



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa


Extensions of an $AC(\sigma)$ functional calculus

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ARTICLE INFO

Article history:

Received 14 March 2008

Available online 6 October 2009

Submitted by G. Corach

Keywords:

Functional calculus

Well-bounded operators

 $AC(\sigma)$ operators

ABSTRACT

On a reflexive Banach space X , if an operator T admits a functional calculus for the absolutely continuous functions on its spectrum $\sigma(T) \subseteq \mathbb{R}$, then this functional calculus can always be extended to include all the functions of bounded variation. This need no longer be true on nonreflexive spaces. In this paper, it is shown that on most classical separable nonreflexive spaces, one can construct an example where such an extension is impossible. Sufficient conditions are also given which ensure that an extension of an AC functional calculus is possible for operators acting on families of interpolation spaces such as the L^p spaces.

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1. Introduction

Given an operator T on a Banach space X , it is often important to be able to identify algebras of functions \mathcal{U} for which one may sensibly assign a meaning to $f(T)$ for all $f \in \mathcal{U}$. In many classical situations, the possession of a functional calculus for a small algebra is enough to ensure an extension of the functional calculus map to a large algebra. For example, many proofs of the spectral theorem for normal operators on a Hilbert space first show that such an operator T must admit a $C(\sigma(T))$ functional calculus, and then proceed to extend this functional calculus to all bounded Borel measurable functions.

Whether a functional calculus for an operator T has a nontrivial extension depends crucially on the space on which T acts. For example, the operator $Tx(t) = tx(t)$ has an obvious $C[0, 1]$ functional calculus on both $X = C[0, 1]$ and on $X = L^\infty[0, 1]$, but only in the latter case does a nontrivial extension exist. Many of the positive theorems that exist in this area come as easy corollaries of theorems which show that a particular functional calculus is sufficient to ensure that an operator admits an integral representation with respect to a family of projections. The integration theory for these families then provides a natural extension of the original functional calculus. Often however, it is sufficient to know that one has a large family of projections which commutes with T , and one is less concerned with the topological properties which are typically required of such families in order to produce a satisfactory integration theory. In this case, what one is interested in is whether one can show that T has a functional calculus for an algebra which contains a large number of idempotent functions.

An operator with a (norm bounded) functional calculus for the absolutely continuous functions on a compact set $\sigma \subseteq \mathbb{C}$ is said to be an $AC(\sigma)$ operator. (We refer the reader to [2] for the definitions of the function spaces $AC(\sigma)$ and $BV(\sigma)$.) In the case where $\sigma \subseteq \mathbb{R}$, such operators have been more commonly referred to as well-bounded operators (see [6]), although we prefer the more descriptive term real $AC(\sigma)$ operators. In this case, as the polynomials are dense in $AC(\sigma)$, the $AC(\sigma)$

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¹ The author would like to acknowledge the receipt of an Australian Postgraduate Award. Some of this research was supported by the Australian Research Council Discovery Grant DP0559097.

functional calculus is necessarily unique. Real $AC(\sigma)$ operators were introduced by Smart and Ringrose [9,8] in order to provide a theory which had similarities to the theory of self-adjoint operators, but which dealt with the conditionally convergent spectral expansions which are more common once one leaves the Hilbert space setting.

If X is reflexive, or more generally if the functional calculus has a certain compactness property, then every real $AC(\sigma)$ operator $T \in B(X)$ admits an integral representation with respect to a spectral family of projections on X . A real $AC(\sigma)$ operator with such a representation is said to be of type (B). The integration theory for spectral families shows that one may extend the functional calculus to the idempotent rich algebra $BV(\sigma)$ of all functions of bounded variation on σ . That is, there exists a norm continuous algebra homomorphism $\Psi : BV(\sigma) \rightarrow B(X)$ such that $\Psi(f) = f(T)$ for all $f \in AC(\sigma)$. The extent to which this theory can be extended to the case where $\sigma \not\subseteq \mathbb{R}$ is not yet known, and so in this paper we shall restrict our attention almost exclusively to case where $\sigma \subseteq \mathbb{R}$. (We refer the reader to [6] for the basic integration theory of real $AC(\sigma)$ operators.)

