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Minimal strong digraphs

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ABSTRACT

We introduce adequate concepts of expansion of a digraph to obtain a sequential construction of minimal strong digraphs. We characterize the necessary and sufficient condition for an external expansion of a minimal strong digraph to be a minimal strong digraph. We prove that every minimal strong digraph of order $n \geq 2$ is the expansion of a minimal strong digraph of order $n - 1$ and we give sequentially generative procedures for the constructive characterization of the classes of minimal strong digraphs. Finally we describe algorithms to compute unlabeled minimal strong digraphs and their isospectral classes.

1. Introduction

In this article, we focus on the study of strongly connected digraphs containing the least possible number of arcs (minimal strong digraphs), that is, strongly connected digraphs which cease to be so if any one of its arcs is suppressed. Minimal strong digraphs can be said to generalize the trees when we consider directed graphs instead of simply graphs. Nevertheless, the structure of minimal strong digraphs is much richer than that of the trees.

We are previously interested in the following nonnegative inverse eigenvalue problem [22]: given real numbers $k_1, k_2, \ldots, k_n$, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with characteristic polynomial $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. The coefficients of the characteristic polynomial are closely related to the cycle structure of the weighted digraph with adjacency matrix A [6], and the irreducible matricial realizations of the polynomial are identified with strongly connected digraphs (henceforth strong digraphs) [4]. The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in any theoretical or constructive characterization of these classes of digraphs. In particular, the characterization of the monic polynomials of degree n with integral coefficients, which are the characteristic polynomials of strong or minimally strong digraphs of order n, is an open problem.

Many classes of connected graphs and digraphs have constructive characterizations. In particular, for (minimal) 2-connected graphs and (minimal) strong digraphs different procedures have been described to construct larger (di)graphs from smaller (di)graphs of these classes [7,19,9,8,14,2]. The common basic idea of these procedures consists of adding paths between qualified vertices in a systematic way.

Bhogadi [2] gives a characterization of Cunningham's decomposition trees for minimal strong digraphs under X-joint (substitution) composition [5]. He uses his characterization to generate all minimal strong digraphs through 12 vertices and all minimal 2-connected graphs through 13 vertices.

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All these procedures have been defined so that the property of minimality is not preserved and the conditions under which minimality is preserved are not characterized.

Zhang and Guo [24] present a method for enumerating all the minimal strong digraphs from the fundamental cycles of a given digraph and they characterize the conditions under which minimality is preserved.

The rest of this paper is organized as follows.

In Section 2, we record basic facts and ideas about the (minimal) strong digraphs.

In Section 3, we introduce two suitable (internal and external) concepts of expansion of a digraph (similar to the operations "subdivision" and "simple path insertion" considered by Hedetniemi [14]) for a sequential construction of minimal strong digraphs. We characterize the necessary and sufficient condition for an external expansion of a minimal strong digraph to be a minimal strong digraph and we show how every minimal strong digraph of order $n \geq 2$ is the expansion of a minimal strong digraph of order $n - 1$.

In Section 4, we propose a sequentially generative procedure for the constructive characterization of the class of minimal strong digraphs.

In Section 5, we implement an algorithm to compute unlabeled minimal strong digraphs following the construction of the previous sections. Another algorithm allows the digraphs and the characteristic polynomials of the isospectral classes of the minimal strong digraphs to be obtained.

2. Basic general ideas

In this paper we use some standard basic concepts and results about graphs as they have been described in [11].

A *digraph* D is a couple $D = (V, A)$, where V is a finite nonempty set and $A \subset V \times V - \{(v, v)\colon v \in V\}$. If $u, v \in V$ we denote (u, v) by uv and we write $D - uv$ and $D + uv$ for the digraphs $(V, A - \{(u, v)\})$ and $(V, A \cup \{(u, v)\})$, respectively. For a vertex $v \in V$, the subdigraph $D - v$ consists of all vertices of D except v and all arcs of D except those incident with v. A *q-cycle* is a directed cycle of length q and it is denoted by C_q. A *directed tree* is the digraph obtained from a tree by replacing each edge $\{u, v\}$ with the two arcs (u, v) and (v, u). We denote a path from the vertex u to the vertex v by u, v-path.

A digraph D is *strongly connected* or (simply) *strong* if every two vertices in D are joined by a path. It is well known that the digraph D is strongly connected if and only if its adjacency matrix M is irreducible [4].

