Deza graphs based on symplectic spaces

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**Abstract**

Let $\mathbb{F}_{2^\nu}$ be the $2\nu$-dimensional symplectic space over a finite field $\mathbb{F}_q$. For $0 \leq s \leq \nu - 1$ and $\nu \geq 2$, let $W_0$ be a fixed subspace of type $(\nu + s, s)$ in $\mathbb{F}_{2^\nu}$, and let $X$ be the set of the 1-dimensional subspaces in $\mathbb{F}_{2^\nu}$ not contained in $W_0$. Denote by $\Gamma$ the graph with the vertex set $X$, in which two vertices $P$ and $Q$ are adjacent if $P + Q$ is non-isotropic. Note that $\Gamma$ is a subgraph of the symplectic graph $Sp(2\nu, q)$. In this paper, we determine an equitable partition of $X$ and compute its quotient matrix. Moreover, we show that $\Gamma$ is a strictly Deza graph, compute its parameters and spectra. Finally, we prove that the (fractional) chromatic number of $\Gamma$ is $q^\nu$.

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1. Introduction

We first recall definitions about symplectic spaces. Notation and terminology will be adopted from Wan’s book [10]. Let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}. $$

The generalized symplectic group of degree $2\nu$ over $\mathbb{F}_q$, denoted by $GSp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrices $T$ over $\mathbb{F}_q$ satisfying $TKT^t = kK$ for some $k \in \mathbb{F}_q^*$, where $T^t$ is the transpose of $T$. $GSp_{2\nu}(\mathbb{F}_q)$ acts on $\mathbb{F}_{2^\nu}$ by the vector matrix multiplication. The row vector space $\mathbb{F}_{2^\nu}$ together with this action is called the $2\nu$-dimensional symplectic space over $\mathbb{F}_q$. The above action induces an action on the set of subspaces of $\mathbb{F}_{2^\nu}$, i.e., a subspace $P$ is carried by $T$ into the subspace $PT$. An $m$-dimensional subspace $P$ in $\mathbb{F}_{2^\nu}$ is said to be of type $(m, s)$ if $PKP^t$ is of rank $2s$. In particular, subspaces of type $(m, 0)$ are called totally isotropic subspaces, and subspaces of type $(2s, s)$ are called non-isotropic subspaces. By
Let $P$ be a subspace of type $(m, s)$ exist in $\mathbb{F}_q^{2v}$ if and only if $2s \leq m \leq v + s$, and $GSp_{2v}(\mathbb{F}_q)$ acts transitively on the set of all subspaces of the same type. For a subspace $P$ of type $(m, s)$, by [10, Corollary 3.9],

$$P^\perp = \{y \in \mathbb{F}_q^{2v} \mid xKy = 0, \forall x \in P\}$$

is a subspace of type $(2v - m, v + s - m)$, which is called the dual subspace of $P$.

**Lemma 1.1.** Let $P_1$ and $P_2$ be two $m \times 2v$ matrices of rank $m$. Then exists a $T \in GSp_{2v}(\mathbb{F}_q)$ such that $P_1 = P_2 T$ if and only if, there exists an element $a \in \mathbb{F}_q^*$ such that $P_1 K P_1^t = a P_2 K P_2^t$.

**Proof.** Suppose that $P_1 K P_1^t = a P_2 K P_2^t$. By [10, Lemma 3.5], there exist the same $m \times m$ nonsingular matrix $Q$ and two $(2v - m) \times 2v$ matrices $Z_1$ and $Z_2$ such that

$$\begin{pmatrix} Q P_1 \\ Z_1 \end{pmatrix} K \begin{pmatrix} Q P_1 \\ Z_1 \end{pmatrix}^t = a \begin{pmatrix} Q P_2 \\ Z_2 \end{pmatrix} K \begin{pmatrix} Q P_2 \\ Z_2 \end{pmatrix}^t,$$

which implies

$$T = \begin{pmatrix} Q P_2 \\ Z_2 \end{pmatrix}^{-1} \begin{pmatrix} Q P_1 \\ Z_1 \end{pmatrix} \in GSp_{2v}(\mathbb{F}_q).$$

Therefore, $P_1 = P_2 T$. The converse is obvious. \qed

The symplectic graph $Sp(2v, q)$ is the graph with the 1-dimensional subspaces of $\mathbb{F}_q^{2v}$ as its vertex set, with two vertices $P$ and $Q$ are adjacent if $P + Q$ is non-isotropic. Tang and Wan [8] proved that $Sp(2v, q)$ is a family of strongly regular graphs, and determined its full automorphism group. Subsequently, Li and Wang [7] proved that the subconstituents of $Sp(2v, q)$ as a subgraph are Deza graphs. Their research stimulates us to study special subgraphs of $Sp(2v, q)$.

For $0 \leq s \leq v - 1$ and $v \geq 2$, let $W_0$ be a fixed subspace of type $(v + s, s)$, and $X$ be the set of the 1-dimensional subspaces in $\mathbb{F}_q^{2v}$ not contained in $W_0$. Define a graph $\Gamma$ with the vertex set $X$, and two vertices $P$ and $Q$ are adjacent if $P + Q$ is non-isotropic. Note that $\Gamma$ is a subgraph of $Sp(2v, q)$.

In this paper we focus on the study of $\Gamma$. In Section 2 we determine an equitable partition of $X$ and compute its quotient matrix. In Section 3 we show that $\Gamma$ is a strictly Deza graph and compute its parameters. In Sections 4 and 5 we determine the spectra and (fractional) chromatic number of $\Gamma$, respectively.

2. Equitable partition

A partition $\pi$ of the vertex set $V$ of a graph $\Delta$ with cells $C_0, C_1, \ldots, C_r$ is equitable if, for any $0 \leq i, j \leq r$, the number of neighbours in $C_j$ of each vertex $u$ in $C_i$ is a constant $b_{ij}$. The directed graph with the $r + 1$ cells of $\pi$ as its vertices and $b_{ij}$ arcs from the $C_i$ to $C_j$ is called the quotient of $\Delta$ over $\pi$, and denoted by $\Delta/\pi$. The adjacency matrix of $\Delta/\pi$ is called the quotient matrix associated with the equitable partition $\pi$ of $\Delta$.