There are many examples of $AC(\sigma)$ operators on nonreflexive spaces which admit a $BV(\sigma)$ functional calculus despite failing to have a spectral family decomposition. It is natural to ask whether there are any nonreflexive spaces on which every $AC(\sigma)$ operator admits an extended functional calculus, or whether there are any easily checked conditions which might ensure that such an extension exists. The aim of this paper is twofold. First we show that if X contains a complemented copy of c_0 or a complemented copy of ℓ^1 , then there is an operator $T \in B(X)$ which admits an $AC(\sigma)$ functional calculus which does not have any extension to $BV(\sigma)$. In the second half of the paper we shall give sufficient conditions for an extension of the functional calculus which apply to linear transformations that act as operators on a range of L^p spaces.

It was shown in [4] that if (Ω, Σ, μ) is a finite measure space and T is a real $AC(\sigma)$ operator on $L^1(\Omega, \Sigma, \mu)$ and $L^p(\Omega, \Sigma, \mu)$ for any $p > 1$ then T admits a spectral family decomposition on $L^1(\Omega, \Sigma, \mu)$ and consequently T has a $BV(\sigma)$ functional calculus on that space. The hypothesis that T be an $AC(\sigma)$ operator on some L^p space other than L^1 is vital here; the operator

$$Tu(t) = tu(t) + \int_0^t u(s) ds, \quad t \in [0, 1],$$

is a real $AC[0, 1]$ operator on $L^1[0, 1]$, but it does not admit a $BV[0, 1]$ functional calculus [5, p. 170]. As we shall show in Section 3, the hypothesis that $\mu(\Omega)$ be finite can be omitted if one only wishes to deduce the existence of a $BV(\sigma)$ functional calculus.

More delicate is the situation for operators acting on L^∞ , and there are several open questions that remain. In practice, however, concrete operators on this space often have additional properties which enable one to establish that an extended functional calculus exists. This will be examined in more detail in Section 4.

Some care needs to be taken in addressing these questions. Even on Hilbert space, extensions need not be unique. For example, the operator T on ℓ^2 ,

$$T(x_0, x_1, x_2, \dots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right)$$

admits an $AC(\sigma(T))$ functional calculus, but both the maps

$$\Phi_1(f)(x_0, x_1, x_2, \dots) = \left(f(0)x_0, f(1)x_1, f\left(\frac{1}{2}\right)x_0, \dots\right),$$

$$\Phi_2(f)(x_0, x_1, x_2, \dots) = \left(\lim_{n \rightarrow \infty} f\left(\frac{1}{n}\right)x_0, f(1)x_1, f\left(\frac{1}{2}\right)x_0, \dots\right)$$

are bounded algebra homomorphisms from $BV(\sigma(T))$ to $B(\ell^2)$ which extend the $AC(\sigma(T))$ functional calculus.

2. Nonreflexive spaces on which an extension need not exist

There are various examples of real $AC(\sigma)$ operators in the literature which do not admit any $BV(\sigma)$ functional calculus extension (see, for example, [5]). We are not aware however of any places where the impossibility of an extension is explicitly proven. Note that this requires more than just showing that the formula defining $f(T)$ for $f \in AC(\sigma)$ doesn't work for $f \in BV(\sigma)$.

Suppose that $T \in B(X)$ is a real $AC(\sigma)$ operator which has a $BV(\sigma)$ functional calculus. Suppose $\lambda \in \sigma$. The following standard calculation is based on results such as Theorem 15.8 of [6], or Theorem 1.4.10 of [7].

