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A tight bound on the number of mobile servers to guarantee transferability among dominating configurations

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ABSTRACT

In this paper, we propose a new framework to provide continuous services to users by a collection of mobile servers distributed over an interconnection network. We model those mobile servers as a subset of nodes, and assume that a user can receive the service if at least one adjacent node (including itself) plays the role of a server; i.e., we assume that the service could not be routed via the interconnection network. The main results obtained in this paper are summarized as follows: For the class of trees consisting of n nodes, $\lfloor n/2 \rfloor$ mobile servers are sometimes necessary and always sufficient to realize continuous services by the mobile servers, and for the class of Hamiltonian graphs with n nodes, $\lceil (n+1)/3 \rceil$ mobile servers are sometimes necessary and always sufficient.
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1. Introduction

In recent years, an increasingly strong requirement for high quality network services provided over an interconnection network such as mobile cellular phone systems and miscellaneous content delivery systems has emerged. In those systems, on-line services should be provided to the users in a transparent manner; i.e., it is strongly required to provide a *common* service to *all* users at *any time*. This motivates the study of a server allocation problem in computer networks; i.e., the problem of finding an allocation of servers to computers that is "good" in terms of several metrics such as the latency of content delivery, minimum bandwidth of content delivery paths, and the maximum number of clients associated to each server. Many such metrics could be treated as a constraint to be satisfied by considering a logical overlay network derived from the given physical network; for example, by logically connecting any two computers whose round trip time (RTT) is smaller than a predetermined threshold, we have a logical network in which at least one end vertex of each edge must be allocated a server to guarantee the content delivery within a given latency, and a similar logical network could also be constructed to guarantee the minimum communication bandwidth of the content delivery paths.

In such logical networks, a given constraint could be naturally represented with the notion of *dominating set*. Given a network $G = (V, E)$ with vertex set V and edge set E, a dominating set for G is a subset of vertices such that for any $u \in V$, either u or at least one of its neighbors is in the subset. The notion of dominating set has been extensively studied in the literature during the past three decades, from various aspects including graph theoretic characterization [2,5,6,12,14,20], computational complexity of finding a dominating set with a minimum cardinality [9,13,16], and polynomial time algorithms for a special class of graphs such as interval graphs and other subclass of perfect graphs [1,3,4,15,17]. It has also been investigated from practical points of view, and it is pointed out by many researchers that the notion of dominating set is closely related with the resource allocation problem in networks [8], and the design of efficient routing schemes for wireless ad hoc networks [7,10,19,18].

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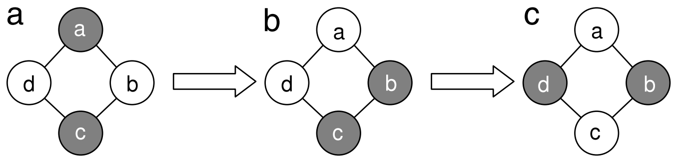

Fig. 1. A sequence of single-step transfers among dominating configurations for a ring with four vertices (dominating vertices are painted gray).

In this paper, we will consider a new model of network services in which each server associated to a vertex can move to any adjacent vertex in a single step. More precisely, each server is realized as a *mobile agent*, and the move of such agents is executed sequentially, i.e., in each step, at most one agent can move to an adjacent vertex. In order to guarantee that every vertex in the graph can receive the same service at any time, the change of the location of mobile agents must be controlled in such a way that the set of vertices associated with the agents forms a dominating set for the given network. We say that the move of an agent is safe if it does not leave any vertex uncovered *after* the move; i.e., vertices can be uncovered *during* a move (since every move of agents considered in this paper is safe, we will omit word "safe" throughout of this paper). Under such model of the mobility of servers, we will consider the following problem:

> *Given two dominating sets A and B for given graph G, can we transfer A to B through a sequence of (safe) moves of the mobile agents*?

Our main results are summarized as follows: (1) For the class of trees consisting of n vertices, $\lfloor n/2 \rfloor$ mobile agents are sometimes necessary and always sufficient to realize mutual transfers among dominating sets, and (2) for the class of Hamiltonian graphs consisting of n vertices, $\lceil (n+1)/3 \rceil$ mobile agents are sometimes necessary and always sufficient. The reader should note that, to the authors' best knowledge, this is the first work to investigate the mutual transferability among dominating sets, although the relation between the notion of dominating set and the server allocation problem has been pointed out by many researchers in the literature.

