Minimum Norm Problems over Transportation Polytopes*

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ABSTRACT

We consider the problem of updating input-output matrices, i.e., for given (m,n) matrices $A \geqslant 0$, $W \geqslant 0$ and vectors $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, find an (m,n) matrix $X \geqslant 0$ with prescribed row sums $\sum_{j=1}^n X_{ij} = u_i$ $(i=1,\dots,m)$ and prescribed column sums $\sum_{i=1}^m X_{ij} = v_j$ $(j=1,\dots,n)$ which fits the relations $X_{ij} = A_{ij} + \lambda_i W_{ij} + W_{ij}\mu_j$ for all i,j and some $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n$. Here we consider the question of existence of a solution to this problem, i.e., we shall characterize those matrices A, W and vectors u, v which lead to a solvable problem. Furthermore we outline some computational results using an algorithm of [2].

1. INTRODUCTION

There are often situations in statistics, input-output analysis, economic theory, and numerical analysis where the data given by a sample must be adjusted for consistency with data obtained from other sources or with deductions obtained from some other theory.

Consider for instance the problem of updating input-output matrices (cf. [1]), i.e., for given *nonnegative* real (m,n) matrices $A = (A_{ij})$ and $W = (W_{ij})$ and vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, find a *nonnegative* real (m,n) matrix $X = (X_{ij})$ with prescribed row sums

$$\sum_{j=1}^{n} X_{ij} = u_i, \qquad i = 1,\dots,m, \tag{1}$$

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and prescribed column sums

$$\sum_{i=1}^{m} X_{ij} = v_j, \qquad j=1,\dots,n, \tag{2}$$

which fits the relations

$$X_{ij} = A_{ij} + \lambda_i W_{ij} + W_{ij}\mu_j \qquad i=1,\dots,m, \quad j=1,\dots,n, \tag{3}$$

for some $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^n$.

The matrix W may depend on the underlying economic theory. With $W = A$ and no nonnegativity condition on X a solution of (1)–(3) is called a *Friedländer solution* (cf. [1], [6]). For *arbitrary* $W \geqslant 0$ we call X a *generalized Friedländer solution* if it satisfies (1), (2), and (3) but is not necessarily nonnegative.

Here we consider the question of the existence of a solution to the problem (1)–(3), i.e., we shall characterize those matrices A, W and vectors u, v which lead to a solvable problem.

We shall show that if the problem is solvable, a solution can easily be obtained using well-known methods. We also demonstrate the connection between (1)–(3) and a quadratic optimization problem with a strictly convex separable objective function subject to linear constraints of transportation type.

Some related work can be found in [1] using the relation

$$X_{ij} = A_{ij} + \lambda_i W_{ij}\mu_j, \qquad i=1,\dots,m, \quad j=1,\dots,n, \tag{3'}$$

instead of (3). In [3] we discuss a more general approach combining the relations (3) and (3′).

Clearly,

$$\sum_{i=1}^{m} u_i = \sum_{j=1}^{n} v_j \tag{4}$$

is a necessary and sufficient condition for (1) and (2) to be consistent; thus we shall assume (4) for the rest of the paper. The coefficient matrix appearing in the constraint set (1) and (2) is known as the *transportation matrix*. Its rank is well known to be $m+n-1$. Its last row is a linear combination of the others and hence can be deleted. We shall denote by T the $(m+n-1, mn)$ matrix so constructed, and similarly denote by v the vector $v' = (v_1, \dots, v_{n-1})$.

Let us represent the matrices X, A, and W as vectors x, a, and w in $\mathbb{R}^{mn}$ by ordering the set $\{(i,j)|1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$ lexicographically. Write $Q=\operatorname{diag}(w)$. Then (1)–(3) carry over to

$$Tx=\binom{u}{v}, \tag{5}$$

$$x=a+QT'\binom{\lambda}{\mu}. \tag{6}$$

Inserting (6) into (5) yields

$$TQT'\binom{\lambda}{\mu}=\binom{u}{v}-Ta. \tag{7}$$

Thus the problem is to characterize those w, a, u, and v which make the system

$$TQT'\binom{\lambda}{\mu}=\binom{u}{v}-Ta, \tag{8a}$$

$$a+QT'\binom{\lambda}{\mu}\geqslant 0 \tag{8b}$$

consistent, i.e., we have to characterize those w, a, u, and v which lead to a generalized Friedländer solution which is nonnegative. We shall first consider the question of the existence of generalized Friedländer solutions.