Let $\chi_L = \chi_{\sigma \cap (-\infty, \lambda]}$ and $\chi_R = \chi_{\sigma \cap (\lambda, \infty]}$ so that $\chi_L + \chi_R = 1$ in $BV(\sigma)$. Let $P = P_\lambda = \chi_L(T)$ and $Q = I - P = \chi_R(T)$. Define subsets $L_\lambda, R_\lambda \subseteq AC(\sigma)$ by

$$L_\lambda = \{f \in AC(\sigma) : f(t) = 0 \text{ for } t \leq \lambda\},$$

$$R_\lambda = \{f \in AC(\sigma) : f(t) = 0 \text{ for } \lambda \leq t\}.$$

Proposition 2.1. *The ranges of the projections P and Q satisfy*

$$PX \subseteq \{x \in X: f(T)x = 0 \text{ for all } f \in L_\lambda\},$$

$$QX \subseteq \{x \in X: f(T)x = 0 \text{ for all } f \in R_\lambda\}.$$

Proof. Note that $f \in L_\lambda \Leftrightarrow f\chi_L = 0 \Leftrightarrow f = f(1 - \chi_L)$. Thus

$$Px = x \Leftrightarrow (I - P)x = 0 \Leftrightarrow (1 - \chi_L)(T)x = 0$$

and so if $Px = x$ and $f \in L_\lambda$, then $f(T)x = f(T)(1 - \chi_L)(T)x = 0$. Since $f \in R_\lambda \Leftrightarrow f\chi_R = 0 \Leftrightarrow f = f(1 - \chi_R)$, the proof for Q is identical. \square

Throughout what follows let $\sigma_0 = \{0\} \cup \{(-1)^k/k\}_{k=1}^\infty$.

Proposition 2.2. *There exists an operator T on c_0 , which admits an $AC(\sigma_0)$ functional calculus but no $BV(\sigma_0)$ functional calculus.*

Proof. First note that the map $U : c_0 \rightarrow C(\sigma_0)$,

$$U(x_0, x_1, x_2, \dots)(t) = \begin{cases} x_0, & t = 0, \\ x_0 + x_k, & t = \frac{(-1)^k}{k}, \end{cases}$$

is an isomorphism, so it suffices to construct an example on $X = C(\sigma_0)$. Define $T \in B(X)$ by $Tx(t) = tx(t)$. Note that $\sigma(T) = \sigma_0$. For $f \in AC(\sigma(T))$, $f(T)x = fx$ and so

$$\|f(T)\| \leq \|f\|_\infty \leq \|f\|_{AC(\sigma_0)}.$$

That is, T has an $AC(\sigma_0)$ functional calculus. Suppose now that this functional calculus can be extended to a $BV(\sigma_0)$ functional calculus. Let $Y = \{x \in C(\sigma_0): x(0) = 0\}$. Define complementary projections $P = \chi_{\sigma_0 \cap [-1, 0]}(T)$ and $Q = I - P = \chi_{\sigma_0 \cap (0, 1/2]}(T)$. Then (as in the proposition)

$$PX \subseteq \{x \in C(\sigma_0): f(T)x = 0 \text{ for all } f \in L_0\}.$$

If $f(T)x = 0$ for all $f \in L_0$, then $x(\frac{1}{2}) = x(\frac{1}{4}) = \dots = 0$, and so the continuity of x implies that $x(0) = 0$. Thus $PX \subseteq Y$. Similar reasoning shows that $QX \subseteq Y$ too. But this implies that every element $x = Px + Qx$ in $C(\sigma_0)$ actually lies in Y which gives the required contradiction. \square

One can construct an example on ℓ^1 in a similar way, although in this case we need to use the less standard space $AC(\sigma_0)$ in order to represent the operator in a simple form.

Lemma 2.3. ℓ^1 is isomorphic to $AC(\sigma_0)$.

Proof. Define $U : AC(\sigma_0) \rightarrow \ell^1$ by

$$U(g) = \left(g(-1), g\left(\frac{1}{2}\right) - g\left(\frac{1}{4}\right), g\left(-\frac{1}{3}\right) - g(-1), g\left(\frac{1}{4}\right) - g\left(\frac{1}{6}\right), \dots \right).$$

It is clear that

$$\|U(g)\|_1 \leq |g(-1)| + \text{var}_{\sigma_0} g \leq \|g\|_{AC(\sigma_0)}.$$

The inverse map is, writing $x = (x_1, x_2, \dots)$,

$$U^{-1}(x)(t) = \begin{cases} \sum_{j=1}^n x_{2j-1}, & t = \frac{-1}{2n-1}, \\ \sum_{j=1}^\infty x_{2j-1}, & t = 0, \\ \sum_{j=1}^\infty x_j - \sum_{j=1}^{n-1} x_{2j}, & t = \frac{1}{2n} \ (n \neq 1), \\ \sum_{j=1}^\infty x_j, & t = \frac{1}{2}. \end{cases}$$