In this section, we give an equitable partition of the vertex set $X$ of $\Gamma$, and determine its quotient matrix.

If $W_0$ is a subspace of type $(v + s, s)$ in $\mathbb{F}_q^{2v}$, by [10, Corollary 3.9], $U_0 = W_0^\perp$ is a $(v - s)$-dimensional totally isotropic subspace. Since $GSp_{2v}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type, we may take

$$U_0 = \begin{pmatrix} 0^{(v-s,s)} \\ I^{(v-s)} \end{pmatrix}, \quad W_0 = \begin{pmatrix} I^{(v)} & 0 \\ 0 & I^{(s)} \end{pmatrix}.$$

Note that the subgroup $G$ of $GSp_{2v}(\mathbb{F}_q)$ fixing $W_0$ is a subgroup of the full automorphism group of $\Gamma$.

We always assume that $e_i$ denotes the $2v$-dimensional row vector whose $i$th component is 1 and other components are 0, and $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ denotes the subspace generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. In particular, if $\alpha = (a_1, \ldots, a_{2v})$, then $[\alpha]$ is denoted by $[a_1, \ldots, a_{2v}]$. 

Lemma 2.1. G acts transitively on $X$. In particular, $\Gamma$ is a vertex transitive graph.

**Proof.** Let

$$W_1 = (I^{(s)} \ 0^{(s,v-s)}) , \quad W_2 = (0^{(s,v)} \ I^{(s)} \ 0^{(s,v-s)}).$$

For any $[\alpha] \in X$,

$$\begin{pmatrix} (W_1 & U_0 & W_2 \alpha) \\ (U_0 & W_2 \alpha) \\ (W_2 \alpha) \end{pmatrix}^{\top} = \begin{pmatrix} 0 & 0 & I^{(s)} & W_1 K \alpha^t \\ 0 & 0 & 0 & U_0 K \alpha^t \\ -I^{(s)} & 0 & 0 & W_2 K \alpha^t \\ \alpha K W_1 \alpha^t & \alpha K U_0 \alpha^t & \alpha K W_2 \alpha^t & 0 \end{pmatrix}.$$

The fact $\alpha K U_0^t \neq 0$ implies that there exists a $(v + s) \times (v + s)$ nonsingular matrix $A$ such that

$$\begin{pmatrix} AW_0 \alpha \\ \alpha \end{pmatrix}^{\top} = \begin{pmatrix} (AW_0 \alpha) \\ \alpha \end{pmatrix}^{\top} = \begin{pmatrix} 0 & 0 & I^{(s)} & 0 \\ 0 & 0 & 0 & \beta^2 \\ -I^{(s)} & 0 & 0 & 0 \\ 0 & -\bar{1} & 0 & 0 \end{pmatrix} \triangleq \Lambda,$$

where $\bar{1} = (1, 0, \ldots, 0)$. Similarly, for any $[\beta] \in X$, there exists a $(v + s) \times (v + s)$ nonsingular matrix $B$ such that

$$\begin{pmatrix} BW_0 \beta \\ \beta \end{pmatrix}^{\top} = A.$$

By Lemma 1.1, there exists a $T \in GSp_{2v} (\mathbb{F}_q)$ such that

$$\begin{pmatrix} AW_0 \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} BW_0 \beta \\ \beta \end{pmatrix},$$

which implies that $T \in G$ and $\alpha T = \beta$. Hence $[\alpha T] = [\beta]$ as desired. \qed

Let $G_{[e_{v+s+1}]}$ denote the stabilizer of $[e_{v+s+1}]$ in $G$. Now we determine the orbits of $G_{[e_{v+s+1}]}$ on $X$.

Lemma 2.2. If two elements $[\alpha]$ and $[\beta]$ of $X$ are in the same orbit of $G_{[e_{v+s+1}]}$, then $[e_{v+s+1}] + [\alpha]$ and $[e_{v+s+1}] + [\beta]$ are subspaces of the same type satisfying

(i) $\dim((e_{v+s+1} + [\alpha]) \cap W_0) = \dim((e_{v+s+1} + [\beta]) \cap W_0);

(ii) $(e_{v+s+1} + [\alpha]) \cap W_0 \subseteq U_0$ if and only if $(e_{v+s+1} + [\beta]) \cap W_0 \subseteq U_0$.

**Proof.** Suppose that $[\alpha]$ and $[\beta]$ fall into the same orbit. Then there exists a $T \in G_{[e_{v+s+1}]}$ such that $[\alpha T] = [\beta]$, which follows that both $[e_{v+s+1}] + [\alpha]$ and $[e_{v+s+1}] + [\beta]$ are of the same type. Clearly, (i) holds. Since

$$U_0 T = W_0^{\perp T}$$

$$= \{ x T \mid x K y^t = 0, \ \forall \ y \in W_0 \}$$

$$= \{ x T \mid (x T) K (y T)^t = 0, \ \forall \ y \in W_0 \}$$

$$= (W_0 T)^{\perp}$$

$$= U_0,$$

(ii) holds. \qed

Theorem 2.3. The orbits of $G_{[e_{v+s+1}]}$ on $X$ have the following represents:

$[e_{v+s+1}], \quad [e_{s+1} + e_{v+s+1}], \quad [e_{s+2} - e_{v+s+1}] \ (s \leq v - 2),$

$[e_{s+1} - e_{v+s} + e_{v+s+1}] \ (s \geq 1), \quad [e_{v+s} - e_{v+s+1}] \ (s \geq 1),$

$[e_{s+1} - e_{v+s+2}] \ (s \leq v - 2), \quad [e_{v+s+2}] \ (s \leq v - 2).$
Proof. For any $[\alpha] \neq [e_{v+s+1}]$, let $\dim(([e_{v+s+1}] + [\alpha]) \cap W_0) = \delta$. Then $\delta = 0$ or $1$. If $\delta = 1$, we always suppose $([e_{v+s+1}] + [\alpha]) \cap W_0 = [e_{v+s+1} + a\alpha]$, where $a \neq 0$. Since $e_{v+s+1} + a\alpha \in W_0$, there exist two $1 \times s$ matrices $A = (a_1, \ldots, a_s)$, $B = (b_1, \ldots, b_s)$ and a $1 \times (v-s)$ matrix $C = (c_1, \ldots, c_{v-s})$ such that

$$e_{v+s+1} + a\alpha = AW_1 + BW_2 + CU_0,$$

where

$$W_1 = (I^{(s)} \ 0^{(s,2,v-s)}) , \quad W_2 = (0^{(s,v)} \ I^{(s)} \ 0^{(s,v-s)}).$$