The remainder of this paper is organized as follows. In Section 2, we introduce necessary notation used throughout of the paper, which includes the definition of transferability between two dominating sets. Section 3 overviews our contribution. The proof of each theorem will be given in Section 4. Finally Section 5 concludes the paper with future work.

2. Preliminaries

Let $G = (V(G), E(G))$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set for G is a subset U of $V(G)$ such that for any vertex $u \in V(G)$, either $u \in U$ or there exists a vertex $v \in U$ such that $\{u, v\} \in E(G)$. In this paper, for technical reasons, we assume that a dominating set is a *multiset*; i.e., it can contain each vertex in $V(G)$ several times. Each element in a dominating (multi)set represents the location of a mobile agent, and in the following, we often identify a mobile agent with a vertex associated with the agent as long as it is clear from the context. Let $\mathcal{D}(G)$ denote an (infinite) set of all dominating (multi)sets for G. A dominating set is said to be minimal if no proper subset is a dominating set (by definition, for any minimal dominating set, the multiplicity of every element must be one). The *domination number* $\gamma(G)$ of G is the size of a minimum dominating set for G, and the *upper domination number* $\Gamma(G)$ of G is the maximum size of a minimal dominating set of G [11].

For any $S_1, S_2 \in \mathcal{D}(G)$, we say that S_1 is *single-step transferable* to S_2, and denote it as $S_1 \to S_2$, if there are two vertices u and v in $V(G)$ such that $S_1 - \{u\} = S_2 - \{v\}$ and $\{u, v\} \in E(G)$. By definition, single-step transferability is a symmetric binary relation, i.e., $S_1 \to S_2$ implies $S_2 \to S_1$. Note that $S_1 \to S_2$ also implies $|S_1| = |S_2|$, and a single-step transfer from S_1 to S_2 is realized by moving a mobile agent located at vertex $u \in S_1$ to its neighbor $v \in S_2$ (note that each vertex can have more than one agents, since each dominating set is assumed to be a multiset). For example, in a ring network consisting of four vertices $\{a, b, c, d\}$, dominating set $\{a, c\}$ is transferred to another dominating set $\{b, c\}$ in a single step by moving an agent located at vertex a to its neighbor b (see Fig. 1, for illustration).

A transitive closure of the relation of single-step transferability naturally defines the notion of transferability, which will be denoted as $S_1 \overset{*}{\to} S_2$, in what follows. Note that every subset of vertices appearing in a transfer from S_1 to S_2 must be a dominating set for G, and that the transferability is a symmetric relation between dominating sets. On the notion of transferability, we have the following lemma which will be frequently used throughout of this paper.

Lemma 1. *Let S_1, S_2 be dominating sets for $G = (V, E)$. If $S_1 \overset{*}{\to} S_2$, then for any S'_1 and S'_2 such that $S_1 \subseteq S'_1 \subseteq V$, $S_2 \subseteq S'_2 \subseteq V$, and $|S'_1| = |S'_2|$, it holds $S'_1 \overset{*}{\to} S'_2$.*

Proof. For each $i \in \{1, 2\}$, let S''_i be a subset of S'_i obtained by removing $|S_i|$ vertices in S_i from S'_i. Note that $|S''_i| = |S'_i| - |S_i|$ and S''_i may intersect with S_i since S'_i. Since S_1 and S_2 are dominating sets for G, we can move agents in S''_1 to S''_2 while keeping each intermediate configuration to be a dominating set for G; i.e., through a sequence of single-step transfers. After obtaining a dominating set consisting of S_1 and S''_2, we can move agents in S_1 to S_2 through a sequence of single-step transfers to obtain S'_2, since $S_1 \overset{*}{\to} S_2$. Hence the lemma follows. $\square$

A set $\mathcal{D}' \subseteq \mathcal{D}(G)$ is said to be *mutually transferable* if it holds $S_1 \xrightarrow{*} S_2$ for any $S_1, S_2 \in \mathcal{D}'$, where a sequence of single-step transfers from S_1 to S_2 can contain a configuration not in $\mathcal{D}'$, although all subsets in it must be an element in $\mathcal{D}(G)$. Note again that if $\mathcal{D}'$ is mutually transferable, then every dominating set in $\mathcal{D}'$ must have the same cardinality.