Note that for this particular set σ_0 , we have that $AC(\sigma_0) = BV(\sigma_0) \cap C(\sigma_0)$. (One can readily verify this using the results from [2].) Thus in order to check that a function g is in $AC(\sigma_0)$, one need only check that it is of bounded variation, and that $\lim_{t \rightarrow 0} g(t)$ exists and equals $g(0)$. It is easy to check then that the image of U^{-1} is inside $AC(\sigma_0)$. Indeed $\|U^{-1}(x)\|_{AC(\sigma_0)} \leq 2\|x\|_1$, and hence U^{-1} is continuous. \square

The argument given in Proposition 2.2 goes through more or less unchanged if one replaces $C(\sigma_0)$ with $AC(\sigma_0)$. Thus there is an example of a real $AC(\sigma)$ operator on ℓ^1 whose functional calculus does not extend to $BV(\sigma)$.

Lemma 2.4. *Suppose that $T \in B(X)$ is a real $AC(\sigma)$ operator whose functional calculus does not extend to a $BV(\sigma)$ functional calculus. Let Y contain a complemented copy of X . Then, for a suitable compact set σ' , there is a real $AC(\sigma')$ operator on Y whose functional calculus also fails to extend to $BV(\sigma')$.*

Proof. Let $\sigma' = \sigma \cup \{\omega\}$ where $\omega = 1 + \max \sigma$. Write $Y = X \oplus Z$ and define $T' \in B(Y)$ by $T' = T \oplus \omega I_Z$. Then T' clearly has an $AC(\sigma')$ functional calculus $f(T') = f(T) \oplus f(\omega)I$. Suppose that this functional calculus admits an extension to $BV(\sigma')$. The important point to note is that the characteristic function $\chi_\sigma \in AC(\sigma')$ and hence the projection onto X , $P = \chi_\sigma(T') = I_X \oplus 0$, commutes with $f(T')$ for all $f \in BV(\sigma')$. This implies that we can write $f(T') = U(f) \oplus V(f)$. Now $BV(\sigma)$ embeds in a natural way into $BV(\sigma')$, and we shall write \tilde{f} for the image of f under this embedding. It is easy to check that the map $\psi : BV(\sigma) \rightarrow B(X)$, $f \mapsto U(\tilde{f})$ is a continuous Banach algebra homomorphism which extends the original functional calculus for T , contradicting our hypothesis. Hence the $AC(\sigma')$ functional calculus for T' cannot extend. \square

Theorem 2.5. *If X contains a complemented copy of c_0 or a complemented copy of ℓ^1 , then there exists a real $AC(\sigma)$ operator on X for which the functional calculus does not extend.*

Remark 2.6. This result bears a resemblance to Theorem 4.4 of [5] which shows that under similar hypotheses on X , there exists a real $AC(\sigma)$ operator on X which is not of type (B). The operators constructed in that paper however, do have a $BV(\sigma)$ calculus. Theorem 4.4 of [5] has been extended to cover an even wider range of nonreflexive spaces [3], but is it not clear to us how one might adapt these construction to the present situation.

The hypotheses of the theorem cover most of the classical separable nonreflexive spaces, but leave the situation for operators on spaces such as ℓ^∞ unclear. At present we do not have any examples of nonreflexive spaces on which every real $AC(\sigma)$ operator does have a $BV(\sigma)$ functional calculus.

3. Extrapolation to L^1

Let (Ω, Σ, μ) be a positive σ -finite measure space, and write L^p for $L^p(\Omega, \Sigma, \mu)$. A linear transformation T defined on equivalence classes of measurable functions $x : \Omega \rightarrow \mathbb{C}$ will be said to define a bounded operator on L^p if $L^p \subseteq \text{Dom}(T)$ and there exists $K_p < \infty$ such that $\|Tx\|_p \leq K_p \|x\|_p$ for all $x \in L^p$. In this case we shall often write T_p for the restriction of T to L^p .