Case 1: $\delta = 1$, $[e_{v+s+1}] + [\alpha]$ is non-isotropic and $e_{v+s+1} + a\alpha \in U_0$. Then $A = B = 0$ and $C \neq 0$. Let $U_0 = [e_{v+s+1} + a\alpha] \oplus U'_0$ and $e_{v+s+1}K\alpha^t = b \neq 0$. Pick

$$A_1 = \begin{pmatrix}
I & W_1Ke_{v+s+1}' & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & U'_0Ke_{v+s+1}' & I & 0 & 0 \\
0 & W_2Ke_{v+s+1}' & 0 & I & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
abl & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Then

$$A_2A_1 \begin{pmatrix}
W_1 \\
U'_0 \\
W_2 \\
e_{v+s+1}
\end{pmatrix} K \begin{pmatrix}
W_1 \\
U'_0 \\
W_2 \\
e_{v+s+1}
\end{pmatrix}^t A_1^t A_2^t = \begin{pmatrix}
0 & 0 & 0 & abl^{(s)} & 0 \\
0 & 0 & 0 & 0 & -ab \\
0 & 0 & 0 & 0 & 0 \\
0 & ab & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \triangleq abA_1.$$

Since

$$\begin{pmatrix}
e_1 \\
\vdots \\
e_s \\
e_{s+1} \\
e_{s+2} \\
\vdots \\
e_{v+s+1}
\end{pmatrix} K \begin{pmatrix}
e_1 \\
\vdots \\
e_s \\
e_{s+1} \\
e_{s+2} \\
\vdots \\
e_{v+s+1}
\end{pmatrix}^t = A_1,$$

by Lemma 1.1, there exists a $T \in GSp_{2v}(\mathbb{F}_q)$ such that

$$A_2A_1 \begin{pmatrix}
W_1 \\
U'_0 \\
W_2 \\
e_{v+s+1}
\end{pmatrix} T = \begin{pmatrix}
W_0 \\
0 \\
0 \\
e_{v+s+1}
\end{pmatrix},$$

which implies that $T \in G_{[e_{v+s+1}]}$ and $(e_{v+s+1} + a\alpha)T = -e_{v+1}$. Hence $[\alpha T] = [e_{v+1} + e_{v+s+1}]$.

Case 2: $\delta = 1$, $[e_{v+s+1}] + [\alpha]$ is totally isotropic and $e_{v+s+1} + a\alpha \in U_0$. Then $A = B = 0$ and $C \neq 0$. By $U_0Ke_{v+s+1}' \neq 0$, $U'_0Ke_{v+s+1}' \neq 0$, which implies $s \leq v-2$. Similar to the proof of Case 1, there exists a $T \in G_{[e_{v+s+1}]}$ such that $[\alpha T] = [e_{s+2} - e_{v+s+1}]$.

Case 3: $\delta = 1$, $[e_{v+s+1}] + [\alpha]$ is non-isotropic and $e_{v+s+1} + a\alpha \notin U_0$. Then $B \neq 0$ or $A \neq 0$; and so $1 \leq s$. Suppose $e_{v+s+1}K\alpha^t = b \neq 0$.

Case 3.1: $B \neq 0$. Then there exists a $b_i \neq 0$. Without loss of generality, suppose $b_1 \neq 0$. Then $W_2 = BW_2 \oplus W'_2 = [b_1e_{v+1} + \cdots + b_se_{v+s}] + W'_2$, where $W'_2 = (0^{(s-1,v)}, I^{(s-1)}, 0^{(s-1,v-s+1)})$. Similarly, we have $W_1 = BW_1 \oplus W'_1 = [b_1e_{v+1} + \cdots + b_se_{v+s}] \oplus W'_1$, where $W'_1 = (I^{(s-1)}, 0^{(s-1,2v-s+1)})$. Let
\[ A_{15} = W'_1 K (e_{v+s+1} + a\alpha)^t, \quad A_{16} = W'_1 K e_{v+s+1}^t, \quad A_{24} = BW'_1 K W'_2, \quad A_{25} = BW'_1 K (e_{v+s+1} + a\alpha)^t, \quad A_{26} = BW'_1 K e_{v+s+1}^t, \quad A_{45} = W'_2 K (e_{v+s+1} + a\alpha)^t, \quad A_{46} = W'_2 K e_{v+s+1}^t \text{ and } B_{25} = -A_{24} A_{15} + A_{25}. \text{ Take} \\
\]

\[
B_1 = \begin{pmatrix}
I & 0 & -A_{16} & 0 & 0 & 0 \\
-A_{24} & 1 & A_{24} A_{16} - A_{26} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{46} & 0 & I \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}^{(v-s-1)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Then

\[
B_1 \left( \begin{array}{cccc}
W'_1 \\
BV'_1 \\
U_0 \\
W'_2 \\
e_{v+s+1} + a\alpha \\
e_{v+s+1}
\end{array} \right) K \left( \begin{array}{cccc}
W'_1 \\
BV'_1 \\
U_0 \\
W'_2 \\
e_{v+s+1} + a\alpha \\
e_{v+s+1}
\end{array} \right)^t B_1 = \left( \begin{array}{cccc}
0 & 0 & 0 & I^{(s-1)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right) \begin{pmatrix} A_{15} & 0 \\
B_{25} & 0 \\
A_{45} & 0 \\
\alpha^{-1} & 0 \\
A_{45} & 0 \\
\alpha^{-1} & 0 \\
\end{pmatrix}.
\]

