A lattice computation of the first moment of the kaon's distribution amplitude

UKQCD Collaboration

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Abstract

We present a lattice computation of the first moment of the kaon's leading-twist distribution amplitude. The results were computed using ensembles with $2 + 1$ dynamical flavours with the domain wall fermion action and Iwasaki gauge action from the RBC and UKQCD joint dataset. The first moment is non-zero because of $SU(3)$-breaking effects, and we find that we are able to measure these effects very clearly. We observe the expected chiral behaviour and finally obtain $\langle \xi \rangle (2 \text{ GeV}) \equiv 3/5 a_K^1 (2 \text{ GeV}) = 0.032(3)$, which agrees very well with results obtained using sum-rules, but with a significantly smaller error. We discuss the systematic uncertainties in detail and explain the prospects for their further reduction. In particular, we are implementing a programme of non-perturbative renormalization to improve the precision on the normalization of the lattice operators (currently performed perturbatively) and are repeating the calculation on a large lattice ($24^3 \times 64$ as compared to the present $16^3 \times 32$ lattice).

1. Introduction

Hadronic light-cone distribution amplitudes are fundamental non-perturbative ingredients in the QCD analysis of hard exclusive processes. Phenomenological applications which require knowledge of the distribution amplitudes include electromagnetic form-factors at large momentum transfer and related processes [1–7]. More recently, following the development of the factorization framework, the distribution amplitudes are also an important component in the phenomenology of exclusive charmless two-body B-decays (i.e., B-decays into two light mesons) [8–14]. These are a particularly important set of processes for CKM-analyses and for studies of CP-violation.

Here, we present the first results from our new lattice project in which we are computing the moments of the light-cone distribution amplitudes for light pseudoscalar and vector mesons.

The subject of this Letter is the first moment of the leading-twist distribution amplitude of the kaon, $\phi_K(u, \mu)$, which parametrizes the overlap of a kaon with longitudinal momentum p with the lowest Fock state consisting of a quark and an anti-quark carrying the momentum fractions up and $\bar{u}p = (1-u)p$, respectively ($u + \bar{u} = 1$). It is defined by the non-local (light-cone) matrix element:

$$\langle 0| \bar{q}(z)\gamma_\rho \gamma_5 \mathcal{P}(z, -z) s(-z) \big| K(p) \rangle \big|_{z^2 = 0}$$

$$\equiv f_K(ip_\rho) \int_0^1 du \, e^{i(u-\bar{u})p \cdot z} \phi_K(u, \mu), \tag{1}$$

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where μ is a renormalization scale and

$$\mathcal{P}(z, -z) = \mathcal{P} \exp\left\{-ig \int_{-z}^{z} dw^\mu A_\mu(w)\right\},\qquad(2)$$

represents the path-ordered exponential from $-z$ to z, so that the bi-local current on the left-hand side of Eq. (1) is gauge invariant. The distribution amplitude is normalized by $\int_0^1 du\, \phi_K(u, \mu) = 1$, and can be expanded in terms of Gegenbauer polynomials $C_n^{3/2}(2u - 1)$,

$$\phi_K(u, \mu) = 6u\bar{u}\left(1 + \sum_{n \geqslant 1} a_n^K(\mu) C_n^{3/2}(2u - 1)\right).\qquad(3)$$

The lowest-order anomalous dimensions of the moments $a_n^K(\mu)$ grow with n and thus higher moments may be suppressed for large values of the renormalization scale μ. In this Letter we present our results for the lowest Gegenbauer moment a_1^K, which is proportional to the average difference of the longitudinal quark and anti-quark momenta of the lowest Fock state:

$$a_1^K(\mu) = \frac{5}{3} \int_0^1 du\,(2u - 1)\phi_K(u, \mu) = \frac{5}{3}\langle 2u - 1\rangle \equiv \frac{5}{3}\langle\xi\rangle(\mu).\qquad(4)$$

$a_1^K = 5/3\langle\xi\rangle$ is obtained from the matrix element of a local operator,

$$\langle 0|\bar{q}(0)\gamma_\rho\gamma_5 \overleftrightarrow{D}_\mu s(0)\big|K(p)\rangle = \langle\xi\rangle f_K p_\rho p_\mu = \frac{3}{5}a_1^K f_K p_\rho p_\mu;\qquad(5)$$

Eq. (5) is the leading term in the Taylor expansion of expression (1) around $z = 0$. Our conventions for the covariant derivatives are $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$, $\overrightarrow{D}_\mu = \overrightarrow{\partial}_\mu + igA_\mu$ and $\overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu - igA_\mu$.

