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Twisted Frobenius–Schur indicators for Hopf algebras ☆

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The classical Frobenius–Schur indicators for finite groups are character sums defined for any representation and any integer $m \geqslant 2$. In the familiar case $m = 2$, the Frobenius–Schur indicator partitions the irreducible representations over the complex numbers into real, complex, and quaternionic representations. In recent years, several generalizations of these invariants have been introduced. Bump and Ginzburg, building on earlier work of Mackey, have defined versions of these indicators which are twisted by an automorphism of the group. In another direction, Linchenko and Montgomery have defined Frobenius–Schur indicators for semisimple Hopf algebras. In this paper, the authors construct twisted Frobenius–Schur indicators for semisimple Hopf algebras; these include all of the above indicators as special cases and have similar properties.

1. Introduction

Classically, the Frobenius–Schur indicator of a character of a finite group is the character evaluated at the sum of squares of the group elements divided by the order of the group. This indicator was introduced by Frobenius and Schur in their investigation of real representations. Indeed, they showed that the only possible values for an irreducible representation are 1, 0, and −1, corresponding to the partition of the irreducible representations into real, complex, and quaternionic representations [FS06]. Higher order versions can be obtained by replacing squares with other powers of group elements.

In recent years, there has been increasing interest in various generalizations of these invariants. In one direction, Bump and Ginzburg [BG04], building on earlier work of Mackey [Mac58] and Kawanaka and Matsuyama [KM90], have defined versions of Frobenius–Schur indicators which are twisted by

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an automorphism of the group. These indicators have applications to the study of multiplicity-free permutation representations, models for finite groups (in the sense of [BGG76]), and Shintani lifting of characters of finite reductive groups.

Another direction involves extending the theory from finite groups to suitable Hopf algebras. In 2000, Linchenko and Montgomery constructed Frobenius–Schur indicators for semisimple Hopf algebras over an algebraically closed field of characteristic zero and proved that the second indicator again only takes the values 0 or ± 1 on irreducible representations [LM00]. The higher indicators were further studied by Kashina, Sommerhäuser, and Zhu, who used them to prove a version of Cauchy's theorem for Hopf algebras, namely that the dimension and exponent of a semisimple Hopf algebra have the same prime factors [KSZ06]. The second indicators have also been used in classifying certain Hopf algebras [Kas03] and in studying possible dimensions of representations [KSZ02]. More recently, Ng and Schauenburg have introduced a categorical definition of Frobenius–Schur indicators for pivotal categories [NS07b] and shown that the two definitions coincide in the case of Hopf algebras [NS08]. This perspective has led to an extension of Cauchy's theorem to spherical fusion categories [NS07a], applications to rational conformal field theory [Ban97,Ban00,NS07a], and some remarkable relations between certain generalizations of these indicators and congruence subgroups of $SL_2(\mathbb{Z})$ [SZ08,NS10].

The goal of this paper is to construct twisted Frobenius–Schur indicators for semisimple Hopf algebras over an algebraically closed field of characteristic zero that include the group and Hopf algebra indicators described above as special cases and have similar properties. Given an automorphism of order n of such a Hopf algebra, we define the *mth twisted Frobenius–Schur* indicator for m any positive multiple of n. This definition is given in Section 2. In the next section, we show that the mth twisted Frobenius–Schur indicator can be realized as the trace of an endomorphism of order m (Theorem 3.5), so that the indicator is a cyclotomic integer. In Section 4, we consider the case of automorphisms of order at most two. We show that the second twisted Frobenius–Schur indicator gives rise to a partition of the simple modules into three classes; this partition involves the relationship between the module and its "twisted dual" (Theorem 4.3). In the final section, we compute a closed formula for the twisted indicator of the regular representation (Theorem 5.1).