Since \( W_0 \) is of type \((v+s,s)\), \( B_{25} \neq 0 \). Take \( B_{25} = 1 \) and

\[
B_2 = \begin{pmatrix}
I & -A_{15} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad B_3 = \left( \begin{array}{cccc}
abl & 0 & 0 & 0 \\
0 & ab & 0 & 0 \\
0 & 0 & abl & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right).
\]

Then

\[
B_2 B_3 B_1 \left( \begin{array}{cccc}
W'_1 \\
V'_1 \\
U_0 \\
W'_2 \\
e_{v+s+1} + a\alpha \\
e_{v+s+1}
\end{array} \right) K \left( \begin{array}{cccc}
W'_1 \\
V'_1 \\
U_0 \\
W'_2 \\
e_{v+s+1} + a\alpha \\
e_{v+s+1}
\end{array} \right)^t B_1 B_2 B_3 = \begin{pmatrix}
0 & 0 & 0 & abl^{(s-1)} & 0 & 0 \\
0 & 0 & 0 & 0 & ab & 0 \\
0 & 0 & 0 & 0 & 0 & ab\bar{e}^{-1} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \triangleq ab A_2.
\]

Since

\[
\left( \begin{array}{cccc}
e_1 \\
\vdots \\
e_{v+s-1} \\
e_{v+s} \\
e_{v+s+1}
\end{array} \right) K \left( \begin{array}{cccc}
e_1 \\
\vdots \\
e_{v+s-1} \\
e_{v+s} \\
e_{v+s+1}
\end{array} \right)^t = A_2,
\]

by Lemma 1.1, there exists a \( T \in G_{[e_{v+s+1}]} \) such that \((e_{v+s+1} + a\alpha)T = -e_{s+1} + e_{v+s-1}\). Hence \([\alpha T] = [e_{s+1} - e_{v+s} + e_{v+s+1}]\).

**Case 3.2:** \( A \neq 0 \). Similar to Case 3.1, there exists a \( T \in G_{[e_{v+s+1}]} \) such that \([\alpha T] = [e_{s+1} - e_{v+s} + e_{v+s+1}]. \)
Case 4: $δ = 1, [e_{v+s+1}] + [α]$ is totally isotropic and $e_{v+s+1} + α E \neq U_0$. Then $B \neq 0$ or $A \neq 0$. Clearly, $s ≥ 1$. Similar to Case 3, there exists a $T \in G[e_{v+s+1}]$ such that $[αT] = [e_{v+s} - e_{v+s+1}]$.

Case 5: $δ = 0$ and $[e_{v+s+1}] + [α]$ is non-isotropic. Then $s ≤ v - 2$ and $[e_{v+s+1}] + [α] + W_0$ is of type $(v + s + 2, s + 2)$, which implies $\text{rank } U_0 K (e_{v+s+1})^t = 2$. Without loss of generality, we may pick $e_{v+s+1} \alpha^t = b \neq 0, U_0 K (e_{v+s+1})^t = E$, where $E = (I^{(2)0)} (v-s-2))$. Take

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I^{(2)} & 0 \\ 0 & 0 & I^{(2)} \end{pmatrix} \quad F_2 = \begin{pmatrix} I^{(v+s+1)} & 0 \\ 0 & b^{-1} \end{pmatrix}$$

Then

$$F_2 F_1 \begin{pmatrix} W_1 \\ U_0 \\ W_2 \end{pmatrix} K \begin{pmatrix} W_1 \\ U_0 \\ W_2 \end{pmatrix} (e_{v+s+1})^t = \begin{pmatrix} 0 & 0 & I^{(s)} \\ 0 & 0 & 0 \\ -I^{(s)} & 0 & 0 \end{pmatrix} \triangleq A_3,$$

Since

$$\begin{pmatrix} e_1 \\ \vdots \\ -e_{s+1} + e_{v+s+2} \end{pmatrix} K \begin{pmatrix} e_1 \\ \vdots \\ -e_{s+1} + e_{v+s+2} \end{pmatrix} = A_3,$$

by Lemma 1.1 there exists a $T \in G[e_{v+s+1}]$ such that $[αT] = [e_{s+1} - e_{v+s+2}]$.

Case 6: $δ = 0$ and $[e_{v+s+1}] + [α]$ is totally isotropic. Then $s ≤ v - 2$. Since $[e_{v+s+1}] + [α] + W_0$ is of type $(v + s + 2, s + 2)$, $\text{rank } U_0 K (e_{v+s+1})^t = 2$. Similar to Case 5, there exists a $T \in G[e_{v+s+1}]$ such that $[αT] = [e_{v+s+2}]$.

By Lemma 2.2 the desired result follows.

**Corollary 2.4.** The orbits of $G[e_{v+s+1}]$ on $X$ are

- $C_0 = \{[e_{v+s+1}]\}$
- $C_1 = \{[a_{s+1} e_{s+1} + \cdots + a_v e_v + e_{v+s+1}] | a_{s+1} \neq 0\}$
- $C_2 = \{[a_{s+2} e_{s+2} + \cdots + a_v e_v + e_{v+s+1}] | (a_{s+2}, \ldots, a_v) \neq (0, \ldots, 0), \quad (s ≤ v - 2)\}$
- $C_3 = \{[a_1 e_1 + \cdots + a_v e_v + e_{v+s+1}] | (a_1, \ldots, a_v) \neq (0, \ldots, 0), \quad (s ≤ v - 2)\}$
- $C_4 = \{[a_1 e_1 + \cdots + a_v e_v + e_{v+s+1}] | (a_1, \ldots, a_v) \neq (0, \ldots, 0) \quad \text{and } a_{v+1} = 0, \quad (1 ≤ s)\}$
- $C_5 = \{[a_1 e_1 + \cdots + a_v e_v + e_{v+s+1}] | (a_1, \ldots, a_v) \neq (0, \ldots, 0) \quad \text{and } a_{v+1} = 0, \quad (1 ≤ s)\}$
- $C_6 = \{[a_1 e_1 + \cdots + a_v e_v + e_{v+s+1}] | (a_1, \ldots, a_v) \neq (0, \ldots, 0) \quad \text{and } a_{v+1} = 0, \quad (1 ≤ s)\}$

**Proof.** It is obvious that $C_0$ is one orbit of this action. Let $C_1$ denote the orbit containing $[e_{s+1} + e_{v+s+1}]$. For each element $[α] \in C_1$, by dim($[e_{v+s+1}] + [α]$) = 1 and $[e_{v+s+1}] + [α] \subseteq U_0, [α]$ has the unique matrix representation of the form $[a_{s+1} e_{s+1} + \cdots + a_v e_v + e_{v+s+1}]$. 
Since \([e_{v+s+1}] + [\alpha]\) is non-isotropic, \(\alpha_{s+1} \neq 0\), which implies that
\[
C_1 = \{ [a_{s+1} e_{s+1} + \cdots + a_s e_s + e_{v+s+1}] \mid a_{s+1} \neq 0 \}.
\]
Similarly, we may determine all the rest \(C_j\)'s.