2. THE FRIEDLÄNDER SOLUTION

We now wish to exhibit the special structure of $D:=TQT'$, the $(m+n-1, m+n-1)$ coefficient matrix of (7). Let $\overline{W}$ be the matrix W without the last column, and denote by R and C the diagonal matrices

$$R=\operatorname{diag}\left(\sum_{j=1}^{n} W_{1j},\ldots,\sum_{j=1}^{n} W_{mj}\right), \tag{9}$$

$$C=\operatorname{diag}\left(\sum_{i=1}^{m} W_{i1},\ldots,\sum_{i=1}^{m} W_{i,n-1}\right), \tag{10}$$

containing the row and column sums of W and $\overline{W}$ respectively as diagonal elements. Then

$$D=\begin{pmatrix} R & \overline{W} \\ \overline{W}' & C \end{pmatrix}. \tag{11}$$

To state the results in a more elegant way we shall first consider some trivial cases and later exclude them. Assume W contains a zero row, say the kth row. Concerning the relation (3) this means we have to fix $X_{kj}=A_{kj}$ ($j=1,\dots,n$). This would be possible if

$$\sum_{j=1}^{n} A_{kj}=u_k. \tag{12}$$

But if (12) holds we can reduce our problem size by deleting row k in X, W, and A and changing v_j to $\tilde{v}_j=v_j-a_{kj}$ ($j=1,\dots,n$). An analogous procedure applies if there is any zero column in W. We therefore assume without loss of generality:

$$W \text{ contains no zero row or column.} \tag{13}$$

Note that (13) implies R and C are positive definite matrices.

We say that a (m,n) matrix B is *disconnected* if there exist permutation matrices P and S such that

$$PBS=\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}; \tag{14}$$

otherwise B is called *connected.* An (n,n) matrix B is *decomposable* if there exists a nonempty proper subset J of $\{1,\dots,n\}$ such that

$$B_{ij}=0 \qquad \text{for} \quad i\notin J, \quad j\in J; \tag{15}$$

otherwise B is called *indecomposable.*

LEMMA 1. *If W is connected, then $\overline{W}'\overline{W}$ is indecomposable.*

Proof. Suppose $B:=\overline{W}'\overline{W}$ is decomposable. Then there exists an index set $J\subseteq\{1,\dots,n-1\}$ with $B_{ij}=0$ for $i\notin J$, $j\in J$. Thus $(W_{,i})'W_{,j}=0$ for all $i\notin J$, $j\in J$ ($W_{,i}$ denotes the ith column of W). Let $I:=\{k\in\{1,\dots,n-1\}\,|\,W_{kj}\neq 0 \text{ for some } j\notin J\}$. Since W is a nonnegative matrix, $W_{kj}=0$ for all

$k \in I$ and $j \in J$. Since W contains no zero columns [cf. (13)], I is a proper subset of $\{1,\ldots,n-1\}$. Thus we obtain

$$W_{ij}=0 \qquad \text{for} \quad \begin{cases} i\in I, & j\in J, \\ i\notin I, & j\notin J, \end{cases}$$

contradicting the connectedness of W. ■

Assume the matrix W is disconnected, i.e.

$$PWS=\begin{pmatrix} W^1 & 0 \\ 0 & W^2 \end{pmatrix}.$$

Let

$$PAS=\begin{pmatrix} A^1 & A^3 \\ A^4 & A^2 \end{pmatrix} \quad \text{and} \quad PXS=\begin{pmatrix} X^1 & X^3 \\ X^4 & X^2 \end{pmatrix};$$

then the condition (3) is

$$X^3=A^3, \qquad X^4=A^4$$

and

$$X^i=A^i+\operatorname{diag}(\lambda^i)\,W^i+W^i\operatorname{diag}(\mu^i), \qquad i=1,2,$$

and obviously the problem may be partitioned into two independent problems.