There are two main issues that need addressing when transferring information about the functional calculus properties of an operator acting on one space L^p to a second space L^q . One concerns the consistency of the functional calculus. As the earlier example shows, the nonuniqueness of extensions means that one cannot expect too much in general.

Suppose that $p, q \neq \infty$. In the case when the extended functional calculus comes from a spectral family representation on each space, then these functional calculi must agree. To see this, suppose that σ_p and σ_q are compact subsets of \mathbb{R} . Note that if T defines an $AC(\sigma_p)$ operator on L^p and an $AC(\sigma_q)$ operator on L^q , then T has an $AC[a, b]$ functional calculus on both spaces for any compact interval $[a, b]$ containing both σ_p and σ_q (see [2, Section 2]). Lemma 3.3 of [4] then can be applied directly to ensure that the spectral families agree on L^p and L^q . A consequence of this is that

- (a) $\sigma(T_p) = \sigma(T_q)$ ($= \sigma$ say), and
- (b) T defines an $AC(\sigma)$ operator on both spaces.

Furthermore, if one uses the spectral family to define $BV(\sigma)$ functional calculi $\Psi_p : BV(\sigma) \rightarrow B(L^p)$ and $\Psi_q : BV(\sigma) \rightarrow B(L^q)$, then $\Psi_p(f)x = \Psi_q(f)x$ for all $f \in BV(\sigma)$ and all $x \in L^p \cap L^q$. (Further details of the constructions using $BV(\sigma)$ rather than $BV[a, b]$ are available in [1].)

The following proposition records some standard facts about operators which act on L^p spaces.

Proposition 3.1. *Let $1 \leq r < s \leq \infty$ and let K be a positive constant. Let S be a linear transformation which, for all $p \in (r, s)$, defines a bounded operator S_p on L^p with $\|S_p\|_p \leq K$. Then*

- (a) *there is a unique operator $U \in B(L^r)$ with $Ux = Sx$ for all $x \in \bigcap_{r \leq p < s} L^p$, and*
- (b) *there is an operator $V \in B(L^s)$ with $Vx = Sx$ for all $x \in \bigcap_{r < p \leq s} L^p$.*

The operator U in (a) satisfies $\|U\|_r \leq K$. The operator V in (b) is unique if $s < \infty$ and can be chosen to satisfy $\|V\|_s \leq K$.

Remark 3.2. In what follows we shall talk about the ‘extension’ of S to L^r or L^s , but it should be noted that this extension need not be proper, nor need it (if $s = \infty$) be unique. In particular, if L^∞ was in the original domain of definition of S , one might have that $V \neq S_\infty$ (see Example 4.1).

Proof. This is a standard exercise except for the case $s = \infty$.

Suppose then that $s = \infty$. We shall use p' to denote the conjugate exponent to p . We shall write $\langle x, y \rangle$ for $\int_{\Omega} x(t)y(t) d\mu$. Let $\mathcal{D}_r = \bigcap_{r < p < \infty} L^p$ and $\mathcal{D}'_r = \bigcap_{1 < q < r'} L^q$. For each $p \in (r, \infty)$, $(S_p)^*$ is a bounded linear operator on $L^{p'}$. A short calculation shows that for all $x \in \mathcal{D}_r$ and $y \in \mathcal{D}'_r$

$$\langle x, (S_{p_1})^* y \rangle = \langle x, (S_{p_2})^* y \rangle$$

for all $p_1, p_2 \in (r, \infty)$. Thus there exists a unique linear mapping $T : \mathcal{D}'_r \rightarrow \mathcal{D}'_r$ such that $\langle Sx, y \rangle = \langle x, Ty \rangle$ for all $x \in \mathcal{D}_r$ and $y \in \mathcal{D}'_r$. Clearly $\|Ty\|_q \leq K\|y\|_q$ for all $y \in \mathcal{D}'_r$ and $q \in (1, r')$.