**Corollary 2.5.** The partition \(\pi\) of \(X\) with cells as orbits of \(G_{[e_{v+s+1}]}\) on \(X\) is equitable, and the following hold:

(i) If \(s = v - 1\), then the adjacency matrix of \(\Gamma/\pi\) is
\[
\begin{pmatrix}
0 & q - 1 & (q^{2v-2} - 1)(q - 1) & 0 \\
1 & q - 2 & (q^{2v-2} - 1)(q - 2) & q^{2v-2} - 1 \\
1 & q - 2 & (q^{2v-2} - q^{2v-3} - 1)(q - 1) + 1 & q^{2v-2} - q^{2v-3} - 1 \\
0 & q - 1 & (q^{2v-2} - q^{2v-3} - 1)(q - 1) & q^{2v-3} - 1
\end{pmatrix}.
\]

(ii) If \(s = 0\), then the adjacency matrix of \(\Gamma/\pi\) is
\[
\begin{pmatrix}
0 & q^{v-1}(q - 1) & 0 & q^{v}(q^{v-1} - 1) & 0 \\
1 & q^{v-1}(q - 2) & q^{v-1} - 1 & q^{v-1}(q - 1)(q^{v-1} - 1) & q^{v-1}(q^{v-1} - 1) \\
0 & q^{v-1}(q - 1) & 0 & q^{v}(q^{v-1} - q^{v-2} - 1) & q^{v-2} \\
1 & q^{v-2}(q - 1)^2 & q^{v-1} - q^{v-2} - 1 & q^{v-1}(q - 1)(q^{v-1} - 1) & q^{v-1}(q^{v-1} - 1) \\
0 & q^{v-2}(q - 1)^2 & q^{v-2}(q - 1) & q^{v-1}(q - 1)(q^{v-1} - 1) & q^{v-1}(q^{v-1} - 1)
\end{pmatrix}.
\]

(iii) If \(1 \leq s \leq v - 2\), then the adjacency matrix of \(\Gamma/\pi\) is
\[
\begin{pmatrix}
0 & q^{v-s-1}(q - 1) & 0 & q^{v-s-1}(q^{2s} - 1)(q - 1) \\
1 & q^{v-s-1}(q - 2) & q^{v-s-1} - 1 & q^{v-s-1}(q^{2s} - 1)(q - 2) \\
0 & q^{v-s-1}(q - 1) & 0 & q^{v-s-1}(q^{2s} - 1)(q - 1) \\
1 & q^{v-s-1}(q - 2) & q^{v-s-1} - 1 & q^{v+s-2}(q - 1)^2 - q^{v-s-1}(q - 2) \\
0 & q^{v-s-1}(q - 1) & 0 & q^{v-s-1}(q^{2s} - 1)(q^{2s} - 1) - q^{v-s-1}(q - 2) \\
1 & q^{v-s-2}(q - 1)^2 & q^{v-s-2} - 1 & q^{v-s-2}(q^{2s} - 1)(q - 1)^2 \\
0 & q^{v-s-2}(q - 1)^2 & q^{v-s-2}(q - 1) & q^{v-s-2}(q^{2s} - 1)(q - 1)^2 \\
0 & q^{v-s-1}(q^{2s} - 1) & q^{v+s-2}(q^{v-s-1} - 1)(q - 1) & q^{v+s-2}(q^{v-s-1} - 1)(q - 1) \\
0 & q^{v-s-1}(q^{2s} - q^{2s-2} - 1) & q^{v+s-2}(q^{v-s-1} - 1)(q - 1) & q^{v+s-2}(q^{v-s-1} - 1) \\
q^{v-s-1}(q^{2s} - q^{2s-2} - 1) & q^{v-s-2}(q^{2s} - 1)(q - 1) & q^{v+s-1}(q^{v-s-1} - 1)(q - 1) & q^{v+s-1}(q^{v-s-1} - 1) \\
q^{v-s-2}(q^{2s} - 1)(q - 1) & q^{v+s-1}(q^{v-s-1} - 1)(q - 1) & q^{v+s-1}(q^{v-s-1} - 1)(q - 1) & q^{v+s-1}(q^{v-s-1} - 1)
\end{pmatrix}.
\]

**Proof.** By [5, Section 9.3] the orbits of \(X\) under \(G_{[e_{v+s+1}]}\) form an equitable partition of \(X\). Since \(b_{01}\) is the number of all the elements \([a_{s+1} e_{s+1} + \cdots + a_s e_s + e_{v+s+1}]\) in \(C_1\), \(b_{01} = |C_1| = q^{v-s-1}(q - 1)\). For \([e_{s+1} + e_{v+s+1}] \in C_1\), \(b_{13} = \text{the number of elements } [a_1 e_t + \cdots + a_{v+s} e_{v+s} + e_{v+s+1}]\) in \(C_3\) satisfying \(\alpha_{s+1} \neq 1\). It follows that \(b_{13} = q^{v-s-1}(q^{2s} - 1)(q - 2)\). Similarly, all the rest \(b_{ij}\)'s may be computed. 

3. Deza graph

As a generalization of strongly regular graphs, Erickson et al. [4] introduced Deza graphs, which were firstly introduced in a slightly more restricted form by Deza and Deza [2]. A regular graph with
degree \( k \) on \( v \) vertices is said to be a \((v, k, b, c)\)-Deza graph if any two distinct vertices \( x \) and \( y \) have \( b \) or \( c \) common adjacent vertices. A Deza graph with diameter two is said to be a strictly Deza graph if it is not strongly regular.

