J. Math. Pures Appl. 80, 3 (2001) 339–372

S0021-7824(00)01180-6/FLA

PINNING PHENOMENA IN THE GINZBURG–LANDAU MODEL OF SUPERCONDUCTIVITY

Amandine AFTALION [a], **Etienne SANDIER** [b], **Sylvia SERFATY** [c]

[a] *Laboratoire d'Analyse Numérique, B.C.187, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France*
[b] *Université François Rabelais, Département de Mathématiques, Parc Grandmont, 37200 Tours, France*
[c] *École Normale Supérieure de Cachan, 61 avenue du Président Wilson, 94235 Cachan Cedex, France*

Received 25 May 2000

ABSTRACT. – We study the Ginzburg–Landau energy of superconductors with a term a_ε modelling the pinning of vortices by impurities in the limit of a large Ginzburg–Landau parameter $\kappa = 1/\varepsilon$. The function a_ε is oscillating between $1/2$ and 1 with a scale which may tend to 0 as κ tends to infinity.

Our aim is to understand that in the large κ limit, stable configurations should correspond to vortices pinned at the minimum of a_ε and to derive the limiting homogenized free-boundary problem which arises for the magnetic field in replacement of the London equation. The method and techniques that we use are inspired from those of Sandier and Serfaty, *Annales Scientifiques de l'ENS* (to appear) (in which the case $a_\varepsilon \equiv 1$ was treated) and based on energy estimates, convergence of measures and construction of approximate solutions. Because of the term $a_\varepsilon(x)$ in the equations, we also need homogenization theory to describe the fact that the impurities, hence the vortices, form a homogenized medium in the material.

Keywords: Superconductivity, Ginzburg–Landau, Pinning, Homogenization

1. Introduction

Superconducting materials have the property of expelling an applied magnetic field. In fact, the behaviour of a superconducting sample varies according to the value of the applied field and the value of the Ginzburg–Landau parameter κ which is characteristic of the material. When κ is large, the superconductors are known as type-II and display vortex patterns for intermediate fields: for high magnetic fields, the material is normal and the magnetic field penetrates into the sample, for low fields, the material is superconducting, that is the magnetic field is expelled from the sample and for intermediate fields, there are vortices. The vortex state is a state where the superconducting and the normal phases coexist: at the center of the vortex, the material is normal and the vortex is circled by a superconducting current carrying a quantized amount of magnetic flux. The motion of vortices generates an electric field hence energy-dissipation. In order to have the desired property of dissipation-free current flow, the vortices have to be held fixed or pinned. In practice, attempts are made to pin vortices either by varying the thickness of the material or by introducing impurities or normal inclusions. Sufficiently strong pinning is

E-mail addresses: Amandine.Aftalion@ens.fr (A. Aftalion), sandier@univ-tours.fr (E. Sandier), serfaty@cmla.ens-cachan.fr (S. Serfaty).

necessary for functional superconductors capable of sustaining strong currents and high magnetic fields. The new high-temperature (high T_c) superconductors are strongly type-II superconductors, that is their phenomenology is dominated by the presence and properties of vortices when an exterior magnetic field is applied. The pinning problem is particularly intricate in high-T_c superconductors where it depends on specific structures such as layering and structural defects.

In this paper, we will be concerned with the case where the vortices are pinned by impurities in the framework of the Ginzburg–Landau model. We will study the behaviour of global minimizers of the Ginzburg–Landau energy when a term modelling the pinning of vortices by impurities is added, in the limit of a large Ginzburg–Landau parameter κ, which describes extreme type-II materials.

1.1. The Ginzburg–Landau model with a pinning term

Recall that in the framework of the Ginzburg–Landau theory (see [33] for more details), the state of the material is completely described by a vector potential A and a complex-valued function u, which can be thought of as a wave-function of the superconducting electrons, and is nondimensionalized such that $|u| \leqslant 1$. The type of material is characterized by the Ginzburg–Landau parameter κ and in the case of type II, κ is large so that we define $\varepsilon = 1/\kappa$, which will be small. The energy is the following:

$$J_\varepsilon(u, A) = \frac{1}{2}\int_\Omega \left|(\nabla - \mathrm{i}A)u\right|^2 + \frac{1}{2\varepsilon^2}\left(a_\varepsilon(x) - |u|^2\right)^2 + |h - h_{\mathrm{ex}}|^2. \tag{1.1}$$

Here, Ω is the domain occupied by the superconductor, $h = \operatorname{curl} A$ is the magnetic field and h_{ex} is the exterior magnetic field which is constant in our problem. A common simplification is to restrict to a two-dimensional problem corresponding to an infinite cylindrical domain of section $\Omega \subset \mathbb{R}^2$ (smooth and simply connected), for an applied field parallel to the axis of the cylinder. Then $A : \Omega \mapsto \mathbb{R}^2$, h is real-valued and all the quantities are translation-invariant.

The energy J_ε that we are going to study here is slightly different from the classical Ginzburg–Landau energy in the sense that there is a term penalizing the variations of the order parameter u. We denote this function by $a_\varepsilon(x)$. In the case originally studied by Ginzburg and Landau, $a_\varepsilon \equiv 1$. In this paper, a typical example for a_ε would be to oscillate between $1/2$ and 1 in the domain, with a typical scale η which may tend to 0 with ε. The minima of a_ε correspond to the impurities in the material. Hence it is expected that these minima will be the pinning sites for the vortices.

The modified Ginzburg–Landau functional (1.1) was first written down by Likharev [20]. Then, this model has been used and developed in [11] and [10]. Review articles on the topic include [4,8,9] and [24]. Computational evidence that the vortices are attracted by the impurities, that is the points of minimum of $a_\varepsilon(x)$ can be found in [10] or [16].

In this paper, we want to address the question of how the term a_ε will modify the properties of the superconductor in the presence of an exterior magnetic field. Recall that in the case where $a_\varepsilon \equiv 1$ and there is no magnetic field, Bethuel, Brezis, Helein [3] studied a functional with a degree boundary condition and provided the understanding of vortices and their energetical cost. Then, various authors [1,2,19] have introduced a fixed weight function (independent of ε) in front of the gradient term of the energy studied by [3]. This is to model variable tickness pinning and is very different from our problem. The method and techniques that we are going to use here are inspired from those of [28] (in which the case $a_\varepsilon \equiv 1$ was treated) and based on energy estimates, convergence of measures and construction of approximate solutions. Because of the term $a_\varepsilon(x)$ in the equations, which can be a rapidly oscillating function, we will also need homogenization theory ([13,17,23]) to describe the fact that the impurities, hence the vortices, form a homogenized medium in the material.

1.2. The equation for the magnetic field

The Ginzburg–Landau equations associated to the functional (1.1) when minimizing for $\{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)\}$ are:

$$\text{(G.L.)} \qquad \begin{cases} -(\nabla - \mathrm{i}A)^2 u = \dfrac{1}{\varepsilon^2} u\big(a_\varepsilon(x) - |u|^2\big), \\ -\nabla^\perp h = \big\langle \mathrm{i}u, (\nabla - iA)u\big\rangle, \end{cases}$$

with the boundary conditions:

$$\begin{cases} h = h_{\mathrm{ex}} & \text{on } \partial\Omega \\ (\nabla u - \mathrm{i}Au) \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\nabla^\perp$ denotes $(-\partial_{x_2}, \partial_{x_1})$, and $\langle z, w\rangle = \mathrm{Re}(z\overline{w})$ for z, w in $\mathbb{C}$. Recall that the problem is invariant under the gauge transformations

$$\begin{cases} u \to u\,\mathrm{e}^{\mathrm{i}\Phi}, \\ A \to A + \nabla\Phi, \end{cases}$$

where $\Phi \in H^2(\Omega, \mathbb{R})$. Physically meaningful quantities are gauge invariant. These include the energy J_ε, the magnetic field h and the superconducting current $j = \langle \mathrm{i}u, (\nabla - \mathrm{i}A)u\rangle$.

Let us describe the properties of a superconductor. These phenomena are described for instance in [33]. The state of the material depends on the applied field h_{ex}. In the absence of pinning, that is when $a_\varepsilon \equiv 1$, there are two critical fields H_{c_1} and H_{c_2} for which a phase transition occurs. Above $H_{c_2} = \mathrm{O}(1/\varepsilon^2)$, superconductivity is destroyed and the material is in the normal phase $(u \equiv 0, h \equiv h_{\mathrm{ex}})$. Below $H_{c_1} = \mathrm{O}(|\log\varepsilon|)$, the material is superconducting everywhere, that is $|u| \sim 1$. This is the Meissner phase characterized by complete expulsion of the magnetic field: in the limit when ε goes to zero, the magnetic field satisfies the London equation:

$$\text{(1.2)} \qquad \begin{cases} -\Delta h + h = 0 & \text{in } \Omega, \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega. \end{cases}$$

Between H_{c_1} and H_{c_2}, the material is in the mixed phase defined by the coexistence of the normal and superconducting phases in the form of vortex filaments: the magnetic field penetrates into the material in the form of flux lines at the center of which u vanishes. The induced magnetic field approximately satisfies:

$$\text{(1.3)} \qquad \begin{cases} -\Delta h + h = 2\pi \sum_i d_i \delta_{p_i} & \text{in } \Omega, \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega, \end{cases}$$

where the p_i's are the centers of the vortices, and the d_i's their degrees, that is the topological degree of the map $u/|u|$. These filaments are of characteristic size ε. They are surrounded by a superconducting region in which $|u| \sim 1$. In order to minimize their repulsion, the flux lines form a triangular lattice, called the “Abrikosov lattice”. With increasing fields, the density of flux lines increase until the vortices overlap and H_{c_2} is reached. The generation of vortices by the external field has been mathematically studied very recently in [29–31,25–27].

In [27], it is proved among other things that, in the limit when ε tends to 0, equation (1.3) is replaced by

$$-\Delta h_* + h_* = \mu_*, \tag{1.4}$$

where μ_* is the density of vortices in units of h_{ex} and $h_* = h/h_{\mathrm{ex}}$. The measure μ_* is supported in an inner region ω depending on the value of h_{ex} and is of uniform density in ω.

Our aim is to give a rigorous proof that in the small ε limit, stable configurations should correspond to vortices pinned at the minimum of a_ε and to derive the limiting homogenized free-boundary problem which arises for the magnetic field in replacement of the London equation (1.4).

Using the second equation in (G.L.), we notice that the energy can be rewritten

$$J_\varepsilon(u,A) = \frac{1}{2}\int_\Omega \frac{1}{|u|^2}|\nabla h|^2 + |h - h_{\mathrm{ex}}|^2 + \frac{1}{2}\int_\Omega \bigl|\nabla |u|\bigr|^2 + \frac{1}{2\varepsilon^2}\bigl(a_\varepsilon(x) - |u|^2\bigr)^2. \tag{1.5}$$

We will show that for a sequence of minimizers $(u_\varepsilon, A_\varepsilon)$, the second integral in (1.5) is negligible. Then, when ε tends to 0, $|u|^2 \sim a_\varepsilon(x)$ outside the vortices, and our main result will state that $h_\varepsilon = \operatorname{curl} A_\varepsilon$ satisfies roughly the following equivalent of (1.3) in the case of pinning:

$$-\operatorname{div}\left(\frac{1}{a_\varepsilon}\nabla h_\varepsilon\right) + h_\varepsilon = 2\pi \sum_i d_i \delta_{p_i}. \tag{1.6}$$

The existence of pinning will modify the locations p_i of the vortices and the value of H_{c_1}.

Since a_ε is a rapidly oscillating function describing impurities, the framework for passing to the limit when ε is small is that of homogenization theory. When passing to the limit in (1.6), we obtain a different limiting operator from (1.4), that is

$$-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla h_*) + h_* = \mu_*, \tag{1.7}$$

where μ_* is a positive measure which is supported in an inner domain ω_Λ and $\boldsymbol{\mathcal{A}}_0$ is the homogenized limit of the matrix $\boldsymbol{\mathcal{A}}_\varepsilon = \frac{1}{a_\varepsilon}\boldsymbol{\mathcal{I}}$ in the sense of H-convergence, see definition below.

DEFINITION 1. – *We say that the family of* 2×2 *matrices* $\boldsymbol{\mathcal{A}}_\varepsilon$ *H-converges to* $\boldsymbol{\mathcal{A}}_0$ *when* ε *tends to* 0*, if and only if, for any* f *in* $H^{-1}(\Omega)$*, the solution* v_ε *in* $H_0^1(\Omega)$ *of*

$$-\operatorname{div}(\boldsymbol{\mathcal{A}}_\varepsilon \nabla v_\varepsilon) + v_\varepsilon = f$$

satisfies

$$v_\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega),$$
$$\boldsymbol{\mathcal{A}}_\varepsilon \nabla v_\varepsilon \rightharpoonup \boldsymbol{\mathcal{A}}_0 \nabla v_0 \quad \text{weakly in } \bigl(L^2(\Omega)\bigr)^2,$$

where v_0 *is the* $H_0^1(\Omega)$ *solution of*

$$-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla v_0) + v_0 = f.$$

We refer to the work of Murat and Tartar [23] for more details on the notion of H-convergence; one can also see [13,17]. In the following, we will always let $\boldsymbol{\mathcal{A}}_\varepsilon = \frac{1}{a_\varepsilon}\boldsymbol{\mathcal{I}}$. Then $\boldsymbol{\mathcal{A}}_0$ is also a diagonal matrix. In the general case, the computation of $\boldsymbol{\mathcal{A}}_0$ is hard and not always known, see [17] for examples. But in some simple cases, this definition allows to compute $\mathcal{A}_0$. For instance, if $a_\varepsilon(x) = a(x/\varepsilon)$, and $a(x) = a_1(x_1)a_2(x_2)$ where a_1 and a_2 are periodic, then

$$\mathcal{A}_0 = \operatorname{diag}\left(\frac{1}{a_1^0}, \frac{1}{a_2^0}\right), \quad \text{with } a_i^0 = \overline{\overline{a_i}\left(\frac{1}{a_j}\right)},$$

where $\overline{a_i}$ denotes the mean of a_i over a period (see [17]). Note that even though the sequence a_ε has no pointwise limit, the limiting problem and $\mathcal{A}_0$ are well defined.

An important property of H-convergence (see [23]) is that if the sequence a_ε is bounded from below and above by positive constants independent of ε, then there exists a subsequence $\mathcal{A}_{\varepsilon'}$ and a matrix $\mathcal{A}_0$ for which $\mathcal{A}_{\varepsilon'}$ H-converges to $\mathcal{A}_0$. For us, it will imply in the following that up to the extraction of a subsequence, the family $\mathcal{A}_\varepsilon$ H-converges to some limit $\mathcal{A}_0$, thus leading to the limiting problem (1.7).

1.3. Main results

Let us now state our hypotheses and results. We assume that h_{ex} is a function of ε and that the following limit exists and is finite:

$$\Lambda = \lim_{\varepsilon\to 0} \frac{|\log\varepsilon|}{h_{\text{ex}}(\varepsilon)}. \tag{1.8}$$

Moreover, we make the following hypotheses on the function $a_\varepsilon(x)$:

(H1) There exists a constant $b_0 > 0$ such that $b_0 \leqslant a_\varepsilon(x) \leqslant 1$.

(H2) There exist a constant C and a sequence $\eta(\varepsilon)$ (which may tend to 0 with ε) such that $1/\eta(\varepsilon) \ll h_{\text{ex}}$ and $|\nabla a_\varepsilon| \leqslant \frac{C}{\eta(\varepsilon)}$.

(H3) There exist a continuous function $b(x)$ and a nonnegative functions $\beta_\varepsilon(x)$ such that $a_\varepsilon(x) = b(x) + \beta_\varepsilon(x)$ and for any $\varepsilon > 0$ and any $x \in \Omega$, $\min_{B(x,\delta(\varepsilon))} \beta_\varepsilon = 0$, where

$$\delta(\varepsilon) \ll \frac{1}{(\log\ |\log\varepsilon|)^{1/2}}.$$

(H4) The family of matrices $\mathcal{A}_\varepsilon$ H-converges to $\mathcal{A}_0$.