There are good (i.e. polynomial bounded) algorithms for computing the connected components of W. Thus we can always partition W into

$$PWS=\begin{bmatrix} W^1 & & 0 \\ & \ddots & \\ 0 & & W^k \end{bmatrix} \tag{16}$$

where the diagonal blocks $W^1,\ldots,W^k$ are each connected. Hence again we shall assume without loss of generality

$$W \text{ is connected.} \tag{17}$$

Consider the partitioning of D given in (11) and denote the Schur complement of R in D (cf. [4]) by $H:=C-\overline{W}'R^{-1}\overline{W}$.

Note that R and C are diagonal matrices containing the row and column sums of W and $\overline{W}$, respectively, as diagonal elements. Thus they depend on W.

THEOREM 2. *D is positive definite iff W is connected. Moreover H is positively invertible, i.e., H^{-1} is a positive matrix iff W is connected.*

Proof. Since $D = TQT'$ and Q is positive semidefinite, D is positive semidefinite. Thus it suffices to show: D is nonsingular iff W is connected. Suppose W is connected. Then $B := C^{-1}\overline{W}'R^{-1}\overline{W}$ is indecomposable (Lemma 1). H^{-1} exists and is positive if $(I-B)^{-1}$ exists and is positive [I denoting an $(n-1, n-1)$ identity matrix]. To prove $(I-B)^{-1}$ exists and is positive for an indecomposable $B \geqslant 0$ it suffices to show

$$(I-B)x \geqslant 0, \quad (I-B)x \neq 0 \qquad \text{for some} \quad x \geqslant 0, \quad x \neq 0 \tag{18}$$

(e.g. [8, p. 107, Theorem 7.4(i)]). To verify (18) we take $x = e$, the vector whose components are all ones, and show $Be \leqslant e$ and $Be \neq e$. Denote by W_i ($W_{,j}$) the ith row (jth column) of W. Then $R^{-1}\overline{W}e \leqslant e$, and since $W_{,n} \neq 0$, we have $R^{-1}\overline{W}e \neq e$; hence $B_i e = C_{ii}^{-1}(W_{,i})'R^{-1}\overline{W}e \leqslant C_{ii}^{-1}(W_{,i})'e = 1$, with a strict inequality for at least one $i \in \{1, \ldots, n-1\}$. Since H is the Schur complement of R in D, D is nonsingular.

Assume W is disconnected, and let P and $\hat{S}$ be permutation matrices such that

$$PW\hat{S} = \begin{pmatrix} W^1 & 0 \\ 0 & W^2 \end{pmatrix}.$$

We may assume that the nth column of W was the only row permuted, i.e. is still the last column of $PW\hat{S}$. Let S be the permutation matrix $\hat{S}$ without the last column and row. Since

$$\begin{aligned} S'HS &= S'CS - S'\overline{W}'R^{-1}\overline{W}S \\ &= S'CS - (S'\overline{W}'P')(PR^{-1}P')(P\overline{W}S), \end{aligned}$$

the permutations leads to a partition of

$$\tilde{C} = S'CS = \begin{pmatrix} \tilde{C}^1 & 0 \\ 0 & \tilde{C}^2 \end{pmatrix},$$

$$\tilde{R}^{-1} = PR^{-1}P' = \begin{bmatrix} (\tilde{R}^1)^{-1} & 0 \\ 0 & (\tilde{R}^2)^{-1} \end{bmatrix},$$

$$\tilde{W} = P\overline{W}S = \begin{pmatrix} \tilde{W}^1 & 0 \\ 0 & \tilde{W}^2 \end{pmatrix}.$$

Hence we obtain

$$\tilde{H}=S'HS=\begin{bmatrix} \tilde{C}^1-(\tilde{W}^1)'(\tilde{R}^1)^{-1}\tilde{W}^1 & 0 \\ 0 & \tilde{C}^2-(\tilde{W}^2)'(\tilde{R}^2)^{-1}\tilde{W}^2 \end{bmatrix}.$$