2. Definition

Let k be an algebraically closed field of characteristic 0, and let H be a semisimple Hopf algebra over k with comultiplication Δ, counit ε, and antipode S. The Hopf algebra H contains a unique two-sided integral Λ normalized so that $\varepsilon(\Lambda) = 1$. We will use the usual Sweedler notation for iterated comultiplication: $\Delta^{m-1}(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_m$. Let $\mathrm{Rep}(H)$ be the category of finite-dimensional left H-modules. All H-modules considered will be objects in $\mathrm{Rep}(H)$. Throughout the paper, an automorphism of H will always refer to a Hopf algebra automorphism (or equivalently, a bialgebra automorphism). In particular, such an automorphism commutes with the antipode.

We are now ready to define the twisted indicators. Let τ be an automorphism of H such that $\tau^m = \mathrm{Id}$ for some $m \in \mathbf{N}$. Let (V, ρ) be an H-module with corresponding character χ.

Definition 2.1. The *mth twisted Frobenius–Schur indicator of* (V, ρ) (or χ) is defined to be the character sum

$$\nu_m(\chi, \tau) = \sum_{(\Lambda)} \chi\big(\Lambda_1 \tau(\Lambda_2) \cdots \tau^{m-1}(\Lambda_m)\big). \tag{2.1}$$

We note that this is only defined for m divisible by the order of τ. We will write $\tilde{\nu}_m(\chi)$ instead of $\nu_m(\chi, \tau)$ when this does not cause confusion.

If $\tau = \mathrm{Id}$, this formula coincides with the definition of Linchenko and Montgomery [LM00]. Moreover, suppose $H = k[G]$ for a finite group G. In this case, $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$, and we recover Bump and Ginzburg's twisted Frobenius–Schur indicators for groups [BG04].

3. A trace formula

In this section, we realize $\tilde{\nu}_m(\chi)$ as the trace of an endomorphism of order m and use this fact to show that the twisted Frobenius–Schur indicators are cyclotomic integers.

We begin by introducing a twisting functor $\mathcal{F}_\tau : \mathrm{Rep}(H) \to \mathrm{Rep}(H)$. Given an H-module V, we set $\mathcal{F}_\tau(V) = V$ as vector spaces with H-action defined by $h \cdot v = \tau(h)v$. Furthermore, if $f : V \to W$ is a morphism of H-modules, then $f : \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(W)$ is also an H-map; we set $\mathcal{F}_\tau(f) = f$. The functor $\mathcal{F}_\tau$ preserves the trivial module, tensor products, and duals:

$$\mathcal{F}_\tau(k) = k, \qquad \mathcal{F}_\tau(V \otimes W) = \mathcal{F}_\tau(V) \otimes \mathcal{F}_\tau(W), \quad \text{and} \quad \mathcal{F}_\tau\big(V^*\big) = \mathcal{F}_\tau(V)^* \tag{3.1}$$

for all $V, W \in \mathrm{Rep}(H)$. In other words, $\mathcal{F}_\tau$ is a strict, rigid, k-linear endomorphism of $\mathrm{Rep}(H)$. Moreover, if σ is another automorphism of H, then $\mathcal{F}_{\sigma\tau} = \mathcal{F}_\tau\mathcal{F}_\sigma$, so $\mathcal{F}_\tau$ is in fact an automorphism.

Let $\mathrm{Aut}_{\mathrm{sr}}(\mathrm{Rep}(H))$ denote the group of strict, rigid, k-linear automorphisms of $\mathrm{Rep}(H)$. Summing up, we obtain

Proposition 3.1. *The map $\tau \mapsto \mathcal{F}_\tau$ is an anti-homomorphism* $\mathrm{Aut}(H) \to \mathrm{Aut}_{\mathrm{sr}}(\mathrm{Rep}(H))$. *In particular, if $\tau^m = \mathrm{Id}$, then $\mathcal{F}_\tau^m = \mathrm{Id}$.*

