shows.

Lemma 5.1 *For every consequence relation* $\vdash$ *from* L *to* M *the following are equivalent:*

(i) (X, Y) *is* $\vdash$*-consistent;*

(ii) $(X, [\Vdash_M]Y)$ *is* $\vdash$*-consistent;*

(iii) $(X[\Vdash_L], Y)$ *is* $\vdash$*-consistent;*

(iv) $(X[\vdash], Y)$ *is* $\Vdash_M$*-consistent;*

(v) $(X, [\vdash]Y)$ *is* $\Vdash_L$*-consistent;*

(vi) $X[\vdash] \cap [\Vdash_M]Y = \emptyset$*;*

(vii) $X[\Vdash_L] \cap [\vdash]Y = \emptyset$*;*

(viii) (X, I) *is* $\vdash$*-consistent for some prime ideal* $I \supseteq Y$*; and*

(ix) (F, Y) *is* $\vdash$*-consistent for some prime filter* $F \supseteq X$*.*

Consistency provides the following bridge between the functors Filt and Idl: The equivalence (4) $\iff$ (5) says, in effect, that consistency acts as a natural transformation between Filt and Idl, both considered as covariant functors to Rel.

The equivalent conditions (8) and (9), on the other hand, correspond to completeness, as mentioned above. We will come back to these conditions when we discuss the topological semantics of coherent sequent calculi.

The machinery provided by the previous lemma, in particular the equivalence of the first four conditions, allows us to improve on Lemma 4.6.

Theorem 5.2 *The functors* Filt *and* Idl *are full and faithful.*

This implies that the image of Filt is equivalent to MLS. Whereas the category MLS is clearly self-dual, this property is not obvious in the case of this full subcategory of SL. It was discovered in 1979 by Jimmie Lawson, [9].

6 Algebraization of observation logic: Proximity lattices

Before we continue with the proof theory of coherent sequent calculi and compatible consequence relations we approach the issue from a completely different angle. To this end we review some of the results reported in [7].

Definition 6.1 *A* strong proximity lattice *is a distributive bounded lattice* $(B; \vee, \wedge, \bot, \top)$ *together with a binary transitive relation* $\prec$ *satisfying* $\prec \circ \prec \; = \; \prec$. *The* algebraic *structure given by the lattice and the* approximation *structure are connected by the following four axioms:*

$$(\vee - \prec)\ (\forall a \in B,\ M \subseteq_{\mathsf{fin}} B)\ M \prec a \iff \bigvee M \prec a$$

$$(\prec - \wedge)\ (\forall a \in B,\ M \subseteq_{\mathsf{fin}} B)\ a \prec M \iff a \prec \bigwedge M$$

$$(\prec - \vee)\ (\forall a, x, y)\ a \prec x \vee y \Rightarrow (\exists x', y')\ x' \prec x, y' \prec y \text{ and } a \prec x' \vee y'$$

$$(\wedge - \prec)\ (\forall a, x, y)\ x \wedge y \prec a \Rightarrow (\exists x', y')\ x \prec x', y \prec y' \text{ and } x' \wedge y' \prec a$$

We use $a \prec M$ *to mean* $a \prec m$*, for all* $m \in M$*, and analogously for* $M \prec a$.

Mappings between proximity lattices are certain relations. In the following definition we use $\circ$ to denote the usual relational product.

Definition 6.2 *A relation* $G \subseteq A \times B$ *between strong proximity lattices* A *and* B *is called* approximable *if it satisfies the following conditions:*

$$(G - \prec)\ G \circ \prec_B = G$$

$$(\prec - G)\ \prec_A \circ\, G = G$$

$$(\vee - G)\ (\forall M \subseteq_{\mathsf{fin}} A,\ b \in B)\ M\ G\ b \iff \bigvee M\ G\ b$$

$$(G - \wedge)\ (\forall a \in B,\ M \subseteq_{\mathsf{fin}} B)\ a\ G\ M \iff a\ G \bigwedge M$$

$$(G - \vee)\ (\forall a \in B,\ M \subseteq_{\mathsf{fin}} B)\ a\ G \bigvee M \Rightarrow (\exists N \subseteq_{\mathsf{fin}} A)\ a \prec_A \bigvee N \text{ and } (\forall n \in N)(\exists m \in M)\ n\ G\ m$$

If a relation satisfies all conditions but $(G - \vee)$ *we call it a* weak approximable relation.