Note that, as we mentioned earlier, it follows from hypothesis (H1) and the compactness of the set of matrices bounded from above and below that there exists a subsequence of $\mathcal{A}_\varepsilon$ which H-converges to $\mathcal{A}_0$ [23]. Our hypothesis (H4) is there to restrict to this subsequence for ease of notation and to impose that the whole sequence converges. Moreover, (H2) means that a_ε can be a constant independent of ε but can also oscillate very quickly with ε (but not too quickly, i.e. not quicker than h_{ex}). Note that in the case where a_ε does not depend on ε, then $\mathcal{A}_\varepsilon = \mathcal{A}_0$ is constant.

Let us emphasize that because $\beta_\varepsilon \geqslant 0$, b can be thought of as the lower envelope of a_ε and the local minima of a_ε are the local minima of b. Hence b will be related to the pinning sites of vortices and the oscillations of a_ε are those of β_ε. Moreover, the hypotheses imply that $b \geqslant b_0$.

First, let us state the result concerning the limiting problem (1.7). We relate h_* and μ_* to the minimum of a variational problem. Let $\mathcal{M}$ denote the space of Radon measures in Ω.

THEOREM 1. – *Let us assume that* (H1) *to* (H4) *are satisfied. Let us define for any* $\Lambda \geqslant 0$,

$$E(f) = \frac{\Lambda}{2}\int_\Omega b(x)\left|-\operatorname{div}(\mathcal{A}_0\nabla f) + f\right| + \frac{1}{2}\int_\Omega \nabla f\cdot\mathcal{A}_0\nabla f + |f-1|^2, \tag{1.9}$$

over

$$V=\left\{f \text{ such that } f-1\in H_0^1(\Omega),\ \text{and } -\operatorname{div}(\mathcal{A}_0\nabla f)+f\in\mathcal{M}\right\}.$$

The minimizer h_* of E over V exists and is unique. It satisfies:

$$(\mathrm{P})\ \begin{cases} h_*-1\in H_0^1(\Omega),\\ \mu_*=-\operatorname{div}(\mathcal{A}_0\nabla h_*)+h_*\in\mathcal{M},\\ h_*\geqslant 1-\dfrac{\Lambda b}{2} \quad \text{in } \Omega,\\ \mu_*\left(h_*-\left(1-\dfrac{\Lambda b}{2}\right)\right)=0 \quad \text{in } \Omega.\end{cases}$$

Moreover $\mu_*\geqslant 0$ and $\mu_*\in H^{-1}(\Omega)$.

Problem (P) is a free-boundary problem, called in the literature an “obstacle problem” (see [18]). Another way of considering problem (P) is to define the subset of Ω

$$\omega_\Lambda=\{x\in\Omega,\ \text{such that } h_*=1-\Lambda b/2\}. \tag{1.10}$$

Then $\mu_*=0$ in $\Omega\setminus\overline{\omega_\Lambda}$, and $h_*=1-\Lambda b/2$ in ω_Λ, $\partial\omega_\Lambda$ being called the “free-boundary”, because ω_Λ is unknown and uniquely determined by the set of equations (P).

Note that if $\mathcal{A}_0$ and b are smooth enough then h_* is $C^{1,\alpha}$ $(\alpha<1)$, μ_* is in L^∞, the free-boundary $\partial\omega_\Lambda$ is regular for almost every Λ (see [5]) and then we can write

$$\mu_*=1-\frac{\Lambda b}{2}+\frac{\Lambda}{2}\operatorname{div}(\mathcal{A}_0\nabla b)\quad \text{in } \omega_\Lambda.$$

Once we have proved Theorem 1 concerning the limiting problem, we can get convergence for any sequence of minimizers $(u_\varepsilon,A_\varepsilon)$ of the energy $J_\varepsilon(u_\varepsilon,A_\varepsilon)$ to $E(h_*)$ in a sense similar to Γ-convergence.

THEOREM 2. – *Let us assume that* (1.8) *and* (H1) *to* (H4) *are satisfied. Let $(u_\varepsilon,A_\varepsilon)$ be a family of minimizers of J_ε, and $h_\varepsilon=\operatorname{curl}A_\varepsilon$ the associated magnetic field. Then, as ε tends to* 0,

$$\frac{h_\varepsilon}{h_{\mathrm{ex}}}\to h_* \quad \text{weakly in } H^1(\Omega),$$

where h_* is the minimizer of E. Moreover,

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon(u_\varepsilon,A_\varepsilon)}{h_{\mathrm{ex}}^2}=E(h_*)=\frac{\Lambda}{2}\int_\Omega b|\mu_*|+\frac12\int_\Omega \nabla h_*\cdot\mathcal{A}_0\nabla h_*+|h_*-1|^2, \tag{1.11}$$

$$\frac{|\nabla h_\varepsilon|^2}{h_{\mathrm{ex}}^2 a_\varepsilon}\to \nabla h_*\cdot\mathcal{A}_0\nabla h_*+\Lambda b\mu_*, \quad \text{in the sense of measures.} \tag{1.12}$$

One can easily notice that if $\Lambda=0$ (i.e. if $h_{\mathrm{ex}}\gg|\log\varepsilon|$), the solution of (P) is $h_*=1$, and $E(h_*)=0$. In this case, Theorem 2 asserts that:

$$\frac{h_\varepsilon}{h_{\mathrm{ex}}}\to 1 \quad \text{strongly in } H^1, \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{\min J_\varepsilon}{h_{\mathrm{ex}}^2}=0.$$

The proof of Theorem 2 is the main part of the paper (see Section 1.6 for a sketch).

1.4. The case $\Lambda > 0$

Let us now present some stronger results in the case where Λ is positive, i.e. h_{ex} is of the order of $|\log\varepsilon|$. The first issue is to determine mathematically the location of vortices. From the physics, we know that vortices are the zeroes of u_ε with non-zero winding number. Instead of defining vortices, we isolate them in disjoint vortex balls covering the set where $|u_\varepsilon|$ is small. The centers of these balls can be thought of as being the centers of the vortices. This method of definition was first introduced by [3]. Here, we use the construction due to E. Sandier [25].

PROPOSITION 1.1. – *Let us assume that $\Lambda > 0$ and that* (H1) *to* (H4) *are satisfied, then there exists ε_0 such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ is a minimizer of J_ε, there exists a family of balls of disjoint closures* (*depending on* ε) $(B_i)_{i\in I_\varepsilon} = (B(p_i, r_i))_{i\in I_\varepsilon}$ *satisfying*:

$$\left\{x \in \Omega, \left|\sqrt{a_\varepsilon(x)} - |u_\varepsilon(x)|\right| \geqslant \frac{1}{|\log\varepsilon|}\right\} \subset \bigcup_{i\in I_\varepsilon} B(p_i, r_i), \tag{1.13}$$

$$\sum_{i\in I_\varepsilon} r_i \leqslant \frac{1}{\mathrm{e}^{\sqrt{|\log\varepsilon|}}}, \tag{1.14}$$

$$\frac{1}{2}\int_{B_i} \frac{|\nabla h_\varepsilon|^2}{|u|^2} \geqslant \pi b(p_i)|d_i||\log\varepsilon|\bigl(1 - \mathrm{o}(1)\bigr), \tag{1.15}$$

where $h_\varepsilon = \operatorname{curl} A_\varepsilon$, *and* $d_i = \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i)$ *if* $\overline{B_i} \subset \Omega$, *and* 0 *otherwise.*

This proposition will be proved at the beginning of Section 2. Here is the meaning of the different inequalities: (1.13) locates the set where $|u_\varepsilon|$ differs from a_ε, which is contained in a union of disjoint balls; these balls represent the vortices or clusters of vortices. (1.14) gives a control on the size of the balls and (1.15) gives a lower bound on the energy, which is the contribution of vortices according to their degree d_i and their location p_i, appearing through the value $b(p_i)$. As opposed to the case of $a_\varepsilon \equiv 1$ (see [28]), the least energy is attained for p_i at the minimum of b.

Using this proposition, Theorem 1 can be made more precise:

THEOREM 3. – *Let us assume that $\Lambda > 0$ and that* (H1) *to* (H4) *are satisfied. For any balls* $B(p_i, r_i)$ *and integers d_i which satisfy* (1.13)–(1.15), *then*

$$\lim_{\varepsilon\to 0} \frac{2\pi}{h_{\mathrm{ex}}} \sum_{i\in I_\varepsilon} d_i a_\varepsilon(p_i) = \int_\Omega b|\mu_*|, \tag{1.16}$$

$$\frac{2\pi}{h_{\mathrm{ex}}} \sum_{i\in I_\varepsilon} d_i \delta_{p_i} \underset{\varepsilon\to 0}{\longrightarrow} \mu_*, \tag{1.17}$$

$$\frac{2\pi}{h_{\mathrm{ex}}} \sum_{i\in I_\varepsilon} |d_i| \delta_{p_i} \underset{\varepsilon\to 0}{\longrightarrow} \mu_*, \tag{1.18}$$

in the sense of measures, where

$$\mu_* = -\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla h_*) + h_*.$$

1.5. Physical interpretations and consequences

Our results show that $h_* h_{\mathrm{ex}}$ is a good approximation of h_ε and that, in the limit $\varepsilon \to 0$, the vortices are scattered in an inner region ω_Λ with density μ_*, where $h_* = 1 - \Lambda b(x)/2$. In the outer region $\Omega \setminus \overline{\omega_\Lambda}$, there are no vortices and h_* satisfies $-\operatorname{div}(\mathcal{A}_0 \nabla h_*) + h_* = 0$. Unlike the case $a_\varepsilon \equiv 1$, the vortex-density in $\overline{\omega_\Lambda}$ is non-uniform in general. Moreover, as Λ decreases, the vortex-region first appears at the minimum of ψ as defined by problem (1.19) below: as in [28], we can derive a necessary and sufficient condition for ω_Λ to be nonempty.

PROPOSITION 1.2. – *Let ψ be the solution of*

$$\begin{cases} -\operatorname{div}(\mathcal{A}_0 \nabla \psi) + \psi = -1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

then

$$\omega_\Lambda \neq \varnothing \Leftrightarrow \lim_{\varepsilon \to 0} \frac{h_{\text{ex}}}{|\log \varepsilon|} \geqslant \frac{1}{2 \max |\psi|}.$$

If we define H_{c_1} as the field such that for $h_{\text{ex}} \leqslant H_{c_1}$, the minimizer of the energy has no vortex (i.e. $|u| \geqslant b_0/2$) and for $h_{\text{ex}} \geqslant H_{c_1}$, there exists a minimizer with vortices; then Proposition 1.2 gives a hint that

$$H_{c_1} \simeq \frac{|\log \varepsilon|}{2 \max |\psi|}.$$

Thus the presence of pinning modifies the values of the first critical field (see [29,26] for the case without pinning). In fact, we could adjust the proof of [26] to obtain: there exists $k_\varepsilon = \mathrm{O}(|\log|\log \varepsilon||)$ such that for ε small enough and

$$h_{\text{ex}} \leqslant \frac{|\log \varepsilon|}{2 \max |\psi|} - k_\varepsilon$$

then any minimizer has no vortex.

Furthermore, the position of the minimum of ψ depends on the pinning potential $a_\varepsilon(x)$. As Λ further decreases, corresponding to h_{ex} increasing, the vortex-region ω_Λ grows, until, for $\Lambda = 0$ ($h_{\text{ex}} \gg |\log \varepsilon|$), $\omega_\Lambda = \Omega$. At this point there are so many vortices that the macroscopic density of vortices and the induced magnetic field are no longer influenced by a_ε. In other words, the strength of flux pinning is 0 for $h_{\text{ex}} \gg |\log \varepsilon|$.

In the case where $a_\varepsilon(x) = a(x)$ is independent of ε, $a(x) = b(x)$ and $\mathcal{A}_0 = a^{-1}\mathcal{I}$. Hence the limiting problem is a London equation with weight. We would like to point out that it is natural to define a vortex velocity by $v = \frac{1}{|u|^2} \nabla h$ (see [15]). In particular

$$v_* = \frac{1}{a} \nabla h_*$$

can be defined as a limiting velocity (per unit of h_{ex}). Note that in ω_Λ, since $h_* = 1 - \frac{1}{2}\Lambda a$, then $v_* = -\frac{1}{2}\Lambda \nabla \log a$. It implies that when a is constant, $v_* = 0$ and there is no mean current in the vortex region. But when a varies spatially, there is a nonzero limiting mean current and a nonzero limiting velocity v_*. Hence $v \simeq h_{\text{ex}} v_*$ that is $\frac{1}{2} \log \kappa \nabla \log a$. This is the result of Chapman and Richardson [11] in the case where the three-dimensional vortex line has no curvature. They describe the phenomenon saying that the variation in a acts as a pinning potential.

When $\Lambda = 0$, the velocity v_* is zero as well. Decreasing Λ means increasing the field. So when a varies spatially, there is a critical exterior magnetic field above which the pinning potential has no role and the current is destroyed.

In the general case where a_ε depends on ε, it would be interesting to prove a convergence of the mean vortex velocity $v_\varepsilon = \frac{1}{|u_\varepsilon|^2} \nabla h_\varepsilon$. Still, one can observe two different effects coming from the presence of pinning in the term $|\nabla h_\varepsilon|^2/a_\varepsilon$ and resulting in the energy $E(h_*)$ in the homogenization process:

– One effect is related to the concentration of energy in the vortices and the location of the vortices. It appears through the term

$$\frac{\Lambda}{2}\int_\Omega b|\mu_*|$$

in the limiting energy E. This term is smaller if μ_* is non-zero at points where b is minimal. (1.16) implies that vortices go to points where $\beta_\varepsilon = 0$. These points will be called pinning sites in the following. Because $\delta(\varepsilon)$ tends to 0, the number of such points is big. The effect on the position of vortices is to see b and the minima of b. Moreover, since (1.17) and (1.18) have the same limit, it means that vortices tend to have positive degrees.

If b does not depend on x then h_* and μ_* are constant in ω_Λ, and there is no change for the location of vortices from the case $a_\varepsilon \equiv 1$. On the other hand, if b is non-uniform, then ∇h_* is non-constant in ω_Λ and there is a pinning current. If for example the domain is a disc and the minima of b, that is the impurities, are located at sites different from the center of the disc, one expects that vortices, or the vortex-region ω_Λ will be closer to the minima of b, but it seems difficult to give a rigorous proof of this qualitative fact.

– The other effect is due to the rapid oscillations of a_ε with ε and comes from the energy outside the vortices, converging to the homogenized term

$$\frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2$$

in E. It changes the equation for the magnetic field h from the usual London equation. If $\beta_\varepsilon \neq 0$, then the homogenization effect can be anisotropic. The size $\delta(\varepsilon)$ (which can be related to η if β_ε is not identically 0) cannot be taken bigger than in (H3), otherwise each pinning site would be too large and the vortices could push one another outside the pinning site.

Let us also point out that we cannot allow stronger oscillations of a_ε than in (H2), because the second integral in (1.5) would become the dominant term. It would be interesting to investigate what happens if (H2)–(H3) are relaxed.

1.6. Main steps of the proof

Let us now state the two steps of the proof of Theorem 2. It is obtained as in [28] by getting first a lower bound on the energy, Proposition 1.3, proved in Section 2, and then an upper bound, Proposition 1.4, proved in Section 3.

PROPOSITION 1.3. – *Let us assume that $\Lambda > 0$ and that* (H1) *to* (H4) *are satisfied. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of J_ε. Then*

$$\liminf_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2} J_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant \frac{\Lambda}{2}\int_\Omega b|\mu_*| + \frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2, \tag{1.20}$$

where h_* is the solution of (P).

