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A free subalgebra of the algebra of matroids

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Abstract

This paper is an initial inquiry into the structure of the Hopf algebra of matroids with restriction–contraction coproduct. Using a family of matroids introduced by Crapo in 1965, we show that the subalgebra generated by a single point and a single loop in the dual of this Hopf algebra is free.

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Keywords: Matroid; Minor algebra; Free algebra

1. Introduction

Major advances in combinatorial theory during recent decades rely upon algebraic structures associated to combinatorial objects, and indeed, often involve studies of combinatorial properties of algebraic systems themselves. In particular, Hopf algebras based on families of combinatorial structures such as posets, graphs, permutations and tableaux play an increasingly prominent role in contemporary combinatorial theory and have been applied to a wide variety of fields. A major exception to this trend occurs in matroid theory, where little attention has been paid to naturally occurring algebraic structures. One such structure, introduced by one of the present authors in [18], is a

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Hopf algebra that may be associated to any family of matroids that is closed under formation of minors and direct sums. This Hopf algebra has as basis the set of isomorphism classes of matroids belonging to the given family, with product induced by the direct sum operation, and coproduct of a matroid $M = M(S)$ given by $\sum_{A\subseteq S} M|A \otimes M/A$, where $M|A$ is the submatroid obtained by restriction to A and M/A is the complementary contraction. A closely related Hopf algebra was constructed by Joni and Rota in [12], as the incidence coalgebra of a hereditary family of geometric lattices. In this case, attention is restricted to simple matroids, and the subsets A appearing in the coproduct are taken to be flats. These Hopf algebras were also briefly considered in connection with the characteristic and Tutte polynomials of matroids in [20] and [21].

Similar constructions have arisen with increasing frequency in recent years, as Hopf algebra techniques have been brought to bear on the study of Feynman diagrams and renormalization processes in Physics [9,14,5], Vassiliev's knot invariants [6–8,13] and graph invariants [11,17]. All of this work has been carried out in the context of graphs, which form an extremely restricted class of matroids, and which have a grossly different classification by isomorphism, save when attention is restricted to 3-connected graphs.

The present article is an initial inquiry into the structure of the matroid Hopf algebra given in [18]. We prove that the subalgebra of the dual algebra generated by "point" and "loop" (the two one-element matroids) is free. (The question of whether of not the corresponding subalgebra, in the context of graphs, is free, which was posed by Lowell Abrams, remains open.) We manage this proof by restricting attention to a class of 2^n mutually nonisomorphic matroids on an n element set which we call "freedom matroids". These matroids are obtained, starting from the empty matroid, by successively adding points, at each stage either in a new dimension or in general position in the top rank. Freedom matroids were introduced by the other present author, in [10], in order to prove that there are at least 2^n nonisomorphic matroids on n elements. The same matroids, presented as transversal matroids, were used in [19] to give a simplified proof of the same result. Several characterizations of freedom matroids were given in [15], where it was also shown that the family of all freedom matroids is closed under formation of minors and duals. In the present paper, we adduce a number of new combinatorial properties of freedom matroids. This work is thus a useful adjunct to recent work that has modeled these, and generalizations of these, matroids in terms of Dyck paths [1] and lattice paths [3], and other work, soon to appear [2,4].

2. Coalgebras of matroids

Throughout this paper, we work over some commutative ring K with unit. All modules, algebras and coalgebras are over K, all maps between such objects are assumed to be K-linear, and all tensor products are taken over K. Given any family of matroids $\mathcal{M}$, we write $\widetilde{\mathcal{M}}$ for the set of isomorphism classes of matroids belonging to $\mathcal{M}$, and denote by $K\{\widetilde{\mathcal{M}}\}$ the free K-module having $\widetilde{\mathcal{M}}$ as basis. For any matroid $M = M(S)$, and $A \subseteq S$, we write $M|A$ for the restriction of M to A, and M/A for the matroid on $S\backslash A$ obtained by contracting A from M.

The following result appeared in [18], as an example of the more general construction of incidence Hopf algebras:

Proposition 2.1. *If $\mathcal{M}$ is a minor-closed family of matroids then $K\{\widetilde{\mathcal{M}}\}$ is a coalgebra, with coproduct δ and counit ϵ determined by*

$$\delta(M) = \sum_{A \subseteq S} M|A \otimes M/A \qquad \textit{and} \qquad \epsilon(M) = \begin{cases} 1, & \textit{if } S = \emptyset, \\ 0, & \textit{otherwise}, \end{cases}$$

for all $M = M(S) \in \mathcal{M}$. If, furthermore, the family $\mathcal{M}$ is closed under formation of direct sums, then $K\{\widetilde{\mathcal{M}}\}$ is a Hopf algebra, with product induced by direct sum.

Whenever $\mathcal{M}$ is minor-closed, we shall write $C(\mathcal{M})$ for the module $K\{\widetilde{\mathcal{M}}\}$ equipped with the above coalgebra structure.

We remark that in the statement of Proposition 2.1, and in all that follows, we do not distinguish notationally between matroids and their isomorphism classes; it will always be clear from the context which is meant. For the purposes of this article, we are interested primarily in the case in which $\mathcal{M}$ is minor-closed and not necessarily closed under direct sums and hence $C(\mathcal{M})$ is only a coalgebra. We do not give a complete proof of the proposition here, but only mention that coassociativity of δ follows directly from the basic identities $(M|T)|U = M|U$, $(M/U)/(T \setminus U) = M/T$ and $(M/U)|(T \setminus U) = (M|T)/U$, which hold for any matroid $M = M(S)$ and $U \subseteq T \subseteq S$.

In the case that $\mathcal{M}$ is closed under formation of direct sums, a formula for the antipode of $\mathcal{M}$ may be deduced from the formula for the antipode of an arbitrary incidence Hopf algebra given in [18].

We will use the following notation for some specific matroids:

$I_n = U_{n,n}$	the *free matroid* of size n
$Z_n = U_{0,n}$	the *zero matroid* of size n
$P_n = U_{1,n}$	the *n-point*
$C_n = U_{n-1,n}$	the *n-circuit*
$I = I_1$	*point*
$Z = Z_1$	*loop*,

where, as usual, $U_{r,n}$ denotes the uniform matroid of rank r on n points.

Example 2.2. Let L be the matroid shown in Fig. 1, consisting of points a, b, c, d, e in the plane, with $\{a, b, c\}$ and $\{a, d, e\}$ collinear. If $\mathcal{M}$ is any minor-closed family containing L, then the coproduct of L in $C(\mathcal{M})$ is given by

$$\begin{aligned} \delta(L) = L \otimes \emptyset + 4(C_3 \oplus I) \otimes Z + C_4 \otimes Z + 2C_3 \otimes P_2 + 8I_3 \otimes Z_2 \\ + 6I_2 \otimes (P_2 \oplus Z) + 4I_2 \otimes P_3 + 4I \otimes N + I \otimes (P_2 \oplus P_2) + \emptyset \otimes L, \end{aligned}$$

where $\oplus$ denotes the direct sum operation on matroids, and N is the three-point line with one of its points doubled.

Example 2.3. The family $\mathcal{I} = \{I_n : n \geq 0\}$ of all free matroids is minor-closed, and the coalgebra $C(\mathcal{I})$ is the free module $K\{I_0, I_1, \ldots\}$, with coproduct and counit given by $\delta(I_n) = \sum_{k=0}^{n} \binom{n}{k} I_k \otimes I_{n-k}$ and $\epsilon(I_n) = \delta_{n,0}$, for all $n \geq 0$. Because $\mathcal{I}$ is also closed

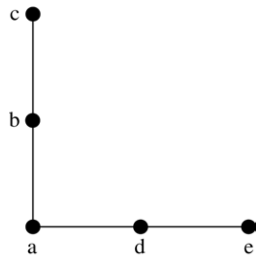

Fig. 1. The matroid 'L' for Example 2.2.

under formation of direct sums, $C(\mathcal{I})$ is in fact a Hopf algebra. Since I_n is equal to the direct sum of n copies of I, we have $I_n = I^n$ in $C(\mathcal{I})$, and thus $C(\mathcal{I})$ is the polynomial Hopf algebra $K[I]$, with coproduct determined by $\delta(I) = I \otimes 1 + 1 \otimes I$.

Similarly, the family $\mathcal{Z} = \{Z_n : n \geq 0\}$ of all zero matroids is closed under formation of minors and direct sums, and $C(\mathcal{Z})$ is equal to the polynomial Hopf algebra $K[Z]$, with $\delta(Z) = Z \otimes 1 + 1 \otimes Z$.

Note that the coproducts in Example 2.3 are cocommutative. This is because the operations of deletion and contraction on free and zero matroids happen to coincide. In fact, these are the only matroids on which these operations coincide; if $\mathcal{M}$ is any minor-closed family that contains matroids outside of $\mathcal{I} \cup \mathcal{Z}$, then the coalgebra $C(\mathcal{M})$ is noncocommutative.

Example 2.4. The class $\mathcal{U}$ of all uniform matroids is minor-closed, and the coproduct on $C(\mathcal{U})$ is given by

$$\delta(U_{r,n}) = \sum_{i=0}^{r} \binom{n}{i} U_{i,i} \otimes U_{r-i,n-i} + \sum_{i=r+1}^{n} \binom{n}{i} U_{r,i} \otimes U_{0,n-i},$$

for all $n \geq r \geq 0$. If we adopt the convention that $U_{k,m} = U_{0,m}$, for $k < 0$ and $U_{k,m} = U_{m,m}$, for $k > m$, then the coproduct on $C(\mathcal{U})$ takes the form

$$\delta(U_{r,n}) = \sum_{i=0}^{n} \binom{n}{i} U_{r,i} \otimes U_{r-i,n-i},$$

for all $n \geq r \geq 0$.