Suppose now that $y \in \bigcap_{1 \leq q < r'} L^q$. Then

$$\|Ty\|_1 = \lim_{q \rightarrow 1^+} \|Ty\|_q \leq \lim_{q \rightarrow 1^+} K\|y\|_q = K\|y\|_1.$$

One may therefore extend T to a bounded linear operator T_1 on all of L^1 . Let $V = (T_1)^* \in B(L^\infty)$. Clearly $\|V\|_\infty \leq K$. We want to show that V is an extension of the linear map S .

Suppose then that $x \in \bigcap_{r < p \leq \infty} L^p$. For any $y \in \bigcap_{1 \leq q < r'} L^q$,

$$\langle Sx, y \rangle = \langle x, Ty \rangle = \langle Vx, y \rangle. \tag{3.1}$$

The norm density of $\bigcap_{1 \leq q < r'} L^q$ in L^1 is now sufficient to deduce that (3.1) is true for all $y \in L^1$, and therefore that $Vx = Sx$. \square

The main issue then in wanting to extend the definition of $f(T)$ from one L^p space to another is showing that one does not lose the property that the map $f \mapsto f(T)$ is an algebra homomorphism.

Theorem 3.3. *Let $1 \leq r < s \leq \infty$ and let T be a linear transformation defining real AC(σ) operators, necessarily of type (B), T_p on L^p for all $p \in (r, s)$. If the AC functional calculi for the operators T_p are uniformly bounded (by M say) for $p \in (r, s)$ then the domain of T can be extended (if necessary) so that T defines a real AC(σ) operator on L^r . Furthermore, the AC(σ) functional calculus for T_r extends to a BV(σ) functional calculus.*

Proof. The hypotheses imply that for each $p \in (r, s)$, T_p has an integral representation with respect to a spectral family and consequently a BV(σ) functional calculus Ψ_p . As noted at the start of this section, these maps can be chosen so that

- (a) $\|\Psi_p\| \leq M$ for all $p \in (r, s)$, and
- (b) $\Psi_p(f)x = \Psi_q(f)x$ for all $p, q \in (r, s)$, all $f \in BV(\sigma)$ and all $x \in L^p \cap L^q$.

Thus, for each $f \in BV(\sigma)$, there is a linear transformation $\Psi(f)$ that defines the operators $\Psi_p(f)$ for all $p \in (r, s)$. By Proposition 3.1, the domain of each $\Psi(f)$ may be extended so that it defines an operator $U_f \in B(L^r)$ with $\|U_f\| \leq M\|f\|_{BV}$. As $L^p \cap L^r$ is dense in L^r , it is easy to verify that the map $f \mapsto U_f$ is an algebra homomorphism from $BV(\sigma)$ into $B(L^r)$. For example, if $f, g \in BV(\sigma)$ and $x \in L^p \cap L^r$,

$$U_{fg}x = \Psi_p(fg)x = \Psi_p(f)\Psi_p(g)x = \Psi_p(f)U_gx = U_fU_gx$$

as $U_gx \in L^p \cap L^r$.

Hence $\Psi_r : BV(\sigma) \rightarrow B(L^r)$, $\Psi_r(f) = U_f$ defines a BV(σ) functional calculus for T on L^r (and this does extend the uniquely determined AC(σ) functional calculus for T_r). \square

We note that in [4, Theorem 4.3], it is shown that the conditions of the above theorem are necessary and sufficient for the extension of the map T to L^r to be a real AC(σ) operator.

As we shall see in the next section, in the L^∞ version of this result the uniform boundedness is not necessary. Example 5.2 of [4] gives an operator which has a BV(σ) functional calculus on ℓ^p for $p \in (1, \infty]$, but for which the AC functional calculus is not uniformly bounded on these spaces.

The Riesz–Thorin interpolation theorem gives the following application of the theorem. This corollary covers, for example, the case where T is a real AC(σ) operator on L^1 and is self-adjoint on L^2 .