The first moment of the kaon's distribution amplitude has in the past been determined mainly from QCD sum rules, and recent results include:

$$a_1^K(1\text{ GeV}) = \begin{cases} 0.05(2) & [15], \\ 0.10(12) & [16], \\ 0.050(25) & [17], \\ 0.06(3) & [18]. \end{cases}\qquad(6)$$

In this work we present the results of a lattice study of this quantity using $N_f = 2 + 1$ dynamical flavours of domain wall fermions [19,20], which have good chiral properties, and the Iwasaki gauge action [21,22]. We used gauge field ensembles from the RBC and UKQCD dataset with three values of the light-quark mass and the calculations are carried out with equal valence and sea quark masses (i.e., with full unitarity). Further details of the simulation can be found in Section 4. We have observed a clear signal for partonic $SU(3)$-breaking effects in the leading-twist kaon distribution amplitude and find that the first moment satisfies the chiral behaviour expected from chiral perturbation theory. For our *best* results we quote

$$\langle\xi\rangle^{\overline{\text{MS}}}(\mu = 2\text{ GeV}) = 0.032 \pm 0.003,$$
$$(a_1^K(2\text{ GeV}) = 0.053(5)),\qquad(7)$$

$$\langle\xi\rangle^{\overline{\text{MS}}}(\mu = 1\text{ GeV}) = 0.040 \pm 0.004,$$
$$(a_1^K(1\text{ GeV}) = 0.066(6)),\qquad(8)$$

in agreement with most previous results. The errors are already small, and, as discussed below, will decrease still further in the near future.

While we were completing this Letter, Braun et al. released the following result from a lattice simulation using improved Wilson fermions [23]:

$$a_1^K(2\text{ GeV}) = 0.0453 \pm 0.0009 \pm 0.0029,\qquad(9)$$

in reasonable agreement with our result.

The plan of the remainder of this Letter is as follows. In the following section we introduce the basic definitions, in particular the ratio of Euclidean lattice correlation functions from which we determine $\langle\xi\rangle$ and which we estimate using a Monte Carlo simulation. Section 3 contains the discussion of the perturbative calculation of the renormalization constants. We present our results for the bare matrix elements in Section 4 and combine them with the renormalization constants and discuss the systematic uncertainties in Section 5, where we present our final result. We end with a brief summary and conclusions (Section 6).

2. $\langle\xi\rangle^{\text{bare}}$ from lattice correlation functions

We start by briefly describing the overall strategy of our calculation of $\langle\xi\rangle$. As we explain in this section, we exploit the fact that $\langle\xi\rangle$ can be obtained directly from a ratio of two Euclidean correlation functions, which we evaluate using a Monte Carlo simulation. The statistical fluctuations are reduced in the ratio and, for each choice of quark masses used in the simulation, we are able to obtain $\langle\xi\rangle$ with good precision.

In constructing the lattice operators, we use the following symmetric left- and right-acting covariant derivatives:

$$\overrightarrow{D}_\mu \psi(x) = \frac{1}{2a}\{U(x, x + \hat{\mu})\psi(x + \hat{\mu}) - U(x, x - \hat{\mu})\psi(x - \hat{\mu})\},\qquad(10)$$

and

$$\bar{\psi}(x)\overleftarrow{D}_\mu = \frac{1}{2a}\{\bar{\psi}(x + \hat{\mu})U(x + \hat{\mu}, x) - \bar{\psi}(x - \hat{\mu})U(x - \hat{\mu}, x)\},\qquad(11)$$

where the U's are the gauge links and $\hat{\mu}$ is a vector of length a in the direction μ (a denotes the lattice spacing).

To illustrate the method, consider the local lattice operators[1]

$$O_{\rho\mu}(x) = \bar{q}(x)\gamma_\rho\gamma_5 \overleftrightarrow{D}_\mu s(x), \qquad A_\rho(x) = \bar{q}(x)\gamma_\rho\gamma_5 s(x),$$
$$P(x) = \bar{q}(x)\gamma_5 s(x),\qquad(12)$$

and define the two-point correlation functions

$$C_{\rho\mu}(t, \vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}}\langle 0|O_{\rho\mu}(t, \vec{x})P^\dagger(0)|0\rangle,\qquad(13)$$

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[1] In the simulation we actually use a smeared pseudoscalar density P in order to improve the overlap with the kaon state. The present discussion holds for both local and smeared pseudoscalar densities.

and

$$C_{A_\nu P}(t, \vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle 0|A_\nu(t, \vec{x}) P^\dagger(0)|0\rangle. \tag{14}$$