3. Main theorems

The first theorem gives a tight bound for the class of trees (proofs of all theorems will be given in the next section).

Theorem 1 (*Trees*). *For any tree T with n vertices, the set of dominating sets for T consisting of k, $k \geq \lfloor n/2 \rfloor$, vertices is mutually transferable, and there is a tree T_0 with n vertices such that $\gamma(T_0) = \lfloor n/2 \rfloor$.*

Next, we provide a lower bound on the number of dominating vertices which is necessary to guarantee the mutual transferability among dominating sets, for all graphs contained in a class of Hamiltonian graphs consisting of n vertices.

Theorem 2 (*Lower Bound*). *Let n be an integer and $k = \lceil (n+1)/3 \rceil - 1$. For any $r \geq 2$, there is a Hamiltonian r-regular graph G with more than n vertices such that the set of dominating sets for G with cardinality k is not mutually transferable.*

It is worth noting that for any Hamiltonian graph G consisting of n vertices, $\gamma(G) \leq \lceil (n+1)/3 \rceil - 1$, since it contains a ring of size n as a subgraph. It is in contrast to the case of trees, since the theorem claims that there is a Hamiltonian r-regular graph G such that k $(= \lceil (n+1)/3 \rceil - 1)$ dominating vertices are not sufficient to guarantee the mutual transferability, while k vertices are sufficient to dominate it. By combining Theorem 2 with the following theorem, we can derive a tightness of the lower bound for the class of Hamiltonian graphs with n vertices.

Theorem 3 (*Hamiltonian Graphs*). *Let G be a Hamiltonian graph with n vertices, and k be an integer such that $k \geq \lceil (n+1)/3 \rceil$. Then, the set of dominating sets of G consisting of k vertices is mutually transferable.*

4. Proofs

4.1. Theorem 1

We prove the upper bound by induction on the number of vertices. The claim immediately holds for $n = 0$ or 1. Assume that the claim holds for each $n \leq t - 1$, and examine the case of $n = t$, where $t \geq 2$. Let T be a rooted tree consisting of t vertices, and S_1 and S_2 be arbitrary dominating sets for T consisting of $k \geq \lfloor t/2 \rfloor$ vertices. Let u be a leaf vertex in T at the deepest level, and v be the unique neighbor of u. Let T' denote a subtree of T obtained by removing v and its all neighbors (with incident edges) from T. If T' contains no vertex, we can realize a transfer from S_1 to S_2 through a dominating set containing v since $\{v\}$ is a minimum dominating set for T. Hence in the following, we assume T' contains at least one vertex. Any dominating set for T containing u is single-step transferable to a dominating set that contains v instead of u, and this transformation allows us to reduce the problem of dominating T by a set with k dominating vertices to the problem of dominating T' by a set with $k - 1 \geq \lfloor (t-2)/2 \rfloor$ dominating vertices. By construction, T' consists of at most $t - 2$ vertices. Hence, by the induction hypothesis and by Lemma 1, S_1 can be transferred to S_2 through a sequence of dominating sets for T of size k, each of which consists of $k - 1$ dominating vertices for T' and vertex v.

To prove the latter half of the claim, we may consider a tree consisting of vertex set $\{0, 1, \ldots, 2m\}$ and edge set $\{(0, i), (i, i+m) \mid 1 \leq i \leq m\}$. Since each leaf must be dominated by different vertex, the domination number of the tree is $m = \lfloor (2m+1)/2 \rfloor$. Hence the theorem follows.

4.2. Theorem 2

The given claim immediately holds for $r = 2$ since it needs $\lceil (3m+1)/3 \rceil - 1 = m$ vertices to dominate a ring with $3m$ vertices, and if $m \geq 2$, there are two dominating sets consisting of m vertices which are not transferable with each other.