We record now a number of basic facts about the strong digraphs that, for simplicity, in the following we write as SC digraphs. In an SC digraph of order $n \geq 2$ the indegree and outdegree of the vertices are bigger than or equal to 1. A vertex is *linear* if it has indegree and outdegree equal to 1.

If we add an arc to the set of arcs of an SC digraph D then the cyclic structure of D is modified. This suggests the introduction of the concept of minimal strong digraph. An SC digraph D is *minimal* if $D - a$ is not strongly connected for every arc $a \in A$. For simplicity, in the following we write minimal strong digraph as MSC digraph.

The set of SC digraphs of order n with vertex set V can be partially ordered by the relation of inclusion among their sets of arcs. Then, the MSC digraphs are the minimal elements of this partially ordered set. Analogously, the set of irreducible (0, 1)-matrices of order n with zero trace can be partially ordered by means of the coordinatewise ordering. The minimal elements of this partially ordered set are *nearly reducible* matrices and so the digraph D is an MSD digraph if and only if its adjacency matrix M is a nearly reducible matrix [4,15]. Hartfiel [13] gives a remarkably canonical form for nearly reducible matrices.

To reduce the cyclic structure of an SC digraph to the structure of an MSC digraph requires to characterize the MSC digraphs and to build the set of SC digraphs starting from the set of MSC digraphs.

If D is an MSC digraph and there is a u, v-path in D, then there cannot be an arc joining the vertex u to the vertex v, that is $uv \notin A$. In general, an arc uv in a digraph D is *transitive* if there is another u, v-path distinct from the arc uv. The semicycle consisting of a u, v-path together with the arc uv is a *pseudocycle*. So an MSC digraph has no transitive arcs or pseudocycles; moreover, this condition characterizes the minimality of the strong connection.

Lemma 1 (*Geller [8], Hedetniemi [14]*). *If D is an SC digraph, then D is minimal if and only if D has no transitive arcs if and only if D has no pseudocycles.*

Consequently, if D is an MSC digraph then so is every strong subdigraph of D.

The *contraction* of a cycle of length k in an SC digraph consists of the reduction of the cycle to a unique vertex, so that $k - 1$ of its vertices and its k arcs are eliminated.

Lemma 2 (*Berge [1]*). *The contraction of a cycle in an MSC digraph preserves the minimality, that is it produces another MSC digraph.*

The size of an SC digraph of order $n \geq 2$ verifies $n \leq |A| \leq n^2 - n$ and the extreme digraphs are the cycle C_n and the complete digraph K_n. The following result was basically obtained by Gupta [10]. Brualdi and Hedrick [3] gave a different proof for a more thorough result. We use Lemma 2 for a shorter proof of the result of Brualdi and Hedrick.

Lemma 3. *The size of an MSC digraph D of order $n \geq 2$ verifies $n \leq |A| \leq 2(n-1)$. The size of D is n if and only if D is an n-cycle. The size of D is $2(n-1)$ if and only if D is a directed tree.*

Proof. It is clear that $n \leq |A|$ and that the cycle C_n is the unique MSC digraph of order n.

Let us see that $|A| \leq 2(n-1)$. We proceed by induction over the order n. If $n = 2$ the unique MSC digraph is the cycle C_2 and the inequality is clear for $|A| = 2$.

Induction hypothesis: we suppose that every MSC digraph of order $n' \leq n$ has at most $2(n'-1)$ arcs.

If the MSC digraph is the cycle C_{n+1} the inequality is clear. If D is an MSC digraph of order $n+1$ distinct from the cycle C_{n+1}, as it is an SC digraph, D contains at least a cycle C_p with $2 \leq p \leq n$. By Lemma 2, the contraction of the cycle C_p produces an MSC digraph D' of order $n+1-(p-1) = n-p+2 \leq n$. By the induction hypothesis, D' has at most $2(n-p+1)$ arcs. Then the number of arcs of the original digraph D will be at most $2(n-p+1)+p = 2n-p+2 \leq 2n$.