Example 2.5. The subclass $\mathcal{C}$ of $\mathcal{U}$ consisting of all circuits and free matroids is minor-closed. The coalgebra $C(\mathcal{C})$ is equal to $K\{I_0, I_1, \ldots, C_1, C_2, \ldots\}$, with coproduct determined by $\delta(I_n) = \sum_{k=0}^{n} \binom{n}{k} I_k \otimes I_{n-k}$, for $n \geq 0$, and $\delta(C_m) = C_m \otimes I_0 + \sum_{k=0}^{m-1} \binom{m}{k} I_k \otimes C_{m-k}$, for all $m \geq 1$.

Given a family $\mathcal{M}$, and $n \geq 0$, we denote by $\mathcal{M}_n$ the set of all matroids belonging to $\mathcal{M}$ whose underlying sets have cardinality n; and for $k, r \geq 0$, we denote by $\mathcal{M}_{r,k}$ the set of all matroids belonging to $\mathcal{M}$ that have rank r and nullity k. Writing $C_n(\mathcal{M})$ and $C_{r,k}(\mathcal{M})$, respectively, for the free modules $K\{\widetilde{\mathcal{M}}_n\}$ and $K\{\widetilde{\mathcal{M}}_{r,k}\}$, we have

$$C(\mathcal{M}) = \bigoplus_{n\geq 0} C_n(\mathcal{M}) = \bigoplus_{r,k\geq 0} C_{r,k}(\mathcal{M}).$$

Proposition 2.6. *If $\mathcal{M}$ is minor-closed, the families of submodules $\{C_n(\mathcal{M}) : n \geq 0\}$ and $\{C_{r,k}(\mathcal{M}) : r, k \geq 0\}$ of $C(\mathcal{M})$, respectively, equip $C(\mathcal{M})$ with the structure of a graded, and bigraded, coalgebra. If $\mathcal{M}$ is also closed under formation of direct sums then $C(\mathcal{M})$ is also thus graded, and bigraded, as a Hopf algebra.*

Proof. The first claim follows immediately from the fact that, for any matroid $M = M(S)$, and $A \subseteq S$, the rank of M is equal to the sum of the ranks of $M|A$ and M/A, and similarly for nullities. The second claim follows from the fact that rank and nullity are additive functions with respect to the disjoint sum operation on matroids. $\square$

Proposition 2.7. *If $\mathcal{M}$ is a minor-closed family and $\mathcal{M}^* = \{M^* : M \in \mathcal{M}\}$ then the map $D_{\mathcal{M}} : C(\mathcal{M}) \to C(\mathcal{M}^*)$, determined by $M \mapsto M^*$, for all $M \in \widetilde{\mathcal{M}}$, is a coalgebra antiisomorphism. In particular, if $\mathcal{M}$ is closed under duality, then $D_{\mathcal{M}}$ is an antiautomorphism of $C(\mathcal{M})$.*

Proof. The map $D_{\mathcal{M}}$ has inverse $D_{\mathcal{M}^*}$, and is thus bijective. For any matroid $M = M(S)$, and $A \subseteq S$, we have the identities $(M|A)^* = M^*/(S \setminus A)$, and $(M/A)^* = M^*|(S \setminus A)$, from which it follows immediately that $\delta(D_{\mathcal{M}}(M)) = (D_{\mathcal{M}} \otimes D_{\mathcal{M}}) \cdot \tau \cdot \delta(M)$, where $\tau : C(\mathcal{M}) \otimes C(\mathcal{M}) \to C(\mathcal{M}) \otimes C(\mathcal{M})$ is the twist map, determined by $M \otimes N \mapsto N \otimes M$, for all $M, N \in \mathcal{M}$. $\square$

For all matroids N_1, N_2 and $M = M(S)$, the *section coefficient* $\binom{M}{N_1, N_2}$ is defined as the number of subsets A of S such that $M|A \cong N_1$ and $M/A \cong N_2$; hence if $\mathcal{M}$ is a minor-closed family, the coproduct on $C(\mathcal{M})$ is determined by

$$\delta(M) = \sum_{N_1, N_2} \binom{M}{N_1, N_2} N_1 \otimes N_2, \tag{2.8}$$

for all $M \in \mathcal{M}$, where the sum is taken over all (isomorphism classes of) matroids N_1 and N_2. We remark that there is no need to restrict the sum in Eq. (2.8) to matroids N_1 and N_2 belonging to $\mathcal{M}$; because the family $\mathcal{M}$ is minor-closed, the section coefficient $\binom{M}{N_1, N_2}$ is zero whenever N_1 or N_2 is outside of $\mathcal{M}$. Another way of viewing this is the following: if $\mathcal{A}$ is the class of all matroids, then the coproduct in $C(\mathcal{A})$ is given by Eq. (2.8); and if $\mathcal{M}$ is any minor-closed class then $C(\mathcal{M})$ is a subcoalgebra of $C(\mathcal{A})$ and thus the coproduct on $C(\mathcal{M})$ is given by the same formula as that for the coproduct on $C(\mathcal{A})$.

Example 2.9. Suppose that $M(S)$ is the matroid shown in Fig. 2, and that $N = P_2 \oplus P_2$ is the matroid consisting of two double points. The section coefficient $\binom{M}{U_{2,3}, N}$ is equal to one (rather than two, as one might first guess) because, although there are two subsets A of S such that $M|A \cong U_{2,3}$, only for $A = \{a, b, c\}$ do we have $M/A \cong N$; the contraction $M/\{a, d, e\}$, is a three point line with one point doubled.

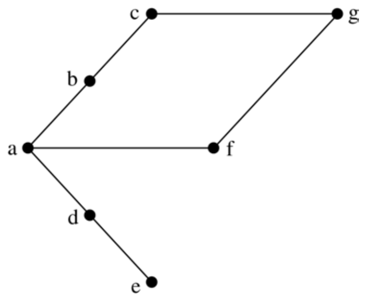

Fig. 2. The matroid 'M' for Example 2.9.

More generally, for matroids $N_1, \ldots, N_k$ and $M = M(S)$, the *multisection coefficient* $\binom{M}{N_1,\ldots,N_k}$ is defined as the number of sequences $(S_0, \ldots, S_k)$ such that $\emptyset = S_0 \subseteq \cdots \subseteq S_k = S$ and $(M|S_i)/S_{i-1} \cong N_i$, for $1 \le i \le k$. Hence the iterated coproduct $\delta^k : C(\mathcal{M}) \to C(\mathcal{M}) \otimes \cdots \otimes C(\mathcal{M})$ is determined by

$$\delta^k(M) = \sum_{N_1,\ldots,N_k} \binom{M}{N_1, \ldots, N_k} N_1 \otimes \cdots \otimes N_k,$$

for all $M \in \mathcal{M}$.

3. Algebras of matroids

For any family of matroids $\mathcal{M}$, we define a pairing $\langle \cdot, \cdot \rangle : K\{\widetilde{\mathcal{M}}\} \times K\{\widetilde{\mathcal{M}}\} \to K$ by setting $\langle M, N \rangle$ equal to the Kronecker delta $\delta_{M,N}$, for all $M, N \in \mathcal{M}$. This pairing determines a pairing of $K\{\widetilde{\mathcal{M}}\} \otimes K\{\widetilde{\mathcal{M}}\}$ with itself, by $\langle M_1 \otimes M_2, N_1 \otimes N_2 \rangle = \langle M_1, N_1 \rangle \cdot \langle M_2, N_2 \rangle$, for all $M_1, M_2, N_1, N_2 \in \mathcal{M}$. If $\mathcal{M}$ is minor-closed, we may thus define a product on $K\{\widetilde{\mathcal{M}}\}$, dual to the coproduct on $C(\mathcal{M})$, by setting

$$\langle N_1 \cdot N_2, M \rangle = \langle N_1 \otimes N_2, \delta(M) \rangle, \tag{3.1}$$

for all $M, N_1, N_2 \in \mathcal{M}$, thus making $K\{\widetilde{\mathcal{M}}\}$ an associative K-algebra, with unit equal to the empty matroid. We denote $K\{\widetilde{\mathcal{M}}\}$, equipped with this algebra structure, by $A(\mathcal{M})$, and note that $A(\mathcal{M})$ is isomorphic to the graded dual algebra of $C(\mathcal{M})$.

Writing $A_n(\mathcal{M})$ and $A_{r,k}(\mathcal{M})$ for the submodules of $A(\mathcal{M})$ generated, respectively, by matroids in $\mathcal{M}$ having n-elements, and those having rank r and nullity k, we have the direct sum decompositions:

$$A(\mathcal{M}) = \bigoplus_{n \ge 0} A_n(\mathcal{M}) = \bigoplus_{r,k \ge 0} A_{r,k}(\mathcal{M}),$$

and it follows from Proposition 2.6 that $A(\mathcal{M})$ is thus both a graded and bigraded algebra. We also have the following result, dual to Proposition 2.7.