PROPOSITION 1.4. – *Let us assume that $\Lambda > 0$ and that* (H1) *to* (H4) *are satisfied. Let μ be a positive Radon measure, and let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of J_ε. Then*

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2} J_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant \frac{\Lambda}{2}\int_\Omega b\,\mathrm{d}\mu + \frac{1}{2}\int_\Omega \nabla h\cdot \boldsymbol{\mathcal{A}}_0 \nabla h + |h-1|^2, \tag{1.21}$$

where h is the solution of

$$\begin{cases} -\operatorname{div}(\boldsymbol{\mathcal{A}}_0\nabla h) + h = \mu & \text{in } \Omega,\\ h = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

Section 2 is devoted to the proof of Proposition 1.3. Let $(u_\varepsilon, A_\varepsilon)$ be a sequence of minimizers and $h_\varepsilon = \operatorname{curl} A_\varepsilon$. The energy $J_\varepsilon(u_\varepsilon, A_\varepsilon)$ gives two contributions: inside the vortex balls and outside. Thus, first we prove Proposition 1.1 where the vortex balls B_i with centers p_i are constructed and where the vortex energy is bounded from below. We define

$$\mu_\varepsilon = \frac{2\pi}{h_{\mathrm{ex}}} \sum_{i\in I_\varepsilon} d_i \delta_{p_i}. \tag{1.23}$$

Then, Proposition 1.1 implies

$$\frac{1}{h_{\mathrm{ex}}^2} \int_{\bigcup_{i\in I} B_i} \frac{1}{|u|^2} |\nabla h_\varepsilon|^2 \geqslant \frac{|\log\varepsilon|}{h_{\mathrm{ex}}} \int_\Omega b|\mu_\varepsilon|, \tag{1.24}$$

which gives the lower bound inside the vortex balls. The next step is to pass to the limit in the energy outside the vortex balls. Letting h_0 be the weak H^1 limit of $h_\varepsilon/h_{\mathrm{ex}}$, we obtain the following, which is similar to a standard result in homogenization theory

$$\liminf_{\varepsilon\to 0} \int_{\Omega\setminus\cup_i B_i} \frac{|\nabla h|^2}{a_\varepsilon h_{\mathrm{ex}}^2} \geqslant \int_\Omega \nabla h_0 \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_0. \tag{1.25}$$

This requires to introduce an auxiliary problem before applying the homogenization theory result and it works because the vortex balls are small and thus can be taken out of the first integral.

Finally we derive from the Ginzburg–Landau equations the crucial fact that h_ε satisfies

$$\frac{1}{h_{\mathrm{ex}}}\left(-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a_\varepsilon}\right) + h_\varepsilon \right) = \mu_\varepsilon + \psi_\varepsilon, \tag{1.26}$$

where ψ_ε tends to 0 and μ_ε defined in (1.23) tends to some μ_0, both convergences being strong in $W^{-1,r}$ for $r < 2$. The notion of H-convergence and a priori estimates allow us to pass to the limit in (1.26) in order to get that the weak H^1 limit of $h_\varepsilon/h_{\mathrm{ex}}$, that we call h_0, solves

$$-\operatorname{div}(\boldsymbol{\mathcal{A}}_0\nabla h_0) + h_0 = \mu_0. \tag{1.27}$$

Combining the lower bounds of the energy inside and outside the vortex balls (1.24)–(1.25), we find

$$\liminf_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2} J_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant E(h_0) \geqslant E(h_*).$$

The last inequality is true because (1.27) implies that h_0 is in V.

Section 3 is devoted to the proof of Proposition 1.4. The proof holds for any positive Radon measure μ. We apply it to μ_* to get that:

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\text{ex}}^2} J_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant E(h_*),$$

which will imply the desired results of convergence.

The upper bound of Proposition 1.4 is obtained by constructing test configurations as follows. First, given a positive Radon measure μ, we construct approximate measures μ_ε which converge weakly to μ:

$$\mu_\varepsilon = \frac{1}{h_{\text{ex}}} \sum_{i=1}^{n_\varepsilon} \mu_\varepsilon^i,$$

where μ_ε^i is the line element on the circle $\partial B(p_\varepsilon^i, \varepsilon)$ normalized so that $\mu_\varepsilon^i(\partial B(p_\varepsilon^i, \varepsilon)) = 2\pi$. The measure μ_ε describes the vortices of our test-configuration. The difficulty is to choose the points p_ε^i satisfying a number of properties. We tile Ω with squares K of size $\delta(\varepsilon)$. In each square, there is at least a point p_K where $\beta_\varepsilon = 0$. We choose n_K points p_ε^i regularly scattered around p_K in a ball of radius $1/h_{\text{ex}}$. The number n_K is chosen depending on $\mu(K)$ so that μ_ε converge to μ. Once the vortices are constructed, the rest follows easily: the magnetic field h_ε is defined to be the solution of

$$\frac{1}{h_{\text{ex}}}\left(-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a_\varepsilon}\right) + h_\varepsilon\right) = \mu_\varepsilon. \tag{1.28}$$

Then, we are the able to construct a configuration $(u_\varepsilon, A_\varepsilon)$ such that $\operatorname{curl} A_\varepsilon = h_\varepsilon$ and u_ε has vortices at the points p_ε^i. Moreover, we obtain

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \approx \frac{1}{2}\int_\Omega \frac{1}{a_\varepsilon}|\nabla h_\varepsilon|^2 + |h_\varepsilon - 1|^2.$$

Finally we are able to show that

$$\limsup_{\varepsilon\to 0} \frac{1}{2h_{\text{ex}}^2}\int_\Omega \frac{1}{a_\varepsilon}|\nabla h_\varepsilon|^2 + |h_\varepsilon - 1|^2 \leqslant \frac{\Lambda}{2}\int_\Omega b \,\mathrm{d}\mu + \frac{1}{2}\int_\Omega \nabla h \cdot \boldsymbol{\mathcal{A}}_0 \nabla h + |h-1|^2,$$

where h solves $-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla h) + h = \mu$ and $h = 1$ on $\partial\Omega$.

2. Lower bound

In the following, we will denote $\nabla_A u = \nabla u - \mathrm{i}Au$. We will often drop the subscripts ε. We consider $(u_\varepsilon, A_\varepsilon)$ a family of minimizers of J_ε, thus a family of solutions of (G.L.). We can state a few a priori bounds. Firstly, by the maximum principle, $|u_\varepsilon| \leqslant \max a_\varepsilon \leqslant 1$. Secondly, by minimality, comparing with $(a_\varepsilon, 0)$, we get

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant J_\varepsilon(a_\varepsilon, 0).$$

But, by hypothesis (H2) on a_ε,

$$J_\varepsilon(a_\varepsilon,0)=\frac{1}{2}\int_\Omega |\nabla a_\varepsilon|^2+\mathrm{O}\big(h_{\mathrm{ex}}^2\big)\leqslant \frac{C}{\eta^2}+\mathrm{O}\big(h_{\mathrm{ex}}^2\big)\leqslant Ch_{\mathrm{ex}}^2.$$

Hence, we have the a-priori estimate

$$J_\varepsilon(u_\varepsilon,A_\varepsilon)\leqslant Ch_{\mathrm{ex}}^2. \tag{2.1}$$

In addition, by applying a gauge-transformation to $(u_\varepsilon, A_\varepsilon)$, we can choose the Coulomb-gauge $\operatorname{div} A_\varepsilon=0$ in Ω, with $A_\varepsilon.n=0$ on $\partial\Omega$. With this choice of gauge, we are easily lead (see [29, 26]) to the a priori bounds

$$\|A_\varepsilon\|_{L^\infty(\Omega)}\leqslant Ch_{\mathrm{ex}}, \tag{2.2}$$

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)}\leqslant Ch_{\mathrm{ex}}. \tag{2.3}$$

We begin with the proof of Proposition 1.1.

2.1. Proof of Proposition 1.1

Step 1. Let (u, A) be an energy-minimizer. Denoting $|u|$ by ρ, since $\int_\Omega |\nabla u|^2\geqslant \int_\Omega |\nabla\rho|^2$, we deduce from (2.1):

$$\int_\Omega |\nabla\rho|^2+\frac{1}{2\varepsilon^2}\big(\rho^2-a_\varepsilon\big)^2\leqslant Ch_{\mathrm{ex}}^2. \tag{2.4}$$

But,

$$\begin{aligned}\int_\Omega |\nabla\rho|^2&=\int_\Omega \big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|^2+\big|\nabla\sqrt{a_\varepsilon}\big|^2-2\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\cdot\nabla\sqrt{a_\varepsilon}\\&\geqslant\int_\Omega \big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|^2-2\big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|\big|\nabla\sqrt{a_\varepsilon}\big|.\end{aligned}$$

Hence, in view of (2.4),

$$\begin{aligned}\int_\Omega \big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|^2&\leqslant Ch_{\mathrm{ex}}^2+\big\|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big\|_{L^2}\big\|\nabla\sqrt{a_\varepsilon}\big\|_{L^2}\\&\leqslant Ch_{\mathrm{ex}}^2+\frac{C}{\eta(\varepsilon)}\big\|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big\|_{L^2},\end{aligned}$$

and, since $1/\eta(\varepsilon)\ll h_{\mathrm{ex}}$,

$$\int_\Omega \big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|^2\leqslant \max\bigg(Ch_{\mathrm{ex}}^2,\frac{C}{\eta^2}\bigg)\leqslant Ch_{\mathrm{ex}}^2.$$

In view of (2.4), we thus have:

$$\frac{1}{2}\int_\Omega \big|\nabla\big(\rho-\sqrt{a_\varepsilon}\big)\big|^2+\frac{1}{2\varepsilon^2}\big(a_\varepsilon-\rho^2\big)^2\leqslant Ch_{\mathrm{ex}}^2\leqslant C|\log\varepsilon|^2. \tag{2.5}$$

Step 2. For any $t \in \mathbb{R}$, let $\Omega_t = \{x \in \Omega / |\rho - \sqrt{a_\varepsilon}|(x) > t\}$ and $\gamma_t = \partial\Omega_t$. Applying the coarea formula and arguing as in Lemma IV.2 of [27],

$$C|\log\varepsilon|^2 \geqslant \int_\Omega \left|\nabla\left(\rho - \sqrt{a_\varepsilon}\right)\right|^2 + \frac{1}{2\varepsilon^2}\left(a_\varepsilon - \rho^2\right)^2 \geqslant \frac{C}{\varepsilon}\int_\Omega \left|\nabla\left(\rho - \sqrt{a_\varepsilon}\right)\right|\left|a_\varepsilon - \rho^2\right|$$
$$\geqslant \frac{C}{\varepsilon}\int_0^{+\infty} r(\gamma_t)t\,\mathrm{d}t.$$

Here, as in [27], $r(\gamma_t)$ is defined as the infimum over all finite coverings of γ_t by balls $B_1, \ldots, B_k$ of the sum $r_1 + \cdots + r_k$ where r_i is the radius of B_i. Combining the previous inequality with the mean-value theorem, we find that there exists a $t \in [0, \frac{1}{|\log\varepsilon|}]$ such that $r(\gamma_t) < C\varepsilon|\log\varepsilon|^3$.

Step 3. The next step is to construct the vortex-balls: starting from the chosen γ_t, covered by balls $B_1, \ldots, B_k$ (whose sum of the radii is controlled by $C\varepsilon|\log\varepsilon|^3$), we use the method of growing and merging of balls used in [25,27]: one needs to grow these balls B_i, keeping a suitable lower bound on the energy they contain, until the desired size is reached, with the desired lower bound. When some balls happen to intersect during the growth process, they are merged into a larger one. We refer the reader to [27], and here we only need to apply the result of Proposition IV.1 of [27] to A_ε and $v = \frac{u}{|u|} = \mathrm{e}^{\mathrm{i}\varphi}$ in $\Omega\setminus\Omega_t$, $\sigma = \mathrm{e}^{-\sqrt{|\log\varepsilon|}}$. We then obtain the existence of balls $B_i = B(p_i, r_i)$ such that (1.13) and (1.14) hold, and

$$\frac{1}{2}\int_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2 + \frac{1}{2}\int_{B_i} |h - h_{\mathrm{ex}}|^2 \geqslant \pi|d_i||\log\varepsilon|\big(1 - \mathrm{o}(1)\big), \tag{2.6}$$

with $d_i = \deg(u, \partial B_i)$ if $\overline{B_i} \subset \Omega$, and 0 otherwise. But we also have, from the Ginzburg–Landau equation $-\nabla^\perp h = \rho^2(\nabla\varphi - A)$, and from $\rho \leqslant 1$,

$$\int_\Omega |\nabla h|^2 = \int_\Omega \rho^4|\nabla\varphi - A|^2 \leqslant \int_\Omega |\nabla_A u|^2 \leqslant Ch_{\mathrm{ex}}^2,$$

hence

$$\int_{B_i} |h - h_{\mathrm{ex}}|^2 \leqslant Cr_i\|h - h_{\mathrm{ex}}\|^2_{L^4(\Omega)} \leqslant Cr_i\|h - h_{\mathrm{ex}}\|^2_{H^1(\Omega)}$$
$$\leqslant Ch_{\mathrm{ex}}^2\,\mathrm{e}^{-\sqrt{|\log\varepsilon|}} = \mathrm{o}(1).$$

Thus, (2.6) becomes

$$\frac{1}{2}\int_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2 \geqslant \pi|d_i||\log\varepsilon|\big(1 - \mathrm{o}(1)\big). \tag{2.7}$$

Now,

$$\frac{1}{2}\int_{B_i\setminus\Omega_t} |\nabla_A u|^2 \geqslant \frac{1}{2}\int_{B_i\setminus\Omega_t} \rho^2|\nabla\varphi - A|^2$$
$$\geqslant \frac{1}{2}\int_{B_i\setminus\Omega_t} a_\varepsilon|\nabla\varphi - A|^2 + \frac{1}{2}\int_{B_i\setminus\Omega_t} \left(\rho^2 - a_\varepsilon\right)|\nabla\varphi - A|^2$$

$$\geqslant \frac{1}{2}\Big(\min_{B_i} a_\varepsilon\Big) \int\limits_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2 - \frac{C}{|\log\varepsilon|} \int\limits_{B_i\setminus\Omega_t} |\nabla\varphi - A|^2,$$

where we have used (1.13). In view of (2.7),

$$\frac{1}{2}\int\limits_{B_i\setminus\Omega_t} |\nabla_A u|^2 \geqslant \pi\Big(\min_{B_i} a_\varepsilon\Big)|d_i||\log\varepsilon|\big(1-\mathrm{o}(1)\big).$$

So, using the hypotheses (H2) and (H3) on a_ε, we are led to the two following lower bounds

$$\frac{1}{2}\int\limits_{B_i\setminus\Omega_t} |\nabla_A u|^2 \geqslant \pi a_\varepsilon(p_i)|d_i||\log\varepsilon|\big(1-\mathrm{o}(1)\big), \tag{2.8}$$

$$\frac{1}{2}\int\limits_{B_i\setminus\Omega_t} |\nabla_A u|^2 \geqslant \pi b(p_i)|d_i||\log\varepsilon|\big(1-\mathrm{o}(1)\big). \tag{2.9}$$

This proves (1.15). □

2.2. Deriving the limiting equation

For any (p_i, d_i) satisfying (1.13)–(1.15), we can define

$$\mu_\varepsilon = \frac{2\pi}{h_{\mathrm{ex}}}\sum_{i\in I_\varepsilon} d_i\delta_{p_i}, \tag{2.10}$$

a measure of vorticity per unit of applied field. We will see that it remains a bounded family of measures.