Let us see that if D is an MSC digraph of order n and size $2(n-1)$ then it is a directed tree. Note that the cycles in a directed tree have length two. We suppose, by reductio ad absurdum, that D has some cycle C_q of length $q > 2$. Let D' be the MSC digraph obtained by the contraction of the cycle C_q in D. The order and the size of D' are $n' = n-(q-1)$ and $m' = 2(n-1)-q$, respectively. Then we have the contradiction $m' \leq 2(n'-1) = 2(n-(q-1)-1) = 2n-2q < 2n-2-q = m'$. □

Brualdi and Hedrick [3] also proved that there exists an MSC digraph of order $n \geq 2$ and size m if and only if $n \leq m \leq 2(n-1)$ and characterized the MSC digraphs of order n and size $2n-3$.

The next theorem was first proved by Dirac [7] and independently by Plummer [19] in the context of minimal two connected graphs and by Berge and by Brualdi and Ryser [4] for minimal strong digraphs. Our proof is a simplification of that given by Berge [1].

Theorem 4. *Every MSC digraph of order $n \geq 2$ has at least two linear vertices.*

Proof. By induction over the order n. If $n = 2$ the unique MSC digraph is the cycle C_2 whose vertices are linear.

Induction hypothesis: we suppose that every MSC digraph of order $n' \leq n$ has at least two linear vertices.

(a) If the MSC digraph is the cycle C_{n+1}, it has $n+1 \geq 3$ linear vertices.
(b) If D is an MSC digraph of order $n+1$ that contains no cycle of length bigger than two then, as it is an SC digraph, it is a directed tree. The extreme vertices (the leaves) of this tree are the linear vertices of D. Because every tree has at least two leaves, then there are at least two linear vertices in D.
(c) If D is an MSC digraph of order $n+1$ that contains a cycle C_p of length p with $3 \leq p < n+1$, then there is at least a vertex v in D that is not in the cycle C_p. By Lemma 2, the contraction of the cycle C_p produces a new MSC digraph D' of order $n+1-(p-1) = n-p+2$ with $2 \leq n-p+2 < n$. By the induction hypothesis, D' has at least two linear vertices that we call u and v. If one of these vertices, let us suppose that u, is the contracted vertex, then in the digraph D there is a unique arc going into the cycle C_p and a unique arc leaving the cycle C_p and, as $p \geq 3$, in C_p there is at least one linear vertex w. Then w and v are two linear vertices in D. If, on the contrary, the linear vertices u and v of D' are distinct from the contracted vertex, then these vertices are also linear in D. □

3. Sequential expansion of MSC digraphs

In this section, we look at that every MSC digraph of order n can be generated from an MSC digraph of order $n-1$. For this purpose, we shall define two different (internal and external) expansion procedures of a digraph consisting in adding a new vertex so that, either the property of being MSC is preserved or the conditions in which the expansion can be carried out while preserving the MSC property are described.

The internal expansion (*one-step expansion* in [12]) of a digraph consists in the substitution of an arc uw by new arcs uv and vw, v being a new vertex in the digraph. More precisely,

Definition 5. The *internal expansion* of the digraph $D = (V, A)$ *by the vertex* $v \notin V$ *over the arc* uw is the digraph $i_{uw}(D) = (V \cup \{v\}, A^*)$ with $A^* = A \cup \{uv, vw\} - \{uw\}$.

The external expansion of a digraph consists in the joining of two vertices u and w (not necessary distinct) of the digraph with a new vertex v by means of the arcs uv and vw. More precisely,

Definition 6. The *external expansion* of the digraph $D = (V, A)$ *by the vertex* $v \notin V$ *from the vertex* $u \in V$ *to the vertex* $w \in V$ is the digraph $e_{uw}(D) = (V \cup \{v\}, A^*)$ with $A^* = A \cup \{uv, vw\}$. Whenever the vertex w coincides with the vertex u we denote $e_{uw}(D)$ by $e_u(D)$ and we call it *external expansion over the vertex u.*

It is easy to prove that the internal expansion of a digraph preserves the SC and MSC properties and that the external expansion preserves the SC property but not the MSC property. The external expansion from the vertex u to the vertex w can produce transitivity in other arcs, including when uw is not an arc of an MSC digraph D, thus losing the property of minimality. Next we characterize the necessary and sufficient condition for an external expansion of an MSC digraph to be an MSC digraph.