Proposition 3.2. *If $\mathcal{M}$ is a minor-closed family and $\mathcal{M}^* = \{M^* : M \in \mathcal{M}\}$ then the map $D : A(\mathcal{M}) \to A(\mathcal{M}^*)$, determined by $M \mapsto M^*$, for all $M \in \widetilde{\mathcal{M}}$, is an* *algebra antiisomorphism. In particular, if $\mathcal{M}$ is closed under duality, then D is an antiautomorphism of $A(\mathcal{M})$.*

By the definition of the pairing, the right-hand side of Eq. (3.1) is the coefficient of the basis element $N_1 \otimes N_2$ in the coproduct $\delta(M)$ which, as noted in Eq. (2.8), is given by the section coefficient $\binom{M}{N_1, N_2}$. Since the left-hand side of (3.1) is the coefficient of the basis element M in the product $N_1 \cdot N_2$, it follows that

$$N_1 \cdot N_2 = \sum_{M \in \widetilde{\mathcal{M}}} \binom{M}{N_1, N_2} M, \tag{3.3}$$

for all $N_1, N_2 \in \mathcal{M}$. We emphasize that, in Eq. (3.3), it is necessary to limit the summation to elements of $\widetilde{\mathcal{M}}$; because $C(\mathcal{M})$ is a subcoalgebra of $C(\mathcal{A})$, where $\mathcal{A}$ is the family of all matroids, it follows that $A(\mathcal{M})$ is a quotient of the algebra $A(\mathcal{A})$. Hence the product of N_1 and N_2 in $A(\mathcal{M})$ is the image of their product in $A(\mathcal{A})$ under the projection homomorphism $A(\mathcal{A}) \to A(\mathcal{M})$, which maps all matroids $M \notin \widetilde{\mathcal{M}}$ to zero.

Example 3.4. Suppose that $\mathcal{M}$ is a minor-closed family containing point I and loop Z. Then $Z \cdot I = I \oplus Z$ in $A(\mathcal{M})$. If $\mathcal{M}$ contains the double point P_2 then $I \cdot Z = I \oplus Z + 2P_2$; otherwise, $I \cdot Z = I \oplus Z$. If $\mathcal{M}$ contains the free matroid I_n then $I^n = n! I_n$, and if $\mathcal{M}$ contains the zero matroid Z_n, we have $Z^n = n! Z_n$ in $A(\mathcal{M})$.

Example 3.5. Suppose that L is the matroid shown in Fig. 1 and that M is the matroid consisting of five points a, b, c, d, e in the plane, with a, b, c collinear. If $\mathcal{M}$ is any minor-closed family that contains L, M and the direct sum $U_{2,3} \oplus P_2$ of the three-point line with a double point, then we have $U_{2,3} \cdot P_2 = M + 2L + (U_{2,3} \oplus P_2)$ in $A(\mathcal{M})$.

Example 3.6. If $\mathcal{M}$ contains the free matroid I_r and zero matroid Z_k, then the product $I_r \cdot Z_k$ in $A(\mathcal{M})$ is given by

$$I_r \cdot Z_k = \sum (\#\text{ of bases of } M) \cdot M,$$

where the sum is over all matroids $M \in \widetilde{\mathcal{M}}$ having rank r and nullity k. On the other hand, for any $M \in \mathcal{M}$ and $k \geq 0$, the product $Z_k \cdot M$ is equal to $\binom{k+\ell}{k} Z_k \oplus M$, where ℓ is the number of loops of M if $Z_k \oplus M \in \mathcal{M}$, and is equal to zero otherwise; so in particular, $Z_k \cdot I_r = Z_k \oplus I_r$ if $\mathcal{M}$ contains $Z_k \oplus I_r$, and $Z_k \cdot I_r = 0$, otherwise.

Example 3.7. Let $\mathcal{C}$ be the minor-closed family consisting of all free matroids I_n and circuits C_k, for $n \geq 0$ and $k \geq 1$. It follows from the coproduct formulas in Example 2.5 that the product in $A(\mathcal{C}) = K[I_0, I_1, \ldots, C_1, C_2, \ldots]$ is determined by

$$I_n \cdot I_m = \binom{n+m}{n} I_{n+m}, \qquad C_k \cdot C_\ell = 0,$$
$$I_n \cdot C_k = \binom{n+k}{n} C_{n+k}, \qquad C_k \cdot I_n = \begin{cases} C_k & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $m, n \geq 0$ and $k, \ell \geq 1$. The dual family $\mathcal{C}^*$ consists of all zero matroids Z_n and multiple points P_k, for $n \geq 0$ and $k \geq 1$. By Proposition 3.2, the product in $A(\mathcal{C}^*)$ is determined by $Z_n \cdot Z_m = \binom{n+m}{n} Z_{n+m}$,

$$P_k \cdot Z_n = \binom{n+k}{n} P_{n+k}, \qquad Z_n \cdot P_k = \begin{cases} P_k & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $P_k \cdot P_\ell = 0$, for all $m, n \geq 0$ and $k, \ell \geq 1$.

4. Orderings of subsets and words

For any set S and $r \geq 0$, we denote by $\mathcal{B}(S)$ and $\mathcal{B}_r(S)$, respectively, the set of all subsets and the set of all r-element subsets of S. In particular, for all $n \geq 0$, we write $\mathcal{B}(n)$ and $\mathcal{B}_r(n)$, respectively, for $\mathcal{B}([n])$ and $\mathcal{B}_r([n])$, where $[n]$ denotes the set $\{1, \ldots, n\}$. Whenever we write a subset of a linearly ordered set S by listing its elements, we shall assume that the list is written in the order induced by S; that is, if S is linearly ordered, and $A = \{a_1, \ldots, a_r\} \subseteq S$, then $a_1 < \cdots < a_r$ in S. Throughout this paper we shall always assume that S, whether linearly ordered or not, is a finite set.

For any linearly ordered S and $r \geq 0$, we define a partial order on $\mathcal{B}_r(S)$ by setting $\{a_1, \ldots, a_r\} \leq \{b_1, \ldots, b_r\}$ if and only if $a_i \leq b_i$ in S, for all $i \in [r]$. Under this ordering, $\mathcal{B}_r(S)$ is a sublattice of the r-fold direct product of linearly ordered sets $S \times \cdots \times S$, and is thus a distributive lattice. The Hasse diagram of $\mathcal{B}_2(\{a, b, c, d, e\})$ is shown in Fig. 3.

We extend the ordering on $\mathcal{B}_r(S)$ to all of $\mathcal{B}(S)$ by setting $B \geq A$ in $\mathcal{B}(S)$ if and only if $B \geq A'$ in some $\mathcal{B}_r(S)$, for some subset A' of A. Hence, if $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_r\}$, then $A \leq B$ if and only if $r \leq k$ and $a_i \leq b_i$, for $1 \leq i \leq r$. Equipped with this ordering, $\mathcal{B}(S)$ is a distributive lattice that contains each $\mathcal{B}_r(S)$ as a sublattice.

Lemma 4.1. *For any linearly ordered set S, the map $\mathcal{B}(S) \to \mathcal{B}(S)$ taking $A \subseteq S$ to its complement in S is a lattice antiautomorphism.*

Proof. Suppose that $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_r\}$ are subsets of the linearly ordered set S such that $A \leq B$ in $\mathcal{B}(S)$, that is, such that $r \leq k$ and $a_i \leq b_i$, for all $i \in [r]$. If $A' = \{s_1, \ldots, s_{n-k}\}$ and $B' = \{t_1, \ldots, t_{n-r}\}$ are the complements of A and B in S, then $n - r \geq n - k$, and $s_j = j + |\{i : a_i < j\}|$ and $t_j = j + |\{i : b_i < j\}|$, for all j. Since $a_i \leq b_i$, for all $i \in [r]$, it follows that $|\{i : a_i < j\}| \geq |\{i : b_i < j\}|$, for all j. Hence $s_j \geq t_j$, for $1 \leq j \leq n - k$, and so $A' \geq B'$ in $\mathcal{B}(S)$. $\square$

For any linearly ordered set S, we denote by S_φ the *reversal* of S, that is, the set S equipped with the opposite ordering: $a \leq b$ in S_φ if and only if $a \geq b$ in S.

Lemma 4.2. *For any linearly ordered set S, the identity map is a lattice antiisomorphism $\mathcal{B}_r(S) \to \mathcal{B}_r(S_\varphi)$.*

Proof. It is immediate from the definition of the ordering on $\mathcal{B}_r(S)$ that $A \leq B$ in $\mathcal{B}_r(S)$ if and only if $A \geq B$ in $\mathcal{B}_r(S_\varphi)$. $\square$

Given a word w on the alphabet $\{0, 1\}$, and $i \in \{0, 1\}$, we denote by $|w|_i$ the number of occurrences of the letter i in w. For all $n \geq 0$, we write $\mathcal{W}_n$ for the set of all words on $\{0, 1\}$ having length n, and let $\mathcal{W}_{n,r} = \{w \in \mathcal{W}_n : |w|_1 = r\}$, for $0 \leq r \leq n$. For any linearly ordered set $S = \{e_1, \ldots, e_n\}$, let $\chi : \mathcal{B}(S) \to \mathcal{W}_n$ be the function which maps

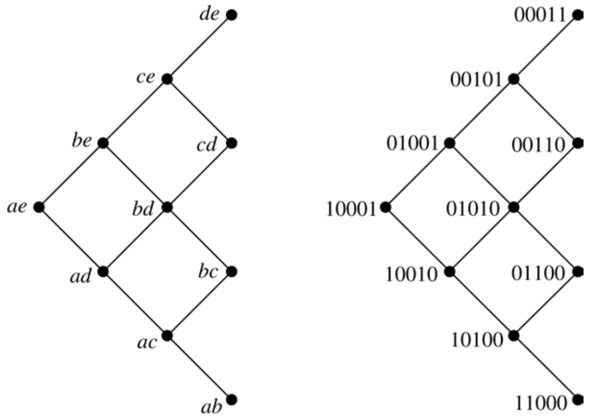

Fig. 3. The lattices $\mathcal{B}_2(a, b, c, d, e)$ and $\mathcal{W}_{5,2}$.