Strong proximity lattices and (weak) approximable relations form a category $\mathsf{SPL}_{(w)}$; the order of approximation $\prec$ on an object acts as identity.

Now, we want to compare this with coherent sequent calculi and consequence relations. It is straight-forward to see from the respective definitions that

$$\Gamma \Vdash \Delta \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \bigwedge \Gamma \prec \bigvee \Delta$$

makes a strong proximity lattice into a coherent sequent calculus and that weak approximable relations likewise translate into compatible consequence relations.

In the other direction, we factor a given $(2,2,0,0)$-algebra with interpolative consequence relation $\Vdash$ by the equations of distributive lattices. A number of calculations show that $\Vdash$ is invariant with respect to these equations and we can therefore without ambiguity define a consequence relation on the quotient algebra by setting

$$[\phi_1], \ldots, [\phi_n] \Vdash_{[]} [\psi_1], \ldots, [\psi_m] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \phi_1, \ldots, \phi_n \Vdash \psi_1, \ldots, \psi_m \,.$$

Note, that for this to work it is essential that the logical rules can be used in both directions. This provides most of the information needed to support the following:

Proposition 6.3 *The categories* MLS *and* SPL_w *are equivalent.*

7 Semantics: Coherent spaces

By going either via strong proximity lattices and using the results in [7], or by further inspection of the lattices of ideals and filters, one obtains the following:

Proposition 7.1 *For a coherent sequent calculus L the continuous lattices* $\mathsf{Filt}(L)$ *and* $\mathsf{Idl}(L)$ *are arithmetic.*

The terminology 'arithmetic' is taken from [3,6] and means that the lattice is distributive and that $x \ll y, z$ implies $x \ll y \wedge z$. The Stone duality of arithmetic lattices is well understood; they are precisely the open-set lattices of *coherent spaces* (sometimes also called *stably locally compact spaces*), [3, Theorem 7.2.19]. As was shown in [7], this duality can be explained at the level of proximity lattices. Again, a similar statement is true for coherent sequent calculi:

Proposition 7.2 *Let L be a coherent sequent calculus and let* $\mathsf{spec}(L)$ *be the set of prime filters on L. The sets*

$$\mathcal{O}_\phi := \{F \in \mathsf{spec}(L) \mid \phi \in F\}$$

with $\phi \in L$ generate a topology on $\mathsf{spec}(L)$. *The resulting space is coherent.*

As in propositional logic, a prime filter on a coherent sequent calculus represents a model. The spectrum $\mathsf{spec}(L)$, then, is the space of all models, and every formula ϕ of L defines a subset of models, namely, those in which ϕ

is true. The definition of the topology on $\mathsf{spec}(L)$ is such that all these extents of formulas are open. In the classical setting of Boolean algebras and Stone spaces the extents are also compact. This is not the case here. However, every formula has a canonical compact subset associated with it:

$$\mathcal{K}_\phi := \{F \in \mathsf{spec}(L) \mid \phi[\Vdash_L] \subseteq F\} .$$

The logic is translated into set-theoretic operations both through the open and the compact interpretation:

Proposition 7.3 *The following are true for a coherent sequent calculus L:*

$$\mathcal{O}_{\phi\wedge\psi} = \mathcal{O}_\phi \cap \mathcal{O}_\psi \;\; \mathcal{O}_{\phi\vee\psi} = \mathcal{O}_\phi \cup \mathcal{O}_\psi$$
$$\mathcal{K}_{\phi\wedge\psi} = \mathcal{K}_\phi \cap \mathcal{K}_\psi \;\; \mathcal{K}_{\phi\vee\psi} = \mathcal{K}_\phi \cup \mathcal{K}_\psi$$

It was shown in [7] that open and compact sets need to be combined in order to reconstruct the logic from the spectrum.

Theorem 7.4 *For a coherent space X the following defines a strong proximity lattice:*

- $B := \{(O, K) \in \Omega(X) \times \mathcal{K}(X) \mid O \subseteq K\}$
- $(O, K) \vee (O, K') := (O \cup O', K \cup K')$
- $(O, K) \wedge (O, K') := (O \cap O', K \cap K')$
- $\bot := (\emptyset, \emptyset)$, $\top := (X, X)$
- $(O, K) \prec (O', K') \quad \stackrel{def}{\Longleftrightarrow} \quad K \subseteq O'$

This is a representation of X, i.e. $X \simeq \mathsf{spec}(B)$. The spectra of strong proximity lattices are precisely the coherent spaces.