Let $(\widetilde{V^{\otimes m}}, \widetilde{\rho^m})$ be the H-module

$$\widetilde{V^{\otimes m}} = V \otimes \mathcal{F}_\tau(V) \otimes \big(\mathcal{F}_\tau(V)\big)^2 \otimes \cdots \otimes \big(\mathcal{F}_\tau(V)\big)^{m-1}. \tag{3.2}$$

To be explicit, $\widetilde{V^{\otimes m}}$ has underlying vector space $V^{\otimes m}$ and action given by

$$\widetilde{\rho^m}(h)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \sum_{(h)} \rho(h_1)v_1 \otimes \rho\big(\tau(h_2)\big)v_2 \otimes \cdots \otimes \rho\big(\tau^{m-1}(h_m)\big)v_m.$$

Let $\alpha : V^{\otimes m} \to V^{\otimes m}$ be the linear map defined by

$$\alpha(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_2 \otimes \cdots \otimes v_m \otimes v_1.$$

Lemma 3.2.

$$\tilde{\nu}_m(\chi) = \mathrm{tr}_{V^{\otimes m}}\big(\alpha \circ \widetilde{\rho^m}(\Lambda)\big).$$

Proof.

$$\begin{aligned}
\tilde{\nu}_m(\chi) &= \sum_{(\Lambda)} \chi\big(\Lambda_1\tau(\Lambda_2)\cdots\tau^{m-1}(\Lambda_m)\big) \\
&= \sum_{(\Lambda)} \mathrm{tr}_V\big(\rho(\Lambda_1)\rho\big(\tau(\Lambda_2)\big)\cdots\rho\big(\tau^{m-1}(\Lambda_m)\big)\big) \\
&= \mathrm{tr}_{V^{\otimes m}}\big(\alpha \circ \big(\rho \otimes \rho\tau \otimes \cdots \otimes \rho\big(\tau^{m-1}\big)\big)\Lambda\big) \\
&= \mathrm{tr}_{V^{\otimes m}}\big(\alpha \circ \widetilde{\rho^m}(\Lambda)\big).
\end{aligned}$$

The third equality uses [KSZ02, Lemma 2.3]. □

It is well known that the integral Λ in H is cocommutative, i.e.,

$$\Delta(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 = \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_1.$$

More generally, $\Delta^m(\Lambda)$ is invariant under cyclic permutations:

$$\Delta^m(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m+1} = \sum_{(\Lambda)} \Lambda_2 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_{m+1} \otimes \Lambda_1. \tag{3.3}$$

Note that if σ is an automorphism of H, then $\sigma(\Lambda) = \Lambda$.

Lemma 3.3.

$$\sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) = \sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1.$$

Proof. By the previous corollary, $\Delta^{m-1}(\Lambda) = \Delta^{m-1}(\tau^{m-1}(\Lambda))$. Since τ^{m-1} is a coalgebra morphism, we get

$$\begin{aligned}\sum_{(\Lambda)} \Lambda_1 \otimes \cdots \otimes \Lambda_m &= \sum_{(\Lambda)} \tau^{-1}(\Lambda)_1 \otimes \cdots \otimes \tau^{-1}(\Lambda)_m \\ &= \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).\end{aligned}$$

Combining this equation with (3.3) gives

$$\sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1 = \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \tau^{-1}(\Lambda_2) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).$$

Applying $(\tau \otimes \tau^2 \otimes \cdots \otimes \tau^m)$, we obtain

$$\sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1 = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m),$$

as desired. □

It is well known that the action of Λ on an H-module W gives a projection onto its invariants. Let $\pi : \widetilde{V^{\otimes m}} \to (\widetilde{V^{\otimes m}})^H$ defined by $\pi(w) = \Lambda \cdot w$ be this projection for $W = \widetilde{V^{\otimes m}}$.