Here q and s represent the light and strange quark fields, respectively. At large Euclidean times t and $T - t$, where T is the length of the lattice in the time direction, the correlation functions (13) and (14) approach

$$\begin{aligned}
&C_{\{\rho\mu\}}(t, \vec{p}) \\
&\quad \to \frac{Z_P f_K^{\text{bare}} e^{-E_K T/2} \sinh((t - T/2)E_K)}{E_K} \\
&\qquad \times (ip_\rho)(ip_\mu)\langle\xi\rangle^{\text{bare}},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&C_{A_\nu P}(t, \vec{p}) \\
&\quad \to \frac{Z_P f_K^{\text{bare}} e^{-E_K T/2} \sinh((t - T/2)E_K)}{E_K} \times (ip_\nu)
\end{aligned} \tag{16}$$

where we have used (4), (5) and defined $Z_P \equiv \langle K(p)|P^\dagger|0\rangle$ and the kaon's bare decay constant $\langle 0|A_\nu|K(p)\rangle \equiv ip_\nu f_K^{\text{bare}}$. The superscript *bare* denotes the fact that the operators are the bare ones in the lattice theory with ultraviolet cut-off a^{-1}. Taking the ratio of the two correlation functions

$$R_{\{\rho\mu\};\nu}(t, \vec{p}) \equiv \frac{C_{\{\rho\mu\}}(t, \vec{p})}{C_{A_\nu P}(t, \vec{p})} \to i\frac{p_\rho p_\mu}{p_\nu}\langle\xi\rangle^{\text{bare}}, \tag{17}$$

allows us readily to extract the bare value of $\langle\xi\rangle$. The braces in the subscripts $\{\rho\mu\}$ indicate that the indices are symmetrized.

In the continuum $\overleftrightarrow{\mathcal{O}}_{\{\rho\mu\}}$ transforms as a second rank Lorentz tensor, whereas on the lattice we need to consider transformation properties under the hypercubic group and discrete symmetries, in particular parity and charge conjugation. We choose to evaluate the matrix elements of $\overleftrightarrow{\mathcal{O}}_{\{\rho\mu\}}$ for $\mu \neq \rho$. These operators transform as a six-dimensional representation of the hypercubic group and lattice symmetries exclude mixing with other operators [24] including operators containing a total derivative (since we are considering matrix elements with a non-zero momentum transfer, operators with a total derivative have to be considered). As can be seen from Eq. (17), in order for the matrix elements to be non-zero, we require both $p_\mu \neq 0$ and $p_\rho \neq 0$. We satisfy this condition by taking $\mu = \nu = 4$, $\rho = 1, 2$ or 3 and $|p_\rho| = 2\pi/L$.

Having obtained $\langle\xi\rangle^{\text{bare}}$, we need to determine the renormalization factor relating the lattice operators $\overleftrightarrow{\mathcal{O}}_{\{\rho\mu\}}$ and A_ν to the corresponding continuum operators in some standard renormalization scheme; here we do this at one-loop order in perturbation theory as explained in the following section.

3. Perturbative renormalization of the lattice operators

The perturbative matching from the lattice to $\overline{\text{MS}}$ schemes is performed by comparing one-loop calculations of the amputated two-point Green function with an insertion of the operator $O_{\{\rho\mu\}}$ in both schemes (which requires the evaluation of the diagrams in Fig. 1), together with appropriate wave function renormalization factors (Fig. 2). Defining $O_{\{\rho\mu\}}^{\overline{\text{MS}}}(\mu) =$
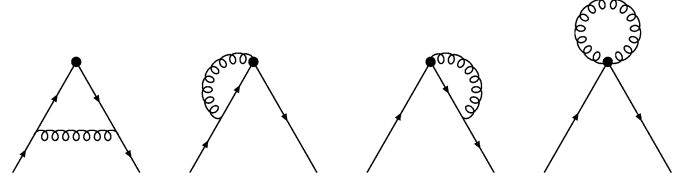

Fig. 1. One-loop vertex diagrams evaluated in the perturbative renormalization of $O_{\{\rho\mu\}}$.
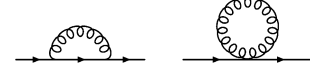

Fig. 2. One-loop diagrams contributing to the quarks' wave function renormalization.