To prove the claim for $r = 3$, let us consider the following graph $G_3 = (V(G_3), E(G_3))$ consisting of 12 vertices, where

$$V(G_3) \stackrel{\text{def}}{=} \{1, 2, \ldots, 12\}, \quad \text{and}$$

$$E(G_3) \stackrel{\text{def}}{=} \{(i, i+1), (i+6, i+7) \mid 1 \leq i \leq 5\} \cup \{(6, 1), (12, 7)\} \cup \{(i, i+6) \mid 1 \leq i \leq 6\}.$$

Note that G_3 is Hamiltonian, cubic, and the domination number of G_3 is four. Now let us consider two dominating sets $S_1 = \{1, 2, 10, 11\}$ and $S_2 = \{4, 5, 7, 8\}$ for G_3. It is obvious that no mobile agent located at a vertex in S_1 can move to an adjacent vertex without leaving a vertex uncovered, since for each vertex in S_1, two adjacent vertices have a unique neighbor in S_1; e.g., vertex 2 dominates vertices 3 and 8 and those two vertices are not dominated by the other vertices. Thus, in order to realize a mutual transfer among dominating sets of G_3, five $(= \lceil (12+1)/3 \rceil)$ vertices are necessary.

The above construction can be directly extended to larger cubic graphs consisting of $12x$ vertices for all $x \geq 1$. More concretely, we may prepare two copies of a ring consisting of $6x$ vertices, and connect corresponding vertices in the copies by $6x$ parallel edges. Similar to the case of G_3, we can show that the resulting graph has two dominating sets of size $4x$ which are not transferable with each other; i.e., $4x+1$ ($=\lceil (12x+1)/3 \rceil$) dominating vertices are necessary to guarantee the mutual transferability among dominating sets.

Finally, an extension to larger r's can be realized in the following manner: For each $r \geq 4$, let us consider the following graph $G_r = (V(G_r), E(G_r))$:

$$\begin{aligned} V(G_r) &= \{(i,j) \mid i \in V(G_3) \text{ and } 1 \leq j \leq r-2\} \quad \text{and} \\ E(G_r) &= \left\{\{(i,j),(i',j)\} \mid (i,i') \in E(G_3) \text{ and } 1 \leq j \leq r-2\right\} \\ &\quad \cup \left\{\{(i,j),(i,j')\} \mid i \in V(G_3),\ j' \neq j \text{ and } 1 \leq j' \leq r-2\right\}. \end{aligned}$$

By construction, G_r is a Hamiltonian r-regular graph consisting of $12(r-2)$ vertices. In addition, by the same reason to the case of G_3, the following two dominating sets for G_r consisting of $4(r-2)$ vertices are not transferable with each other:

$$\begin{aligned} S_1 &= \{(1,j),(2,j),(10,j),(11,j) \mid 1 \leq j \leq r-2\} \\ S_2 &= \{(4,j),(5,j),(7,j),(8,j) \mid 1 \leq j \leq r-2\}. \end{aligned}$$

Hence the theorem follows.

4.3. Theorem 3

The proof of Theorem 3 consists of two parts. In the first part, we show that the claim holds if the given Hamiltonian graph is a ring. The second part provides a transfer of a dominating set for a Hamiltonian graph to a dominating set for a Hamiltonian cycle contained in it. By combining those two results, we can conclude that for any Hamiltonian graph, the set of dominating sets for the graph consisting of k, $k \geq \lceil (n+1)/3 \rceil$, vertices is mutually transferable.

4.3.1. Ring

A ring network consisting of n vertices, denoted as C_n, is a graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{\{i, i+1\} \mid 0 \leq i \leq n-2\} \cup \{\{n-1, 0\}\}$. In what follows, we denote $i \pmod n$ by i for brevity.

Lemma 2 (*Rings*). *The set of dominating sets for C_n consisting of k, $k \geq \lceil (n+1)/3 \rceil$, vertices is mutually transferable.*

Proof. At first, we introduce a potential function defined as follows:

$$\Phi(D) \stackrel{\text{def}}{=} \sum_{i \in D} i,$$

where D is a dominating set for C_n; i.e., $\Phi(D)$ is the sum of (the indices of) the vertices contained in D. For example, the potential of $\{1, 3\}$ is calculated as $1 + 3 = 4$. For any $k \geq \lceil (n+1)/3 \rceil$, a dominating set D^* of minimum potential is represented as follows:

$$D^* = \left\{ n-3, n-6, \ldots, n-3\lfloor n/3 \rfloor, \underbrace{0, 0, \ldots, 0}_{k-\lfloor n/3 \rfloor} \right\}.$$