Proposition 3.4. *The linear automorphism α restricts to an automorphism of $(\widetilde{V^{\otimes m}})^H$.*

Proof. It is enough to show that $(\pi \circ \alpha)(w) = (\alpha \circ \pi)(w)$ for $w = v_1 \otimes \cdots \otimes v_m$. Computing gives

$$\begin{aligned}(\pi \circ \alpha)(w) &= (\pi \circ \alpha)(v_1 \otimes \cdots \otimes v_m) \\ &= \pi(v_2 \otimes \cdots \otimes v_m \otimes v_1) \\ &= \sum_{(\Lambda)} \rho(\Lambda_1) v_2 \otimes \rho\big(\tau(\Lambda_2)\big) v_3 \otimes \cdots \otimes \rho\big(\tau^{m-1}(\Lambda_m)\big) v_1,\end{aligned}$$

and

$$\begin{aligned}(\alpha \circ \pi)(v) &= \alpha\big(\Lambda \cdot (v_1 \otimes \cdots \otimes v_m)\big) \\ &= \alpha\Bigg(\sum_{(\Lambda)} \rho(\Lambda_1)v_1 \otimes \rho\big(\tau(\Lambda_2)\big)v_2 \otimes \cdots \otimes \rho\big(\tau^{m-1}(\Lambda_m)\big)v_m\Bigg) \\ &= \sum_{(\Lambda)} \rho\big(\tau(\Lambda_2)\big)v_2 \otimes \cdots \otimes \rho\big(\tau^{m-1}(\Lambda_m)\big)v_m \otimes \rho(\Lambda_1)v_1.\end{aligned}$$

By Lemma 3.3, these two expressions are equal. □

Theorem 3.5. *For any $V \in \operatorname{Rep}(H)$ with character χ, the mth twisted Frobenius–Schur indicator satisfies*

$$\tilde{\nu}_m(\chi) = \operatorname{tr}(\alpha|_{(\widetilde{V^{\otimes m}})^H}).$$

Proof. By Proposition 3.4, the image of α is contained in $(\widetilde{V^{\otimes m}})^H$. Moreover, its restriction to $(\widetilde{V^{\otimes m}})^H$ coincides with the restriction of α. The result now follows by Lemma 3.2. □

Corollary 3.6. *Let ζ_m be a primitive mth root of 1, then*

$$\tilde{\nu}_m(\chi) \in \mathbb{Z}[\zeta_m].$$

Proof. The operator α is of order m, so its eigenvalues are mth roots of unity. It is now immediate from the theorem that the twisted indicators are cyclotomic integers. □

As we will see below, when $m = 2$, the twisted Frobenius–Schur indicators are actually in $\mathbb{Z}$.

4. Twisted second Frobenius–Schur indicators

In this section, we will show that the second twisted Frobenius–Schur indicator gives rise to a partition of the irreducible H-modules into three classes, depending on the relationship between the module and its *twisted dual*. We also compute the indicators for all automorphisms of H_8 – the smallest semisimple Hopf algebra that is neither commutative nor cocommutative.

4.1. Twisted duals and the partition of the simple modules

Let τ be an automorphism such that $\tau^2 = \mathrm{Id}$. We will let $T = \tau S$ denote the corresponding anti-involution. Note that $TS = ST$. Let (V, ρ) be a finite-dimensional H-module with character χ. Using (2.1) for $m = 2$, we have

$$\tilde{\nu}_2(\chi) = \sum_{(\Lambda)} \chi\big(\Lambda_1 TS(\Lambda_2)\big).$$

Definition 4.1. The *twisted duality functor* $(-)^\dagger : \operatorname{Rep}(H) \to \operatorname{Rep}(H)$ is the composition of $\mathcal{F}_\tau$ and the duality functor.

In other words, $V^\dagger$ is the dual space V^* equipped with the H-module structure given by

$$(h \cdot f)(v) = f\big(T(h) \cdot v\big),$$

for all $h \in H$, $f \in V^*$ and $v \in V$. If $f : V \to W$ is an H-map, then $f^\dagger : W^\dagger \to V^\dagger$ is just the usual dual map.