$Z_{O_{\{\rho\mu\}}} O_{\{\rho\mu\}}^{\text{latt}}(a)$, the renormalization factor is given by

$$\begin{aligned}
Z_{O_{\{\rho\mu\}}} = \frac{1}{(1 - w_0^2)Z_w}\bigg[&1 + \frac{g^2 C_F}{16\pi^2}\bigg(-\frac{8}{3}\ln(\mu^2 a^2) + \Sigma_1^{\overline{\text{MS}}} \\
&- \Sigma_1 + V^{\overline{\text{MS}}} - V\bigg)\bigg].
\end{aligned} \tag{18}$$

In this expression, $(1 - w_0^2)Z_w$ is a characteristic normalization factor for the physical quark fields in the domain wall formalism. It is a common factor in the numerator and denominator of the ratio $R_{\{\rho\mu\};\nu}$ as are the contributions from the wave function renormalization. Z_w represents an additive renormalization of the large Dirac mass or domain wall height $M = 1 - w_0$ which can be rewritten in multiplicative form at one-loop as

$$Z_w = 1 + \frac{g^2 C_F}{16\pi^2}z_w, \qquad z_w = \frac{2w_0}{1 - w_0^2}\Sigma_w. \tag{19}$$

The one-loop correction z_w becomes very large for certain choices of M [25,26], including that used in our numerical simulations, so that some form of mean-field improvement is necessary, as discussed below.[2]

The terms $\Sigma_1^{\overline{\text{MS}}}$ and Σ_1 come from quark wave function renormalization. The terms $V^{\overline{\text{MS}}}$ and V come from the one-loop corrections to the amputated two-point function. They are given by "vertex" and "sail" diagrams, plus an operator tadpole diagram in the lattice case. Using naive dimensional regularization (NDR) in Feynman gauge with a gluon mass infrared (IR) regulator,

$$\Sigma_1^{\overline{\text{MS}}} = \frac{1}{2}, \qquad V^{\overline{\text{MS}}} = -\frac{25}{18}. \tag{20}$$

The lattice contribution Σ_1 has been evaluated for domain wall fermions with the Iwasaki gluon action in Feynman gauge and a gluon mass IR regulator in [26]. We have calculated the lattice vertex term V for the same action, gauge and IR regulator to complete the evaluation of $Z_{O_{\{\rho\mu\}}}$. The perturbative calculation is explained in [25–27] and the form of the Iwasaki gluon propagator can be found in [28]. Values for V are given as

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[2] The factor $1/(1 - w_0^2)Z_w$ cancels however, in the evaluation of the ratio $Z_{O_{\{\rho\mu\}}}/Z_A$, where Z_A is the renormalization constant for the axial current.

Table 1

Constants needed for the perturbative renormalization of the operator $O_{\{\rho\mu\}}$ using domain wall fermions and the Iwasaki gauge action ($c_1 = -0.331$). M is the domain wall height and $\delta\Sigma_1 + \delta V = \Sigma_1^{\overline{MS}} - \Sigma_1 + V^{\overline{MS}} - V$, while other quantities are defined in the text. Σ_1 and V are dependent on the gauge and the infrared regulator: Feynman gauge and a gluon mass are used here. $z_w^{(MF)}$ and Σ_1 are extracted from the results in [26], while V has been calculated as part of this work

M	z_w	z_w^{MF}	Σ_1	V	$\delta\Sigma_1 + \delta V$
0.1	−243.86	−86.579	4.6519	−4.6297	−0.9110
0.2	−113.29	−39.501	4.5193	−4.5614	−0.8468
0.3	−69.404	−23.830	4.4093	−4.5101	−0.7881
0.4	−47.077	−15.949	4.3158	−4.4678	−0.7369
0.5	−33.278	−11.142	4.2354	−4.4311	−0.6932
0.6	−23.648	−7.8365	4.1665	−4.3980	−0.6574
0.7	−16.300	−5.3538	4.1079	−4.3673	−0.6295
0.8	−10.263	−3.3459	4.0593	−4.3381	−0.6101
0.9	−4.9617	−1.6078	4.0204	−4.3097	−0.5996
1.0	0.0	0.0	3.9915	−4.2816	−0.5988
1.1	4.9442	1.5902	3.9731	−4.2529	−0.6090
1.2	10.192	3.2748	3.9664	−4.2232	−0.6321
1.3	16.136	5.1900	3.9727	−4.1916	−0.6700
1.4	23.346	7.5350	3.9943	−4.1571	−0.7261
1.5	32.784	10.648	4.0343	−4.1182	−0.8050
1.6	46.322	15.194	4.0974	−4.0728	−0.9135
1.7	68.294	22.720	4.1905	−4.0176	−1.0618
1.8	111.69	37.901	4.3249	−3.9462	−1.2676
1.9	241.55	84.270	4.5209	−3.8447	−1.5651

a function of M in Table 1. Chiral symmetry of the domain wall action implies that these results also apply for the operator which is $O_{\{\rho\mu\}}$ without the γ_5. We have confirmed that our results reproduce those found by Capitani [27] if we replace the gluon propagator for the Iwasaki gauge action by the propagator corresponding to the standard plaquette action. This provides a powerful check of our calculation. We note that the perturbative renormalization factor for the same operator using overlap fermions and the Lüscher–Weisz gauge action can be found in [29].