LEMMA 2.1. – *If $\Lambda > 0$, and $(u_\varepsilon, A_\varepsilon)$ is a family of minimizers of J_ε with $h_\varepsilon = \operatorname{curl} A_\varepsilon$, we can extract a sequence $\varepsilon_n \to 0$ such that there exists $h_0 - 1 \in H_0^1(\Omega)$, and $\mu_0 \in \mathcal{M}$ with*

$$\frac{h_{\varepsilon_n}}{h_{\mathrm{ex}}} - 1 \rightharpoonup h_0 - 1 \quad \textit{in } H_0^1(\Omega),$$
$$\mu_{\varepsilon_n} \to \mu_0 \quad \textit{in the sense of measures.}$$

Proof. – As seen in the previous proof, since $(u_\varepsilon, A_\varepsilon)$ is a solution of the second Ginzburg–Landau equation

$$\int\limits_\Omega |\nabla h_\varepsilon|^2 \leqslant \int\limits_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 \leqslant C h_{\mathrm{ex}}^2$$

and

$$\int\limits_\Omega |h_\varepsilon - h_{\mathrm{ex}}|^2 \leqslant C h_{\mathrm{ex}}^2.$$

Hence, $h_\varepsilon/h_{\mathrm{ex}} - 1$ is bounded in $H_0^1(\Omega)$, and we can find a sequence $\varepsilon_n \to 0$ such that $h_{\varepsilon_n}/h_{\mathrm{ex}}$ converges weakly in H_0^1 to some $h_0 - 1$. On the other hand, from Proposition 1.1,

$$C h_{\mathrm{ex}}\frac{|\log\varepsilon|}{\Lambda} \geqslant J_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant \sum_{i\in I_\varepsilon} \pi|d_i|b(p_i)|\log\varepsilon|\big(1-\mathrm{o}(1)\big)$$
$$\geqslant b_0\sum_i \pi|d_i||\log\varepsilon|\big(1-\mathrm{o}(1)\big),$$

where b_0 is given by hypothesis (H1) on a_ε. Hence,

$$\frac{1}{2}\int_\Omega |\mu_{\varepsilon_n}| = \frac{\pi\sum_i |d_i|}{h_{\rm ex}} \leqslant C,$$

thus (μ_{ε_n}) is a bounded sequence of measures, and extracting again if necessary, we can assume that μ_{ε_n} converges to some μ_0 in the sense of measures. □

PROPOSITION 2.1. – *Let μ_0 and h_0 be the measures and fields defined in Lemma* 2.1. *Then there exists $r_0 < 2$ such that $\mu_0 \in W^{-1,r}(\Omega)$ $\forall r \in (r_0, 2)$, and h_0 is the unique solution in $W^{1,r}$ of*

$$\begin{cases} -\operatorname{div}(\mathcal{A}_0\nabla h_0) + h_0 = \mu_0 & \text{in } \Omega,\\ h_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

The proof of this proposition requires the following lemma, a slight refinement of the result stated in [26], Lemma II.3.

LEMMA 2.2. – *Under the hypotheses of Lemma* 2.1*, for any $q > 2$,*

$$\frac{1}{h_{\rm ex}}\operatorname{curl}\frac{(\mathrm{i}u_\varepsilon, \nabla u_\varepsilon)}{a_\varepsilon} - \mu_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{strongly in } \big(W_0^{1,q}(\Omega)\big)'.$$

Proof. – Denote $\tilde\Omega = \Omega\setminus\bigcup_i B_i$. On $\tilde\Omega$, $|u_\varepsilon| \geqslant b_0 > 0$ and $v_\varepsilon = u_\varepsilon/|u_\varepsilon|$ is well-defined. Let $q > 2$, and $\xi \in W_0^{1,q}$. We need to show that:

$$\left|\frac{1}{h_{\rm ex}}\int_\Omega \xi\operatorname{curl}\frac{(\mathrm{i}u_\varepsilon, \nabla u_\varepsilon)}{a_\varepsilon} - \frac{2\pi}{h_{\rm ex}}\sum_i d_i\xi(p_i)\right| \leqslant \mathrm{o}(1)\|\xi\|_{W_0^{1,q}(\Omega)}.$$

Dropping again some of the subscripts, we have

$$\frac{1}{h_{\rm ex}}\int_\Omega \xi\operatorname{curl}\frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon} = -\frac{1}{h_{\rm ex}}\int_\Omega \nabla^\perp\xi\cdot\frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon}. \tag{2.12}$$

Then, the method consists in splitting this integral into the integral over the vortex-balls (which is going to be negligible because the balls are small enough) and the integral over $\tilde\Omega$, the complement of the balls.

Step 1. We prove that

$$\left|\int_{\bigcup_i B_i} \frac{1}{h_{\rm ex}}\nabla^\perp\xi\cdot\frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon}\right| = \mathrm{o}(1)\|\nabla\xi\|_{L^q(\Omega)}. \tag{2.13}$$

Indeed, since $a_\varepsilon \geqslant b_0 > 0$,

$$\left|\int_{\bigcup_i B_i} \frac{1}{h_{\rm ex}}\nabla^\perp\xi\cdot\frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon}\right| \leqslant \frac{1}{b_0}\frac{\|\nabla u\|_{L^2(\Omega)}}{h_{\rm ex}}\|\nabla\xi\|_{L^q}\left(\operatorname{vol}\left(\bigcup_i B_i\right)\right)^{1/p},$$

where $1/p + 1/q = 1/2$ and we have used Hölder's inequality twice. Using (2.3),

$$\left| \int_{\bigcup_i B_i} \frac{1}{h_{\rm ex}} \nabla^\perp \xi \cdot \frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon} \right| \leqslant C \left(\sum_i r_i^2 \right)^{1/p} \|\nabla \xi\|_{L^q(\Omega)}.$$

In addition, $(\sum_i r_i^2)^{1/p} \leqslant (\sum_i r_i)^{2/p} = \mathrm{o}(1)$ since we know that $\sum_i r_i \to 0$. Therefore, (2.13) is proved.

Step 2. We observe that

$$\frac{1}{h_{\rm ex}} \int_{\tilde\Omega} \nabla^\perp \xi \cdot \frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon} = \frac{1}{h_{\rm ex}} \int_{\tilde\Omega} \frac{|u|^2}{a_\varepsilon} (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi$$

$$= \frac{1}{h_{\rm ex}} \int_{\tilde\Omega} (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi + \frac{1}{h_{\rm ex}} \int_{\tilde\Omega} \left(\frac{|u|^2}{a_\varepsilon} - 1 \right) (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi. \tag{2.14}$$

We claim that

$$\frac{1}{h_{\rm ex}} \left| \int_{\tilde\Omega} \left(\frac{|u|^2}{a_\varepsilon} - 1 \right) (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi \right| \leqslant \mathrm{o}(1) \|\nabla \xi\|_{L^q}. \tag{2.15}$$

Indeed,

$$\frac{1}{h_{\rm ex}} \left| \int_{\tilde\Omega} \left(\frac{|u|^2}{a_\varepsilon} - 1 \right) (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi \right| \leqslant \frac{1}{b_0 h_{\rm ex}} \left| \int_{\tilde\Omega} \left(|u|^2 - a_\varepsilon \right) |\nabla v| |\nabla \xi| \right|$$

$$\leqslant C \frac{\|\nabla v\|_{L^2(\tilde\Omega)}}{h_{\rm ex}} \|\nabla \xi\|_{L^q(\Omega)} \left\| |u|^2 - a_\varepsilon \right\|_{L^p(\Omega)},$$

with $1/p + 1/q = 1/2$. From the a priori estimate (2.1),

$$\int_\Omega \left(|u|^2 - a_\varepsilon \right)^p \leqslant C \int_\Omega \left(|u|^2 - a_\varepsilon \right)^2 \leqslant C \varepsilon^2 h_{\rm ex}^2 = \mathrm{o}(1),$$

hence, using $\|\nabla v\|_{L^2(\tilde\Omega)} \leqslant C \|\nabla u\|_{L^2(\Omega)} \leqslant C h_{\rm ex}$, we obtain (2.15). Combining (2.12)–(2.15), we have

$$\frac{1}{h_{\rm ex}} \int_\Omega \operatorname{curl} \frac{(\mathrm{i}u, \nabla u)}{a_\varepsilon} \xi = \frac{1}{h_{\rm ex}} \int_{\tilde\Omega} (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi + \mathrm{o}(1) \|\xi\|_{W_0^{1,q}}. \tag{2.16}$$

Step 3. We evaluate $\int_{\tilde\Omega} (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi$. Noticing that $\operatorname{curl} (\mathrm{i}v, \nabla v) \equiv 0$ on $\tilde\Omega$, we have

$$\int_{\tilde\Omega} (\mathrm{i}v, \nabla v) \cdot \nabla^\perp \xi = \int_{\partial\tilde\Omega} \xi \left(\mathrm{i}v, \frac{\partial v}{\partial \tau} \right) = \sum_i \int_{\partial B_i \cap \Omega} \xi \left(\mathrm{i}v, \frac{\partial v}{\partial \tau} \right).$$

There remains to prove that

$$\sum_i \int_{\partial B_i \cap \Omega} \xi \left(\mathrm{i}v, \frac{\partial v}{\partial \tau} \right) = 2\pi \sum_i d_i \xi(a_i) + \mathrm{o}(h_{\rm ex}) \|\xi\|_{W_0^{1,q}(\Omega)}. \tag{2.17}$$

Let f be a C^1 function defined on $\mathbb{R}_+$ such that

$$\begin{cases} f(x) = x & \text{for } x \leqslant b_0/2, \\ f(x) = 1 & \text{for } x \geqslant b_0, \\ |f'(x)| \leqslant C & \text{for any } x \geqslant 0. \end{cases} \tag{2.18}$$

We can define the complex-valued function:

$$w = f(|u|)v. \tag{2.19}$$

It has a meaning everywhere by setting $w = u$ where $|u| \leqslant b_0/2$. Then, it is easy to check that

$$|\nabla w| \leqslant C|\nabla u| \quad \text{in } \Omega, \tag{2.20}$$

and

$$\sum_i \int_{\partial B_i \cap \Omega} \xi \left(\mathrm{i}v, \frac{\partial v}{\partial \tau} \right) = \sum_i \int_{\partial B_i \cap \Omega} \xi \left(\mathrm{i}w, \frac{\partial w}{\partial \tau} \right). \tag{2.21}$$

Using Stokes theorem, we have

(2.22)

$$\left| \sum_i \int_{\partial B_i} \big(\xi - \xi(p_i)\big) \left(\mathrm{i}w, \frac{\partial w}{\partial \tau} \right) \right| = \left| \sum_i \int_{B_i} \nabla^{\perp}\xi \cdot (\mathrm{i}w, \nabla w) + \big(\xi - \xi(p_i)\big) \operatorname{curl}(\mathrm{i}w, \nabla w) \right|.$$

But, on the one hand,

$$\begin{aligned} \frac{1}{h_{\mathrm{ex}}} \left| \sum_i \int_{b_i} \nabla^{\perp}\xi \cdot (\mathrm{i}w, \nabla w) \right| &\leqslant C \frac{\|\nabla w\|_{L^2}}{h_{\mathrm{ex}}} \|\nabla \xi\|_{L^q} \left(\sum_i \operatorname{vol}(B_i) \right)^{1/p} \\ &\leqslant C \frac{\|\nabla u\|_{L^2}}{h_{\mathrm{ex}}} \|\nabla \xi\|_{L^q} \left(\sum_i r_i^2 \right)^{1/p} \\ &\leqslant \mathrm{o}(1) \|\nabla \xi\|_{L^q} \end{aligned} \tag{2.23}$$

as in the proof of (2.13). On the other hand, using the fact that, since $q > 2$, $W_0^{1,q}$ embeds in $C^{0,\beta}$ for some $\beta < 1$, and $|\operatorname{curl}(\mathrm{i}w, \nabla w)| \leqslant C|\nabla w|^2 \leqslant C|\nabla u|^2$, we have:

$$\begin{aligned} \left| \sum_i \frac{1}{h_{\mathrm{ex}}} \int_{\partial B_i} \big(\xi - \xi(p_i)\big) \operatorname{curl}(\mathrm{i}w, \nabla w) \right| &\leqslant \left(\max_i r_i \right)^{\beta} \|\xi\|_{C^{0,\beta}(\Omega)} \sum_i \int_{U_i} \frac{|\nabla u|^2}{h_{\mathrm{ex}}} \\ &\leqslant \mathrm{e}^{-\beta\sqrt{|\log \varepsilon|}} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{h_{\mathrm{ex}}} \|\xi\|_{W_0^{1,q}} \\ &\leqslant h_{\mathrm{ex}} \, \mathrm{e}^{-\beta\sqrt{|\log \varepsilon|}} \|\xi\|_{W_0^{1,q}} = \mathrm{o}(1) \|\xi\|_{W_0^{1,q}}, \end{aligned} \tag{2.24}$$

since $h_{\mathrm{ex}} \leqslant C|\log \varepsilon|$. As in [26], the proof remains valid even if B_i intersects $\partial\Omega$. Combining (2.23), (2.24), (2.21), and (2.22), (2.17) is proved. Consequently, in view of (2.16), we can conclude that

$$\left| \frac{1}{h_{\mathrm{ex}}} \int_{\Omega} \xi \operatorname{curl} \frac{(\mathrm{i}u, \nabla u)}{a_{\varepsilon}} - \frac{2\pi}{h_{\mathrm{ex}}} \sum_i d_i \xi(p_i) \right| \leqslant \mathrm{o}(1) \|\xi\|_{W_0^{1,q}} :$$

hence that $\frac{1}{h_{\rm ex}}\operatorname{curl}\frac{(iu,\nabla u)}{a_\varepsilon}-\mu_\varepsilon\to 0$ strongly in $(W_0^{1,q})'$ as stated. □

Proof of Proposition 2.1. – For the sake of simplicity, we write ε instead of ε_n.

Step 1. We prove that h_ε satisfies

$$\frac{1}{h_{\rm ex}}\left(-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a_\varepsilon}\right)+h_\varepsilon\right)=f_\varepsilon, \tag{2.25}$$

with $f_\varepsilon=\mu_\varepsilon+\psi_\varepsilon$, where $\psi_\varepsilon\to 0$ strongly in $(W_0^{1,q})'$ for $q>2$. Indeed, we start from the second Ginzburg–Landau equation:

$$-\nabla^\perp h_\varepsilon=(iu_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon),$$

divide it by a_ε and take the curl:

$$-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a_\varepsilon}\right)=\operatorname{curl}\left(\frac{(iu_\varepsilon,\nabla u_\varepsilon)}{a_\varepsilon}-A_\varepsilon\frac{|u_\varepsilon|^2}{a_\varepsilon}\right),$$

hence

$$-\operatorname{div}\left(\frac{\nabla h_\varepsilon}{a_\varepsilon}\right)+h_\varepsilon=\operatorname{curl}\frac{(iu_\varepsilon,\nabla u_\varepsilon)}{a_\varepsilon}+\operatorname{curl}\left(A_\varepsilon\left(1-\frac{|u_\varepsilon|^2}{a_\varepsilon}\right)\right). \tag{2.26}$$

Now consider a test-function $\xi\in W_0^{1,q}(\Omega)$, $q>2$,

$$\begin{aligned}\left|\int_\Omega\xi\operatorname{curl}\left(A_\varepsilon\left(1-\frac{|u|^2}{a_\varepsilon}\right)\right)\right|&=\left|\int_\Omega\nabla^\perp\xi\cdot A_\varepsilon\left(1-\frac{|u|^2}{a_\varepsilon}\right)\right|\\&\leqslant C\|A_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\xi\|_{L^2(\Omega)}\left\|a_\varepsilon-|u|^2\right\|_{L^2(\Omega)}.\end{aligned}$$

The a-priori bound (2.2), $\|A_\varepsilon\|_{L^\infty(\Omega)}\leqslant O(h_{\rm ex})$ and the energy bound, $\|a_\varepsilon-|u|^2\|_{L^2}\leqslant C\varepsilon h_{\rm ex}$, yield

$$\left|\int_\Omega\xi\operatorname{curl}\left(A_\varepsilon\left(1-\frac{|u|^2}{a_\varepsilon}\right)\right)\right|\leqslant o(1)\|\nabla\xi\|_{L^2}.$$

Consequently, $\operatorname{curl}(A_\varepsilon(1-\frac{|u|^2}{a_\varepsilon}))\to 0$ strongly in $(W_0^{1,q})'$ for $q>2$. Combining this with (2.26) and Lemma 2.2, we get the desired result.