Theorem 7. *Let $D = (V, A)$ be an MSC digraph and let u, w be vertices such that $uw \notin A$. The external expansion $e_{uw}(D)$ of D by the vertex $v \notin V$ from the vertex u to the vertex w is an MSC digraph if and only if the digraph $D + uw$ has no transitive arcs distinct from uw.*

Proof. Clearly uw is a transitive arc of the digraph $D + uw$ because D is an SC digraph. If there exists a transitive arc pq distinct from uw in $D + uw$, then there is a longer p, q-path that includes the arc uw. This path has the form $p \dots uw \dots q$ where p and u may coincide or q and w may coincide, but not both simultaneously. Then the path $p \dots uvw \dots q$ makes the arc pq transitive in the digraph $e_{uw}(D)$. In fact, for every $pq \in A$, the arc pq is transitive in $D + uw$ if and only if pq is transitive in $e_{uw}(D)$ if and only if $e_{uw}(D)$ is not MSC. □

The following result is the base of a possible generative construction of MSC digraphs of order $n \geq 2$ starting from MSC digraphs of order $n - 1$. In fact, we prove a stronger result; more exactly, we prove that every linear vertex of an MSC digraph originates in the (internal or external) expansion of an MSC digraph. Thus, if an MSC digraph D has $p \geq 2$ linear vertices, then we can obtain p distinct "reductions" with one vertex less than D, though some might be isomorphic.

Theorem 8. *Let $D^* = (V, A^*)$ be an MSC digraph of order $n \geq 3$ and $v \in V$ a linear vertex in D^*. Then there exists an MSC digraph $D = (V - \{v\}, A)$ whose (internal or external) expansion by the vertex v is the digraph D^*.*

Proof. As v is a linear vertex there are two unique vertices u and w such that $uv \in A^*$ and $vw \in A^*$.

(a) If $u = w$, then $A = A^* - \{uv, vu\}$ and $D = (V - \{v\}, A) = D^* - v$ is obviously MSC. By construction, the external expansion of the digraph D by the vertex v over the vertex u is the digraph D^*.

(b) If $u \neq w$, as there are no transitive arcs in D^*, then $uw \notin A^*$.

  (b_1) We suppose that no u, w-path distinct from the path uvw exists in D^*. In this case we replace the arcs uv, vw in D^* by the new arc uw, more precisely, we take $A = A^* \cup \{uw\} - \{uv, vw\}$. The new digraph $D = (V - \{v\}, A)$ is by construction SC and, as there are no u, w-paths in D, the arc uw is not transitive and then D is also minimal. By construction, the internal expansion of the digraph D by the vertex v over the arc uw is the digraph D^*.

  (b_2) If there exists any u, w-path distinct from the path uvw in D^*, then we make $A = A^* - \{uv, vw\}$. The u, w-path ensures the strong connection of the new digraph $D = (V - \{v\}, A) = D^* - v$ which is minimal because there are no transitive arcs in D^* and therefore neither in D. By construction, the external expansion of the digraph D by the vertex v from the vertex u to the vertex w is the digraph D^*. □

Definition 9. The SC digraph D is a *reduction* of the SC digraph D^* if D^* is an internal or external expansion of D.

From the above Theorems 4 and 8 one can also deduce the following consequences:

Corollary 10. *Every MSC digraph of order $n \geq 3$ can be reduced to the cycle C_2 by a sequence of $n - 2$ reductions.*

It is possible to define procedures for the reduction of an MSC digraph to obtain different classes of MSC digraphs such as a tree T of cycles of distinct lengths, and this tree T can be reduced to one cycle (whose length is bounded by the biggest of the lengths of the cycles in T), or one path of cycles C_2 or one star of cycles C_2. All of them can finally be reduced to one cycle C_2 and this to a unique vertex.

Remark. Following Lemma 2, we can make reductions preserving the MSC property through the contraction of cycles. A procedure could be determined by the length of the cycles. The minimal number of contractions of cycles to reduce an MSC digraph to a vertex is the cyclomatic number $|A| - |V| + 1$ [1].

4. Construction of MSC and SC digraphs

In the previous section we saw, on the one hand, that the internal expansion of an MSC digraph of order n on any one of its arcs produces an MSC digraph of order $n + 1$, and on the other hand (Theorem 7), we saw under which conditions the external expansion of an MSC digraph of order n over pairs of non-adjacent vertices produces an SC digraph of order $n + 1$ preserving the minimality. We also saw (Theorem 8) how every MSC digraph of order $n + 1$ can be obtained by (internal or external) expansion of an MSC digraph of order n. These three results suggests a sequentially generative procedure for the construction of the set of MSC digraphs of order $n + 1$ starting from the set of MSC digraphs of order n. In Fig. 1 we describe the three first steps of this process.