Our numerical simulations use $M = 1.8$. For this value of M, with the Iwasaki gluon action, the one-loop coefficient z_w in the physical quark normalization can be extracted from Σ_w in Table III of [26]. This gives $z_w \approx 112$, making it clear that mean-field improvement is necessary. We follow the prescription described in [26].

The first step is to define a mean-field value for the domain wall height,

$$M^{MF} = M - 4\left(1 - P^{1/4}\right) \tag{21}$$

where $P = 0.58813(4)$ is the average plaquette value in our simulations. This leads to

$$M^{MF} = 1.3029. \tag{22}$$

The physical quark normalization factor becomes $[1 - (w_0^{MF})^2]Z_w^{MF}$, with

$$Z_w^{MF} = 1 + \frac{g^2 C_F}{16\pi^2} z_w^{MF},$$

$$z_w^{MF} = \frac{2w_0^{MF}}{1 - (w_0^{MF})^2}\left(\Sigma_w + 32\pi^2 T_{MF}\right) = 5.2509, \tag{23}$$

where $T_{MF} = 0.0525664$ [26] is a mean-field tadpole factor and Σ_w is evaluated at M^{MF}, leading to $z_w^{MF} = 5.2509$. Likewise, $\Sigma_1 = 3.9731$ and $V = -4.1907$ in Eq. (18) are evaluated at M^{MF} and the mean-field improved renormalization factor for our simulations becomes:

$$Z_{O_{\{\rho\mu\}}} = \frac{1}{0.9082}\left[1 - \frac{g^2 C_F}{16\pi^2} 5.2509\right]$$
$$\times \left[1 + \frac{g^2 C_F}{16\pi^2}\left(-\frac{8}{3}\ln(\mu^2 a^2) - 0.6713\right)\right]. \tag{24}$$

We make two choices for the mean-field improved $\overline{MS}$ coupling. The first uses the measured plaquette value, P, according to [26]

$$\frac{1}{g_{\overline{MS}}^2(\mu)} = \frac{P}{g^2} + d_g + c_p + \frac{22}{16\pi^2}\ln(\mu a), \tag{25}$$

where $d_g = 0.1053$ and $c_p = 0.1401$ for the Iwasaki gauge action and $\beta = 6/g^2 = 2.13$ in our simulations. The second choice is the usual continuum $\overline{MS}$ coupling. At $\mu a = 1$, we find $\alpha_{\overline{MS}}(\text{plaq}) = 0.1752$ and $\alpha_{\overline{MS}}(\text{ctm}) = 0.3385$. This disparity in the values of the two couplings is further motivation for the programme of non-perturbative renormalization which we are currently undertaking. With these two choices of coupling, our value for the renormalization factor becomes

$$Z_{O_{\{\rho\mu\}}} = \begin{cases} 0.9811 & \text{plaquette coupling,} \\ 0.8719 & \text{continuum } \overline{MS}. \end{cases} \tag{26}$$

We also evaluate the mean-field improved expression for the axial vector current [26], interpolating to our mean-field M^{MF}, and obtain

$$Z_A = \begin{cases} 0.7947 & \text{plaquette coupling,} \\ 0.6514 & \text{continuum } \overline{MS}. \end{cases} \tag{27}$$

The ratio of the two renormalization factors is

$$\frac{Z_{O_{\{\rho\mu\}}}}{Z_A} = \begin{cases} 1.2346 & \text{plaquette coupling,} \\ 1.3384 & \text{continuum } \overline{MS}. \end{cases} \tag{28}$$

For the purposes of this Letter we include the spread of results in Eq. (28) as the estimate of our current systematic uncertainty of the renormalization factor.[3] This uncertainty will be significantly reduced as we complete our programme of non-perturbative renormalization. $\langle\xi\rangle^{\text{bare}}$ should be multiplied by the factor on the right-hand side of Eq. (28) to obtain the result in the $\overline{MS}$ scheme. For this factor we take

$$\frac{Z_{O_{\{\rho\mu\}}}}{Z_A} = 1.28 \pm 0.05. \tag{29}$$

4. Numerical simulation and results

The numerical results presented here are based on Monte Carlo estimates of correlation functions evaluated on representative sets of UKQCD/RBC gauge field configurations that were generated with $N_f = 2 + 1$ flavours of dynamical domain wall