Step 2. We prove that f_ε converges to μ_0, the weak limit of μ_ε, in $W^{-1,r}(\Omega)$ for any $r<2$. Indeed, from the upper bound on the energy, we know that $\frac{1}{a_\varepsilon h_{\rm ex}}\nabla h_\varepsilon$ is bounded in $L^2(\Omega)$, hence, in view of (2.25), f_ε is bounded in H^{-1}, hence in $W^{-1,p}$ for $p<2$. But, on the other hand, $f_\varepsilon=\mu_\varepsilon+\psi_\varepsilon$, with ψ_ε bounded in $W^{-1,p}$ for $p<2$, hence μ_ε remains bounded in $W^{-1,p}$ for $p<2$. Furthermore, μ_ε is also bounded in the sense of measures, therefore we can apply a theorem of Murat (see [22] and the annex of the paper by Brezis who gives a simpler proof) which asserts that such a μ_ε, bounded in the sense of measures and in $W^{-1,p}$ for $p<2$, is necessarily compact in $W^{-1,r}$ for $r<p$. Since this is also the case for ψ_ε, which converges to zero, this implies that f_ε is compact in $W^{-1,r}$ for $r<2$. In addition, its limit in the sense of distributions is μ_0, hence it must converge to μ_0 in $W^{-1,r}$.

Step 3. We wish to pass to the limit in (2.25), but it is not possible directly because the H-convergence requires a right-hand side in H^{-1}. So we are going to pass to the limit in the duality sense for a fixed right-hand side. Let $g \in W^{-1,q}$ for $q > 2$. Using the hypothesis (H1) on a_ε, (which implies in particular the uniform ellipticity of $\frac{1}{a_\varepsilon}\mathcal{I}$), we can apply a theorem of Meyers [21]: there exists a $q_0 > 2$, such that if g is in $W^{-1,q}$ with $2 < q \leqslant q_0$, then equation

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla v_\varepsilon}{a_\varepsilon}\right) + v_\varepsilon = g & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.27}$$

has a unique solution v_ε in $W_0^{1,q}$. Thus, we have

$$_{W_0^{1,q'}}\left\langle \frac{h_\varepsilon}{h_{\text{ex}}} - 1, g \right\rangle_{W^{-1,q}} = {}_{W^{-1,q'}}\langle f_\varepsilon - 1, v_\varepsilon \rangle_{W_0^{1,q}}, \tag{2.28}$$

where $1/q' + 1/q = 1$, and we want to pass to the limit.

More precisely, Meyers' theorem yields that the operator R_ε which maps g to v_ε, is a bounded linear operator from $W^{-1,q}$ to $W_0^{1,q}$ (for $2 < q \leqslant q_0$), hence up to extraction of a subsequence, v_ε has a weak limit v_0 in $W_0^{1,q}$. We assumed in hypothesis (H4) that $\frac{1}{a_\varepsilon}\mathcal{I}$ H-converges to $\boldsymbol{\mathcal{A}}_0$. By the definition of H-convergence (see [23]), and since $W_0^{1,q} \subset H_0^1$, this implies that v_0 is the solution of:

$$\begin{cases} -\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla v_0) + v_0 = g & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.29}$$

Since this possible weak limit v_0 is unique, the whole sequence v_ε converges to v_0 weakly in $W_0^{1,q}$. In addition, f_ε converges strongly to μ_0 in $W^{-1,q'}$, thus we have

$$_{W^{-1,q'}}\langle f_\varepsilon - 1, v_\varepsilon \rangle_{W_0^{1,q}} \to \langle \mu_0 - 1, v_0 \rangle.$$

On the other hand, $\frac{h_\varepsilon}{h_{\text{ex}}} - 1$ converges weakly to $h_0 - 1$ in H_0^1. Thus,

$$_{W_0^{1,q'}}\left\langle \frac{h_\varepsilon}{h_{\text{ex}}} - 1, g \right\rangle_{W^{-1,q}} \to \langle h_0 - 1, g \rangle.$$

Therefore, we can pass to the limit in (2.28), and we are led to

$$_{W_0^{1,q'}}\langle h_0 - 1, g \rangle_{W^{-1,q}} = {}_{W^{-1,q'}}\langle \mu_0 - 1, v_0 \rangle_{W_0^{1,q}}. \tag{2.30}$$

Meyers' aforementioned theorem, also yields that for $q_0' \leqslant q' < 2$, (2.11) has a unique solution in $W^{1,q'}$. Since (2.30) holds for any g in $W^{-1,q}$, it implies that h_0 is this solution. □

2.3. Deriving a lower bound outside the vortex balls

Next, we would like to deduce from (2.11) a lower bound like

$$\liminf_{\varepsilon \to 0} \int_{\Omega \setminus \cup_i B_i} \frac{|\nabla h|^2}{a_\varepsilon h_{\text{ex}}^2} \geqslant \int_{\Omega} \nabla h_0 \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_0.$$

But this is impossible to derive straightforwardly because the domain of integration in the left-hand side integral is not Ω. To remedy this, we replace h_ε by an auxiliary field $\overline{h_\varepsilon}$, a sort of truncated of h_ε in the balls. This is a trick that was already used in [27] Proposition IV.1, Step 1.

LEMMA 2.3. – *There exists $\overline{h_\varepsilon}$ such that $\overline{h_\varepsilon}-1\in H_0^1(\Omega)$ and*

(1) $\frac{\overline{h_\varepsilon}}{h_{\text{ex}}}-1\rightharpoonup h_0-1$ *in* $H_0^1(\Omega)$,

(2)

$$\int_{\Omega\setminus\bigcup_i B_i}\frac{|\nabla h|^2}{a_\varepsilon}+\int_\Omega |h_\varepsilon-h_{\text{ex}}|^2\geqslant\int_\Omega \frac{|\nabla\overline{h_\varepsilon}|^2}{a_\varepsilon}+|\overline{h_\varepsilon}-h_{\text{ex}}|^2-\mathrm{o}(1),$$

(3)

$$\liminf_{\varepsilon\to 0}\int_\Omega\frac{|\nabla\overline{h_\varepsilon}|^2}{a_\varepsilon}\geqslant\int_\Omega\nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0.$$

Proof. – We consider $\overline{A_\varepsilon}$ a solution of the following minimization problem:

$$\min_{A\in H^1(\Omega,\mathbb{R}^2),\,\operatorname{div}A=0}\int_{\Omega\setminus\bigcup_i B_i}a_\varepsilon|\nabla\varphi-A|^2+\int_\Omega|\operatorname{curl}A-h_{\text{ex}}|^2,\tag{2.31}$$

where $\nabla\varphi$ denotes the gradient of the phase of u_ε which is well-defined in $\Omega\setminus\bigcup_i B_i$. If we write $\overline{h_\varepsilon}=\operatorname{curl}\overline{A_\varepsilon}$, and we test (2.31) with h_ε, we have

$$\int_{\Omega\setminus\bigcup_i B_i}a_\varepsilon|\nabla\varphi-\overline{A_\varepsilon}|^2+\int_\Omega|\overline{h_\varepsilon}-h_{\text{ex}}|^2\leqslant\int_{\Omega\setminus\bigcup_i B_i}a_\varepsilon|\nabla\varphi-A_\varepsilon|^2+\int_\Omega|h_\varepsilon-h_{\text{ex}}|^2\leqslant Ch_{\text{ex}}^2.\tag{2.32}$$

In addition, $\overline{h_\varepsilon}$ and $\overline{A_\varepsilon}$ satisfy the following equations:

$$\begin{cases}-\nabla^\perp\overline{h_\varepsilon}=a_\varepsilon(\nabla\varphi-\overline{A_\varepsilon}) & \text{in }\Omega\setminus\bigcup_i B_i,\\ \overline{h_\varepsilon}=cst=c_i & \text{on }B_i,\forall i,\\ \overline{h_\varepsilon}=h_{\text{ex}} & \text{on }\partial\Omega.\end{cases}\tag{2.33}$$

Thus, it satisfies

$$-\operatorname{div}\left(\frac{\nabla\overline{h_\varepsilon}}{a_\varepsilon h_{\text{ex}}}\right)+\frac{\overline{h_\varepsilon}}{h_{\text{ex}}}=\nu_\varepsilon,\tag{2.34}$$

where ν_ε is the measure defined by:

$$\forall\xi\in W_0^{1,q}(\Omega),\ (q>2),\qquad\int_\Omega\nu_\varepsilon\xi=\sum_i\frac{1}{h_{\text{ex}}}\int_{\partial B_i}\xi\frac{\partial\varphi}{\partial\tau}+\sum_i\frac{1}{h_{\text{ex}}}\int_{B_i}c_i\xi.\tag{2.35}$$

On the other hand, using Cauchy–Schwartz inequality,

$$\left|\frac{1}{h_{\text{ex}}}\sum_i\int_{B_i}c_i\xi\right|=\left|\frac{1}{h_{\text{ex}}}\int_{\bigcup_i B_i}\overline{h_\varepsilon}\xi\right|\leqslant\|\xi\|_{L^\infty}\left\|\frac{\overline{h_\varepsilon}}{h_{\text{ex}}}\right\|_{L^2}\left(\sum_i r_i\right)^{1/2}.$$

In view of (2.32), $\|\frac{\overline{h_\varepsilon}}{h_{\mathrm{ex}}}\|_{L^2}$ is bounded, and $(\sum_i r_i)^{1/2} \leqslant \sum_i r_i \to 0$ from Proposition 1.1. Hence,

$$\left|\frac{1}{h_{\mathrm{ex}}}\sum_i \int_{B_i} c_i \xi\right| = \mathrm{o}(1)\|\xi\|_{L^\infty}.$$

On the other hand, the same proof as for Lemma 2.2 shows that

$$\left|\sum_i \frac{1}{h_{\mathrm{ex}}}\int_{\partial B_i} \frac{\partial\varphi}{\partial\tau}\xi - \int_\Omega \xi \,\mathrm{d}\mu_\varepsilon\right| = \mathrm{o}(1)\|\xi\|_{W_0^{1,q}}.$$

Hence, in view of (2.35), $\nu_\varepsilon - \mu_\varepsilon$ converges strongly to 0 in $(W_0^{1,q})'$. The same argument as in Proposition 2.1 allows to conclude from (2.34) that

$$\frac{\overline{h_\varepsilon}}{h_{\mathrm{ex}}} - 1 \rightharpoonup h_0 - 1 \quad \text{in } H_0^1(\Omega),$$

using the uniqueness of the solution of (2.11).

Using (2.32) and (2.33), we get

$$\begin{aligned}\int_\Omega \frac{|\nabla\overline{h_\varepsilon}|^2}{a_\varepsilon} + |\overline{h_\varepsilon} - h_{\mathrm{ex}}|^2 &= \int_{\Omega\setminus\bigcup_i B_i} a_\varepsilon|\nabla\varphi - \overline{A_\varepsilon}|^2 + \int_\Omega |\overline{h_\varepsilon} - h_{\mathrm{ex}}|^2 \\ &\leqslant \int_{\Omega\setminus\bigcup_i B_i} a_\varepsilon|\nabla\varphi - A_\varepsilon|^2 + \int_\Omega |h_\varepsilon - h_{\mathrm{ex}}|^2.\end{aligned}$$

As in the proof of Proposition 1.1, we have

$$\int_{\Omega\setminus\bigcup_i B_i} a_\varepsilon|\nabla\varphi - A_\varepsilon|^2 \leqslant \int_{\Omega\setminus\bigcup_i B_i} \frac{|\nabla h_\varepsilon|^2}{a_\varepsilon} + \mathrm{o}(1).$$

Thus, assertion (2) is proved. In addition, $\overline{h_\varepsilon}/h_{\mathrm{ex}} - 1$ is bounded in $H_0^1(\Omega)$ and the convergence to $h_0 - 1$ is weak in H_0^1. There remains to prove the third assertion. But it is a classical result in homogenization theory (see [17]) that, since $\overline{h_\varepsilon}/h_{\mathrm{ex}} - 1 \rightharpoonup h_0 - 1$ in $H_0^1(\Omega)$ and $\frac{1}{a_\varepsilon}\mathcal{I}$ H-converges to $\boldsymbol{\mathcal{A}}_0$,

$$\liminf_{\varepsilon\to 0}\int_\Omega \frac{1}{a_\varepsilon}\left|\nabla\left(\frac{\overline{h_\varepsilon}}{h_{\mathrm{ex}}}\right)\right|^2 \geqslant \int_\Omega \nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0.$$

This completes the proof of the lemma. □

We recall that we defined E in (1.9).

LEMMA 2.4. – *With the same notations,*

$$\liminf_{\varepsilon\to 0}\frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \frac{\Lambda}{2}\int_\Omega b|\mu_0| + \frac{1}{2}\int_\Omega \nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0 + |h_0 - 1|^2 = E(h_0).$$

Proof. – The energy can easily be bounded from below as follows, splitting between the contribution inside the vortex-balls and the contribution outside:

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h - h_{\mathrm{ex}}|^2$$
$$\geqslant \frac{1}{2}\int_{\bigcup_{i\in I} B_i} |\nabla_A u|^2 + \frac{1}{2}\int_{\Omega\setminus\cup_i B_i} \rho^2|\nabla\varphi - A|^2 + \frac{1}{2}\int_\Omega |h - h_{\mathrm{ex}}|^2.$$

As previously, since for the energy-minimizers $-\nabla^\perp h = (\mathrm{i}u, \nabla_A u)$, and $|\rho^2 - a_\varepsilon| \leqslant C/|\log\varepsilon|$ in $\Omega\setminus\bigcup_i B_i$, we have

$$\int_{\Omega\setminus\bigcup_i B_i} \rho^2|\nabla\varphi - A|^2 = \int_{\Omega\setminus\bigcup_i B_i} \frac{|\nabla h|^2}{a_\varepsilon}\big(1 - \mathrm{o}(1)\big).$$

Therefore, in view of Proposition 1.1,

$$J_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant \pi\sum_i |d_i|b(p_i)|\log\varepsilon|\big(1 - \mathrm{o}(1)\big) + \int_{\Omega\setminus\cup_i B_i} \frac{|\nabla h|^2}{a_\varepsilon}\big(1 - \mathrm{o}(1)\big) + \int_\Omega |h - h_{\mathrm{ex}}|^2,$$

and with assertion (2) of Lemma 2.3,

$$\frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \frac{1}{2}\frac{|\log\varepsilon|}{h_{\mathrm{ex}}}\int_\Omega b|\mu_\varepsilon| + \frac{1}{h_{\mathrm{ex}}^2}\int_\Omega \frac{|\nabla\overline{h_\varepsilon}|^2}{a_\varepsilon} + \int_\Omega \left|\frac{\overline{h_\varepsilon}}{h_{\mathrm{ex}}} - 1\right|^2 - \mathrm{o}(1).$$

We thus obtain, using assertion (3) of Lemma 2.3 that:

$$\liminf \frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \liminf \frac{1}{2}\left(\frac{|\log\varepsilon|}{h_{\mathrm{ex}}}\int_\Omega b|\mu_\varepsilon|\right) + \int_\Omega \nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0 + |h_0 - 1|^2. \tag{2.36}$$

Similarly, using (2.8), we obtain

$$\liminf \frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \liminf \frac{1}{2}\left(\frac{|\log\varepsilon|}{h_{\mathrm{ex}}}\int_\Omega a_\varepsilon|\mu_\varepsilon|\right) + \int_\Omega \nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0 + |h_0 - 1|^2. \tag{2.37}$$

Then, using the weak convergence of μ_ε to μ_0 in $\mathcal{M}$, and the weak lower semi-continuity of $\mu \mapsto \int_\Omega b|\mu|$, we conclude from (2.36) that

$$\liminf \frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \frac{\Lambda}{2}\int_\Omega b|\mu_0| + \int_\Omega \nabla h_0\cdot\boldsymbol{\mathcal{A}}_0\nabla h_0 + |h_0 - 1|^2 = E(h_0). \qquad \square$$

The final convergence result will then follow from the combination of this result with the upper bound of Section 3, leading to the fact that necessarily h_0 has to be h_*, the minimizer of E, and $\mu_0 = \mu_*$.