$A \subseteq S$ to the word $x_1 \ldots x_n$, where

$$x_i = \begin{cases} 1, & \text{if } e_i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Note that χ maps each $\mathcal{B}_r(S)$ bijectively onto $\mathcal{W}_{n,r}$ and that, under the natural identification of $\mathcal{W}_n$ with the set of functions $S \to \{0, 1\}$, the function χ simply maps subsets of S to their characteristic functions.

Define maps $\pi_k : \mathcal{W}_{n,r} \to [n]$, for $1 \le k \le r$, by letting $\pi_k(w)$ be the position of the kth 1 in $w \in \mathcal{W}_{n,r}$. It follows that, for $S = \{e_1, \ldots, e_n\}$, the map $\pi : \mathcal{W}_{n,r} \to \mathcal{B}_r(S)$ which is inverse to χ is given by $\pi(w) = \{e_{\pi_1(w)}, \ldots, e_{\pi_r(w)}\}$, for all $w \in \mathcal{W}_{n,r}$. We define a partial order on $\mathcal{W}_{n,r}$ by setting $v \le w$ if and only if $\pi_k(v) \le \pi_k(w)$, for $1 \le k \le r$. For example, the Hasse diagram of the lattice $\mathcal{W}_{5,2}$ is given in Fig. 3.

Lemma 4.3. *For any linearly ordered set S, and $1 \le r \le n = |S|$, the map $\chi : \mathcal{B}_r(S) \to \mathcal{W}_{n,r}$ is a lattice isomorphism.*

Proof. It is immediate from the definition of χ that $A \le B$ in $\mathcal{B}_r(S)$ if and only if $\pi_k(\chi(A)) \le \pi_k(\chi(B))$, for $1 \le k \le r$. $\square$

Lemma 4.4. *For all $v = x_1 \cdots x_r$ and $w = y_1 \cdots y_r$ in $\mathcal{W}_{n,r}$, the inequality $v \le w$ holds if and only if $|x_1 \cdots x_k|_1 \ge |y_1 \cdots y_k|_1$, for $1 \le k \le r$.*

Proof. The proof is immediate from the definitions. $\square$

5. Freedom matroids

By a *flag* on a finite set S we shall mean a sequence $(S_0, \ldots, S_r)$ of subsets of S such that $S_r = S$ and S_{i-1} is a proper subset of S_i, for $1 \le i \le r$. We do not require S_0 to be empty.

Proposition 5.1. *For any flag* $(S_0, \ldots, S_r)$ *on a set* S*, the family*

$$\mathcal{I} = \{I \subseteq S : |I \cap S_i| \leq i, \textit{for all } i\}$$

is the collection of independent sets of a matroid $M(S_0, \ldots, S_r)$*, of rank* r*, on* S*.*

Proof. It is clear that $\mathcal{I}$ contains the empty set and is closed under formation of subsets. Now suppose that $I, J \in \mathcal{I}$ with $|I| < |J|$. If $|I \cap S_i| < i$ for all i, then for any $x \in J \setminus I$ we have $|(I \cup x) \cap S_i| \leq i$ for all i, and hence $I \cup x \in \mathcal{I}$. So we suppose that there exists some i such that $|I \cap S_i| = i$, and let m be the maximal such i. Note that $m < r$, since $m = |I \cap S_m| \leq |I| < |J| = |J \cap S_r| \leq r$.

Now, since $|J \cap S_m| \leq m = |I \cap S_m|$, and $|J| > |I|$, we must have $|J \cap S'_m| > |I \cap S'_m|$, where S'_m denotes the complement of S_m in S, and hence the set $(J \setminus I) \cap S'_m$ is nonempty. Let x be any element of $(J \setminus I) \cap S'_m$. For $m < i \leq r$, we have $|I \cap S_i| < i$, and thus $|(I \cup x) \cap S_i| \leq i$. Since $x \notin S_m$ we have $(I \cup x) \cap S_i = I \cap S_i$, and so $|(I \cup x) \cap S_i| \leq i$, for all $i \leq m$. Thus $I \cup x \in \mathcal{I}$. $\square$

We refer to the matroid $M(S_0, \ldots, S_r)$ as the *freedom matroid* (see [16]) defined by the flag $(S_0, \ldots, S_r)$. Note that it follows immediately from the definition that each S_k is a flat of rank k in $M(S_0, \ldots, S_r)$.

If M is a matroid on S and $e \in S$, we denote by $M \setminus e$ and M/e the matroids obtained from M by, respectively, deleting and contracting e.

Proposition 5.2. *For any freedom matroid* $M = M(S_0, \ldots, S_r)$ *and* $e \in S$*, the deletion* $M \setminus e$ *and contraction* M/e *are given by*

$$M \setminus e = M(T_0, \ldots, T_r) \qquad \textit{and} \qquad M/e = M(T_0, \ldots, T_{k-2}, T_k, \ldots, T_r),$$

where $T_i = S_i \setminus e$*, for all* i*, and* $k = \min\{i : x \in S_i\}$*.*

Proof. The independent sets of $M \setminus e$ are the subsets of S that do not contain e and contain no more than i elements of each S_i, which are precisely the independent subsets of $M(T_0, \ldots, T_r)$.

If e is a loop in M, then $M/e = M \setminus e = M(T_0, \ldots, T_r)$, which agrees with the expression for M/e given in the proposition, since $k = 0$ in this case. If e is not a loop, then A is independent in M/e if and only if $e \notin A$ and $A \cup e$ is independent in M, that is $|(A \cup e) \cap S_i| \leq i$, for all i; in other words, $|A \cap T_i| \leq i$, for $i < k$, and $|A \cap T_i| \leq i - 1$, for $i \geq k$. Since $T_{k-1} \subseteq T_k$, the condition $|A \cap T_k| \leq k - 1$ implies that $|A \cap T_{k-1}| \leq k - 1$ and hence the latter inequality is redundant. Thus A is independent in M/e if and only if $|A \cap T_i| \leq i$ for $0 \leq i \leq k - 2$ and $|A \cap T_i| \leq i - 1$, for $k \leq i \leq r$; equivalently, if and only if A is independent in $M(T_0, \ldots, T_{k-2}, T_k, \ldots, T_r)$. $\square$

Corollary 5.3 (*[15]*). *The class of freedom matroids is minor-closed.*

We now characterize the closure operators and closed sets of freedom matroids. We begin with the following proposition.

Proposition 5.4. *The closure of an independent set* A *in a freedom matroid* $M = M(S_0, \ldots, S_r)$ *is given by* $c\ell_M(A) = A \cup S_m$*, where* $m = \max\{i : |A \cap S_i| = i\}$*.*

Proof. First note that $|A \cap S_0| = 0$, because A is independent, and thus such m exists. Now, since $|A \cap S_m| = m$, the set $A \cup x$ is dependent for all $x \in S_m \setminus A$, and thus $S_m \subseteq c\ell_M(A)$. On the other hand, for any $y \notin A \cup S_m$, the set $A \cup y$ is independent, since $|(A \cup y) \cap S_i| = |A \cap S_i| \leq i$, for $i \leq m$ and $|(A \cup y) \cap S_i| \leq 1 + |A \cap S_i| \leq i$, for $i > m$; hence $c\ell_M(A) \subseteq A \cup S_m$. $\square$

We may thus find the closure of an arbitrary set A in a freedom matroid by applying Proposition 5.4 to any maximal independent subset B of A and using the fact that $c\ell(B) = c\ell(A)$.

Proposition 5.5. *A set $F \subseteq S$ is closed in $M(S_0, \dots, S_r)$ if and only if $F = A \cup S_m$, for some $m \geq 0$ and $A \subseteq S \backslash S_m$ such that $|A \cap S_i| < i - m$, for all $i > m$; in which case the rank of F is $m + |A|$.*

Proof. Suppose that F is closed and that B is a basis for F. By Proposition 5.4, $F = c\ell(B) = B \cup S_m$ for some m such that $|B \cap S_m| = m$ and $|B \cap S_i| < i$, for all $i > m$. Letting $A = B \backslash S_m$, we thus have $F = A \cup S_m$ and $|A \cup S_i| < i - m$, for all $i > m$.

On the other hand, suppose that $F = A \cup S_m$ for some $m \geq 0$ and $A \subseteq S \backslash S_m$, such that $|A \cup S_i| < i - m$, for all $i > m$. Let B be a basis for S_m. Since A is disjoint from S_m, and thus also from B, and $|B| = m$, it follows from the above inequality that $|(A \cup B) \cap S_i| \leq i$, for $i > m$, and hence that $A \cup B$ is independent. Since $m = \max\{i : |(A \cup B) \cap S_i| = i\}$, it follows from Proposition 5.4 that $A \cup S_m = c\ell(A \cup B)$, and is thus closed. $\square$

Note that if we are given a closed set F in $M(S_0, \dots, S_r)$, we can express F as $A \cup S_m$, according to Proposition 5.5, by letting $m = \max\{i : S_i \subseteq F\}$, and taking $A = F \backslash S_m$.