In general, for an MSC digraph $D = (V, A)$ of order n and size m, the n-th iteration is performed as follows:

(a) an internal expansion over each one of its m arcs;
(b) an external expansion over each one of its n vertices;
(c) an external expansion from a vertex u to a vertex w, such that $uw \notin A$, whenever the digraph $D = (V, A \cup \{uw\})$ has no transitive arcs distinct from uw (Theorem 7).

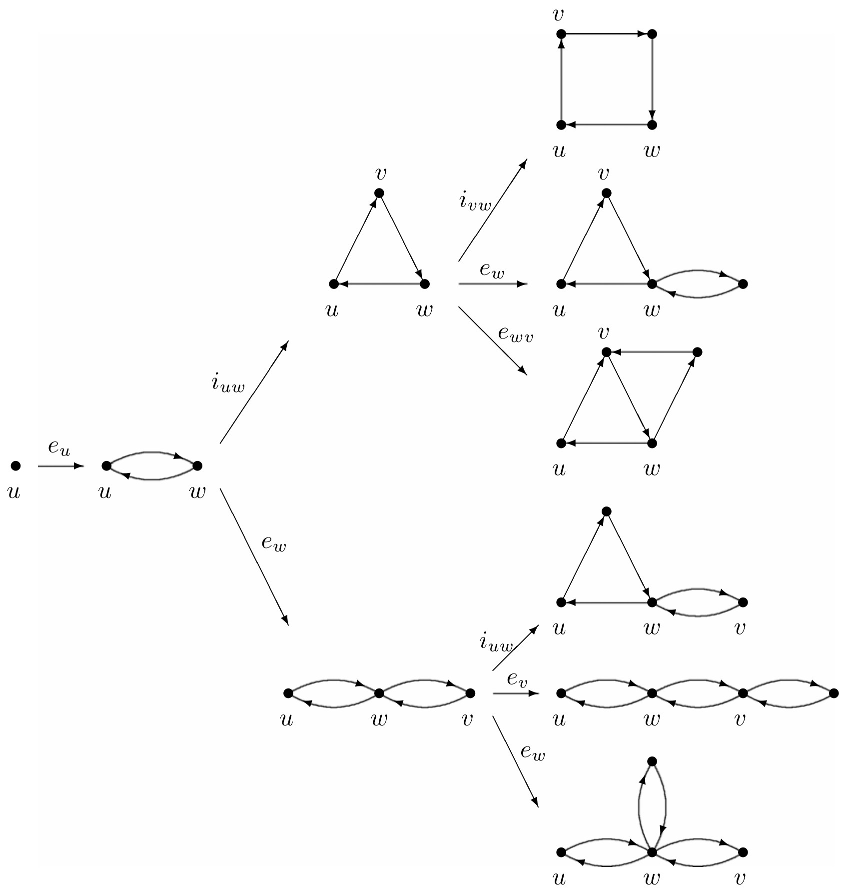

Fig. 1. Sequential generative construction of MSC digraphs.

Note that isomorphic digraphs can be obtained at each step (a), (b) and (c) separately, but also in relation to each other.

To build the set of SC digraphs of order n from the set of MSC digraphs of order n is sufficient to add any set of transitive arcs.

The above procedures are useful for building and cataloging the sets of MSC digraphs and SC digraphs of order n but do not give close formulas for the numbers, UMS(n) and $US(n)$, of unlabeled MSC and SC digraphs of order n, respectively.

Labeled strong digraphs were first counted by Liskovec [16], who gives recurrent formulas for the number, $S(n)$, of labeled strong digraphs of order n and for the number, $S(n, m)$, of labeled strong digraphs of order n and size m. He also shows the asymptotic behavior $S(n) \approx 2^{n(n-1)}$ and $US(n) \approx 2^{n(n-1)}/n!$ Liskovec formulas were simplified by Wright [23], while Robinson [20] gives a natural combinatorial explanation of the simplified equation of Wright.

Unlabeled strong digraphs were enumerated "in a somewhat cumbersome manner" by Liskovec [17] and Robinson [21] "outlined" a method for enumerating them.

The numbers, $MS(n)$ and UMS(n), of labeled and unlabeled MSC digraphs of order n are unknown.