3. Upper bound

In this section we prove Proposition 1.4. First we remark that if h is the solution of $-\operatorname{div}(\mathcal{A}\nabla h) + h = \mu$ with boundary value 1, then

$$h(x) - 1 = \int G(x, y)\, \mathrm{d}(\mu - 1)(y),$$

where $G(\cdot, y)$ is the solution of $-\operatorname{div}(\mathcal{A}\nabla h) + h = \delta_y$ vanishing on $\partial\Omega$ and $\mu - 1$ denotes the difference between the measure μ and the Lebesgue measure in Ω. From this it follows easily that

$$\int_\Omega \nabla h \cdot \mathcal{A}\nabla h + |h-1|^2 = \iint G(x, y)\, \mathrm{d}(\mu - 1)(x)\, \mathrm{d}(\mu - 1)(y). \tag{3.1}$$

This last expression will be the one we use.

To prove Proposition 1.4 we will then need some properties of the Green functions G_ε, G_0 associated to the operators $-\operatorname{div}(\boldsymbol{\mathcal{A}}_\varepsilon \nabla u) + u$ and $-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla u) + u$ respectively. These properties will be proved at the end of this section.

LEMMA 3.1. – *Let $a_\varepsilon = b + \beta_\varepsilon$ be a sequence of functions satisfying* (H1) *to* (H4), *and $\boldsymbol{\mathcal{A}}_0$ be the homogenized limit of the matrices $\boldsymbol{\mathcal{A}}_\varepsilon = {a_\varepsilon}^{-1}\boldsymbol{\mathcal{I}}$ as ε goes to zero. For any $y \in \Omega$, let $G_\varepsilon(., y)$ (resp. $G_0(\cdot, y)$) be the solution of $-\operatorname{div}(\boldsymbol{\mathcal{A}}_\varepsilon \nabla G_\varepsilon) + G_\varepsilon = \delta_y$ (resp. $-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla G_0) + G_0 = \delta_y$) that vanishes on $\partial\Omega$.*

The following properties hold:

(1) *$G_\varepsilon(x, y)$, $G_0(x, y)$ are positive functions, and symmetric in x and y.*

(2) *Δ denoting the diagonal in $\mathbb{R}^2$, there exists $C > 0$ such that $G_\varepsilon(x, y)$, $G_0(x, y)$ are bounded by*

$$C\big(\big|\log|x - y|\big| + 1\big)$$

for all $x, y \in \overline{\Omega} \times \overline{\Omega} \setminus \Delta$.

(3) *For any compact $K \subset \Omega$, there exists $C > 0$ such that for any $x, y \in \Omega$*

$$G_\varepsilon(x, y) + \frac{a_\varepsilon(x)}{2\pi} \log|x - y| \leqslant \frac{C}{\eta(\varepsilon)},$$

where $\eta(\varepsilon)$ is defined in (H3).

(4) *G_ε converges to G_0 locally uniformly in $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$.*

PROPOSITION 3.1. – *Assume that $\Lambda > 0$ and that* (H1) *to* (H4) *are satisfied. Let μ be a positive Radon measure with support in $\overline{\Omega}$ and $(p_\varepsilon^i)_{1 \leqslant i \leqslant n_\varepsilon}$ be families of points in Ω such that $\forall i \neq j$*

$$\big|p_\varepsilon^i - p_\varepsilon^j\big| > 4\varepsilon, \qquad d\big(p_\varepsilon^i, \partial\Omega\big) > \alpha_0 > 0, \tag{3.3}$$

where α_0 is independent of ε,

$$\frac{2\pi}{h_{\mathrm{ex}}} \sum_{i=1}^{n_\varepsilon} \delta_{p_\varepsilon^i} \longrightarrow \mu, \quad \textit{in the sense of measures}, \tag{3.4}$$

and

$$\lim_{\varepsilon\to 0}\left(\sum_{\substack{i\neq j\\ |p_\varepsilon^i-p_\varepsilon^j|<\alpha}}\frac{|\log|p_\varepsilon^i-p_\varepsilon^j||}{h_{\mathrm{ex}}^2}\right)\underset{\alpha\to 0}{\longrightarrow}0. \tag{3.5}$$

Then there exist configurations $(v_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ *such that*

$$\limsup_{\varepsilon\to 0}\frac{J_\varepsilon(v_\varepsilon,B_\varepsilon)}{h_{\mathrm{ex}}^2}\leqslant\frac{\Lambda}{2}\limsup_{\varepsilon\to 0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}+\frac{1}{2}\iint G_0\,\mathrm{d}(\mu-1)\,\mathrm{d}(\mu-1), \tag{3.6}$$

where G_0 *is defined in Lemma* 3.1.

This proposition states that under reasonable hypotheses on points p_ε^i, one can construct a good test configuration with prescribed vortices at p_ε^i. Moreover, (3.4) implies that $n_\varepsilon/h_{\mathrm{ex}}$ is bounded. The following proposition asserts that the construction of points p_ε^i is possible.

PROPOSITION 3.2. – *Assume that* $\Lambda>0$ *and that* (H1) *to* (H4) *are satisfied. Then given any positive Radon measure* μ *of the form* $\sigma(x)\,dx$ *where* σ *is a positive continuous function compactly supported in* Ω*, there exist families of points* $(p_\varepsilon^i)_{1\leqslant i\leqslant n_\varepsilon}$ *satisfying* (3.3), (3.4), (3.5) *and such that*

$$\limsup_{\varepsilon\to 0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}\leqslant\int_\Omega b(x)\,\mathrm{d}\mu(x). \tag{3.7}$$

The proof of Proposition 1.4 follows easily from these two propositions. First, taking any positive Radon measure μ supported in $\overline{\Omega}$, we may approach it in the weak-* topology by measures $\mu_n=\sigma_n(x)\,\mathrm{d}x$ where $\sigma_n\in C_c(\Omega)$ is a positive function such that $\overline{\lim}\, I(\mu_n)\leqslant I(\mu)$. This is done using a mollifier and convolution. Applying Propositions 3.1 and 3.2, we may construct test-configurations $(v_\varepsilon^n, B_\varepsilon^n)_{\varepsilon>0}$ such that

$$\limsup_{\varepsilon\to 0}\frac{J_\varepsilon(v_\varepsilon^n,B_\varepsilon^n)}{h_{\mathrm{ex}}^2}\leqslant\frac{\Lambda}{2}\int b(x)\,\mathrm{d}\mu_n(x)+\frac{1}{2}\iint G_0\,\mathrm{d}(\mu_n-1)\,\mathrm{d}(\mu_n-1).$$

Therefore the same inequality is satisfied if we replace $(v_\varepsilon^n, B_\varepsilon^n)$ by the minimizing configuration $(u_\varepsilon, A_\varepsilon)$. This proves that for each n,

$$\limsup_{\varepsilon\to 0}\frac{J_\varepsilon(u_\varepsilon,A_\varepsilon)}{h_{\mathrm{ex}}^2}\leqslant I(\mu_n),$$

and then,

$$\limsup_{\varepsilon\to 0}\frac{J_\varepsilon(u_\varepsilon,A_\varepsilon)}{h_{\mathrm{ex}}^2}\leqslant\frac{\Lambda}{2}\int_\Omega b\,\mathrm{d}\mu+\frac{1}{2}\iint G_0(x,y)\,\mathrm{d}(\mu-1)(x)\,\mathrm{d}(\mu-1)(y). \tag{3.8}$$

Using (3.1) we get the conclusion of Proposition 1.4.

3.1. Proof of Proposition 3.1

The method for constructing a test configuration $(v_\varepsilon, B_\varepsilon)$ with prescribed vortices $(p_\varepsilon^i)_{1\leqslant i\leqslant n_\varepsilon}$ follows closely that of [28]. First we define h_ε to be the solution of:

$$\begin{cases} -\operatorname{div}(\boldsymbol{\mathcal{A}}_\varepsilon \nabla h_\varepsilon) + h_\varepsilon = \sum_{i=1}^{n_\varepsilon} \mu_\varepsilon^i & \text{in } \Omega, \\ h_\varepsilon = h_{\text{ex}} & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where μ_ε^i is the line element on the circle $\partial B(p_\varepsilon^i, \varepsilon)$ normalized so that $\mu_\varepsilon^i(\partial B(p_\varepsilon^i, \varepsilon)) = 2\pi$.

Then we let B_ε be any vector field such that $\operatorname{curl} B_\varepsilon = h_\varepsilon$. Finally, we define $v_\varepsilon = \rho_\varepsilon \,\mathrm{e}^{\mathrm{i}\varphi_\varepsilon}$ as follows: first we let

$$\rho_\varepsilon(x) = \begin{cases} 0 & \text{if } \left|x - p_i^\varepsilon\right| \leqslant \varepsilon \text{ for some } i, \\ \sqrt{a_\varepsilon(x)}\dfrac{|x - p_i^\varepsilon| - \varepsilon}{\varepsilon} & \text{if } \varepsilon < |x - a_i^\varepsilon| < 2\varepsilon \text{ for some } i, \\ \sqrt{a_\varepsilon(x)} & \text{otherwise,} \end{cases} \tag{3.10}$$

and for any $x \in \Omega_\varepsilon = \Omega \setminus \bigcup_i B(p_i^\varepsilon, \varepsilon)$,

$$\varphi_\varepsilon(x) = \oint_{(x_0,x)} \big(B_\varepsilon - \boldsymbol{\mathcal{A}}_\varepsilon \nabla^\perp h_\varepsilon\big).\tau \,\mathrm{d}\ell, \tag{3.11}$$

where x_0 is a base point in Ω_ε, (x_0, x) is any curve joining x_0 to x in Ω_ε and τ is the tangent vector to the curve. From (3.9), we see that this definition of $\varphi_\varepsilon(x)$ does not depend modulo 2π on the particular curve (x_0, x) chosen. The fact that φ_ε is not defined on $\bigcup_i B(p_i^\varepsilon, \varepsilon)$ is not important since ρ_ε is zero there. Thus, φ_ε satisfies

$$-\boldsymbol{\mathcal{A}}_\varepsilon \nabla^\perp h_\varepsilon = \nabla\varphi_\varepsilon - B_\varepsilon \quad \text{in } \Omega_\varepsilon. \tag{3.12}$$

Having defined $v_\varepsilon = \rho_\varepsilon \,\mathrm{e}^{\mathrm{i}\varphi_\varepsilon}$, we estimate $J_\varepsilon(v_\varepsilon, B_\varepsilon)$. Recall that

$$J_\varepsilon(v_\varepsilon, B_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon|^2 + {\rho_\varepsilon}^2|\nabla\varphi_\varepsilon - B_\varepsilon|^2 + |h_\varepsilon - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2}\big(a_\varepsilon - {\varphi_\varepsilon}^2\big)^2. \tag{3.13}$$

Using the fact that $|\nabla a_\varepsilon| \ll h_{\text{ex}}$ (hypothesis (H2)) and that the number of points p_ε^i is less than Ch_{ex} – which follows from (3.4) – it is not difficult to check that

$$\frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(a_\varepsilon - {\rho_\varepsilon}^2\big)^2 \ll h_{\text{ex}}^2. \tag{3.14}$$

Also, from (3.10), (3.12),

$${\rho_\varepsilon}^2|\nabla\varphi_\varepsilon - B_\varepsilon|^2 \leqslant a_\varepsilon|\nabla\varphi_\varepsilon - B_\varepsilon|^2 = \nabla h_\varepsilon \cdot \boldsymbol{\mathcal{A}}_\varepsilon \nabla h_\varepsilon$$

in Ω_ε. Therefore, replacing in (3.13) and in view of (3.14)

$$\limsup_{\varepsilon\to 0} \frac{J_\varepsilon(v_\varepsilon, B_\varepsilon)}{h_{\text{ex}}^2} \leqslant \limsup_{\varepsilon\to 0} \frac{1}{2h_{\text{ex}}^2}\int_\Omega \nabla h_\varepsilon \cdot \boldsymbol{\mathcal{A}}_\varepsilon \nabla h_\varepsilon + |h_\varepsilon - h_{\text{ex}}|^2. \tag{3.15}$$

Because h_ε is the solution of (3.9), we may rewrite the right-hand side of this inequality as

$$\limsup_{\varepsilon\to 0}\frac{1}{2}\iint G_\varepsilon(x,y)\,\mathrm{d}(\mu_\varepsilon-1)(x)\,\mathrm{d}(\mu_\varepsilon-1)(y),$$

where

$$\mu_\varepsilon=\frac{1}{h_{\mathrm{ex}}}\sum_{i=1}^{n_\varepsilon}\mu_\varepsilon^i, \tag{3.16}$$

and μ_ε^i is defined in (3.9). It follows from (3.4), (3.9) and (3.16) that $\mu_\varepsilon\to\mu$ as $\varepsilon\to 0$. Thus, to finish the proof of the proposition, it remains to show that:

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\frac{1}{2}\iint G_\varepsilon\,\mathrm{d}(\mu_\varepsilon-1)\,\mathrm{d}(\mu_\varepsilon-1)\\&\quad\leqslant\frac{\Lambda}{2}\limsup_{\varepsilon\to 0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}+\frac{1}{2}\iint G_0\,\mathrm{d}(\mu-1)\,\mathrm{d}(\mu-1).\end{aligned}\tag{3.17}$$

Proof of (3.17). – Let $\alpha>0$ and let $\Delta_\alpha=\{(x,y)\mid |x-y|<\alpha\}$. Recall that $\mu_\varepsilon\to\mu$. Hence, it follows that $(\mu_\varepsilon-1)\otimes(\mu_\varepsilon-1)\to(\mu-1)\otimes(\mu-1)$ as $\varepsilon\to 0$. But from Lemma 2.1, G_ε tends to G_0 uniformly in $\overline{\Omega}\times\overline{\Omega}\setminus\Delta_\alpha$, therefore

$$\lim_{\varepsilon\to 0}\frac{1}{2}\iint_{\overline{\Omega}\times\overline{\Omega}\setminus\Delta_\alpha}G_\varepsilon\,\mathrm{d}(\mu_\varepsilon-1)\,\mathrm{d}(\mu_\varepsilon-1)=\frac{1}{2}\iint_{\overline{\Omega}\times\overline{\Omega}\setminus\Delta_\alpha}G_0\,\mathrm{d}(\mu-1)\,\mathrm{d}(\mu-1).\tag{3.18}$$

Now we treat the integral on Δ_α. More precisely we prove that

$$\limsup_{\varepsilon\to 0}\iint_{\Delta_\alpha}G_\varepsilon\,\mathrm{d}(\mu_\varepsilon-1)\,\mathrm{d}(\mu_\varepsilon-1)\leqslant\frac{\Lambda}{2}\limsup_{\varepsilon\to 0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}+\mathrm{o}_\alpha(1),\tag{3.19}$$

where $\lim_{\alpha\to 0}\mathrm{o}_\alpha(1)=0$. Adding (3.18), (3.19) and letting $\alpha\to 0$ yields (3.17). We are left with proving (3.19). First we use the bound $|G_\varepsilon(x,y)|<C|\log|x-y||$ from which one easily gets

$$\iint_{\Delta_\alpha}G_\varepsilon\,\mathrm{d}(\mu_\varepsilon-1)\,\mathrm{d}(\mu_\varepsilon-1)\leqslant\iint_{\Delta_\alpha}G_\varepsilon\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}\mu_\varepsilon+C\alpha^2|\log\alpha|.$$