Here $\Omega(X)$ denotes the set of open subsets of X and $\mathcal{K}(X)$ the set of those compact subsets which are *saturated* in the sense that they equal the intersection of their neighborhoods.

Section 5 helps to explain the meaning of the open and compact sets O and K making up the tokens (O, K) in this theorem. An open set O can be seen to represent positive information, and a compact set K to represent the negative information $X - K$. The constraint $O \subseteq K$ avoids self-contradiction of tokens.

It was shown in [7] that the category of proximity lattices and approximable relations is equivalent to the category of coherent spaces and continuous functions. The question arises how *weak* approximable relations (or, equivalently, compatible consequence relations) can be captured on the side of the spectrum.

Given a compatible consequence relation $\vdash$ between coherent sequent calculi L and M, one can define a relation $R_\vdash$ between the two spectra as follows:

$$F \; R_\vdash \; G \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad F[\vdash] \subseteq G$$

where $F \in \mathsf{spec}(L)$, $G \in \mathsf{spec}(M)$. One can then show that the composition of consequence relations via $\mathsf{Cut\text{-}Comp}$ translates into ordinary composition of relations

$$R_{\vdash \circ \vdash'} = R_{\vdash} \circ R_{\vdash'} \ .$$

In order to understand the resulting relations better, one might also look at the *function* $(\cdot)[\vdash]$ associated with $\vdash$

$$F \mapsto F[\vdash] \ .$$

However, the arguments here are prime filters whereas the result can be an arbitrary filter. Composition, therefore, requires an extra twist: Given two functions $f\colon \mathsf{spec}(L) \to \mathsf{Filt}(M)$, $g\colon \mathsf{spec}(M) \to \mathsf{Filt}(N)$ define their composition by

$$g \circ f = \bigcap \{g(G) \mid G \in \mathsf{spec}(M) \text{ and } f(F) \subseteq G\}$$

This definition exploits, among other things, the fact that every filter equals the intersection of all prime filters it is contained in.

The formulation of these results in topological terms is somewhat more familiar. We use the fact that $\mathsf{spec}(L)$ is a coherent space X, that $\mathsf{Idl}(L)$ is isomorphic to the open set lattice of X, an that $\mathsf{Filt}(L)$ corresponds to the lattice $(\mathcal{K}(X), \supseteq)$ of compact saturated subsets of X. We equip $\mathcal{K}(X)$ with the Scott-topology derived from the order $\supseteq$ on filters.

Proposition 7.5 *Let L and M be coherent sequent calculi with spectra X and Y. The compatible consequence relations from L to M are in a bijection with the continuous maps from X to $\mathcal{K}(Y)$.*

We can reformulate this further: For a coherent space X the ordered set $(\mathcal{K}(X), \supseteq)$ of compact saturated subsets is always an arithmetic lattice. Equipped with the Scott-topology it is therefore again coherent, [3, Section 7.2.7]. A continuous function between coherent spaces lifts to a mapping between the compact set lattices. It has been shown in [8, Lemma 2] that this mapping is Scott-continuous. It follows that $\mathcal{K}$ defines an endofunctor on COH. This functor is also part of a monad. Its 'unit' takes a point x to $\uparrow x$, and its 'multiplication' maps an element $\mathcal{A} \in \mathcal{K}(\mathcal{K}(X))$ to $\bigcup \mathcal{A} \in \mathcal{K}(X)$ (see [10, Proposition 7.21]). As our main result of this section we get that the category of compatible theories is exactly the Kleisli category of this monad:

Theorem 7.6 *The categories MLS and SPL_w are equivalent to the Kleisli category $\mathsf{COH}_{\mathcal{K}}$ of the Smyth power monad $(\mathcal{K}, \uparrow, \bigcup)$.*

8 Cut elimination

The famous Cut Elimination Theorem of Gentzen [5] states that every valid sequent in the sequent calculus can be derived without employing the cut rule. Sequents in our setting, however, are not about absolute validity but about derivability of sequents from assumed sequents. The analogous theorem for this situation says that in every such derivation cuts between arbitrary sequents can be eliminated in favour of cuts between assumed sequents. We will exhibit a similar result which applies to the rule Cut-Comp. We have argued in Section 2 that it is necessary and appropriate to include with every classical rule its inverse, e.g., we not only allow

$$\frac{\phi, \psi, \Gamma \vdash \Delta}{\phi \wedge \psi, \Gamma \vdash \Delta} \quad \text{but also} \quad \frac{\phi \wedge \psi, \Gamma \vdash \Delta}{\phi, \psi, \Gamma \vdash \Delta}$$