The matrix $\tilde{B}^1:=(\tilde{C}^1)^{-1}(\tilde{W}^1)'(\tilde{R}^1)^{-1}\tilde{W}^1$ has all row sums 1, since $\tilde{R}^1$ already contains the row sums of W^1. Thus $(I-\tilde{B}^1)e=0$ and $I-\tilde{B}^1$ is singular. Hence $\tilde{H}$ can not be nonsingular, which proves that D can not be nonsingular. ■

Consider an element $D_{ik}=\sum_{j=1}^{mn}T_{ij}Q_{jj}T_{kj}$ of the product $D=TQT'$. It is not hard to see that all columns of T (rows of T') for which the corresponding diagonal element Q_{jj} in Q is zero have no influence on the elements of D. Eliminate all columns of T for which $Q_{jj}=0$ and call the resulting matrix $\tilde{T}$. Eliminate all zero rows and columns of Q and denote this resulting matrix by $\tilde{Q}$. Then we still obtain

$$D=\tilde{T}\tilde{Q}\tilde{T}'. \tag{19}$$

Theorem 2 states that if W is connected we can compute a generalized Friedländer solution by solving the equation

$$D\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\begin{pmatrix}u\\ v\end{pmatrix}-Ta$$

[cf. (8a)] and may exploit the fact that D is positive definite using standard numerical techniques (cf. [9]).

3. NONNEGATIVE FRIEDLÄNDER SOLUTIONS

Actually $W_{ij}=0$ [i.e. $Q_{kk}=0$ for $k=(i-1)m+j$] means that X_{ij} has to be fixed at the value of A_{ij} [cf. (3)]. Hence it is not worth while to take these variables into account. We could eliminate them from the beginning. Clearly, we would then have to correct the prescribed row and column sums u and v accordingly:

$$u_i:=u_i-\sum_{\{j\mid W_{ij}=0\}}A_{ij}, \qquad i=1,\dots,m,$$

$$v_j:=v_j-\sum_{\{i\mid W_{ij}=0\}}A_{ij}, \qquad j=1,\dots,n.$$

Thus [recall (19)] we may assume that W is a positive matrix. This in turn makes Q a positive definite diagonal matrix.

Consider the strictly convex separable quadratic optimization problem

$$\begin{array}{ll} \min & q'y+\tfrac{1}{2}y'Q^{-1}y \\ \text{subject to} & Ty=\binom{u}{v}, \end{array} \tag{20}$$

where $q:=-Q^{-1}a$. Clearly (20) has an unique solution. Note also that the matrix T has full row rank, and hence contains a basis. Thus (20) can be expressed as an unconstrained optimization problem, and the first order conditions equal the equations (8a). Thus obviously the system (8) has a solution iff (20) has a *nonnegative* minimizer. Therefore actually we are looking for conditions on Q, a, u, and v which guarantee that the minimizer of (20) is nonnegative. Since D is nonsingular, (8) is consistent iff

$$a+QT'D^{-1}\left[\binom{u}{v}-Ta\right]\geqslant 0 \tag{21}$$

holds. Consider again the partition of

$$D=\begin{pmatrix} R & \overline{W} \\ \overline{W}' & C \end{pmatrix},$$

and recall that we denote by $H=C-\overline{W}'R^{-1}\overline{W}$ the Schur complement of R in D. Then it is well known (cf. [3]) that D^{-1} may be partitioned as

$$D^{-1}=\begin{bmatrix} R^{-1}(I_m+\overline{W}H^{-1}\overline{W}'R^{-1}) & -R^{-1}\overline{W}H^{-1} \\ -H^{-1}\overline{W}'R^{-1} & H^{-1} \end{bmatrix} \tag{22}$$

Setting $s=\binom{s_1}{s_2}$ and collecting terms appropriately, we easily obtain

$$D^{-1}s=\begin{pmatrix} R^{-1}\overline{W} \\ -I_{n-1} \end{pmatrix}H^{-1}(\overline{W}'R^{-1}s_1-s_2)+\begin{pmatrix} R^{-1}s_1 \\ 0 \end{pmatrix}. \tag{23}$$

We shall first consider an easily handled special case: $W_{ij}=1$ for all $i=1,\dots,m$ and $j=1,\dots,n$. This amounts to considering the optimization problem

$$\begin{array}{ll} \min & \|x-a\|_2 \\ \text{subject to} & Tx=\binom{u}{v}, \end{array} \tag{24}$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm. Here we shall give an explicit formula for the optimal solution x. Obviously this leads to a necessary and sufficient condition for the solvability of our original problem in this special case.