Therefore (3.19) will follow if we prove

$$\limsup_{\varepsilon\to 0}\iint_{\Delta_\alpha}G_\varepsilon\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}\mu_\varepsilon\leqslant\frac{\Lambda}{2}\limsup_{\varepsilon\to 0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}+\mathrm{o}_\alpha(1).\tag{3.20}$$

To prove this, we come back to the definition of μ_ε. From this definition, we have

$$\iint_{\Delta_\alpha}G_\varepsilon\,\mathrm{d}\mu_\varepsilon\,\mathrm{d}\mu_\varepsilon\leqslant\frac{1}{h_{\mathrm{ex}}^2}\Bigg(\sum_{\substack{1\leqslant i\neq j\leqslant n_\varepsilon\\|p_\varepsilon^i-p_\varepsilon^j|<2\alpha}}\iint G_\varepsilon\,\mathrm{d}\mu_\varepsilon^i\,\mathrm{d}\mu_\varepsilon^j+\sum_{i=1}^{n_\varepsilon}\iint G_\varepsilon\,\mathrm{d}\mu_\varepsilon^i\,\mathrm{d}\mu_\varepsilon^i\Bigg).\tag{3.21}$$

Let us first estimate the first sum on the right-hand side. If $x \in \operatorname{Supp}\mu_\varepsilon^i = \partial B(p_\varepsilon^i, \varepsilon)$, $y \in \operatorname{Supp}\mu_\varepsilon^j$ and $i \neq j$, since $|p_\varepsilon^i - p_\varepsilon^j| > 4\varepsilon$, then $|x - y| > \frac{1}{2}|p_\varepsilon^i - p_\varepsilon^j|$. Using the bound $|G_\varepsilon(x, y)| < C|\log|x - y||$ together with the fact that $|p_\varepsilon^i - p_\varepsilon^j| < 2\alpha$ and α is small enough, we get

$$\iint G_\varepsilon \,\mathrm{d}\mu_\varepsilon^i \,\mathrm{d}\mu_\varepsilon^j < C\bigl|\log\bigl|p_\varepsilon^i - p_\varepsilon^j\bigr|\bigr|.$$

Then, by hypothesis (3.5),

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2} \sum_{\substack{1\leqslant i\neq j\leqslant n_\varepsilon \\ |p_\varepsilon^i - p_\varepsilon^j|<2\alpha}} \iint G_\varepsilon \,\mathrm{d}\mu_\varepsilon^i \,\mathrm{d}\mu_\varepsilon^j \leqslant \mathrm{o}_\alpha(1). \tag{3.22}$$

As for the second sum in the right-hand side of (3.21), we use property (3) in Lemma 3.1 to get that for any $1 \leqslant i \leqslant n_\varepsilon$, and any $x, y \in \operatorname{Supp}\mu_\varepsilon^i$,

$$G_\varepsilon(x, y) + \frac{a_\varepsilon(x)}{2\pi} \log|x - y| < \frac{C}{\eta(\varepsilon)} \ll |\log\varepsilon|. \tag{3.23}$$

But $x \in \operatorname{Supp}\mu_\varepsilon^i$ is equivalent to $|x - p_\varepsilon^i| = \varepsilon$. Then property (H2) of a_ε implies that $a_\varepsilon(x) \approx a_\varepsilon(p_\varepsilon^i)$ as $\varepsilon \to 0$. Replacing in (3.23) and integrating with respect to $\mu_\varepsilon^i \otimes \mu_\varepsilon^i$ yields

$$\iint G_\varepsilon \,\mathrm{d}\mu_\varepsilon^i \,\mathrm{d}\mu_\varepsilon^i \leqslant 2\pi a_\varepsilon\bigl(p_\varepsilon^i\bigr)|\log\varepsilon|\bigl(1 + \mathrm{o}_\varepsilon(1)\bigr)$$

and then, summing over $1 \leqslant i \leqslant n_\varepsilon$ and dividing by h_{ex},

$$\limsup_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2} \sum_{i=1}^{n_\varepsilon} \iint G_\varepsilon \,\mathrm{d}\mu_\varepsilon^i \,\mathrm{d}\mu_\varepsilon^i \leqslant \frac{\Lambda}{2} \limsup_{\varepsilon\to 0} \frac{2\pi \sum_{i=1}^{n_\varepsilon} a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}. \tag{3.24}$$

Here we have used the fact that $|\log\varepsilon| \sim \Lambda h_{\mathrm{ex}}$. Thus (3.20) is proved and the proposition follows. □

3.2. Proof of Proposition 3.2

Let $\mu = \sigma(x)\,\mathrm{d}x$, $C = \|u\|_\infty$ and $\alpha_0 = \operatorname{dist}(\operatorname{supp}\mu, \partial\Omega)$. Also, let

$$\widetilde{\Omega} = \bigl\{x \in \Omega \mid d(x, \partial\Omega) > \alpha_0/2\bigr\}. \tag{3.25}$$

Recall that from hypothesis (H3) on a_ε there exists a positive function $\delta(\varepsilon)$ such that

$$\delta(\varepsilon) \ll \frac{1}{(\log\,|\log\varepsilon|)^{1/2}}, \quad \text{and for any } x \in \Omega,\ \min_{B(x,\delta(\varepsilon))} \beta_\varepsilon = 0. \tag{3.26}$$

For any $\varepsilon > 0$, we tile $\mathbb{R}^2$ with open squares of sidelength $2\delta(\varepsilon)$ and let $\mathcal{K}(\varepsilon)$ be the family of those squares that are entirely inside $\widetilde{\Omega}$. We denote by c_K the center of a square K. Since μ is absolutely continuous with respect to the Lebesgue measure, we have $\mu(K) \leqslant C\delta^2$.

Now the family of points $(p_\varepsilon^i)_{1\leqslant i\leqslant n_\varepsilon}$ is defined as follows: for any $K\in\mathcal{K}(\varepsilon)$, we let

$$n(K,\varepsilon)=\left[\frac{h_{\mathrm{ex}}(\varepsilon)\mu(K)}{2\pi}\right], \tag{3.27}$$

where $[x]$ is the biggest integer no greater than x. Using (3.26) there is a point $p_K\in B(c_K,\delta)$ such that $\beta_\varepsilon(p_K)=0$ (p_K is a pinning site). We now pick $n(K,\varepsilon)$ points evenly scattered in the ball $B(p_K,1/h_{\mathrm{ex}})$, and we call $\mathcal{P}(K,\varepsilon)$ their union. By evenly scattered we mean that for any $p,q\in\mathcal{P}(K,\varepsilon)$,

$$|p-q|\geqslant\frac{C}{h_{\mathrm{ex}}\sqrt{n(K,\varepsilon)}}. \tag{3.28}$$

We let

$$n_\varepsilon=\sum_{K\in\mathcal{K}(\varepsilon)}n(K,\varepsilon),\quad\text{and}\quad\mathcal{P}(\varepsilon)=\bigcup_{K\in\mathcal{K}(\varepsilon)}\mathcal{P}(K,\varepsilon)=\left(p_\varepsilon^i\right)_{1\leqslant i\leqslant n_\varepsilon} \tag{3.29}$$

be our family of points. We now check that this family satisfies (3.3), (3.4), (3.5) and (3.7).

(3.3) is clear from (3.28) if p_ε^i, p_ε^j belong to the same pinning site. It is even more true if p_ε^i, p_ε^j do not belong to the same site since in this case their mutual distance is at least $2\delta(\varepsilon)\gg\varepsilon$. Moreover from (3.25) we have $d(p_\varepsilon^i,\partial\Omega)>\alpha_0/2$.

For (3.4), let

$$\mu_\varepsilon=\frac{2\pi}{h_{\mathrm{ex}}}\sum_{i=1}^{n_\varepsilon}\delta_{p_\varepsilon^i} \tag{3.30}$$

and f be a continuous function in $\overline{\Omega}$. We let $\gamma_\varepsilon=\sup_{K\in\mathcal{K}(\varepsilon)}\sup_{x,y\in K}|f(x)-f(y)|$. Then since the size of the squares in $\mathcal{K}(\varepsilon)$ tends to zero with ε, so does γ_ε. Let K_ε be the union of the squares in $\mathcal{K}(\varepsilon)$, then for ε small enough $\operatorname{supp}\mu\subset K_\varepsilon$ and

$$\left|\int f\,\mathrm{d}\mu-\int f\,\mathrm{d}\mu_\varepsilon\right|\leqslant\|f\|_\infty\sum_{K\in\mathcal{K}(\varepsilon)}\left|\mu(K)-\mu_\varepsilon(K)\right|+\gamma_\varepsilon(\mu_\varepsilon+\mu)(K_\varepsilon).$$

It is clear that the second term on the right-hand side goes to zero with ε. For the first term we note that from (3.27), (3.30), we have $|\mu(K)-\mu_\varepsilon(K)|\leqslant 2\pi/h_{\mathrm{ex}}$ while the number of squares in $\mathcal{K}(\varepsilon)$ is of the order of $1/\delta^2$. From (3.26) it then follows that $\sum_{K\in\mathcal{K}(\varepsilon)}|\mu(K)-\mu_\varepsilon(K)|$ tends to zero with ε. We thus have $\lim_{\varepsilon\to0}\int f\,\mathrm{d}\mu_\varepsilon=\int f\,\mathrm{d}\mu$ and (3.4) follows.

We easily deduce (3.7) from (3.4). Indeed from (H2) and the fact that each point is at a distance at most $1/h_{\mathrm{ex}}$ from a pinning site, we get that $a_\varepsilon(p)\approx b(p)$ as $\varepsilon\to0$, uniformly in $p\in\mathcal{P}(\varepsilon)$. Moreover, since $n_\varepsilon/h_{\mathrm{ex}}$ is bounded,

$$\lim_{\varepsilon\to0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}a_\varepsilon(p_\varepsilon^i)}{h_{\mathrm{ex}}}=\lim_{\varepsilon\to0}\frac{2\pi\sum_{i=1}^{n_\varepsilon}b(p_\varepsilon^i)}{h_{\mathrm{ex}}}=\int b(x)\,\mathrm{d}\mu(x),$$

by the convergence of μ_ε to μ.

It remains to prove (3.5). We split the sum in (3.5) as follows: let $\mathcal{I}(\varepsilon)$ be the set of pairs of indices (i,j) such that $1\leqslant i\neq j\leqslant n_\varepsilon$ and p_ε^i, p_ε^j belong to the same square of the subdivision

$\mathcal{K}(\varepsilon)$. Let $\mathcal{J}(\varepsilon)$ be pairs (i, j) such that p_ε^i, p_ε^j belong to different squares. Then

$$\sum_{\substack{i\neq j\\ |p_\varepsilon^i - p_\varepsilon^j|<\alpha}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right| = \sum_{\substack{(i,j)\in\mathcal{I}(\varepsilon)\\ |p_\varepsilon^i - p_\varepsilon^j|<\alpha}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right| + \sum_{\substack{(i,j)\in\mathcal{J}(\varepsilon)\\ |p_\varepsilon^i - p_\varepsilon^j|<\alpha}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right|. \tag{3.31}$$

The first sum in (3.31) is estimated as follows. For every $K \in \mathcal{K}(\varepsilon)$, $\mu(K) < C\delta^2$ thus the number of points of $\mathcal{P}(\varepsilon)$ in K is less than $C\delta^2 h_{\mathrm{ex}}$. The number of squares being of the order of δ^{-2}, the cardinal of $\mathcal{I}(\varepsilon)$ is less than $C\delta^2 h_{\mathrm{ex}}^2$. Using (3.26), (3.27) and (3.28), we find

$$\sum_{\substack{(i,j)\in\mathcal{I}(\varepsilon)\\ |p_\varepsilon^i - p_\varepsilon^j|<\alpha}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right| \leqslant C h_{\mathrm{ex}}^2 \delta^2 \log\ |\log\varepsilon| \ll h_{\mathrm{ex}}^2. \tag{3.32}$$

To treat the second sum in (3.31), we note that if K and K' are distinct squares in $\mathcal{K}(\varepsilon)$ and $p \in K$, $q \in K'$ then

$$\forall x \in K,\ \forall y \in K', \quad |x-y| \leqslant 4|p-q|.$$

Thus we may write, using the fact that $\mu(K) < C\delta^2$:

$$\sum_{\substack{i\neq j\\ p_\varepsilon^i\in K,\, p_\varepsilon^j\in K'}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right| \leqslant C h_{\mathrm{ex}}^2 \iint_{K\times K'} \left(\left|\log|x-y|\right| + 1\right) \mathrm{d}x\,\mathrm{d}y.$$

Summing over pairs of squares $K, K' \in \mathcal{K}(\varepsilon)$ such that $K \times K'$ intersects $\{(x, y) \mid |x-y| < \alpha\}$ we get for ε small enough

$$\sum_{\substack{(i,j)\in\mathcal{J}(\varepsilon)\\ |p_\varepsilon^i - p_\varepsilon^j|<\alpha}} \left|\log\left|p_\varepsilon^i - p_\varepsilon^j\right|\right| \leqslant C h_{\mathrm{ex}}^2 \iint_{|x-y|<2\alpha} \left(\left|\log|x-y|\right| + 1\right) \mathrm{d}x\,\mathrm{d}y. \tag{3.33}$$

Summing (3.32), (3.33), dividing by h_{ex}^2 and letting ε and then α tend to zero yields (3.5). Proposition 3.2 is proved. □

3.3. Proof of Lemma 3.1

The fact that G_ε and G_0 are positive is a simple consequence of the maximum principle, that they are symmetric is standard and follows from Green's identity.

The inequality

$$G_\varepsilon(x, y), G_0(x, y) < -C \log|x-y| + C$$

is a well known property of Green functions for elliptic operators in divergence form, a proof can be found in [32].

To prove property (3), we let:

$$v_\varepsilon(x, y) = G_\varepsilon(x, y) + \frac{a_\varepsilon(y)}{2\pi} \log|x-y|$$

and L_ε be the operator $u \mapsto -\operatorname{div}(\boldsymbol{\mathcal{A}}_\varepsilon \nabla u) + u$. Then letting $f_\varepsilon = L_\varepsilon v_\varepsilon(\cdot, y)$, we have

$$f_\varepsilon(x, y) = -\frac{a_\varepsilon(y)}{2\pi} \nabla \frac{1}{a_\varepsilon(x)} \cdot \nabla_x \log|x-y| - \frac{a_\varepsilon(y)}{2\pi} \log|x-y|. \tag{3.34}$$

Thus for any $1 \leqslant q < 2$, there is a C independent of y and ε, such that $\|f_\varepsilon(\cdot, y)\|_{L^q} \leqslant C/\eta(\varepsilon)$. On the other hand, $v_\varepsilon(\cdot, y)$ is bounded in $W^{1,q}(\Omega)$ independently of ε and y (see [32]).

Now, Theorem 2 of [21] implies that there exist $p > 2$ and $p' < 2$ such that if u satisfies $L_\varepsilon u = f$, then for any compact $K \subset \Omega$,

$$\|\nabla u\|_{L^p(K)} \leqslant C(K)\big(\|\nabla u\|_{L^{p'}(\Omega)} + \|f\|_{W^{-1,p}(\Omega)}\big).$$

We may choose $q < 2$ such that $W^{-1,p} \subset L^q$ and $p' < q$. Thus, we find that $v_\varepsilon(\cdot, y)$ is bounded in $W^{1,p}(K)$ by $C/\eta(\varepsilon)$. Since $p > 2$, this yields the uniform bound $\forall x \in K, \forall y \in \Omega$,

$$|v_\varepsilon(x, y)| \leqslant \frac{C(K)}{\eta(\varepsilon)},$$

i.e. property (3).