Note that $\Phi(D^*) = n \times \lfloor n/3 \rfloor - \frac{3}{2}(\lfloor n/3 \rfloor \times (\lfloor n/3 \rfloor + 1))$, and D^* is a unique dominating set to have potential $\Phi(D^*)$ among all dominating sets consisting of k vertices. Now, let us prove that any dominating set D for C_n consisting of k vertices can always be transferred to D^* (recall that $D_1 \stackrel{*}{\to} D^*$ and $D_2 \stackrel{*}{\to} D^*$ implies $D_1 \stackrel{*}{\to} D_2$). To prove this claim, it is enough to show that for any D ($\neq D^*$), there is a dominating set D' such that $D \to D'$ and $\Phi(D) > \Phi(D')$, since D^* is the unique dominating set of a minimum potential.

Let $d_1, d_2, \ldots, d_k$ be a non-increasing sequence of elements in D. Note that $d_k \leq 2$ since otherwise, vertex 1 cannot be dominated. First, consider the case of $d_k = 0$. Since $D \neq D^*$, it holds either: (1) $d_1 \neq n-3$, or (2) there is $2 \leq j \leq k$ such that $d_{j-1} = n-3(j-1)$ and $d_j \neq n-3j$. In the first case, since vertex $n-2$ must be dominated by d_1, it must hold either $d_1 = n-1$ or $n-2$. In each case, by moving mobile agent d_1 to vertex $d_1 - 1$, we have a dominating set D' such that $\Phi(D') = \Phi(D) - 1$ (note that such a move does not leave any vertex uncovered since we are assuming that $d_k = 0$). A similar claim holds for the second case.

Next, consider the case of $d_k = 1$. Let $D'' = \{n-2, n-5, \ldots, n-3\lfloor n/3 \rfloor + 1, \underbrace{1, 1, \ldots, 1}_{k-\lfloor n/3 \rfloor}\}$ be a dominating set which is obtained by "rotating" D^*. If the given D disagrees with D'' at a position in the non-increasing order of the elements, we can apply the same argument to the case of $d_k = 0$ to obtain D' such that $\Phi(D') = \Phi(D) - 1$. On the other hand, if D completely agrees with D'', then we may simply move mobile agent d_k to vertex 0 to obtain a D' such that $\Phi(D') = \Phi(D) - 1$, since: (1) D'' contains at least two 1's for $n \equiv 0 \pmod 3$; (2) D'' contains 1 and 2 for $n \equiv 1 \pmod 3$; and (3) D'' contains 1 and 3 for $n \equiv 2 \pmod 3$. The case of $d_k = 2$ can also be proved in a similar way. Hence the lemma follows. $\square$

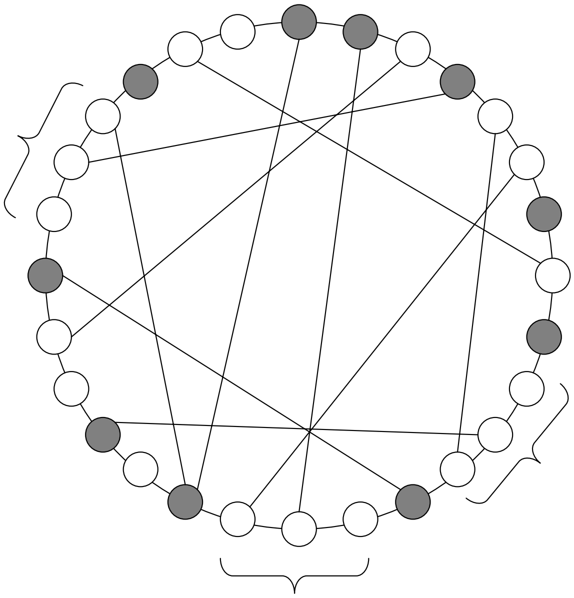

Fig. 2. An example of Hamiltonian graph G.

As for the number of single-step transfers, we have the following claim.