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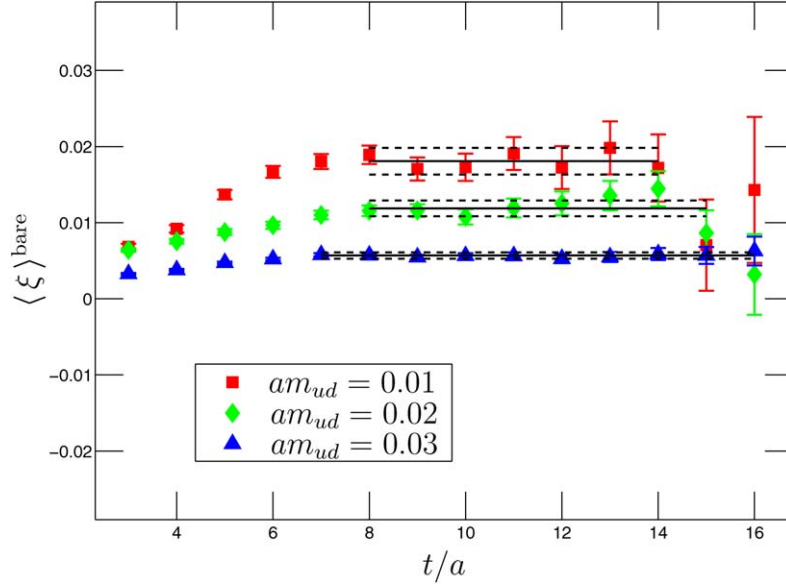[3] We mention in passing that using the bare coupling, $\frac{Z_{O_{\{\rho\mu\}}}}{Z_A} = 1.26$.

Fig. 3. Jack-knife results for $\langle\xi\rangle^{\text{bare}}$ as a function of the time. The ranges over which we fit and the corresponding results are indicated by the black lines.

fermions [19,20] with Iwasaki Gauge action [21,22] using the QCDOC computer [30–33]. The hadronic spectrum and other properties of these configurations have been studied in detail and the results will be presented in Ref. [34]. We wish to thank our collaborators in the RBC and UKQCD Collaborations for access to several preliminary results which we require for this study. These are specifically m_{res} in the chiral limit, the bare kaon pseudoscalar masses, and the inverse lattice spacing.

The preliminary nature of these intermediate values does not introduce a significant uncertainty in our results compared to our statistical errors and the systematic errors due to the chiral extrapolation and renormalization. Indeed we shall only quote the lattice spacing to two significant figures and demonstrate the insensitivity of our results to this. However, we caution the reader that definitive results for these intermediate quantities will be found in the forthcoming publication.

The lattice volume is $(L/a)^3 \times T/a = 16^3 \times 32$ and the length of the fifth dimension is $L_s = 16$. The choice of bare parameters is $\beta = 2.13$ for the bare gauge coupling, $am_s = 0.04$ for the strange quark mass (which has been tuned to correspond to the physical value) and $am_q = 0.03, 0.02, 0.01$ for the bare light-quark masses. With this choice of simulation parameters the lattice spacing is $a^{-1} = 1.6$ GeV [34]. Due to the remnant chiral symmetry breaking the quark mass has to be corrected additively by the residual mass in the chiral limit, $am_{\text{res}} = 0.003$ [34].

Statistical errors for observables have been estimated both with the jack-knife and with a direct estimation of the integrated autocorrelation time as suggested in [35].

4.1. Bare correlation functions

For each value of the light-quark mass we computed the correlation functions on 300 gauge configurations separated by 10 trajectories in the Monte Carlo history. On each configuration we averaged the results obtained from 4 sources for the light-

est quark mass ($m_q a = 0.01$) and 2 sources for the remaining two masses ($m_q a = 0.02$ and 0.03). The sources were chosen to be at the origin and $(8, 8, 8, 16)$ for all three masses, and in addition at $(4, 4, 4, 8)$ and $(12, 12, 12, 24)$ for $m_q a = 0.01$. In order to improve the overlap with the ground state, at the source where we insert the density $P^\dagger$, we employed gauge invariant Jacobi smearing [36] (radius 4 and 40 iterations) with APE-smeared links in the covariant Laplacian operator (4 steps and smearing factor 2) [37,38].

The preliminary kaon masses corresponding to the simulated bare light-quark masses are $am_K^{0.03} = 0.4164(10)$, $am_K^{0.02} = 0.3854(10)$, and $am_K^{0.01} = 0.3549(14)$ [34].