Corollary 5.6. *If F is any flat of rank k in $M(S_0, \dots, S_r)$, then $|F| \leq |S_k|$.*

Proof. By Proposition 5.5, if F is a flat of rank k in $M(S_0, \dots, S_r)$ then $F = S_m \cup A$, for some m and $A \subseteq S \backslash S_m$ with $|A| = k - m$. Since $|S_k| - |S_m| \geq k - m$, it follows that $|F| = |S_m| + |A| = |S_m| + k - m \leq |S_k|$. $\square$

6. Freedom matroids on ordered sets

In the case that S is linearly ordered it is convenient to consider flags $(S_0, \dots, S_r)$ such that each S_i is an initial segment in the ordering of S. In this case, the flag $(S_0, \dots, S_r)$ is determined by S together with the set $\{1 + \max S_i : 0 \leq i \leq r - 1\}$. Hence if S is linearly ordered and we are given a subset $T = \{t_1, \dots, t_r\}$ of S, we may obtain a flag $(T_0, \dots, T_r)$ on S by setting $T_r = S$ and $T_i = \{s \in S : s < t_{i+1}\}$, for $0 \leq i \leq r - 1$. We denote the freedom matroid $M(T_0, \dots, T_r)$ by $M_T(S)$, or simply M_T, when the set S is understood. If $T \subseteq [n]$ and $S = \{e_1, \dots, e_n\}$, we also write $M_T(S)$ for the matroid $M_{\alpha(T)}(S)$, where $\alpha : \mathcal{B}(n) \to \mathcal{B}(S)$ is the natural bijection $i \mapsto e_i$.

Proposition 6.1. *If S is linearly ordered and $T \subseteq S$, then the family of independent sets of $M_T = M_T(S)$ is given by $\{A \subseteq S : A \geq T$ in $\mathcal{B}(S)\}$. If $|T| = r$, then the family of bases of M_T is given by $\{B : B \geq T$ in $\mathcal{B}_r(S)\}$.*

Proof. Suppose that $T = \{t_1, \ldots, t_r\}$ and $A = \{a_1, \ldots, a_k\}$ in $\mathcal{B}(S)$. Since $T_r = S$, we have $A = A \cap T_r$, and thus $|A \cap T_r| \leq r$ if and only if $k \leq r$. Now for $0 \leq i \leq r$, we have $A \cap T_i = \{a_j \in A : a_j < t_{i+1} \text{ in } S\}$; therefore, since $a_1 < \cdots < a_k$ and $t_1 < \cdots < t_r$, it follows that $|A \cap T_i| \leq i$ if and only if $a_{i+1} \geq t_{i+1}$. Hence A is independent in M_T if and only if $A \geq T$ in $\mathcal{B}(S)$. □

Example 6.2. Suppose that $S = \{a, b, c, d, e, f, g\}$ and $T = \{b, e, f\}$. Then $M_T = M(T_0, T_1, T_2, T_3)$, where $T_0 = \{a\}$, $T_1 = \{a, b, c, d\}$, $T_2 = \{a, b, c, d, e\}$ and $T_3 = S$. The bases of M_T are the sets $\{b, e, f\}, \{c, e, f\}, \{d, e, f\}, \{b, e, g\}, \{c, e, g\}, \{d, e, g\}, \{b, f, g\}, \{c, f, g\}, \{d, f, g\}$ and $\{e, f, g\}$.

Proposition 6.3. *For any linearly ordered S, and $T \subseteq S$, the dual $M_T(S)^*$ of the matroid $M_T(S)$ is equal to $M_{T'}(S_\varphi)$, where T' is the complement of T in S and S_φ is the reversal of S. In particular, the class of freedom matroids is closed under duality.*

Proof. Suppose that $|S| = n$ and $|T| = r$. It follows from Proposition 6.1 that the set of bases of $M_T(S)^*$ is given by $\{B' : B \geq T \text{ in } \mathcal{B}_r(S)\}$, which, according to Lemma 4.1, is equal to $\{C : C \leq T' \text{ in } \mathcal{B}_{n-r}(S)\}$. By Lemma 4.2, we have $C \leq T'$ in $\mathcal{B}_{n-r}(S)$ if and only $C \geq T'$ in $\mathcal{B}_{n-r}(S_\varphi)$, and hence the result follows from Proposition 6.1. □

The following Lemma, which is a corollary of Proposition 6.1, will be used in the next section.

Lemma 6.4. *Suppose that $M(S) = M(S_0, \ldots, S_r)$ is a freedom matroid, where S is linearly ordered and each S_i is an initial segment in S, and let $A \subseteq S$ and $a \in A$. If $b \in S \setminus A$ satisfies $b > a$ in S, then $\rho((A \setminus a) \cup b) \geq \rho(A)$.*

Proof. Let B be a maximal independent subset of A that contains a. Since $b > a$ in S, it follows that $(B \setminus a) \cup b > B$ in $\mathcal{B}(S)$. Hence, by Proposition 6.1, the set $(B \setminus a) \cup b$ is independent in M, and so $\rho((A \setminus a) \cup b) \geq \rho(A)$. □

Recall from Section 4 that, given a word $w \in \mathcal{W}_{n,r}$, and $1 \leq k \leq r$, we denote by $\pi_k(w)$ the position of the kth 1 in w, and for $S = \{e_1, \ldots, e_n\}$, the bijection $\pi : \mathcal{W}_{n,r} \to \mathcal{B}_r(S)$ is given by $\pi(w) = \{e_{\pi_1(w)}, \ldots, e_{\pi_r(w)}\}$. We thus may define a mapping $w \mapsto M_w$ from $\mathcal{W}_{n,r}$ to the set of rank r freedom matroids on S by setting $M_w = M_{\pi(w)}(S)$, for all $w \in \mathcal{W}_{n,r}$.

Example 6.5. If $S = \{a, b, c, d, e, f, g, h, i, j, k, l\}$ and $w = 001011001000$, then $\pi(w) = \{c, e, f, i\}$. The sets S_i may be read off from the following table:

$w:$	0	0	1	0	1	1	0	0	1	0	0	0
$S_0:$	a	b										
$S_1:$	a	b	c	d								
$S_2:$	a	b	c	d	e							
$S_3:$	a	b	c	d	e	f	g	h				
$S_4:$	a	b	c	d	e	f	g	h	i	j	k	l,

and $M_w = M_{\{c,e,f,i\}}$ is the freedom matroid $M(S_0, S_1, S_2, S_3, S_4)$.

When freedom matroids were first introduced, in [10], they were given the following recursive construction by single-element extensions: If w is the empty word, then M_w is the empty matroid, and for $w = vx$, where $|x| = 1$, M_w is obtained from M_v as follows:

(i) If $x = 1$, add a point independently to M_v in a new dimension, that is, let $M_w = M_v \oplus I$.
(ii) If $x = 0$, add a point e to M_v in general position in the top rank, that is, let M_w be the free extension of M_v by e.

Example 6.6. If $w = 001001010010$ and $S = \{a, b, c, d, e, f, g, h, i, j, k, l\}$, then M_w consists of loops a and b, together with a triple point $\{c, d, e\}$, collinear with distinct points f and g, this line being coplanar with general points h, i, j, with two additional points k and l in general position in 3-space.

7. Matroids and words

Suppose that M is a matroid of rank r on an n-element set S, having rank function ρ. We associate to any maximal chain $\emptyset = A_0 \subset \cdots \subset A_n = S$ in the Boolean algebra 2^S the word $x_1 \cdots x_n \in \mathcal{W}_{n,r}$ defined by $x_i = \rho(A_i) - \rho(A_{i-1})$, for all $i \in [n]$. If the set $S = \{e_1, \ldots, e_n\}$ is linearly ordered, then there is a distinguished maximal chain $A_0 \subset \cdots \subset A_n$ in 2^S, given by $A_i = \{e_1, \ldots, e_i\}$, for all $i \in [n]$. The word $w_{M(S)} = x_1 \cdots x_n$ associated to this chain is thus determined by

$$x_i = \begin{cases} 0, & \text{if } e_i \in c\ell(\{e_1, \ldots, e_{i-1}\}), \\ 1, & \text{otherwise,} \end{cases}$$

for all $i \in [n]$. We refer to $w_{M(S)}$ as the *distinguished word* of $M(S)$. Note that $w_{M(S)}$ is also determined by the equality $|x_1 \cdots x_i|_1 = \rho(\{e_1, \ldots, e_i\})$, for all $i \in [n]$.

Lemma 7.1. *For any matroid $M(S)$ of rank r, with S linearly ordered of cardinality n, the word $w = w_{M(S)}$ is determined by the condition that $\pi(w) = \min\{B \in \mathcal{B}_r(S) :$ B is a basis for $M\}$.*

Proof. Suppose $S = \{e_1, \ldots, e_n\}$, and that the 1's in w occur in positions $i_1, \ldots, i_r$, so that $\pi(w) = \{e_{i_1}, \ldots, e_{i_r}\}$. Since e_{i_k} is not in the closure of $\{e_1, \ldots, e_{i_k-1}\}$, for all $k \in [r]$, it follows that $\pi(w)$ is independent, and thus is a basis for M. If $B = \{b_1, \ldots, b_r\} \subseteq S$ is such that $k \le i_k$, for some $k \in [r]$, then $\{b_1, \ldots, b_k\} \subseteq \{e_1, \ldots, e_{i_k-1}\}$, which has rank $k - 1$, and so B is not a basis for M. Hence any basis B of M satisfies $B \ge \pi(w)$ in $\mathcal{B}_r(S)$. $\square$

If $S = \{e_1, \ldots, e_n\}$ is linearly ordered, then the symmetric group Σ_n acts naturally on S by $\sigma(e_i) = e_{\sigma(i)}$, for all $i \in [n]$, and thus we can identify Σ_n with the group Σ_S of permutations of S. For any σ in Σ_S (or in Σ_n), we denote by S_σ the underlying set of S equipped with the linear order (or *reorder*) given by $\sigma(e_1) < \cdots < \sigma(e_n)$. Hence, $a \le b$ in S if and only if $\sigma(a) \le \sigma(b)$ in S_σ, and so $\sigma : S \to S_\sigma$ is a poset isomorphism. The natural map $\mathcal{B}(S) \to \mathcal{B}(S_\sigma)$, given by $A \mapsto \sigma(A)$, for all $A \subseteq S$, and also denoted by σ, is also a poset isomorphism. We denote by π_σ the map $\mathcal{W}_{n,r} \to \mathcal{B}_r(S_\sigma)$, which takes a word to the subset of S_σ corresponding to positions of its 1's. Note that π_σ is equal to the composition $\sigma\pi$.