In this section we prove that \( \Gamma \) is a strictly Deza graph, and compute all its parameters.

We will denote the adjacent vertices \( u \) and \( v \) by \( u \sim v \). The common neighborhood \( N_{u,v} \) of vertices \( u \) and \( v \) consists of all the vertices of \( \Gamma \) adjacent to both \( u \) and \( v \).

**Lemma 3.1.** For any two adjacent vertices \([\alpha]\) and \([\beta]\) of \( \Gamma \),

\[
|N_{[\alpha],[\beta]}| = (q^{2v-2} - q^{v+s-2})(q - 1) \quad \text{or} \quad q^{2v-2}(q - 1) - q^{v+s-1}.
\]

**Proof.** By Theorem 2.3, we may take \([\alpha] = [e+s+1]\) and \([\beta] = [e+s+1 + e(v+s+1)]\). For each vertex \([\gamma]\) adjacent to \([e+s+1], x_{s+1} \neq 0\), which implies that \([\gamma]\) has the unique matrix representation of the form

\[
[x_1, \ldots, x_s, 1, x_{s+2}, \ldots, x_{2v}],
\]

where \((x_{s+1}, \ldots, x_{2v}) \neq (0, \ldots, 0)\). If \([\gamma] \sim [e+s+1 + e(v+s+1)], then \(x_{s+1} - 1 \neq 0\), which implies that

\[
|N_{[e+s+1],[e+s+1 + e(v+s+1)]}| = q^{2v-2}(q - 1) - q^{v+s-1}.
\]

If \([\gamma] \sim [e+s+1 - e(v+s+1)], then \(x_{s+1} - x_{s+1} - 1 \neq 0\), which implies that

\[
|N_{[e+s+1],[e+s+1 - e(v+s+1)]}| = (q^{2v-2} - q^{v+s-2})(q - 1).
\]

Hence, the desired result follows. \(\Box\)

**Lemma 3.2.** For any two non-adjacent vertices \([\alpha]\) and \([\beta]\),

\[
|N_{[\alpha],[\beta]}| = (q^{2v-2} - q^{v+s-2})(q - 1) \quad \text{or} \quad q^{2v-2}(q - 1) - q^{v+s-1}.
\]

**Proof.** By Theorem 2.3, we may take \([\alpha] = [e(v+s+1)]\) and \([\beta]\) is one of the following forms

\[
[e+s+1 - e(v+s+1)] \quad (s \leq v-2), \quad [e(v+s) - e(v+s+1)] \quad (s \geq 1), \quad [e(v+s+2)] \quad (s \leq v-2).
\]

Each vertex \([\gamma]\) adjacent to \([e(v+s+1)]\) has the unique matrix representation of the form \((1)\). If \([\gamma] \sim [e+s+2 - e(v+s+1)], then \(x_{s+2} - 1 \neq 0\), which implies that

\[
|N_{[e(v+s+1)],[e+s+2 - e(v+s+1)]}| = q^{2v-2}(q - 1) - q^{v+s-1}.
\]

If \([\gamma] \sim [e(v+s) - e(v+s+1)], then \(x_{s+1} \neq 1\), which implies that

\[
|N_{[e(v+s+1)],[e(v+s) - e(v+s+1)]}| = (q^{2v-2} - q^{v+s-2})(q - 1).
\]

If \([\gamma] \sim [e(v+s+2)], then \(x_{s+2} \neq 0\), which implies that

\[
|N_{[e(v+s+1)],[e(v+s+2)]}| = (q^{2v-2} - q^{v+s-2})(q - 1).
\]

Hence, the desired result follows. \(\Box\)

**Theorem 3.3.** \( \Gamma \) is a strictly Deza graph with parameters \((v, k, b, c)\), where \(v = q^{v+s}(q^{v-s} - 1)/(q - 1)\), \(k = q^{v+s-1}(q^{v-s} - 1)\), \(b = (q^{2v-2} - q^{v+s-2})(q - 1)\) and \(c = q^{2v-2}(q - 1) - q^{v+s-1}\).

**Proof.** By \([10, \text{Theorem 1.7}]\),

\[v = |X| = \left[\frac{2v}{1}\right] - \left[\frac{v + s}{1}\right] = \frac{q^{v+s}(q^{v-s} - 1)}{q - 1}.
\]

Let \([\alpha] = [x_1, \ldots, x_{2v}]\) be any vertex adjacent to \([e(v+s+1)]\). Then \(x_{s+1} \neq 0\). By Lemma 2.1 \( \Gamma \) is regular of degree \( k = q^{v-s-1}(q^{v-s} - 1) \). By Lemmas 3.1 and 3.2, the desired result follows. \(\Box\)
Remarks. For $1 \leq s \leq v - 1$ and $v \geq 2$, let $X'$ be the set of 1-dimensional subspaces of $W_0$ not contained in $U_0$. Define a graph $\Gamma'$ with the vertex set $X'$, and two vertices $P$ and $Q$ are adjacent if $P + Q$ is non-isotropic. Then $\Gamma'$ is a strictly Deza or strongly regular graph according to $s \geq 2$ or $s = 1$, respectively. If $s = v - 1$, then $\Gamma'$ and $\Gamma''$ are the first and second subconstituents of $Sp(2v, q)$, respectively.

4. Spectra

Let $A$ be the adjacency matrix of a graph $\Delta$. If $\theta_0 > \theta_1 > \cdots > \theta_t$ are all the distinct eigenvalues of $A$ and $m_i$ is the multiplicity of $\theta_i$, $0 \leq i \leq t$, then the spectra of $\Delta$ is

$$\begin{pmatrix}
\theta_0 & \theta_1 & \cdots & \theta_t \\
m_0 & m_1 & \cdots & m_t
\end{pmatrix}.$$ 

In this section we shall determine the spectra of $\Gamma'$.

Applying the following useful proposition, Bang, Fujisaki and Koolen [1] determined the spectra of the local graphs of the twisted Grassmann graphs, which were first introduced by Dam and Koolen [9].