In order to extract $\langle\xi\rangle$ from the ratio $R_{\{\rho\mu\};\nu}(t, \vec{p})$ defined in (17) we need $|\vec{p}| \neq 0$. Since hadronic observables with larger lattice momenta have larger lattice artefacts and statistical errors, we restrict the choice of indices to $\rho = \nu = 4$[4] and $\mu = 1, 2, 3$ with $|\vec{p}| = 2\pi/L$ ($p_\mu = \pm 2\pi/L$ with the remaining two components of $\vec{p}$ equal to 0). $\langle\xi\rangle^{\text{bare}}$ is then obtained from the correlation function at large times:

$$R_{\{4k\};4}(t, p_k = \pm 2\pi/L) = \pm i \frac{2\pi}{L} \langle\xi\rangle^{\text{bare}},$$

$$|\vec{p}| = \frac{2\pi}{L}, \quad k = 1, 2, 3. \tag{30}$$

The plot in Fig. 3 shows our results for $\langle\xi\rangle$ as a function of t obtained from the ratio $R_{\{4k\};4}(t, p_k = \pm 2\pi/L)$ for the three values of the mass of the light quark. The results have been averaged over the three values for k and the 6 equivalent lattice momenta with $|\vec{p}| = 2\pi/L$ and combining the results at t with those at $T - t - 1$. There are clear plateaus, demonstrating that the $SU(3)$-breaking effects are measurable and $\langle\xi\rangle$ can be determined. The results from the fits for $\langle\xi\rangle^{\text{bare}}$ are summarized in Table 2.

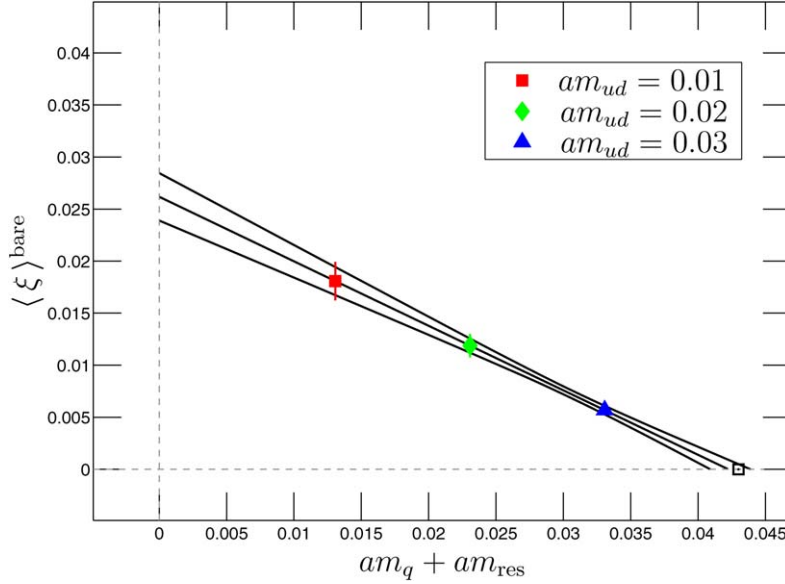[4] The index 4 corresponds to the time-direction.

Fig. 4. Linear chiral extrapolation for $\langle \xi \rangle^{\text{bare}}$.

Table 2
Summary of results for the bare values of the 1st moment of the kaon's distribution amplitude. The result we obtain after the linear chiral extrapolation is quoted in the right-most column

am_{ud}	0.03	0.02	0.01	χ-limit
$\langle u - \bar{u} \rangle$	0.0057(4)	0.0119(10)	0.0181(18)	0.0262(23)

4.2. Chiral extrapolation

For the pion the first moment a_1^π vanishes since isospin symmetry induces invariance under $u \leftrightarrow (1 - u)$. For a non-degenerate quark–antiquark pair (such as the kaon) flavour symmetry breaking implies that the first moment of the distribution amplitude is non-zero. The leading $SU(3)$-violating effects for the kaon's distribution amplitude have been studied at next-to-leading order in chiral perturbation theory (χPT) in [39]. No chiral logarithms appear and the prediction for the mass-dependence is

$$\langle \xi \rangle = \frac{8B_0}{f^2}(m_s - m_{ud})b_{1,2}, \qquad (31)$$

where f and B_0 are conventional χPT parameters and $b_{1,2}$ is a Wilson coefficient as introduced in [39].