Corollary 1. *Let S_1, S_2 be dominating sets for C_n consisting of k, $\lceil (n+1)/3 \rceil \leq k \leq n$, vertices. Then, S_1 can be transferred to S_2 through $O(n^2)$ single-step transfers.*

4.3.2. Preprocessing for reduction

Let $G = (V, E)$ be a Hamiltonian graph with n vertices, and R be a Hamiltonian cycle in it. In what follows, edges contained in R will be referred to as *ring edges* and the other edges in G will be referred to as *chord edges*. Let $S \subseteq V$ be a dominating set for G with at least $\lceil (n+1)/3 \rceil$ vertices. Fig. 2 illustrates an example of Hamiltonian graph, where vertices painted gray form a dominating set for the graph. In the following, we will transfer S to a dominating configuration for R by consecutively removing chord edges and by moving mobile agents accordingly.

In the first step of the transfer, we apply the following operation whenever possible:

Operation 1: If the removal of a chord edge does not violate the condition of domination for its end vertices, then remove it.

Let G' be the resulting graph. Fig. 3 illustrates the resultant graph derived from the graph shown in Fig. 2. Note that S is a dominating set for G', and G' contains at most $|V - S| - 2 \leq n - \lceil (n+1)/3 \rceil - 2$ chord edges, since there are at most $|V - S|$ vertices to be dominated by the vertices in S, and at least two of them have already been dominated via ring edges. In addition, for any chord edge in G', exactly one of the end vertices must be a member of S and the other vertex must be connected with exactly one chord edge (otherwise, Operation 1 can be applied to remove a chord edge).

As the next step, we consider a subgraph G'' of G' that is obtained by removing all ring edges incident to the vertices dominated via chord edges. Fig. 4 shows an example of the resulting graph. By construction, G'' is either a ring, or a forest such that every leaf is a member in $V - S$ and every vertex with degree more than two is a member in S (in what follows, we call such a vertex "branch" vertex). If G'' is a ring, S is a dominating set for R. Thus, in the following, we assume that G'' is a forest, without loss of generality. Since $|S| \geq \lceil (n+1)/3 \rceil$, in at lease one of the resultant trees, the number of dominating vertices exceeds one third of the number of the vertices. Let T be one such tree and S_T $(\subseteq S)$ be the set of dominating vertices contained in T.

In the following, we will show that S_T can be transferred to a dominating configuration for T in which at least one leaf is a dominating one. Note that the proof of the above claim completes the proof of Theorem 3 since it implies that at least one chord edge can always be removed from G' and the same argument holds for the resultant graph as long as there remains a chord edge in it; i.e., we could transfer the given configuration S for G to a dominating configuration for R (note that in the sequence of reductions, we will have to replace G'' with a new subgraph after removing a chord edge from G').

4.3.3. Transfer of S_T

Tree T contains exactly two leaf vertices dominated via ring edges. Let $u_1, u_2, \ldots, u_m$ be the sequence of vertices on the path connecting those two leaf vertices and which lies along the ring R, i.e., u_1 and u_m are vertices dominated via ring edges and are connected with vertices dominated via chord edges in G''.

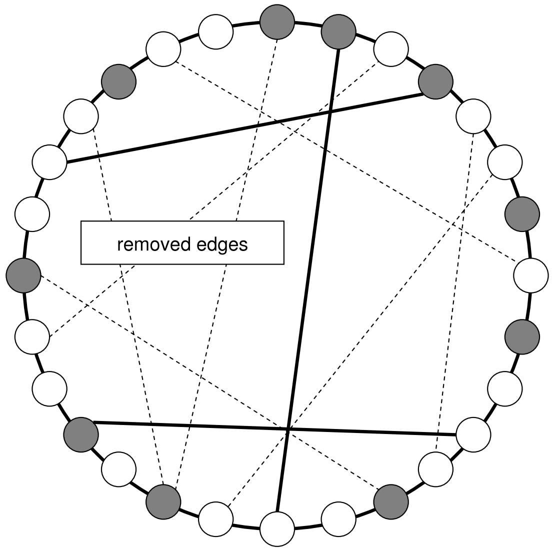

Fig. 3. The resultant graph G' after applying Operation 1 to G.