Corollary 3.4. *If, for some $s > 1$, a linear map T defines real AC(σ) operators T_1 and T_s on L^1 and L^s respectively, then the operator T_1 on L^1 has a BV(σ) functional calculus.*

Proof. The hypotheses imply that T defines a real AC(σ) operator on L^p for all $p \in (1, s)$ with a uniform bound on the AC(σ) functional calculus on these spaces. The result now follows by Theorem 3.3. \square

4. Operators on L^∞

The situation for operators on L^∞ is quite different to that for operators on L^1 . The main problem is not that operators cannot be extended to L^∞ , but rather that the extensions need not be unique. In particular, transformations that give rise to different operators on L^∞ may give identical operators on another L^p space.

Example 4.1. Choose a Banach limit in $L \in (l^\infty)^*$. For $x \in l^\infty$, define $Tx := (Lx, 0, 0, \dots)$. Let $1 \leq r < \infty$; then T defines a bounded linear operator on l^r and on l^∞ . On l^r , T is the zero operator; on l^∞ , $\|T\|_\infty = 1$. The (nonproper) extension of T to l^p for $r < p < \infty$ is the zero operator and applying Proposition 3.1 to this map on l^p for $r \leq p < \infty$ yields the zero operator on l^∞ . Of course, the zero operator is a real $AC(\{0\})$ operator with a $BV(\{0\})$ functional calculus. The original map T is not a real $AC(\sigma)$ operator on l^∞ however, since it is a nonzero nilpotent operator.

Theorem 4.2. Let $1 < r < \infty$ and let T be a linear transformation defining real $AC(\sigma)$ operators (of type (B)) T_p on L^p for all $p \in (r, \infty)$. If the $AC(\sigma)$ functional calculi for the operators T_p are uniformly bounded (by M say) for $p \in (r, \infty)$ then there exists an operator $\tilde{T} \in B(L^\infty)$ such that

- (a) \tilde{T} is a real $AC(\sigma)$ operator,
- (b) $\tilde{T}x = Tx$ for all $x \in \bigcap_{r < p < \infty} L^p$, and
- (c) the $AC(\sigma)$ functional calculus for \tilde{T} extends to a $BV(\sigma)$ functional calculus.

Proof. As in Theorem 3.3, for each $f \in BV(\sigma)$ there exists a linear transformation $\Psi(f)$ defining operators $\Psi_p(f)$ for all $p \in (r, \infty)$, and $\Psi_p : BV(\sigma) \rightarrow B(L^p)$ is a functional calculus for T_p .

Using the construction in the proof of Proposition 3.1, we extend the domain of each map $\Psi(f)$ so that it defines an operator $V_f \in B(L^\infty)$. By this construction, each V_f is the adjoint of an operator $U_f \in B(L^1)$, and it may be readily verified that for each $f, g \in BV(\sigma)$, $U_{f+g} = U_f + U_g$ and $U_{fg} = U_f U_g$. It follows that the map $f \mapsto V_f$ is an algebra homomorphism: for example, for $f, g \in BV(\sigma)$, $y \in L^1$, $x \in L^\infty$,

$$\langle y, V_{fg}x \rangle = \langle U_{fg}y, x \rangle = \langle U_g U_f y, x \rangle = \langle U_g U_f y, x \rangle = \langle U_f y, V_g x \rangle = \langle y, V_f V_g x \rangle,$$

so that $V_{fg} = V_f V_g$. Let $\tilde{T} = V_e$ where $e(z) = z$. Again the construction ensures that \tilde{T} satisfies (a) and (b).

It follows then that $\Psi_\infty : BV(\sigma) \rightarrow B(L^\infty)$, $\Psi_\infty(f) = V_f$, defines a $BV(\sigma)$ functional calculus for \tilde{T} on L^∞ . \square

Remark 4.3. As was shown in Example 4.1, on an infinite measure space \tilde{T} may be a different operator to T_∞ . In the case that (Ω, Σ, μ) is a finite measure space, property (b) ensures that this does not occur.

From a practical point of view, the problem is more often to determine whether a given real $AC(\sigma)$ operator on L^∞ has a $BV(\sigma)$ functional calculus. There is in fact no known example of a real $AC(\sigma)$ operator without a BV functional calculus on any L^∞ space. A candidate for such an operator is

$$Tx(t) = tx(t) + \int_t^1 x(s) ds, \quad x \in L^\infty[0, 1].$$

Showing that this operator does not have a $BV[0, 1]$ functional calculus would require a different sort of proof to those provided in Section 2 since the functional calculus *does* extend to the algebra of left-continuous functions of bounded variation whose continuous singular part is zero. In particular, and unlike the examples in Section 2, the functional calculus can be extended to include the characteristic functions $\chi_{[0, \lambda]}$. It might be noted that this extension is not constructive. Further details can be found in [6, Chapter 15].