THEOREM 3. *Let*

$$x^0 := a + T'(TT')^{-1}(t - Ta), \qquad \text{where} \quad t = \binom{u}{v}.$$

Then x^0 solves (uniquely)

$$\min \quad \{\|x-a\|_2 \,|\, Tx = t\}.$$

In particular the updating problem [cf. (1)–(3)] is solvable if and only if $x^0 \geqslant 0$. Moreover, the inverse of (TT') has the form

$$(TT')^{-1} = \left(\begin{array}{cccc|cccc} \frac{n-1}{nm}+1 & \frac{n-1}{nm} & \cdots & \frac{n-1}{nm} & -1 & -1 & \cdots & -1 \\ \frac{n-1}{nm} & \frac{n-1}{nm}+1 & \cdots & \frac{n-1}{nm} & -1 & -1 & \cdots & -1 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \frac{n-1}{nm} & \frac{n-1}{nm} & \cdots & \frac{n-1}{nm}+1 & -1 & -1 & \cdots & -1 \\ \hline -\frac{n}{m} & -\frac{n}{m} & \cdots & -\frac{n}{m} & \frac{2}{m} & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots & 1 & \ddots & & \vdots \\ -\frac{n}{m} & -\frac{n}{m} & \cdots & -\frac{n}{m} & \vdots & & \ddots & 1 \\ -\frac{n}{m} & -\frac{n}{m} & \cdots & -\frac{n}{m} & 1 & \cdots & 1 & \frac{2}{m} \end{array}\right)$$

Thus $x^0 = (x_{ij}^0)$ can be expressed as

$$x_{ij}^0 := \frac{n-m-1}{m^2} c_n - \frac{1}{m^2} \sum_{k=1}^{n-1} c_k + c_j + \frac{1}{n} r_i + A_{ij},$$

where $r_i := u_i - \sum_{j=1}^{n} A_{ij}$ and $c_j := v_j - \sum_{i=1}^{m} A_{ij}$.

Proof. Let us denote by $E_{n,m}$ an (n,m) matrix whose entries are all ones. Since $W_{ij}=1$ for all i and j, we obtain $C=\mathrm{diag}(m,\dots,m)$ and $R=\mathrm{diag}(n,\dots,n)$. Therefore we have

$$\begin{aligned} H &= C-\overline{W}'R^{-1}\overline{W} \\ &= mI_{n-1}-E_{n-1,m}\left(\frac{1}{n}I_m\right)E_{m,n-1} \\ &= mI_{n-1}-\frac{m}{n}E_{n-1,n-1}=m\left(I_{n-1}-\frac{1}{n}E_{n-1,n-1}\right). \end{aligned}$$

Using $F:=(1/m)(I_{n-1}+E_{n-1,n-1})$, one can easily check that

$$HF=I_{n-1}+E_{n-1,n-1}-\frac{1}{n}E_{n-1,n-1}-\frac{1}{n}E_{n-1,n-1}E_{n-1,n-1}=I_{n-1},$$

which gives $F=H^{-1}$.