Lemma 4.2. *There is an equality of functors $(-)^{\dagger\dagger} = (-)^{**}$. In particular, $(-)^{\dagger}$ is an involutory auto-equivalence of* $\operatorname{Rep}(H)$.

Proof. Eq. (3.1) implies

$$V^{\dagger\dagger} = \mathcal{F}_\tau\big(\big(\mathcal{F}_\tau\big(V^*\big)\big)^*\big) = \mathcal{F}_\tau^2\big(V^{**}\big) = V^{**}$$

for any module V. It is immediate that $f^{\dagger\dagger} = f^{**}$ for any H-map f. $\square$

The lemma shows that the usual evaluation map $\Psi : V \to V^{\dagger\dagger}$ given by $\Psi(v)(f) = f(v)$ is a canonical isomorphism of H-modules. We now define the transpose endomorphism on $\operatorname{Hom}(V^\dagger, V)$ via $f \mapsto \Psi^{-1} \circ f^\dagger$. (Since $\operatorname{Hom}(V^\dagger, V)$ and $\operatorname{Hom}(V^*, V)$ share the same underlying vector space, this is just the usual transpose.) In general, transposition is not H-linear. However, it is immediate that it restricts to give an endomorphism of $\operatorname{Hom}_H(V^\dagger, V)$. In particular, we can consider symmetric and skew-symmetric H-maps $V^\dagger \to V$:

$$\operatorname{Sym}_H\big(V^\dagger, V\big) = \big\{f \in \operatorname{Hom}_H\big(V^\dagger, V\big) \bigm| f^t = f\big\}$$

and

$$\operatorname{Alt}_H\big(V^\dagger, V\big) = \big\{f \in \operatorname{Hom}_H\big(V^\dagger, V\big) \bigm| f^t = -f\big\}.$$

We can now state the main theorem of this section.

Theorem 4.3. *Let V be an irreducible representation with character χ. Then the following properties hold*:

(1) $\tilde{\nu}_2(\chi) = 0, 1,$ *or* $-1, \forall \chi \in \operatorname{Irr}(H)$.
(2) $\tilde{\nu}_2(\chi) \neq 0$ *if and only if $V \cong V^\dagger$. Moreover, $\tilde{\nu}_2(\chi) = 1$ (resp. -1) if and only if there is a symmetric (resp. skew-symmetric) nonzero intertwining map $V \to V^\dagger$.*

Remark 4.4. This result is well-known in two special cases. If we let $T = S$ (i.e., $\tau = \mathrm{Id}$), then we recover Theorem 3.1 in [LM00]. On the other hand, when H is a group algebra, this is a theorem of Sharp [Sha60] and Kawanaka and Matsuyama [KM90]. See also [KS08].

We will provide some preliminary results before proving the theorem.

There is a canonical H-isomorphism $Q : \widetilde{V^{\otimes 2}} \to \operatorname{Hom}(V^\dagger, V)$ given by

$$\begin{aligned}
\widetilde{V^{\otimes 2}} &= V \otimes \mathcal{F}_\tau(V) \\
&\cong V \otimes \mathcal{F}_\tau(V)^{**} \\
&\cong \operatorname{Hom}\big(\mathcal{F}_\tau(V)^*, V\big) \\
&= \operatorname{Hom}\big(V^\dagger, V\big).
\end{aligned}$$

As a linear map, Q is just the usual isomorphism $V \otimes V \to \operatorname{Hom}(V^*, V)$ with $Q(v \otimes w)(\phi) = \phi(w)v$ for $v, w \in V$ and $\phi \in V^*$. Thus, $Q \circ \alpha = (-)^t \circ Q$. Taking H-invariants and applying Proposition 3.4, we obtain the following lemma.