It is now the right moment to make a distinction between these two kinds of rules. Call a rule *positive* if it introduces a connective into a sequent and *negative* otherwise. The positive rules of Section 2 are precisely those which are read from top to bottom. If R is any relation between finite sets of elements of $(2,2,0,0)$-algebras denote with R^+ the smallest such relation which contains R and is closed under application of positive rules.

For algebras themselves we say that $B \subseteq L$ is a *generating set* if the smallest subalgebra B^+ of L containing B is L itself. This generation process can also be described by finitary rules:

$$\frac{x \in B}{x \in B^+} \quad \frac{x, y \in B^+}{x \vee y \in B^+} \quad \frac{x, y \in B^+}{x \wedge y \in B^+} \quad \frac{}{\top, \bot \in B^+}$$

That is to say, the elements x of B^+ are precisely those for which $x \in B^+$ can be derived using these rules.

If $B \subseteq L$ and $C \subseteq M$ and if $\vdash$ is a consequence relation from L to M, write $\vdash|_{B,C}$ to abbreviate $\vdash \cap\, \mathfrak{P}_f(B) \times \mathfrak{P}_f(C)$, the restriction of $\vdash$ to sequents made up entirely from the respective generators.

We now come to the first important lemma relating sets of generators for algebras and freely generated consequence relations.

Lemma 8.1 *If $B \subseteq L$ and $C \subseteq M$ are generating sets, and $\vdash$ is a consequence relation, then $\vdash = (\vdash|_{B,C})^+$.*

The lemma shows that we can restrict our attention to the behaviour of consequence relations on generators for the algebras involved. In the remainder of this section we examine how far this idea can be pushed. We start with the composition of consequence relations via Cut-Comp.

Lemma 8.2 *If C is a generating set for M and $\vdash, \vdash'$ are consequence relations from L to M, and M to N, respectively, then*

$$\vdash \circ \vdash' \;=\; \vdash|_{\cdot,C} \circ \vdash|_{C,\cdot}$$

Theorem 8.3 (Cut Elimination) *Let $B \subseteq L$, $C \subseteq M$ and $D \subseteq N$ be sets of generators. Then for any consequence relations $\vdash$ and $\vdash'$ between L, M and N it is the case that*

$$\vdash \circ \vdash' = (\vdash|_{B,C} \circ \vdash'|_{C,D})^+$$

The category MLS has compatible consequence relations as its arrows. The question arises, in how far the defining properties of these can be read off from a relation between set of generators. As it turns out, it is very difficult to derive general rules in this direction. We therefore assume that all algebras are totally free over their sets of generators, that is, that they are term algebras. This restriction is not too severe as formulas in logic are usually freely defined anyway.

Lemma 8.4 *Let B and C be sets of generators for term algebras L and M, and let R be a relation between B and C. Then R^+ is closed under application of all rules, not just the positive ones.*

Another condition that is needed for compatibility and identities in MLS is that of interpolation.

Lemma 8.5 *Let L and M be term algebras over B and C and let R be a relation on finite subsets of B and C. Suppose further that $\phi, \Gamma\ R\ \Delta$ implies that $\phi \Vdash_L \psi$ and $\psi, \Gamma\ R\ \Delta$ for some $\psi \in B$ and similarly for the dual condition. Then R^+ has interpolants.*

The following is basically Gentzen's Cut Elimination Theorem in the context of derivations with assumed sequents.

Lemma 8.6 *Let L be the term algebra over B and let R be a binary relation on finite subsets of B satisfying that if $\Gamma\ R\ \Delta, \phi$ and $\phi, \Pi\ R\ \Lambda$; then $\Gamma, \Pi\ R\ \Delta, \Lambda$. Then R^+ is closed under* Cut.

The previous three lemmas together entail the following result:

Theorem 8.7 *A binary relation $\Vdash$ on finite sets from a term algebra L over generators B is a coherent sequent calculus if and only if $\Vdash|_{B,B}$ has interpolants and is closed under* Cut.