**Proposition 4.1** ([1, Theorem 4.2]). Let $\Delta$ be a graph with the vertex set $V$. Let $H$ be a subgroup of the automorphism group $\text{Aut}(\Delta)$ of $\Delta$. For a vertex $x \in V$, let $A_x$ be the quotient matrix associated with the equitable partition of $V$ consisting of orbits under the action of $H_x$, the stabilizer of $x$ in $H$. Then

$$\{\theta \mid \theta \text{ is an eigenvalue of } \Delta\} = \bigcup_{x \in V}\{\eta \mid \eta \text{ is an eigenvalue of } A_x\}.$$ 

**Lemma 4.2.** The distinct eigenvalues of $\Gamma'$ are $q^{v+s-1}(q^{v-s} - 1)$, $q^{v-1}$, 0 and $-q^{v-1}$.

**Proof.** If $s = v - 1$, by Corollary 2.5(i) $\Gamma'/\pi$ has spectra

$$\begin{pmatrix}
q^{2v-2}(q - 1) & q^{v-1} & 0 & -q^{v-1} \\
1 & 1 & 1 & 1 \\
\end{pmatrix}.$$ 

If $s = 0$, by Corollary 2.5(ii) $\Gamma'/\pi$ has spectra

$$\begin{pmatrix}
q^{v-1}(q^v - 1) & q^{v-1} & 0 & -q^{v-1} \\
1 & 1 & 1 & 2 \\
\end{pmatrix}.$$ 

If $1 \leq s \leq v - 2$, by Corollary 2.5(iii) $\Gamma'/\pi$ has spectra

$$\begin{pmatrix}
q^{v+s-1}(q^{v-s} - 1) & q^{v-1} & 0 & -q^{v-1} \\
1 & 2 & 2 & 2 \\
\end{pmatrix}.$$ 

By Proposition 4.1, the desired result follows. \(\square\)

**Theorem 4.3.** The spectra of $\Gamma'$ is

$$\begin{pmatrix}
q^{v+s-1}(q^{v-s} - 1) & q^{v-1} & 0 & -q^{v-1} \\
1 & q(q^{v-s} - 1)(q^v - q^2 - q + 1) & q^{v+s} - q^{2s} - q + 1 & q(q^{v-s} - 1)(q^v - q^2 + q - 1) \\
\end{pmatrix}$$ 

**Proof.** By Theorem 3.3,

$$\sum_{i=0}^{3} m_i = \frac{q^{v+s-1}(q^{v-s} - 1)}{q - 1},$$

$$\sum_{i=0}^{3} m_i \theta_i = 0,$$

$$\sum_{i=0}^{3} m_i \theta_i^2 = \nu k = \frac{q^{2v+2s-1}(q^{v-s} - 1)^2}{q - 1}.\quad (2)$$
Since $\Gamma$ is $k$-regular, the multiplicity of the eigenvalue $k$ is one. By (2),

\[
\begin{align*}
  m_1 + m_2 + m_3 &= \frac{q^{v+s}(q^{v-s} - 1)}{(q - 1)} - 1, \\
  m_1 - m_3 &= -q^s(q^{v-s} - 1), \\
  m_1 + m_3 &= \frac{q^{2s}(q^{v-s} - 1)^2}{q - 1}.
\end{align*}
\]

The above system of linear equations has the following unique solution:

\[
\begin{align*}
  m_1 &= \frac{q^s(q^{v-s} - 1)(q^v - q^s - q + 1)}{2(q - 1)}, \\
  m_2 &= \frac{q^{v+s} - q^{2s} - q + 1}{q - 1}, \\
  m_3 &= \frac{q^s(q^{v-s} - 1)(q^v - q^s + q - 1)}{2(q - 1)},
\end{align*}
\]

as desired. \(\Box\)

5. Chromatic number

We denote by $\chi(\Delta)$ the chromatic number of a graph $\Delta$, i.e., the least number of colors that can be assigned to the vertices of $\Delta$ in such a way that adjacent vertices are assigned different colors. The graph $\Delta$ has an $a/b$-coloring if, to each vertex of $\Delta$, one can assign a $b$-element subset of $\{1, 2, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets. Define $\chi^*(\Delta) = \inf\{\frac{a}{b} \mid \Delta \text{ can be } a/b \text{-colored}\}$, which is called fractional chromatic number of $\Delta$. It is well known that $\chi^*(\Delta) \leq \chi(\Delta)$.

Tang and Wan [8] determined the chromatic number of $Sp(2\nu, q)$. Subsequently, Li and Wang [7] determined the chromatic number of the subconstituents of $Sp(2\nu, q)$. In this section we show that the (fractional) chromatic number of $\Gamma$ is $q^\nu$.

**Proposition 5.1** ([3]). There exist maximal totally isotropic subspaces $V_i$, $i = 1, \ldots, q^\nu + 1$, of $F_q^{2\nu}$ such that

\[ F_q^{2\nu} = V_1 \cup \cdots \cup V_{q^\nu + 1}, \]

where $V_i \cap V_j = \{0\}$ for all $i \neq j$.

**Lemma 5.2.** $\Gamma$ is $q^\nu$-partite. That is, there exist subsets $X_1, \ldots, X_{q^\nu}$ of $X$ such that

\[ X = X_1 \cup \cdots \cup X_{q^\nu}, \]

and there are no adjacent vertices in each subset. Moreover, $|X_i| = q^s(q^{v-s} - 1)/(q - 1)$.

**Proof.** Since $GSp_{2\nu}(F_q)$ acts transitively on each set of subspaces of the same type, without loss of generality, we may assume that

\[ F_q^{2\nu} = V_1 \cup \cdots \cup V_{q^\nu + 1} \]

as in **Proposition 5.1** and $U_0 \subseteq V_{q^\nu + 1}$. Set $Y_i = \{[\alpha] \mid 0 \neq \alpha \in V_i\}$, $i = 1, \ldots, q^\nu + 1$, and

\[ V = \{[\alpha] \mid 0 \neq \alpha \in F_q^{2\nu}\}. \]

Then

\[ V = Y_1 \cup \cdots \cup Y_{q^\nu + 1}. \]

(3)

Let $X_i = Y_i \cap X$, $i = 1, \ldots, q^\nu + 1$. Then

\[ X = X \cap V = X_1 \cup \cdots \cup X_{q^\nu} \cup X_{q^\nu + 1}. \]
For any $[\alpha] \in X$, $U_0 K \alpha^i \neq 0$, which implies that $X_{q^{v+1}} = \emptyset$. Then
\[
X = X_1 \cup \cdots \cup X_{q^v},
\]
and there are no adjacent vertices in each $X_i$ since $V_i$ is totally isotropic. Hence $\Gamma$ is $q^v$-partite.