Proving an “ L^∞ version” of Corollary 3.4 is problematic. Given a linear transformation T which defines real $AC(\sigma)$ operators on L^∞ and L^r for some $r < \infty$, extrapolating the operators $\Psi_p(f) = f(T_p)$ ($r \leq p < \infty$) to L^∞ using Theorem 4.2 may not give a homomorphism which even matches the $AC(\sigma)$ functional calculus for T_∞ .

Example 4.4. As a variant of Example 4.1, consider the linear transformation $Sx := (Lx, Lx, Lx, \dots)$. In this case S_p is a real $AC(\{0, 1\})$ operator on each l^p space. Indeed each S_p is of type (B). For $p < \infty$, the $BV(\sigma)$ functional calculus for S_p is given by $f(S_p) = f(0)I$. This has many extensions to l^∞ , only one of which is the $BV(\sigma)$ functional calculus for T_∞ , $f(S_\infty) = f(0)(I - S_\infty) + f(1)S_\infty$. Note that this example shows that the spectral consistency results listed at the start of Section 3 do not hold when one of the spaces is an L^∞ space. In particular, in this example $\sigma(S_p) \neq \sigma(S_\infty)$.

We finish with two positive results which cover a wide range of concrete examples.

Proposition 4.5. *Let $1 \leq r < \infty$ and let T be a linear transformation defining real AC(σ) operators $T_r \in B(L^r)$ and $T_\infty \in B(L^\infty)$. If $T_\infty = S^*$ for some operator $S \in L^1$, then T_∞ has a BV(σ) functional calculus.*

Proof. By the Riesz–Thorin interpolation theorem, T defines real AC(σ) operators $T_p \in B(L^p)$ for all $p \in [r, \infty)$, with a uniform bound on the AC functional calculi of these operators. Using Proposition 3.1 to construct an operator $U^* \in B(L^\infty)$, we see that $U = S$, as the extension of the family $(T_p)^*$ to L^1 is unique. Thus $U^* = T_\infty$, and by Theorem 4.2, T_∞ is a real AC(σ) operator with a BV functional calculus. \square

Proposition 4.5 would apply, for example, to the case where $Tx = Ax$ for some self-adjoint infinite matrix A acting on $x \in \ell^2$. If this matrix multiplication defines a real AC(σ) operator on ℓ^∞ and is bounded on ℓ^2 , then T satisfies the hypotheses of Proposition 4.5 (since every self-adjoint operator on ℓ^2 is an AC(σ) operator) and so T must admit a BV(σ) functional calculus on ℓ^∞ .

Proposition 4.6. *Suppose that (Ω, Σ, μ) is a finite measure space. Let $1 \leq r < \infty$ and let T be a linear transformation defining real AC(σ) operators $T_r \in B(L^r)$ and $T_\infty \in B(L^\infty)$. Then T_∞ has a BV(σ) functional calculus.*

Proof. By interpolation T defines a real AC(σ) operator of type (B) on L^p for all $p \in [r, \infty)$. Let Ψ_p denote the BV(σ) functional calculus for T_p . Then there exists a constant M such that $\|\Psi_p(f)\|_p \leq M\|f\|_{\text{BV}}$ for all $p \in [r, \infty)$. Suppose now that $f \in \text{BV}(\sigma)$. As $L^\infty \subseteq \bigcap_{r \leq p < \infty} L^p$, there exists an operator $U_f \in B(L^\infty)$ such that $\Psi_p(f)x = U_f x$ for all $x \in L^\infty$. Further $\|U_f\|_\infty \leq M\|f\|_{\text{BV}}$. As in the proof of Theorem 4.2, one can show that the map $f \mapsto U_f$ is a BV(σ) functional calculus for T_∞ . \square

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