We plot our results for $\langle \xi \rangle^{\text{bare}}$ as a function of the light-quark mass in Fig. 4. We take into account the remnant chiral symmetry breaking by defining the chiral limit at the point $am_q + am_{\text{res}} = 0$. The linear behaviour predicted in Eq. (31) is well satisfied (with a tiny χ^2/d.o.f. of about 10^{-5}). Moreover the line passes through $\langle \xi \rangle^{\text{bare}} = 0$ at a value of the light-quark mass (denoted by the open square in Fig. 4) which is consistent with the mass of the strange quark, as expected for the $SU(3)$ symmetric case ($am_{ud} = am_s = 0.04$). More specifically, the intercept of the linear fit with the $\langle \xi \rangle^{\text{bare}} = 0$ axis occurs at $am_{ud} = 0.0391^{+0.0017}_{-0.0013}$.

From the linear fit[5] we obtain $\langle \xi \rangle^{\text{bare}} = 0.0262(23)$ in the chiral limit and in the next section we combine this result with the renormalization constant in Eq. (29) to arrive at our final result.

5. Systematic uncertainties and our final result

To obtain the final result for the $\langle \xi \rangle$ in the $\overline{\text{MS}}$ scheme at $\mu \simeq 1.6$ GeV we multiply the result obtained from the bare operators with cut-off $a^{-1} = 1.6$ GeV, $\langle \xi \rangle^{\text{bare}} = 0.0262(23)$, by the ratio of renormalization factors in Eq. (29), $\frac{Z_{O_{\{\rho\mu\}}}}{Z_A} = 1.28(5)$:

$$\langle \xi \rangle^{\overline{\text{MS}}}(\mu = 1.6 \text{ GeV}) = 0.034 \pm 0.003. \qquad (32)$$

In order to be able to compare our result with previous calculations we evolve it to renormalization scales of 1 and 2 GeV using the three-loop anomalous dimension [40],[6] obtaining:

$$\langle \xi \rangle^{\overline{\text{MS}}}(\mu = 2 \text{ GeV}) = 0.032 \pm 0.003, \qquad (33)$$

$$\langle \xi \rangle^{\overline{\text{MS}}}(\mu = 1 \text{ GeV}) = 0.040 \pm 0.004. \qquad (34)$$

The error in the renormalization factor due to the uncertainty in the lattice spacing is negligible. For example if we conservatively allow the lattice spacing to vary between 1.58 and 1.62 GeV, the contribution to the relative error on $\langle \xi \rangle^{\overline{\text{MS}}}$ is less than 0.2%.

Among the uncertainties which, at this stage at least, we are not in a position to check numerically are the continuum extrapolation, finite-volume effects and the fact that the strange quark

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[5] We have also performed quadratic fits to the chiral behaviour but the results do not change in any significant way.

[6] It should however, be recalled, that we have calculated the renormalization constant at $\mu = a^{-1}$ at one-loop order. For the precision of our calculation, it would therefore have been sufficient to have used the two-loop anomalous dimension.

mass ($m_s a = 0.04$) is only approximately tuned to its physical value. The lattice artefacts are formally of $O(a^2 \Lambda_{\text{QCD}}^2) \simeq 2.5\%$. We would expect the finite volume effects to be small and are currently checking this with a simulation on a $24^3 \times 64$ lattice. The strange quark mass appears to be well tuned [34] so again we expect the contribution to the error from this uncertainty to be very small. Thus we expect the errors from these three sources to be sufficiently small not to change the errors quoted in Eqs. (33) and (34) which we take to be our best estimates. We are also carrying out a systematic programme of non-perturbative renormalization which will enable us to reduce the uncertainty in the renormalization constants.

6. Summary and conclusions

In this Letter we have presented the first results from our major lattice study of the leading-twist light-cone distribution amplitudes of the light mesons. We demonstrate that the $SU(3)$-breaking effects which lead to a non-zero value for the first moment of the kaon's distribution amplitude are sufficiently large to be calculable in lattice computations and satisfy the expected chiral behaviour. Our results for the first moment are presented in Eqs. (33) and (34).

We are in the process of implementing a number of improvements, including non-perturbative renormalization and a simulation on a larger lattice ($24^3 \times 64$) which will help reduce these two sources of systematic error (or in the latter case give us confidence in our expectation that the finite-volume errors are indeed small). We will also produce results for the second moment of the pion's and kaon's distribution amplitudes and for those of the ρ and K^* vector mesons. In order to reduce the lattice artefacts we will investigate the use of partially twisted boundary conditions [41–43] which will allow us to calculate the observables at smaller values of lattice momenta.

While we were completing this Letter, Ref. [23] appeared with the results of an $N_f = 2$ study of the distribution amplitude, using $O(a)$ improved Wilson fermions and the plaquette Wilson gauge action. For the first moment of the kaon's distribution amplitude the calculation was performed at $\beta = 5.29$ corresponding to a lattice spacing of about 2.6 GeV. The result of Ref. [23] is given in Eq. (9) above.

Acknowledgements

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