For any two distinct subspaces $V_i, V_j$, $1 \leq i \neq j \leq q^v$, since $U_0 + V_i$ and $U_0 + V_j$ are of type $(2v-s, v-s)$, there exists a $T \in \text{GSP}_{2v}(\mathbb{F}_q)$ such that $U_0 T = U_0$ and $V_i T = V_j$. Note that for any $[\alpha] \in V_i$, $[\alpha] \in X_j$ if and only if $U_0 K \alpha^i \neq 0$. Therefore, $X_j = \{[\alpha] \mid U_0 K \alpha^i \neq 0 \text{ and } \alpha \in V_i\}$, which follows that $\{|\alpha T| \mid U_0 K \alpha^i \neq 0 \text{ and } \alpha \in V_i\}$, $|X_j| \leq |X_i|$. Hence $|X| = |X_i|$, which follows that $|X_i| = |X|/q^v = q^v(q^{v-s} - 1)/(q - 1)$. \hfill $\square$

**Lemma 5.3.** For $0 \leq s \leq v - 1$, let $W$ be any maximal totally isotropic subspace of $\mathbb{F}_q^{2v}$, and $U$ be a $(v - s)$-dimensional totally isotropic subspace of $\mathbb{F}_q^v$ intersecting trivially with $W$. Then the number of $1$-dimensional subspaces $[\alpha]$ of $W$ satisfying $\alpha K U^\perp \neq 0$ is $q^v(q^{v-s} - 1)/(q - 1)$.

**Proof.** Since $W + U$ is of type $(2v-s, v-s)$ and $\text{GSP}_{2v}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type, we may assume that $W = [e_{q+1}, \ldots, e_{2v}], U = [e_{q+1}, \ldots, e_{v}]$. Let $\alpha = a_{v+1} e_{q+1} + \cdots + a_{2v} e_{2v}; a_{v+1}, e_{v+1} \in \mathbb{F}_q; 1 \leq i \leq v$. Then the number of $\alpha \in W$ satisfying $\alpha K U^\perp \neq 0$ is $q^v$. Hence the number of $1$-dimensional subspaces $[\alpha]$ of $W$ satisfying $\alpha K U^\perp \neq 0$ is $q^v(q^{v-s} - 1)/(q - 1)$. \hfill $\square$

**Theorem 5.4.** $\chi(\Gamma) = q^v$.

**Proof.** By **Lemma 5.2**, $\chi(\Gamma) \leq q^v$. Suppose that $\chi(\Gamma) = n < q^v$. Then there exist subsets $X_1, X_2, \ldots, X_n$ of $X$ such that
\[
X = X_1 \cup \cdots \cup X_n,
\]
where there are no adjacent vertices in each subset; and so
\[
\sum_{i=1}^n |X_i| = q^{v+s}(q^{v-s} - 1)/q - 1 > n \cdot q^v(q^{v-s} - 1)/q - 1,
\]
which implies that there exists some $X_i$ such that $|X_i| > q^v(q^{v-s} - 1)/(q - 1)$. Let $W_i$ be the subspace generated by all $\alpha$ such that $[\alpha] \in X_i$. Then $W_i$ is maximal totally isotropic. Note that $\alpha K U^\perp_0 \neq 0$ for each $[\alpha] \in X_i$, a contradiction to **Lemma 5.3**. Hence $\chi(\Gamma) = q^v$. \hfill $\square$

Now we compute the fractional chromatic number of $\Gamma$.

**Proposition 5.5** ([5, Corollary 7.5.2]). If $\Delta$ is a vertex-transitive graph with the vertex set $V$, then
\[
\chi^*(\Delta) = \frac{|V|}{\alpha(\Delta)},
\]
where $\alpha(\Delta)$ is the size of the largest independent set in $\Delta$, in which no two vertices are adjacent.

**Lemma 5.6.** $\alpha(\Gamma) = q^v(q^{v-s} - 1)/(q - 1)$.

**Proof.** By **Lemma 5.2**, $\alpha(\Gamma) \geq q^v(q^{v-s} - 1)/(q - 1)$. Suppose that $\alpha(\Gamma) > q^v(q^{v-s} - 1)/(q - 1)$ and $S$ is an independent set with size $\alpha(\Gamma)$. Let $W$ be the subspace generated by all $\alpha$ such that $[\alpha] \in S$. Then $W$ is maximal totally isotropic. Let $X_W = \{[\alpha] \mid 0 \neq \alpha \in W\}$ and $X_0 = \{[\alpha] \mid 0 \neq \alpha \in W_0\}$. Then $S \subseteq X_W, S \cap X_0 = \emptyset$; and so $S \subseteq X_W \setminus (X_W \cap X_0)$. Since $\dim(W \cap W_0) \geq s, |X_W \cap X_0| \geq (q^v - 1)/(q - 1)$, which follows that $\alpha(\Gamma) \leq (q^v - q^s)/(q - 1)$, a contradiction. Hence the desired result follows. \hfill $\square$

**Theorem 5.7.** $\chi^*(\Gamma) = q^v$.

**Proof.** Combining **Proposition 5.5** and **Lemma 5.6**, the desired result follows. \hfill $\square$

**Remarks.** Similar to symplectic graphs, Wan and his students [6,11,12] determined the full automorphism groups of orthogonal graphs and unitary graphs, which are also strongly regular graphs. We shall discuss subgraphs of these graphs, and determine their full automorphism groups in future.
Acknowledgements

This research is partially supported by NCET-08-0052, NSF of China (10871027, 10971052), and Hunan Provincial Natural Science Foundation of China (09JJ3006).

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