Given $A, B \subseteq S$ of equal cardinality, with complements A' and B' in $[n]$, the *shuffle* $\sigma_{A,B} \in \Sigma_S$ is the unique permutation of S which maps B onto A, and thus also B' onto A', whose restrictions to B and B' are order-preserving. For example, if $A = \{4, 7\}$ and $B = \{1, 5\}$ in $S = [7]$, then $\sigma_{A,B} = 4123756$ (where $\sigma = \sigma_1 \cdots \sigma_n \in \Sigma_n$ is the usual word notation for permutations, indicating that $\sigma(i) = \sigma_i$, for all i), or in cycle notation, $\sigma_{A,B} = (1432)(576)$.

Lemma 7.2. *Suppose that S is linearly ordered, and that $A \geq B$ in $\mathcal{B}(S)$, where $|A| = |B|$, and let $\sigma = \sigma_{A,B} \in \Sigma_S$ be the shuffle. If $C \subseteq S$ satisfies $C \geq A$ in $\mathcal{B}(S)$, then $C \geq A$ in $\mathcal{B}(S_\sigma)$.*

Proof. Suppose that the complements of $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ in S are $A' = \{a'_1, \ldots, a'_k\}$ and $B' = \{b'_1, \ldots, b'_k\}$, respectively, so that the shuffle $\sigma = \sigma_{A,B}$ is given by $b_i \mapsto a_i$ and $b'_j \mapsto a'_j$, for all $i \in [r]$ and $j \in [k]$. Since $\sigma : \mathcal{B}(S) \to \mathcal{B}(S_\sigma)$ is an isomorphism, it follows that for any $C \subseteq S$, we have $C \geq A$ in $\mathcal{B}(S_\sigma)$ if and only if $\sigma^{-1}(C) \geq \sigma^{-1}(A) = B$ in $\mathcal{B}(S)$. Now suppose that $C = \{c_1, \ldots, c_m\} \geq A$ in $\mathcal{B}(S)$, so that $m \leq r$ and $c_i \geq a_i$, for all $i \in [m]$. Since $A \geq B$ in $\mathcal{B}(S)$, it follows from Lemma 4.1 that $A' \leq B'$ in $\mathcal{B}(S)$. Hence $\sigma^{-1}(a) \leq a$, for all $a \in A$, and $\sigma^{-1}(a') \geq a'$, for all $a' \in A'$. Consider $c_i \in C$. If $c_i \in A'$, then $\sigma^{-1}(c_i) \geq c_i \geq a_i \geq b_i$. On the other hand, if $c_i \in A$, then $c_i = a_j$, for some $j \geq i$ (since $c_i \geq a_i$), and so $\sigma^{-1}(c_i) = \sigma^{-1}(a_j) = b_j \geq b_i$. Hence $\sigma^{-1}(C) \geq B$ in $\mathcal{B}(S)$, and therefore $C \geq A$ in $\mathcal{B}(S_\sigma)$. □

For any matroid $M(S)$ of rank r, where S is linearly ordered of cardinality n, we define a mapping $\lambda_M : \Sigma_S \to \mathcal{W}_{n,r}$ (or equivalently, $\lambda_M : \Sigma_n \to \mathcal{W}_{n,r}$) by setting $\lambda_M(\sigma) = w_{M(S_\sigma)}$, for all $\sigma \in \Sigma_S$. Note that, in particular, if $\iota \in \Sigma_S$ is the identity permutation, then $\lambda_M(\iota) = w_{M(S)}$ is the distinguished word of $M(S)$. We emphasize that the map λ_M depends not only on the matroid $M = M(S)$, but on the linear ordering of S.

For example, if M is the matroid on $S = \{a, b, c, d, e, f, g\}$ shown in Fig. 2, and $\sigma \in \Sigma_7$ is the permutation 6237154, then $\lambda_M(\sigma) = 1110010$.

Proposition 7.3. *Suppose that $M(S)$ is a rank r matroid, with S an n-element linearly ordered set. If $v \leq w_{M(S)}$ in $\mathcal{W}_{n,r}$, then $\lambda_M(\sigma_{A,B}) = v$, where $A = \pi(w_{M(S)})$ and $B = \pi(v)$.*

Proof. By Lemma 7.1, $A = \pi(w_{M(S)})$ is the minimum basis of M in $\mathcal{B}_r(S)$. Since $A \geq B = \pi(v)$ in $\mathcal{B}_r(S)$, it follows from Lemma 7.2 that A is also the minimum basis of M in $\mathcal{B}_r(S_\sigma)$, where σ is the shuffle $\sigma_{A,B}$. Since $A = \sigma(B) = \sigma(\pi(v)) = \pi_\sigma(v)$, it thus follows from Lemma 7.1 that $v = w_{M(S_\sigma)}$, that is, $\lambda_M(\sigma) = v$. □

Corollary 7.4. *For any rank r matroid M on an n-element linearly ordered set, the image of λ_M is an order ideal in $\mathcal{W}_{n,r}$.*

Proof. The proof is immediate from Proposition 7.3. □

It was shown in [10] (Theorem: "Existence of a matroid with a given first word") that in the case in which $M = M_w$ is a freedom matroid, the word w is the maximum among words associated to M by the map λ_M. The following theorem is a strengthening of this result, giving a characterization of the words in the image of λ_M whenever M is a freedom matroid.

Theorem 7.5. *If M is the freedom matroid M_w for some $w \in \mathcal{W}_{n,r}$, then the image of $\lambda_M : \Sigma_n \to \mathcal{W}_{n,r}$ is the principal order ideal $\{v \in \mathcal{W}_{n,r} : v \le w\}$.*

Proof. Suppose that $M = M(S) = M_w$, where $S = \{e_1, \ldots, e_n\}$ and $w = x_1 \cdots x_n$ belongs to $\mathcal{W}_{n,r}$. It follows that $M = M(S_0, \ldots, S_r)$, where $S_r = S$, and $S_{k-1} = \{e_1, \ldots, e_{\pi_k(w)-1}\}$, for $1 \le k \le r$. For any $\sigma \in \Sigma_n$, the word $\lambda_M(\sigma) = y_1 \cdots y_n$ is determined by the condition that $|y_1 \cdots y_i|_1 = (\{e_{\sigma(1)}, \ldots, e_{\sigma(i)}\})$, for $1 \le i \le n$, and by Corollary 5.6, if $\rho(\{e_{\sigma(1)}, \ldots, e_{\sigma(i)}\}) = k$, for some i, then $i \le |S_k| = \pi_{k+1}(w) - 1$. Since $\pi_{k+1}(w)$ is the position of the $(k+1)$st one in w, it follows that $|x_1 \cdots x_i|_1 \le k = |y_1 \cdots y_i|_1$. Hence, by Lemma 4.4, we have $\lambda_M(\sigma) \le w$. The result thus follows from Corollary 7.4. □

Example 7.6. Suppose that $M(S) = U_{2,4} \oplus P_2$ is the matroid consisting of a four-point line and a double point. The image of λ_M in $\mathcal{W}_{6,3}$ (given any linear ordering on S) is the order ideal {111000, 110100, 101100, 110010}, which has maximal elements 110010 and 101100, and thus is not principal. Hence, it follows from Theorem 7.5 that M is not a freedom matroid.

Corollary 7.7 ([10]). *There are precisely 2^n nonisomorphic freedom matroids (and thus at least 2^n nonisomorphic matroids) on an n-element set.*

Proof. Given a matroid M on S, the definition of λ_M depends on a choice of ordering of S, but the image of λ_M depends only on the isomorphism class of M. Hence, by Theorem 7.5, if $v \ne w$, then the freedom matroids M_v and M_w are not isomorphic. □

Recall that the *Bruhat order* (or *strong Bruhat order*) on Σ_n is determined by the condition that σ covers $\tau = \tau_1 \cdots \tau_n$ in Σ_n if and only if σ may be obtained from τ by reversing a single pair (τ_i, τ_j), such that $i < j$ and $\tau_i < \tau_j$ and the number of inversions of σ is one greater than the number of inversions of τ. Under the assumptions $i < j$ and $\tau_i < \tau_j$, the exchange (τ_i, τ_j) increases the number of inversions by one if and only if, for all k with $i < k < j$, either $\tau_k < \tau_i$ or $\tau_k > \tau_j$, which, in particular, is the case if either $j = i + 1$ or $\tau_j = \tau_i + 1$. For example, in the Bruhat order on Σ_4, the permutation 1423 is covered by 4123, 2413 and 1432. Reversing the pair (1, 3) in 1423 creates three new inversions, so that, even though 3421 is greater than 1423, it is not a cover. The identity permutation is the minimum element of Σ_n, and the flip map $\varphi = n(n-1)\cdots 1$ is the maximum element.