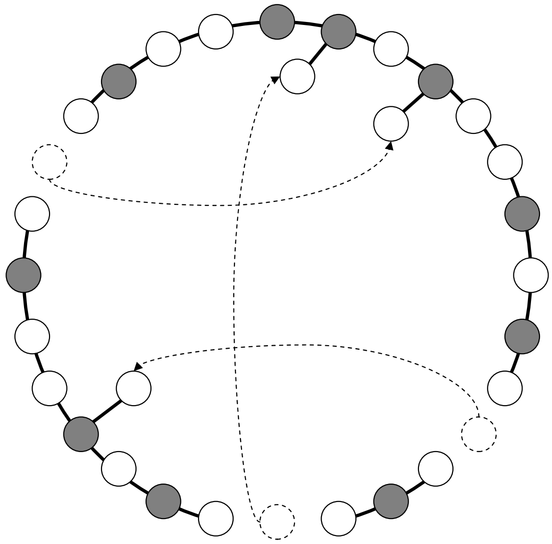

Fig. 4. An example of graph G''.

If T contains no branch vertices, i.e., it is a path, then the claim obviously holds since we can transfer S_T to a configuration in which either u_1 or u_m is a dominating vertex. Thus the following remark holds.

Remark 1. We may assume that T contains at least one branch.

Let u_i be the first branch in T; i.e., vertices $u_1, u_2, \ldots, u_{i-1}$ form a path connecting to u_i. Note that $u_i \in S$. Here, we may assume $i = 2$, without loss of generality, for the following two reasons:

Case 1: If $i \neq 3\ell + 2$ for any $\ell \geq 1$, or if $i = 3\ell + 2$ for some $\ell \geq 1$ and the path connecting u_1 and u_{i-1} contains at least $\ell + 1$ dominating vertices, then since $u_i \in S_T$, there exists at least one vertex on the path which is dominated by two vertices in S_T. Let u_j be one of such vertices with the smallest index j; i.e., u_j is dominated by two vertices u_{j-1} and u_{j+1} in S_T, and for any $j' < j$, $u_{j'} \in S_T$ or $u_{j'}$ has a unique neighbor in S_T. By moving a mobile agent from u_j to u_{j-1}, we can reduce the smallest index by three without leaving any vertex uncovered. Thus, by repeating the same operation at most $\lceil j/3 \rceil$ times, we can transfer S_T to a dominating configuration containing vertex u_1.

Case 2: If $i = 3\ell + 2$ for some $\ell \geq 1$ and the path connecting u_1 and u_{i-2} are dominated by exactly ℓ vertices, then we can reduce T to a smaller tree by removing vertices $u_1, u_2, \ldots, u_{i-2}$ since: (1) the number of dominating vertices in the resulting subtree exceeds one third of the number of vertices in the subtree, and (2) if we can move a mobile agent to a leaf in the subtree then we can move a mobile agent to a leaf in the original tree T. In fact, if we can move a mobile agent to vertex u_{i-1}, which is a leaf in the subtree but is not a leaf in the original tree, then since the path connecting u_1 and u_{i-1} now contains $\ell + 1$ dominating vertices, the procedure used in Case 1 can be applied to obtain a domination configuration containing vertex u_1.

Remark 2. We may assume that vertex u_2 is the first branch in T.

In addition, if T contains exactly one branch vertex, by the same reason to above, (1) we can transfer S_T to a configuration in which u_m is a dominating vertex, or (2) we can reduce T to a star-shaped tree with at least four vertices centered at u_2 that is dominated by at least two vertices, i.e., we can transfer S_T to a configuration in which at least one leaf vertex is a dominating one. Thus in the following, we assume that T contains at least two branch vertices.

Remark 3. We may assume that T contains at least two branches, let u_j ($\neq u_2$) denote the second branch.

For each x satisfying $1 \leq x \leq m-1$, let T_x and $\overline{T}_x$ be two subtrees of T obtained by removing edge (u_x, u_{x+1}) from T, where T_x and $\overline{T}_x$ denote a subtree containing vertices u_x and u_{x+1}, respectively.