To prove property (4), we note that for any $\alpha > 0$, $L_\varepsilon G_\varepsilon(\cdot, y) = 0$ in $\Omega \setminus B(y, \alpha)$ while $G_\varepsilon(\cdot, y)$ is bounded in $W^{1,q}(\Omega)$ independently of ε and y (see [32]). Using the aforementioned result of [21], we find that $G_\varepsilon(\cdot, y)$ is bounded in $W^{1,p}_{\text{loc}}(\Omega \setminus B(y, \alpha))$, for some $p > 2$, independently of y and ε, thus G_ε converges locally uniformly in $\Omega \times \Omega \setminus \Delta$, where Δ is the diagonal. The limit is necessarily G_0, since $G_0(\cdot, y)$ satisfies $L_0 G_0(\cdot, y) = -\operatorname{div} \boldsymbol{\mathcal{A}}_0 \nabla_x G_0 + G_0 = \delta_y$ and L_ε H-converges to L_0. Lemma 2.1 is proved. □

4. Convergence results

We can then proceed as in the rest of Section III in [28].

PROPOSITION 4.3. – *The minimum of E is uniquely achieved by $h_* \in C^{1,\gamma}(\Omega)$ $(\forall \gamma < 1)$ satisfying*:

$$\begin{cases} h_* \geqslant 1 - \dfrac{\Lambda b}{2} & \textit{in } \Omega, \\ h_* = 1 & \textit{on } \partial\Omega, \\ \mu_* := -\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla h_*) + h_* \geqslant 0, & \\ \left(h_* - \left(1 - \dfrac{\Lambda b}{2}\right)\right)\mu_* = 0. & \end{cases} \tag{4.1}$$

As in [28], we divide the proof of this proposition into several lemmas.

LEMMA 4.1. – *Let μ_*^+ and μ_*^- be the positive and negative parts of the measure μ_*. Then*

$$h_* = 1 - \frac{\Lambda b}{2} \quad \mu_*^+ \textit{ a.e.},$$

$$h_* = 1 + \frac{\Lambda b}{2} \quad \mu_*^- \textit{ a.e.},$$

$$1 - \frac{\Lambda b}{2} \leqslant h_* \leqslant 1 + \frac{\Lambda b}{2}.$$

Proof. – As in [28], the minimum of E is achieved by some h_*, by lower semi-continuity. Performing variations $(1+tf)\mu_*$ where $f \in C^0(\Omega)$, and looking at the first order in $t \to 0$, we find similarly as in [28] that

$$\frac{\Lambda b}{2}|\mu_*| + (h_* - 1)\mu_* = 0.$$

Hence,

$$h_* = 1 - \frac{\Lambda b}{2} \quad \mu_*^+ \text{ a.e.},$$
$$h_* = 1 + \frac{\Lambda b}{2} \quad \mu_*^- \text{ a.e.}$$

As in [28], considering variations $\mu_* + \nu$, where $\nu \in \mathcal{M} \cap H^{-1}$ and ν and μ_* are mutually singular, we are led to $1 - \frac{\Lambda b}{2} \leqslant h_* \leqslant 1 + \frac{\Lambda b}{2}$. □

LEMMA 4.2. – *μ_* is a positive measure.*

Proof. –

$$\int_\Omega \mu_*(h_* - 1)_+ = \int_\Omega \mu_*^+(h_* - 1)_+ - \int_\Omega \mu_*^-(h_* - 1)_+.$$

Since $(h_* - 1)_+ = 0$ μ_*^+-a.e., we have

$$\begin{aligned}\int_\Omega \mu_*(h_* - 1)_+ &= -\int_\Omega \mu_*^-(h_* - 1)_+ \\ &= \int_\Omega \big(-\operatorname{div}(\boldsymbol{\mathcal{A}}_0 \nabla h_*) + h_*\big)(h_* - 1)_+ \\ &= \int_{h_*>1} \nabla h_* \cdot (\boldsymbol{\mathcal{A}}_0 \nabla h_*) + h_*(h_* - 1) \geqslant 0,\end{aligned}$$

because $\boldsymbol{\mathcal{A}}_0$ is a symmetric positive matrix (this follows from the compactness of the set of matrices bounded from above and below). We deduce that

$$\int_\Omega \mu_*^-(h_* - 1)_+ = 0,$$

but since $h_* - 1 = \Lambda b/2$, μ_*^- a.e., we have

$$\int_\Omega \frac{\Lambda b}{2}\mu_*^- = 0,$$

hence $\mu_*^- = 0$, and $\mu_* \geqslant 0$. □

Thus, h_* satisfies all the properties listed in (4.1).

We can now complete the convergence results. From the upper bound of Proposition 1.4 and Lemma 2.4, we deduce that for our family of minimizers $(u_\varepsilon, A_\varepsilon)$,

$$\min_V E = E(h_*) \geqslant \liminf_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}^2} \geqslant E(h_0) \geqslant E(h_*).$$

h_* being the unique minimizer of E, we conclude that $h_0 = h_*$ and thus $\mu_0 = \mu_*$. We also obtain

$$\lim_{\varepsilon\to 0} \frac{J_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} = E(h_*). \tag{4.2}$$

Since the possible limits are unique, the whole family $h_\varepsilon/h_{\mathrm{ex}}$ converges to h_*, and the same for μ_ε.

In view of (2.37), we have

$$\liminf_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \liminf_{\varepsilon\to 0} \frac{1}{2}\left(\frac{|\log\varepsilon|}{h_{\mathrm{ex}}} \int_\Omega a_\varepsilon|\mu_\varepsilon|\right) + \frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2$$
$$\geqslant \frac{\Lambda}{2}\int_\Omega b|\mu_*| + \frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2,$$

while

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \leqslant \frac{\Lambda}{2}\int_\Omega b|\mu_*| + \frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2.$$

Thus, we deduce that

$$\lim_{\varepsilon\to 0}\int_\Omega a_\varepsilon|\mu_\varepsilon| = \int_\Omega b\mu_*.$$

On the other hand,

$$\liminf_{\varepsilon\to 0}\int_\Omega a_\varepsilon|\mu_\varepsilon| \geqslant \liminf_{\varepsilon\to 0}\int_\Omega b|\mu_\varepsilon| \geqslant \int_\Omega b|\mu_*|,$$

hence $\int_\Omega b|\mu_\varepsilon| \to \int_\Omega b\mu_*$, while $\int_\Omega b\mu_\varepsilon \to \int_\Omega b\mu_*$. We conclude that $\int_\Omega b(|\mu_\varepsilon| - \mu_\varepsilon) \to 0$ and thus $|\mu_\varepsilon|$ and μ_ε have the same limiting measure μ_*. This proves (1.16), (1.17), and (1.18).

Following [28], Section IV, we can also prove easily the following:

PROPOSITION 4.2. – *If $\Lambda = 0$, then $h_* = 1$ and $\frac{h_\varepsilon}{h_{\mathrm{ex}}} - 1 \to 0$ strongly in $H_0^1(\Omega)$. If $\Lambda > 0$, then $\frac{h_\varepsilon}{h_{\mathrm{ex}}} - 1 \rightharpoonup h_* - 1$ in $H_0^1(\Omega)$, the convergence is not strong and*

$$\frac{|\nabla h_\varepsilon|^2}{h_{\mathrm{ex}}^2 a_\varepsilon} \to \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + \Lambda b\mu_* \quad \text{in } \mathcal{M}.$$

Proof. – First, it is easy to get, as seen in Lemma 2.4 for example, that

$$\int_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 \geqslant \int_\Omega \frac{|\nabla h_\varepsilon|^2}{a_\varepsilon}\bigl(1 - \mathrm{o}(1)\bigr),$$

thus, we have

$$\liminf_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{\mathrm{ex}}^2} \geqslant \liminf_{\varepsilon\to 0} \frac{1}{h_{\mathrm{ex}}^2}\left(\frac{1}{2}\int_\Omega \frac{|\nabla h_\varepsilon|^2}{a_\varepsilon} + |h_\varepsilon - h_{\mathrm{ex}}|^2\right) \tag{4.3}$$

$$\geqslant \frac{\Lambda}{2}\int_\Omega b\mu_* + \frac{1}{2}\int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + |h_* - 1|^2. \tag{4.4}$$

The case $\Lambda = 0$ follows easily from the upper bound $\min J_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant o(h_{ex}^2)$ of Section 2 combined with (4.4).

The convergence of h_ε/h_{ex} to h_* is weak in H^1, in general, thus strong in $L^2(\Omega)$, and

$$\lim_{\varepsilon\to 0} \int_\Omega \left| \frac{h_\varepsilon}{h_{ex}} - 1 \right|^2 = \int_\Omega |h_* - 1|^2.$$

Combining this to the convergence result (4.2), we have

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int_\Omega \frac{|\nabla h_\varepsilon|^2}{h_{ex}^2 a_\varepsilon} = \frac{\Lambda}{2} \int_\Omega b\mu_* + \frac{1}{2} \int_\Omega \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_*. \tag{4.5}$$

Then, we argue as in [28], Proposition IV.1. Roughly speaking, one considers any open set $U \subset \Omega$, and gets a lower bound

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_U \frac{|\nabla h_\varepsilon|^2}{h_{ex}^2 a_\varepsilon} &= \liminf_{\varepsilon\to 0} \int_{U\cap(\cup_i B_i)} \frac{|\nabla h_\varepsilon|^2}{h_{ex}^2 a_\varepsilon} + \int_{U\setminus \cup_i B_i} \frac{|\nabla h_\varepsilon|^2}{h_{ex}^2 a_\varepsilon} \\ &\geqslant \Lambda \int_U b|\mu_\varepsilon| + \int_U \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* \geqslant \Lambda \int_U b\mu_* + \int_U \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_*.\end{aligned}$$

Since this is true for any $U \subset \Omega$, comparing this to (4.4) and (4.5), we obtain as in [28],

$$\frac{|\nabla h_\varepsilon|^2}{h_{ex}^2 a_\varepsilon} \to \nabla h_* \cdot \boldsymbol{\mathcal{A}}_0 \nabla h_* + \Lambda b \mu_* \quad \text{in } \mathcal{M}. \qquad \square$$

This completes the proof of Theorems 1, 2 and 3.

Acknowledgments

The authors are very grateful to François Murat for taking time explaining the basis of homogenization and pointing out the good references. They would also like to thank very much Jon Chapman for fruitful discussions on pinning models and Alano Ancona for pointing out references on Green functions.

REFERENCES

[1] N. ANDRÉ, I. SHAFRIR, Asymptotic behavior of minimizers for the Ginzburg–Landau functional with weight. I, II, *Arch. Rational Mech. Anal.* 142 (1) (1998) 45–73, 75–98.

[2] A. BEAULIEU, R. HADIJI, A Ginzburg–Landau problem with weight having minima on the boundary, *Proc. Roy. Soc. Edinburgh Sect. A* 128 (6) (1998) 1181–1215.

[3] F. BETHUEL, H. BREZIS, F. HÉLEIN, *Ginzburg–Landau Vortices*, Birkhäuser, 1994.

[4] G. BLATTER, M.V. FEIGEL'MAN, V.B. GESHKENBEIN, A.I. LARKIN, V.M. VINOKUR, Vortices in high temperature superconductors, *Rev. Mod. Physics* 66 (1994) 112.5.

[5] A. BONNET, R. MONNEAU, Existence of a smooth free-boundary in a superconductor with a Nash–Moser inverse function theorem argument, to appear in *Interfaces and Free Boundaries*.

[6] F. BETHUEL, T. RIVIÈRE, Vortices for a variational problem related to superconductivity, *Annales IHP, Analyse Non Linéaire* 12 (1995) 243–303.

[7] A. BOSSAVIT, *Notes sur les modèles macroscopiques de supraconducteurs en vue du calcul numérique*, Pont-à-Mousson, 1994.

[8] S.J. CHAPMAN, Macroscopic models of superconductivity, *Proc. NATO ASI*, 1999.

[9] S.J. CHAPMAN, Macroscopic models of superconductivity, *Proc. 4th ICIAM*, 1999.

[10] S.J. CHAPMAN, Q. DU, M.D. GUNZBURGER, A Ginzburg–Landau type model of superconducting/normal junctions including Josephson junctions, *Eur. J. Appl. Math.* 6 (1995) 97–114.

[11] S.J. CHAPMAN and G. RICHARDSON, Vortex pinning by inhomogeneities in type II superconductors, preprint.

[12] S.J. CHAPMAN, J. RUBINSTEIN, M. SCHATZMAN, A mean-field model of superconducting vortices, *Eur. J. Appl. Math.* 7 (2) (1996) 97–111

[13] D. CIORANESCU, P. DONATO, *Introduction to Homogenization*, Oxford University Press, 1999.

[14] D. CIORANESCU, F. MURAT, Un terme étrange venu d'ailleurs, in: *Nonlinear Partial Differential Equations and Their Applications, Coll. de France Semin., Vol. II*, Res. Notes Math., Vol. 60, 1982, pp. 98–138.

[15] M. CYROT, D. PAVUNA, *Introduction to Superconductivity and High-T_c Materials*, World Scientific.

[16] Q. DU, M.D. GUNZBURGER, J.S. PETERSON, Computational simulations of type II superconductivity including pinning phenomena, *Ph. Rev. B* 51 (22) (1995) 16194–16203.

[17] V.V. JIKOV, S.M. KOZLOV, O.A. OLEINIK, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, 1994.

[18] D. KINDERLEHRER, G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Pure and Applied Mathematics, Vol. 88, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York–London, 1980.

[19] L. LASSOUED, P. MIRONESCU, Ginzburg Landau type energy with discontinuous constraint, Preprint.

[20] K. LIKHAREV, Superconducting weak links, *Rev. Mod. Phys.* 51 (1979) 101–159.

[21] N.G. MEYERS, An L^p estimate for the gradient of solutions of second order elliptic equations, *Ann. Sc. Norm. Sup. Pisa* 17 (1963) 189–206.

[22] F. MURAT, L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* 60 (1981) 309–322.

[23] F. MURAT, L. TARTAR, *H-convergence, Topics in the Mathematical Modelling of Composite Materials*, Progress in Nonlinear Differential Equations and Their Applications, Birkhaüser, Boston, 1994.

[24] L.M. PISMEN, *Vortices in Nonlinear Fields*, International Series of Monographs on Physics, Vol. 100, Oxford Science Publications, 1999.

[25] E. SANDIER, Lower bounds for the energy of unit vector fields and applications, *J. Functional Analysis* 152 (2) (1998) 379–403.

[26] E. SANDIER, S. SERFATY, Global minimizers for the Ginzburg–Landau functional below the first critical magnetic field, *Annales IHP, Analyse Non Linéaire* 17 (1) (2000) 119–145.

[27] E. SANDIER, S. SERFATY, On the energy of type-II superconductors in the mixed phase, to appear in *Reviews in Math. Phys.*

[28] E. SANDIER, S. SERFATY, A rigorous derivation of a free-boundary problem arising in superconductivity, to appear in *Annales Scientifiques de l'ENS*.

[29] S. SERFATY, Local minimizers for the Ginzburg–Landau energy near critical magnetic field, Part I, *Comm. Contemporary Mathematics* 1 (2) (1999) 213–254.

[30] S. SERFATY, Local minimizers for the Ginzburg–Landau energy near critical magnetic field, Part II, *Comm. Contemporary Mathematics* 1 (3) (1999) 295–333.

[31] S. SERFATY, Stable configurations in superconductivity: uniqueness, multiplicity and vortex-nucleation, *Arch. Rat. Mech. Anal.* 149 (4) (1999) 329–365.

[32] G. STAMPACCHIA, *Equations elliptiques du second ordre à coefficients discontinus*, Presses de l'Université de Montréal, Montréal, Québec, 1966.

[33] M. TINKHAM, *Introduction to Superconductivity*, McGraw-Hill, 1996, 2nd ed.