Proposition 7.8. *If $M = M_w$ for any $w \in \mathcal{W}_{n,r}$, and Σ_n is given the Bruhat order, then $\lambda_M : \Sigma_n \to \mathcal{W}_{n,r}$ is an order-reversing map.*

Proof. Suppose that $M_w = M(S) = M(S_0, \ldots, S_r)$, where S is linearly ordered and each S_i is an initial segment in S. Suppose that τ covers σ in the Bruhat order on Σ_n and let $S_\sigma = \{e_1, \ldots, e_n\}$ and $S_\tau = \{f_1, \ldots, f_n\}$, so that $e_k = f_k$ for all but two indices i and j, where

$$i < j, \qquad e_i < e_j, \qquad f_j = e_i, \qquad \text{and} \qquad f_i = e_j.$$

Letting $E_k = \{e_1, \ldots, e_k\}$ and $F_k = \{f_1, \ldots, f_k\}$, for all $k \in [n]$, we have $E_k = F_k$, for $1 \le k < i$ and $j < k \le n$, and since $e_j > e_i$ in S, it follows from Lemma 6.4 that $\rho(F_k) \ge \rho(E_k)$, for $i \le k \le j$. Letting $\lambda_M(\sigma) = x_1 \cdots x_n$ and $\lambda_M(\tau) = y_1 \cdots y_n$, we thus have $|x_1 \cdots x_k|_1 = \rho(E_k) \leq \rho(F_k) = |y_1 \cdots y_k|_1$, for all $k \in [n]$, and hence $\lambda_M(\sigma) \geq \lambda_M(\tau)$, by Lemma 4.4. $\square$

Example 7.9. Suppose that $S = \{a, b, c, d\}$ and $M(S) = M_{0101}$, so that a is a loop, $\{b, c\}$ a double point and d an isthmus in M. The image of $\lambda_M : \Sigma_4 \to \mathcal{W}_{4,2}$ is the order ideal $\{1100, 0110, 1001, 1010\}$, and under λ_M, the two permutations in the interval $[1234, 1324]$ of Σ_4 map to 0101, the four permutations in the interval $[1243, 1432]$ map to 0110, the four permutations in the interval $[2134, 3214]$ map to 1001, the set $\{\sigma : \sigma \geq 2143$ and either $\sigma \leq 3241$ or $\sigma \leq 4132\}$ maps to 1010, and the interval $[2413, 4321]$ maps to 1100.

8. The algebra of freedom matroids

We now consider the algebra $A(\mathcal{F})$ corresponding to the minor-closed class $\mathcal{F}$ of freedom matroids. Throughout this section we shall assume that the ring K is a field of characteristic zero. The set $\{M_w : w \in \mathcal{W}\}$, where $\mathcal{W}$ is the set of all words on $\{0, 1\}$, is a K-vector space basis for $A(\mathcal{F})$, and the product is given by

$$M_u \cdot M_v = \sum_{w \in \mathcal{W}} \binom{w}{u, v} M_w,$$

where $\binom{w}{u,v}$ denotes the section coefficient $\binom{M_w}{M_u, M_v}$. As is the case for any matroid algebra, $A(\mathcal{F})$ is bigraded by rank and nullity, and so $A(\mathcal{F}) = \bigoplus_{r,k \geq 0} A_{r,k}(\mathcal{F})$, where $A_{r,k}(\mathcal{F})$ has basis $\{M_w : w \in W_{r+k,r}\}$, and the section coefficient $\binom{w}{u,v}$ is zero whenever $w \notin \mathcal{W}_{|u|+|v|,|u|_1+|v|_1}$.

In the proof of our main theorem below, we make use of the *incidence algebra* of the lattice $\mathcal{W}_{n,r}$. In general, the incidence algebra $I(P)$ of a locally finite poset P is the K-vector space of all functions $f : P \times P \to K$ such that $f(x, y) = 0$, whenever $x \not\leq y$, equipped with the *convolution* product:

$$(fg)(x, z) = \sum_{x \leq y \leq z} f(x, y)g(y, z),$$

for all $f, g \in I(P)$, and $x \leq z$ in P. The convolution identity $\delta \in I(P)$ is given by $\delta(x, y) = \delta_{x,y}$, for all $x \leq y$ in P. An element $f \in I(P)$ is invertible if and only if $f(x, x)$ is a unit in K, for all $x \in P$, in which case the convolution inverse f^{-1} is determined recursively by $f^{-1}(x, x) = f(x, x)^{-1}$, for all $x \in P$, and

$$\begin{aligned} f^{-1}(x, z) &= f(z, z)^{-1} \sum_{x \leq y < z} f^{-1}(x, y) f(y, z) \\ &= f(x, x)^{-1} \sum_{x < y \leq z} f(x, y) f^{-1}(y, z), \end{aligned}$$

for all $x < z$ in P.

Recall that the matroids consisting of a single point and a single loop are denoted by I and Z, respectively, and note that $I = M_1$ and $Z = M_0$ are the freedom matroids corresponding to words of length one.

Theorem 8.1. *The algebra $A(\mathcal{F})$ is free, generated by I and Z.*

Proof. For any word $w = x_1 \cdots x_n$ in $\mathcal{W}$, we denote by P_w the product $M_{x_1} \cdots M_{x_n}$ in $A(\mathcal{F})$. Since $A(\mathcal{F})$ is graded it suffices to show that the set $\{P_w : w \in \mathcal{W}_{n,r}\}$ is a basis for $A_{r,n-r}(\mathcal{F})$, for all $n \geq r \geq 0$. Given words $w, v \in \mathcal{W}_{n,r}$, with $w = x_1 \cdots x_n$, we write $c(w, v)$ for the multisection coefficient $\binom{v}{x_1,\ldots,x_n}$. Observe that $c(w, v)$ is equal to the number of permutations $\sigma \in \Sigma_n$ such that $\lambda_{M_v}(\sigma) = w$, and hence Theorem 7.5 implies that $c(w, v)$ is nonzero if and only if $w \leq v$ in the lattice ordering of $\mathcal{W}_{n,r}$. We thus have

$$P_w = \sum_{v \geq w} c(w, v) M_v, \tag{8.2}$$

for all $w \in \mathcal{W}_{n,r}$, where all coefficients are nonzero. Because $c(w, v) = 0$, whenever $w \not\leq v$, the function c belongs to the incidence algebra of $\mathcal{W}_{n,r}$. Since $c(w, w) \neq 0$ for all w, and K is a field of characteristic zero, it follows that c has a convolution inverse c^{-1}, and therefore

$$M_w = \sum_{v \geq w} c^{-1}(w, v) P_v,$$

for all $w \in \mathcal{W}_{n,r}$. Hence the linear endomorphism of $A_{r,n-r}(\mathcal{F})$ determined by $M_w \mapsto P_w$, for all $w \in \mathcal{W}_{n,r}$, is invertible, and so $\{P_w : w \in \mathcal{W}_{n,r}\}$ is a basis for $A_{r,n-r}(\mathcal{F})$. $\square$

Note that, since $P_v \cdot P_w = P_{vw}$ in $A(\mathcal{F})$, for all $v, w \in \mathcal{W}$, Theorem 8.1 can be restated as the fact that the map $P_w \mapsto w$ defines an isomorphism from $A(\mathcal{F})$ onto the free algebra $K\{\mathcal{W}\} = K\langle\{0, 1\}\rangle$, which has concatenation of words as product.

The use of incidence algebras in the proof of Theorem 8.1 can be avoided as follows: Choose an ordering $w_1, \ldots, w_m$ of $\mathcal{W}_{n,r}$ such that $i \leq j$, whenever $w_i \leq w_j$ in $\mathcal{W}_{n,r}$ (such as the opposite of lexicographic order) and set $c_{ij} = c(w_i, w_j)$, for all $i \leq j$ in $[m]$. Then $P_{w_i} = \sum_{j=1}^{m} c_{ij} M_{w_j}$, for all i, and by Theorem 7.5, the matrix $C = (c_{ij})_{1 \leq i,j \leq m}$ is upper-triangular, with nonzero entries along the main diagonal. Since K is a characteristic zero field, C is thus invertible, and hence the set $\{P_{w_i} : 1 \leq i \leq m\}$ is a basis for $A_{r,n-r}(\mathcal{F})$.

Corollary 8.3. *If $\mathcal{M}$ is any minor-closed family that contains the class $\mathcal{F}$ of freedom matroids, then the subalgebra of $A(\mathcal{M})$ generated by I and Z is free.*

Proof. For each word $w = x_1 \cdots x_n \in \mathcal{W}$, let Q_w denote the product $M_{x_1} \cdots M_{x_n}$ in $A(\mathcal{M})$. Since $\mathcal{F} \subseteq \mathcal{M}$, the algebra $A(\mathcal{F})$ is a quotient of $A(\mathcal{M})$, where the canonical homomorphism $\pi : A(\mathcal{M}) \mapsto A(\mathcal{F})$ maps every freedom matroid in $\mathcal{M}$ to itself and every nonfreedom matroid to zero. Since $\pi(Q_w) = P_w$, for all $w \in \mathcal{W}$ and, by Theorem 8.1, the P_w are linearly independent in $A(\mathcal{F})$, it follows that the Q_w are linearly independent in $A(\mathcal{M})$. Hence the subalgebra of $A(\mathcal{M})$ generated by I and Z is free. $\square$

Example 8.4. If $S = \{a, b, c, d\}$, then the basis $\{M_w : w \in \mathcal{W}_{4,2}\}$ of $A_{2,2}(\mathcal{F})$ consists of the following matroids:

$M_{1100} = U_{2,4}$	a, b, c, d collinear
M_{1010}	$\{a, b\}$ a double-point, collinear with points c and d
$M_{1001} = P_3 \oplus I$	$\{a, b, c\}$ a triple-point , d a distinct point
$M_{0110} = Z \oplus U_{2,3}$	a a loop , b, c, d collinear
$M_{0101} = I \oplus P_2 \oplus Z$	a a loop , $\{b, c\}$ a double-point, d a distinct point
$M_{0011} = Z_2 \oplus I_2$	a and b loops, c and d distinct points.