Let $p:=\overline{W}'R^{-1}s_1-s_2$, where

$$s=\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}=\begin{pmatrix} u \\ v \end{pmatrix}-Ta$$

[cf. (23)]; then

$$p_j=\sum_{k=1}^{m}\frac{1}{n}r_k-c_j \qquad j=1,\dots,n-1$$

and

$$H^{-1}p=\frac{1}{m}(p+E_{n-1,n-1}p)=\frac{1}{m}\left(p+I_{n-1}\sum_{j=1}^{n-1}p_j\right),$$

or

$$\begin{aligned} (H^{-1}p)_j &= \frac{1}{m}\left(p_j+\frac{n-1}{n}\sum_{i=1}^{m}r_i-\sum_{k=1}^{n-1}c_k\right) \\ &= \frac{1}{m}\left[\sum_{i=1}^{m}r_i-\left(c_j+\sum_{k=1}^{n-1}c_k\right)\right] \\ &= \frac{1}{m}\left(\sum_{k=1}^{n}c_k-\sum_{k=1}^{n-1}c_k-c_j\right) \qquad \text{[cf. (4)]} \\ &= \frac{1}{m}(c_n-c_j), \qquad j=1,\dots,n-1. \end{aligned}$$

Using Eq. (22), we obtain easily the elements of $D^{-1}=(TT')^{-1}$ as stated in the theorem. Computing $D^{-1}s$ via (23), we have the following interim results:

$$R^{-1}\overline{W}H^{-1}p=\frac{1}{m}\left(\sum_{k=1}^{n-1}\frac{1}{m}(c_n-c_k)\right)e$$

$$=\frac{1}{m^2}\left((n-1)c_n-\sum_{j=1}^{n-1}c_j\right)e=\frac{1}{m^2}\left(nc_n-\sum_{k=1}^{n}c_k\right)e,$$

$$-I_{n-1}H^{-1}p=c-\frac{1}{m}c_n e.$$

Thus we obtain

$$D^{-1}s=\begin{pmatrix} R^{-1}\overline{W} \\ -I_{n-1}\end{pmatrix}H^{-1}p+\begin{pmatrix} R^{-1}r \\ 0\end{pmatrix}$$

$$=\begin{bmatrix} \dfrac{1}{m^2}\left(nc_n-\sum_{k=1}^{n}c_k\right)e+\dfrac{1}{n}r \\ c-\dfrac{1}{m}c_n e\end{bmatrix}$$

and therefore

$$x_{ij}=\frac{1}{m^2}\left(nc_n-\sum_{k=1}^{n}c_k\right)+c_j-\frac{1}{m}c_n+\frac{1}{n}r_i+A_{ij}$$

$$=c_n\left(\frac{n}{m^2}-\frac{1}{m}\right)-\frac{1}{m^2}\sum_{k=1}^{n}c_k+c_j+\frac{1}{n}r_i+A_{ij}$$

$$=\frac{n-m-1}{m^2}c_n-\frac{1}{m^2}\sum_{k=1}^{n-1}c_k+c_j+\frac{1}{n}r_i+A_{ij},$$

which proves the theorem. ∎

We shall now treat the general case, i.e. $W\geqslant 0$ arbitrary, but connected. Let us consider the term

$$p:=\overline{W}'R^{-1}s_1-s_2, \qquad s=\begin{pmatrix}s_1\\ s_2\end{pmatrix}=\begin{pmatrix}u\\ v\end{pmatrix}-Ta,$$

which occurs in (23). s_1 stands for the differences of the row sums of a solution X and the matrix A, and similarly s_2 for the differences of the column sums of a solution X and A. Thus p contains as components the deviation of the column sum differences from a weighted arithmetic mean of the row differences. In case $W_{ij}=1$ for all i and j, we obtain for instance

$$p_k=\frac{1}{m}\sum_{i=1}^{m}(s_1)_i-(s_2)_k$$

$$=\frac{1}{m}\sum_{j=1}^{n}(s_2)_j-(s_2)_k \qquad [\text{cf. } (4)].$$

For vectors $x\in\mathbb{R}^k$ and (m,n) matrices W, we use the maximum norms

$$\|x\|_\infty:=\max\{|x_j|\,|\,j=1,\dots,k\} \quad \text{and} \quad \|W\|_\infty:=\max\left\{\sum_{j=1}^{n}|W_{ij}|\,\middle|\,i=1,\dots,m\right\}.$$

Let W_{hn} be an element of W with $W_{hn}R_{hk}^{-1}:=\min\{W_{in}R_{ii}^{-1}|i=1,\dots,m\}$; then we have the following result:

THEOREM 4. *Assume there is at least one column of W which has a positive inner product with every other column and*

$$\left(\frac{r_i}{R_{ii}}+\frac{A_{ij}}{W_{ij}}\right)-\|p\|_\infty W_{hn}\frac{\|W\|_\infty}{\|W'\|_\infty}\geqslant 0 \tag{26}$$

holds for all i and j where $W_{ij}>0$. Then the nonnegative input-output updating problem [*cf.* (1)–(3)] *is solvable.*

In practical applications such as estimating input-output matrices, (26) is usually fulfilled, since $r\geqslant 0$ and $\|p\|_\infty$ is near zero. Thus Theorem 4 explains the well-known empirical observation that the generalized Friedländer solution is already nonnegative and thus solves the estimating problem (1)–(3).

Proof. We have already shown that it is enough to prove $QT'D^{-1}s+a\geqslant 0$ [cf. (25)]. From (23) we conclude

$$D^{-1}s=\begin{pmatrix}R^{-1}\overline{W}\\-I_{n-1}\end{pmatrix}H^{-1}p+\begin{pmatrix}R^{-1}s_1\\0\end{pmatrix}.$$

Consider again the matrix $B:=C^{-1}\overline{W}'R^{-1}\overline{W}$, which we introduced in the proof of Theorem 1, and write $z^0:=C^{-1}p$. Then we shall prove

$$H^{-1}p\geqslant\frac{-\|z^0\|_\infty}{1-\|B\|_\infty}e, \tag{27}$$

where e again is a vector having components all ones. If W has one column (e.g. the last column) which has a positive inner product with every other column, it is obviously connected. Thus H^{-1} exists. But, as easily verified in the proof of Theorem 1, $Be<e$ holds; this gives $\|B\|_\infty<1$. Let $z^k:=B^kz^0$ for $k\in\mathbb{N}$. Since $\sum_{k=0}^{\infty} B^k$ converges to $(I_{n-1}-B)^{-1}$, we obtain

$$\begin{aligned} H^{-1}p=(I_{n-1}-B)^{-1}C^{-1}p=\sum_{k=0}^{\infty} B^kz^0&=\sum_{k=0}^{\infty} z_k \\ &\geqslant \sum_{k=0}^{\infty} -\|z^k\|_\infty e \\ &\geqslant \sum_{k=0}^{\infty} -\|B^k\|_\infty \|z^0\|_\infty e \\ &\geqslant \left(-\|z^0\|_\infty \sum_{k=0}^{\infty} \|B\|_\infty^k\right)e \\ &=\frac{-\|z^0\|_\infty}{1-\|B\|_\infty}e, \end{aligned}$$

which proves (27). Thus

$$D^{-1}s\geqslant\frac{-\|z^0\|_\infty}{1-\|B\|_\infty}\begin{pmatrix} R^{-1}\overline{W}e \\ -e\end{pmatrix}+\begin{pmatrix} R^{-1}r \\ 0\end{pmatrix}.$$

Since $(R^{-1}\overline{W}e)_i=1-W_{in}(We)_i^{-1}=1-W_{in}R_{ii}^{-1}$, the inequality above yields

$$(T'D^{-1}s)_{ij}\geqslant\begin{cases} \dfrac{\|z^0\|_\infty}{1-\|B\|_\infty}W_{in}R_{ii}^{-1}+r_iR_{ii}^{-1}, & j\neq n, \\ \dfrac{-\|z^0\|_\infty}{1-\|B\|_\infty}(1-W_{in}R_{ii}^{-1})+r_iR_{ii}^{-1}, & j=n, \end{cases} \tag{28}$$

for all i,j. Let $x_{ij}:=W_{ij}(T'D^{-1}s)_{ij}+A_{ij}$. Since $A\geqslant 0$, (25) has no solution iff $x_{ij}>0$ for all i,j with $W_{ij}\neq 0$. Thus we have to verify