5. Algorithms

In this section we implement two algorithms. The first one computes unlabeled MSC digraphs, following the construction described in the previous section. With this algorithm we were able to calculate all MSC digraphs up to order 14 on a personal computer. This extends Bhogadi's results to order 13 and 14 and proves the efficiency of our method. We now present a general description of the algorithm.

Input:

(1) The order n of the MSC digraphs to be computed.
(2) The list L_{n-1} of all unlabeled MSC digraphs of order $n - 1$.

Output: A sorted list L_n of all unlabeled MSC digraphs of order n.

Table 1
Number of unlabeled MSC digraphs of order n and m arcs.

$m \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3		1											
4		1	1										
5			2	1									
6			2	4	1								
7				7	6	1							
8				3	27	9	1						
9					23	70	12	1					
10					6	131	169	16	1				
11						66	559	344	20	1			
12						11	571	1970	662	25	1		
13							191	3479	5 874	1 159	30	1	
14							23	2229	17 109	15 526	1 947	36	1
15								541	18 509	69 845	37 072	3 086	42
16								47	8 226	120 582	246 971	81 561	4 743
17									1 514	87 963	646 339	773 413	167 500
18									106	28 879	732 150	2 954 946	2 191 491
19										4 217	385 484	4 974 754	11 819 034
20										235	98 146	3 973 379	28 600 421
21											11 724	1 587 924	33 313 635
22											551	324 638	19 785 730
23												32 527	6 234 794
24												1 301	1 052 874
25													90 285
26													3 159
UMS(n)	1	2	5	15	63	288	1526	8627	52 021	328 432	2 160 415	14 707 566	103 263 709

Algorithm.

(1) Set $L = [\ ]$.
(2) For every $g_{n-1} = (V, A) \in L_{n-1}$:
  (a) For all $uw \in A$:
    – Set $g_n = i_{uw}(g_{n-1})$.
    – Compute $c\_g_n = \text{CanonicalForm}(g_n)$
    – If $c\_g_n \notin L_n$ add the digraph c_g_n to L_n.
  (b) For all $u \in V$:
    – Set $g_n = e_u(g_{n-1})$.
    – Compute $c\_g_n = \text{CanonicalForm}(g_n)$
    – If $c\_g_n \notin L_n$ add the digraph c_g_n to L_n.
  (c) For all $u \neq w$ such that $uw \notin A$ and $e_{uw}(g_{n-1})$ is minimal:
    – Set $g_n = e_{uw}(g_{n-1})$.
    – Compute $c\_g_n = \text{CanonicalForm}(g_n)$
    – If $c\_g_n \notin L_n$ add the digraph c_g_n to L_n.

In this algorithm there are three essential procedures. The first one computes a canonical form of a digraph and it is necessary to detect isomorphic digraphs. Both procedures can be solved by using the software package *nauty* [18]. However, for MSC digraphs, we can consider another efficient method. Compute a vertex set partition $V_1, \ldots, V_k$ in such a way that, given two arbitrary subsets V_i and V_j, all vertices of V_i have the same number of arcs with the end vertex in V_j. Finally, obtain a canonical form from this partition. If the canonical form computing has complexity $O(f(n))$ then the overall complexity of this procedure is $O(n^2|L_{n-1}|f(n))$.

Let $D = (V, A)$ be an MSC digraph and let u, w be vertices such that $uw \notin A$. The second procedure determines whether the external expansion $e_{uw}(D)$ is minimal, by using the characterization of Theorem 7. For every arc $xz \in D + uw$, with $xz \neq uw$, we have to compute whether xz is transitive. Each case can be solved in $O(n)$ time, checking if there is a path from x to z in the digraph $(D + uw) - xz$. Thus, this procedure has complexity $O(n^2)$ and, considering all cases, the overall complexity is $O(n^3|L_{n-1}|)$.

The last procedure updates the sorted list of digraphs L_n. It is a well-known problem that can be solved in logarithmic time. However, the size of the list increases very quickly. Therefore, it is necessary to store the list on a hard disk. Then the overall complexity of this procedure is $O(n^2|L_{n-1}|\log(n^2|L_{n-1}|))$ because there are $O(n^2|L_{n-1}|)$ updates.