Let $k = \lfloor j/3 \rfloor$. Note that subtree T_{j-1} must contain at least k dominating vertices under S_T, since $k-1$ vertices are not enough to dominate $j-2$ vertices from u_1 to u_{j-2}. If T_{j-1} contains more than k dominating vertices, then we can transfer S_T to a configuration in which one leaf connected with u_2 is a dominating vertex by applying at most $O(j)$ single-step transfers. In the following, we will show that if T_{j-1} contains exactly k dominating vertices, then we can identify a subtree T' of T such that: (1) T' contains at most one branch, (2) the number of dominating vertices in T' exceeds one third of the number of vertices in it, and (3) the move of a mobile agent to a leaf in T' causes a removal of chord edge in G''. Note that the proof of the above claim completes the proof of Theorem 3.

The above claim is an immediate consequence of the following lemma, since the number of branches in tree T is finite.

Lemma 3. *Recall that u_j is the second branch in T and that $k = \lfloor j/3 \rfloor$. If T_{j-1} contains exactly k dominating vertices under S_T, then either $\overline{T}_{j-1}$ or $\overline{T}_{j-2}$ is a subtree of T such that:* (1) *the number of dominating vertices in the subtree under S_T exceeds one third of the number of vertices in it, and* (2) *the move of a mobile agent to a leaf in the subtree causes a removal of chord edge in G''.*

Proof. At first, consider the case of $j \not\equiv 2 \pmod 3$. In this case, we can transfer the configuration of mobile agents in T_{j-1} such that u_{j-2} is contained in the resulting dominating configuration. Note that under the resulting dominating set S'_T, u_{j-1} is dominated by u_{j-2}; i.e., edge $\{u_{j-1}, u_j\}$ is not used to dominate vertex u_{j-1}. Since u_2 is a branch, T_{j-1} consists of at least j ($\geq 3k$) vertices, i.e., dominating vertices contained in $\overline{T}_{j-1}$ exceeds one third of the number of vertices in the subtree. In addition, each leaf in $\overline{T}_{j-1}$ is a leaf in the original tree T. Thus, by assumption, $\overline{T}_{j-1}$ is a subtree satisfying the claim.

Next, consider the case of $j \equiv 2 \pmod 3$. Since T_{j-1} contains k dominating vertices, each vertex in T_{j-1} must have a unique dominating vertex in S_T; i.e., S_T contains k vertices $u_2, u_5, u_8, \ldots, u_{j-3}$ on the path connecting u_1 and u_{j-1}. Now consider subtree T_{j-2}. In the subtree, at least $j-1$ ($= j-2+1$) vertices are dominated by $(j-2)/3$ vertices. Thus, by assumption, $\overline{T}_{j-2}$ is a tree such that: (1) the number of dominating vertices in the tree under S'_T exceeds one third of the number of the vertices, and (2) the move of a mobile agent to a leaf in $\overline{T}_{j-2}$ causes a removal of chord edge, since if we can move a mobile agent to leaf u_{j-1} of tree $\overline{T}_{j-2}$ then by symmetry, the same is true for other leaves connected to vertex u_j.

Hence, the lemma follows. □

In the above algorithm, the removal of a chord edge takes $O(n)$ single-step transfers. Since there are $O(n)$ chord edges in the worst case, by Corollary 1, we have the following claim.

Corollary 2. *Let G be a Hamiltonian graph with n vertices, and let S_1, S_2 be dominating sets for G consisting of k, $\lceil (n+1)/3 \rceil \leq k \leq n$, vertices. Then, S_1 can be transferred to S_2 through $O(n^2)$ single-step transfers.*

5. Concluding remarks

In this paper, we proposed a new framework to provide continuous services to users by a collection of mobile servers, and proved several tight bounds on the number of mobile servers to guarantee the mutual transferability among dominating configurations.

There remain several interesting open problems listed below:

- How can we extend the discussion to other classes of graphs? We have known that a lower bound for general graphs is the number of vertices connected to a leaf vertex, but we did not derive any upper bound for graphs that are not Hamiltonian but could be covered by two or more cycles.
- How can we reduce the number of single-step transfers connecting two dominating configurations? It could be asymptotically bounded as $O(n^2)$, but it is not clear if it could be reduced to $o(n^2)$.
- Is it possible to construct a distributed scheme that could be executed autonomously with no global information about the overall configuration of the underlying network?

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