As an example application of these results, consider the construction of a binary product in MLS. Given L and M, take $B(L \times M)$ to be the disjoint union of L and M. Here we denote members of $B(L\times M)$ by $\pi\colon \phi$ for $\phi \in L$ and $\pi'\colon \psi$ for $\psi \in M$. Take $L \times M$ to be the term algebra generated by $B(L \times M)$. To define an identity arrow for $L \times M$, it suffices to describe its behavior on $B(L \times M)$. For this, we specify

$$\frac{\Gamma \Vdash_L \Delta}{\pi\colon \Gamma \Vdash_{L\times M} \pi\colon \Delta} \qquad \frac{\Gamma \Vdash_M \Delta}{\pi'\colon \Gamma \Vdash_{L\times M} \pi'\colon \Delta}$$

where $\pi\colon \{\gamma_0, \dots, \gamma_{n-1}\}$ is $\{\pi\colon \gamma_0, \dots, \pi\colon \gamma_{n-1}\}$. Clearly, the relation $\Vdash_{L\times M}$ has interpolants and is closed under Cut precisely because $\Vdash_L$ and $\Vdash_M$ are. So $\Vdash^+_{L\times M}$ is a coherent sequent calculus. We claim that this is the identity for a product of $\Vdash_L$ and $\Vdash_M$.

The projection from $L \times M$ to L is defined by behavior on the generators $B(L \times M)$ and L. Namely,

$$\frac{\Gamma \Vdash_L \Delta}{\pi\colon \Gamma \vdash_\pi \Delta}\ .$$

The projection to M is defined similarly. Again, the conditions of compatibility for $\vdash_\pi$ and $\vdash_{\pi'}$ can be read directly from the coherence of $\vdash_L$ and $\Vdash_M$ so $\vdash^+_\pi$ and $\vdash^+_{\pi'}$ are compatible consequence relations. Now suppose that $\vdash_f$ and $\vdash_g$

are compatible consequence relations from N to L and N to M, respectively. Define $\vdash_{\langle f,g\rangle}$ by

$$\frac{\Gamma \vdash_f \Delta}{\Gamma \vdash_{\langle f,g\rangle} \pi{:}\,\Delta} \qquad \frac{\Gamma \vdash_g \Lambda}{\Gamma \vdash_{\langle f,g\rangle} \pi'{:}\,\Lambda}$$

Now, $\vdash^+_{\langle f,g\rangle} \circ \vdash^+_\pi = (\vdash_{\langle f,g\rangle} \circ \vdash_\pi)^+$, so we have only to note that the right hand sides of sequents in $\vdash_{\langle f,g\rangle}$ are only of two forms: $\pi{:}\,\Delta$ for $\Delta \subseteq_{\mathsf{fin}} L$ and $\pi'{:}\,\Lambda$ for $\Lambda \subseteq_{\mathsf{fin}} M$. Furthermore, the left hand sides of sequents in $\vdash_\pi$ are all of the form $\pi{:}\,\Gamma$ for $\Gamma \subseteq_{\mathsf{fin}} L$. That is, $\vdash_{\langle f,g\rangle} \circ \vdash_\pi$ is exactly $\vdash \circ \Vdash_M = \vdash_f$. Thus $(\vdash_{\langle f,g\rangle} \circ \vdash_\pi)^+ = \vdash_f$. Evidently, $\vdash^+_{\langle f,g\rangle}$ is the unique consequence relation for which $\vdash_f = (\vdash_{\langle f,g\rangle} \circ \vdash_\pi)^+$ and $\vdash_g = (\vdash_{\langle f,g\rangle} \circ \vdash_{\pi'})^+$ as any such relation is determined by its behavior on $B(L \times M)$ and must agree with $\vdash_{\langle f,g\rangle}$ on $B(L \times M)$.

A similar argument shows that by taking L_1 to be the free algebra on no generators, we get a terminal object. Furthermore, the construction above obviously extends to arbitrary products: Given an indexed set of coherent sequent calculi $\{L_i\}_{i\in I}$, define $\prod_{i\in I} L_i$ as the algebra freely generated by the disjoint union $\coprod_{i\in I} L_i$, writing $\pi^i{:}\,\phi$ when $\phi \in L_i$. Then $\Vdash_{\prod_{i\in I} L_i}$ is defined exactly as before as are $\Vdash_{\pi^i}$ and $\vdash_{\langle f_i\rangle_{i\in I}}$. Also, because MLS is self dual, the exact same constructions yield coproducts.

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