We summarize the results of the computation in Table 1. For every n from 1 to 14, it includes the total number, UMS(n), of unlabeled MSC digraphs of order n. We also classify the MSC digraphs of a given order by the number m of their arcs. When the number of arcs is equal to $2n - 2$ the digraphs become directed trees, changing n, the following sequence of unlabeled trees is obtained: 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159....

Table 2
Isospectral classes of MSC digraphs of order n and m arcs.

$m \setminus n$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3		1											
4		1	1										
5			2	1									
6			2	4	1								
7				6	6	1							
8				3	18	9	1						
9					16	35	12	1					
10					6	62	65	16	1				
11						43	172	103	20	1			
12						11	227	395	160	25	1		
13							115	801	791	227	30	1	
14							22	769	2 290	1 423	319	36	1
15								319	3 530	5 567	2 411	424	42
16								42	2 645	12 437	11 942	3 807	559
17									848	14 978	36 638	23 583	5 805
18									102	8 812	64 337	93 732	43 070
19										2 349	61 376	228 358	217 303
20										204	29 317	318 654	695 323
21											6 401	244 989	1 351 485
22											488	95 369	1 517 405
23												17 660	949 476
24												1 078	307 783
25													48 567
26													2 723
Sum	1	2	5	14	47	161	614	2446	10 387	46 023	213 260	1 027 691	5 139 542
Total	1	2	5	14	47	161	604	2360	9 796	42 510	193 891	922 109	4 560 898
Δ	0	0	0	0	0	0	10	86	591	3 513	19 369	105 582	578 644

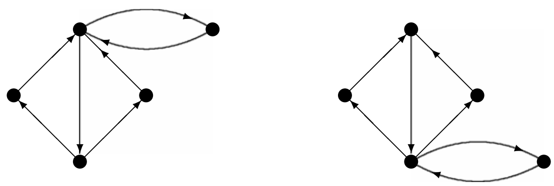

Fig. 2. Non-isomorphic isospectral MSC digraphs.

The other implemented algorithm computes the isospectral classes of the MSC digraphs. It determines the digraphs and the characteristic polynomial of each class. If Gauss's algorithm is used in order to compute characteristic polynomials, then the overall complexity is $O(n^3|L_n|)$. Table 2 includes the obtained results. Observe that, for $n \geq 8$, there are isospectral classes realized by MSC digraphs with a different number of arcs. In order to explain this fact, we have included three summary rows. The first one is the sum of the numbers of the isospectral classes in the number of possible arcs, the second one includes the total number of isospectral classes of a given order and the last one is the difference between them.

Finally, we remark that, from this table, we can extract the following sequences of isospectral classes:

1. For MSC digraphs: 1, 2, 5, 14, 47, 161, 604, 2360, 9796, 42 510, 193 891, 922 109, 4 560 898. . . .
2. For trees: 1, 1, 2, 3, 6, 11, 22, 42, 102, 204, 488, 1078, 2723. . . .

Remark. With respect to our initial motivation of the nonnegative inverse eigenvalue problem, in the context of (minimal) strong digraphs, and to the open problem mentioned in the Introduction, we can conclude that the characterization of the monic polynomials of degree n with integral coefficients, which are the characteristic polynomials of MSC digraphs of order n, has been indirectly solved in this paper in the sense that the above algorithms allow the class of characteristic polynomials of MSC digraphs of order n and the sets of MSC digraphs with equal characteristic polynomial to be cataloged.

Fig. 2 shows the first pair of non-isomorphic MSC digraphs having the same characteristic polynomial, in this case $x^5 - x^3 - 2x^2$.

It is well known that there exist classes of isospectral trees which are as large as desired [6]. So, classes of MSC digraphs (in particular directed trees) can be also be built which can be any size with the same characteristic polynomial.

It is also well known that the isospectrality relationship does not preserve the connectivity of graphs [6]. Only the first of the SC digraphs of Fig. 3 is minimal but both have equal characteristic polynomial $x^5 - 3x^2$, so the isospectrality relationship does not preserve the minimality of the strong connection either.

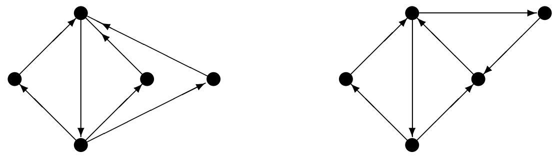

Fig. 3. MSC and SC isospectral digraphs.

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