$$(T'D^{-1}s)_{ij}+A_{ij}W_{ij}^{-1}>0 \tag{29}$$

for all i,j with $W_{ij}\neq 0$. Using the assumption (26) we obtain

$$\frac{\|z^0\|_\infty}{1-\|B\|_\infty}W_{in}R_{ii}^{-1}+r_iR_{ii}^{-1}+A_{ij}W_{ij}^{-1}\geq r_iR_{ii}^{-1}+A_{ij}W_{ij}^{-1}$$

$$\geq \|p\|_\infty W_{hn}\|W\|_\infty\|W'\|_\infty^{-1},$$

$$\geq 0$$

which proves (29) for all i,j with $j\neq n$:

$$\|B\|_\infty=\max\{(Be)_i|i=1,\dots,n-1\}$$

$$=1-\min\left\{C_{ii}^{-1}\sum_{k=1}^{m}W_{ki}W_{kn}R_{kk}^{-1}\,\middle|\,i=1,\dots,n-1\right\}$$

$$\leq 1-\min\left\{W_{hn}R_{hh}^{-1}C_{ii}^{-1}\sum_{k=1}^{m}W_{ki}\,\middle|\,i=1,\dots,n-1\right\}$$

$$=1-W_{hn}R_{hh}^{-1}.$$

This now proves the theorem, again using the assumption (26):

$$x_{in}\geq -\|z^0\|_\infty(1-\|B\|_\infty)^{-1}(1-W_{in}R_{ii}^{-1})+r_iR_{ii}^{-1}+A_{in}W_{in}^{-1}$$

$$\geq -\|z^0\|_\infty R_{hh}W_{hn}^{-1}(1-W_{in}R_{ii}^{-1})+r_iR_{ii}^{-1}+A_{in}W_{in}^{-1}$$

$$\geq -\|C^{-1}p\|_\infty\|W\|_\infty W_{hn}^{-1}+r_iR_{ii}^{-1}+A_{in}W_{in}^{-1}$$

$$\geq -\|C^{-1}\|_\infty\|p\|_\infty\|W\|_\infty W_{hn}^{-1}+r_iR_{ii}^{-1}+A_{in}W_{in}^{-1}$$

$$\geq r_iR_{ii}^{-1}+A_{in}W_{in}^{-1}-\|p\|_\infty\|W\|_\infty\|W'\|_\infty^{-1}W_{hn}$$

$$\geq 0.$$

■

4. COMPUTING THE NONNEGATIVE FRIEDLÄNDER SOLUTION

We have used the strictly convex separable quadratic optimization problem

$$\min \quad q'y+\tfrac{1}{2}y'Q^{-1}y,$$
$$Ty=\binom{u}{v}, \tag{30}$$
$$y\geq 0.$$

[cf. (21)] to solve the nonnegative Friedländer problem using an iterative method such as the Hildreth [7] method.

Specifically, we have estimated various input-output matrices of the German economy. Here A denotes some input-output matrix of the years 1956–1966, and as weights for the objective function we have considered the following three possibilities:

$$\begin{aligned} w_{ij} &= 1 && \text{for all } (i,j), \\ w_{ij} &= a_{ij} && \text{for all } (i,j), \\ w_{ij} &= a_{ij}^2 && \text{for all } (i,j). \end{aligned}$$

We counted the number of samples where the Friedländer solution was already optimal. These are listed in the column "No. of FO" of Table 1.

TABLE 1

Size of A	No. of test prob.	Weights w	No. of LB or UB	No. of FO	CPU time (sec)
12×12	13	1	0	2	7
	13	a	0	9	5.6
	12	a^2	0	8	8.4
	10	1	30	0	11
	10	a	30	1	10.3
	10	a^2	30	3	12
56×56	12	1	0	2	8.2
	12	a	0	5	5.3
	12	a^2	0	5	7.3
	5	1	100	0	15.2
	5	a	100	0	15.8
	5	a^2	100	2	10.3

All computing times include computer input/output operations (reading and writing), which are roughly 4 to 6 seconds and were performed on an IBM 370/168 computer. A more detailed description of the algorithm can be found in [2]. Cottle [5] also developed an algorithm for solving the structured problem (30) using linear complementarity techniques.

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