Listing $\mathcal{W}_{4,2}$ in opposite lexicographic order, $\mathcal{W}_{4,2} = \{w_1, w_2, w_3, w_4, w_5, w_6\} = \{1100, 1010, 1001, 0110, 0101, 0011\}$, the matrix C of multisection coefficients c_{ij} is given by

$$\begin{array}{c} \\ 1100 \\ 1010 \\ 1001 \\ 0110 \\ 0101 \\ 0011 \end{array} \begin{array}{c} \begin{array}{cccccc} 1100 & 1010 & 1001 & 0110 & 0101 & 0011 \end{array} \\ \begin{pmatrix} 24 & 20 & 12 & 12 & 8 & 4 \\ 0 & 4 & 6 & 6 & 6 & 4 \\ 0 & 0 & 6 & 0 & 4 & 4 \\ 0 & 0 & 0 & 6 & 4 & 4 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}.$$

So, for example, $P_{1001} = I \cdot Z \cdot Z \cdot I$ is equal to $6M_{1001} + 4M_{0101} + 4M_{0011}$ in $A(\mathcal{F})$. Observe that c_{34} is the only zero entry above the main diagonal C, which corresponds to the fact that $w_3 = 1001$ and $w_4 = 0110$ are the only two noncomparable elements of the lattice $\mathcal{W}_{4,2}$. Also note that, since the matrix entry $c(v, w)$ is equal to the number of orderings of the underlying set of M_w with corresponding word equal to v, the sum of the entries in each column of C is equal to 4!.

Example 8.5. Suppose that $\mathcal{M}$ is any minor-closed class containing all freedom matroids and the smallest nonfreedom matroid $D = P_2 \oplus P_2$, consisting of two double-points, and let $PL(\mathcal{M})$ be the subalgebra of $A(\mathcal{M})$ generated by I and Z. The matrix expressing the basis $\{Q_w : w \in \mathcal{W}_{4,2}\}$ of $PL(\mathcal{M}) \cap A_{2,2}(\mathcal{M})$ in terms of the basis $\widetilde{\mathcal{M}}_{2,2} = \{D\} \cup \{M_w : w \in \mathcal{W}_{4,2}\}$ of $A_{2,2}(\mathcal{M})$ is given by

$$\begin{array}{c} \\ 1100 \\ 1010 \\ 1001 \\ 0110 \\ 0101 \\ 0011 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1100 & 1010 & D & 1001 & 0110 & 0101 & 0011 \end{array} \\ \begin{pmatrix} 24 & 20 & 16 & 12 & 12 & 8 & 4 \\ 0 & 4 & 8 & 6 & 6 & 6 & 4 \\ 0 & 0 & 0 & 6 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 6 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}.$$

In this context, Corollary 8.3 amounts to the observation that this matrix contains as a submatrix the nonsingular matrix C in the previous example, and thus has independent rows.

We now turn our attention to the coalgebra $C(\mathcal{F})$ of freedom matroids. Recall from Section 2 that $C(\mathcal{F})$ has as basis the set $\tilde{\mathcal{F}} = \{M_w : w \in \mathcal{W}\}$ of all isomorphism classes of freedom matroids, and has coproduct determined by Eq. (2.8), so that

$$\delta(M_w) = \sum_{u,v\in\mathcal{W}} \binom{w}{u,\, v} M_u \otimes M_v,$$

for all $w \in \mathcal{W}$. Hence if we define a coproduct on the vector space $K\{\mathcal{W}\}$, having all 0,1-words as basis, by $\delta(w) = \sum_{u,v} \binom{w}{u,v} u \otimes v$, then $K\{\mathcal{W}\}$ and $C(\mathcal{F})$ are isomorphic coalgebras via the mapping $M_w \mapsto w$. For example,

$$\begin{aligned}\delta(1010) = {} & 1010 \otimes \emptyset + 2(101 \otimes 0) + 2(110 \otimes 0) + 10 \otimes 10 \\ & + 5(11 \otimes 00) + 2(1 \otimes 100) + 2(1 \otimes 010) + \emptyset \otimes 1010.\end{aligned}$$

It is then an interesting exercise to give a description of this coproduct solely in terms of the combinatorics of words.

Let $\{P'_w : w \in \mathcal{W}\}$ be the basis of $C(\mathcal{F})$ which is dual to the basis $\{P_w : w \in \mathcal{W}\}$ of $A(\mathcal{F})$ via the pairing defined in the beginning of Section 3, that is, such that $\langle P'_w, P_v\rangle = \delta_{w,v}$, for all $v, w \in \mathcal{W}$. Eq. (8.2) means that $\langle M_v, P_w\rangle = c(w, v)$, for all $v, w \in \mathcal{W}$, and so we have

$$M_w = \sum_{v\in\mathcal{W}} \langle M_w, P_v\rangle P'_v = \sum_{v\le w} c(v, w) P'_v$$

for all $w \in \mathcal{W}$. Hence if $|w| = n$, and we write λ for λ_{M_w}, we have

$$M_w = \sum_{\sigma\in\Sigma_n} P'_{\lambda(\sigma)}.$$

For example, referring to the matrix C in Example 8.4, we see that $M_{0110} = 12P'_{1100} + 6P'_{1010} + 6P'_{0110}$ in $C(\mathcal{F})$.

Corollary 8.6. *The coalgebra $C(\mathcal{F})$ has basis $\{P'_w : w \in \mathcal{W}\}$ and coproduct given by*

$$\delta(P'_w) = \sum_{uv=w} P'_u \otimes P'_v,$$

for all $w \in \mathcal{W}$.

Proof. The result follows immediately from Theorem 8.1 by duality. □

Corollary 8.6 can be restated as saying that the map determined by $P'_w \mapsto w$ is a coalgebra isomorphism from $C(\mathcal{F})$ onto the cofree coalgebra $K\{\mathcal{W}\}$, which has the deconcatenation coproduct $\delta(w) = \sum_{uv=w} u \otimes v$.

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References

[1] F. Ardila, The Catalan matroid, Journal of Combinatorial Theory A 104 (2003) 49–62.

[2] J. Bonin, A. de Mier, Lattice path matroids: structural properties, preprint arxiv:math.co/043337, March, 2004.

[3] J. Bonin, A. de Mier, M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials, Journal of Combinatorial Theory A 104 (2003) 63–94.
[4] J. Bonin, O. Gimenez, Multi-path matroids (in preparation).
[5] D. Broadhurst, D. Kreimer, Renormalization automated by Hopf algebra, Journal of Symbolic Computation 27 (6) (1999) 581–600.
[6] S.V. Chmutov, S.V. Duzhin, S.K. Lando, Vassiliev knot invariants I. Introduction, Advances in Soviet Mathematics 21 (1994) 117–126.
[7] S.V. Chmutov, S.V. Duzhin, S.K. Lando, Vassiliev knot invariants II. Intersection graph conjecture for trees, Advances in Soviet Mathematics 21 (1994) 127–134.
[8] S.V. Chmutov, S.V. Duzhin, S.K. Lando, Vassiliev knot invariants III. Forest algebra and weighted graphs, Advances in Soviet Mathematics 21 (1994) 135–145.
[9] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem I: The Hopf algebra structure of graphs and the main theorem, Communications in Mathematical Physics 210 (1) (2000) 249–273.
[10] H. Crapo, Single-element extensions of matroids, Journal of Research National Bureau of Standards 69B (1965) 55–66.
[11] J. Ellis-Monaghan, New results for the Martin polynomial, Journal of Combinatorial Theory B 74 (1998) 326–352.
[12] S.A. Joni, G.-C. Rota, Coalgebras and bialgebras in combinatorics, Studies in Applied Mathematics 61 (1979) 93–139.
[13] M. Kontsevich, Vassiliev's knot invariants, Advances in Soviet Mathematics 16 (1993) 137–150.
[14] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Advances in Theoretical and Mathematical Physics 2 (2) (1998) 303–334.
[15] J. Oxley, K. Prendergast, D. Row, Matroids whose ground sets are domains of functions, Journal of the Australian Mathematical Society A 32 (3) (1982) 380–387.
[16] U.S. Representatives, W. Jones (R-N.C.), B. Ney (R-Ohio), Committee on House Administration, Press release, http://www.house.gov/ney/freedomfriespr.htm, 2003, March.
[17] I. Sarmiento, Hopf algebras and the Penrose polynomial, Freie Universität, Berlin, 1999 (preprint).
[18] W. Schmitt, Incidence Hopf algebras, Journal of Pure and Applied Algebra 96 (1994) 299–330.
[19] D.J.A. Welsh, A bound for the number of matroids, Journal of Combinatorial Theory 6 (1969) 313–316.
[20] W. Kook, V. Reiner, O. Stanton, A convolution formula for the Tutte polynomial, Journal of Combinatorial Theory B 76 (1999) 297–300.
[21] J.P.S. Kung, A multiplication identity for characteristic polynomials of matroids, Advances in Applied Mathematics 32 (2004) 319–326.