Lemma 4.5. *There is a commutative diagram of H-maps*

$$\begin{tikzcd} \mathrm{Hom}_H\big(V^\dagger, V\big) \arrow[r, "(-)^t"] & \mathrm{Hom}_H\big(V^\dagger, V\big) \\ \big(\widetilde{V^{\otimes 2}}\big)^H \arrow[u, "Q"] \arrow[r, "\alpha"] & \big(\widetilde{V^{\otimes 2}}\big)^H \arrow[u, "Q"'] \end{tikzcd}$$

Let β be the restriction of the transpose map to $\mathrm{Hom}_H(V^\dagger, V)$. The lemma says that β is a conjugate of $\alpha|_{(\widetilde{V^{\otimes m}})^H}$. Since $\beta^2 = \mathrm{Id}$, the eigenspace decomposition of β gives

$$\mathrm{Hom}_H\big(V^\dagger, V\big) = \mathrm{Sym}_H\big(V^\dagger, V\big) \oplus \mathrm{Alt}_H\big(V^\dagger, V\big). \tag{4.1}$$

Proposition 4.6. *Let V be an H-module. Then,*

$$\tilde{\nu}_2(\chi) = \dim \mathrm{Sym}_H\big(V^\dagger, V\big) - \dim \mathrm{Alt}_H\big(V^\dagger, V\big).$$

Proof. By (4.1), the right side of this equation is $\mathrm{tr}(\beta)$. The assertion follows since $\tilde{\nu}_2(\chi) = \mathrm{tr}(\beta)$ by Theorem 3.5 and Lemma 4.5. □

Remark 4.7. The standard decomposition of $\mathrm{Hom}(V^\dagger, V)$ into symmetric and skew-symmetric linear maps is not necessarily an H-decomposition. In fact, even when $\tau = \mathrm{Id}$, one need not get an H-decomposition unless H is cocommutative.

Proof of Theorem 4.3. Since V is simple, it follows from Lemma 4.2 that $V^\dagger$ is also simple. By Schur's lemma, $\dim \mathrm{Hom}_H(V^\dagger, V) \leqslant 1$. If $V^\dagger \ncong V$, then $\dim \mathrm{Hom}_H(V^\dagger, V) = 0$, so $\tilde{\nu}_2(\chi) = 0$ by Proposition 4.6. Otherwise, $V^\dagger \cong V$, and Proposition 4.6 shows that $\tilde{\nu}_2(\chi)$ equals 1 or -1 depending on the parity of any such isomorphism. □

Remark 4.8. One can also prove Theorem 4.3 using the orthogonality relations for irreducible characters instead of Theorem 3.5. Recall that if the irreducible characters of H are given by $\chi_1, \ldots, \chi_n$, then

$$\sum_{(\Lambda)} \chi_i(\Lambda_1)\chi_j\big(S(\Lambda_2)\big) = \delta_{ij}.$$

(This is the dual statement of Theorem 7.5.6 in [DNR00].) Given a module (V, ρ), the twisted dual $(V^\dagger, \widetilde{\rho})$ satisfies $\widetilde{\rho}(h) = \rho(T(h))^t$. Using this, one computes

$$\tilde{\nu}_2(\chi) = \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} \widetilde{\rho}\big(S(\Lambda_2)\big)_{mm'}.$$

Now, assume that V is simple. If $V \ncong V^\dagger$, then this expression is 0 by the orthogonality relations. Otherwise, there exists a nonzero intertwiner $\varphi \in \mathrm{Hom}_H(V^\dagger, V)$, so that $\widetilde{\rho}(h) = \varphi^{-1}\rho(h)\varphi$; moreover, φ is symmetric or skew-symmetric. A calculation using the orthogonality relations for matrix elements given in [Lar71] shows that the above expression reduces to the parity of φ.

Table 1
Characters for the irreducible representations of H_8.

	1	x	y	xy	z	xz	yz	xyz
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	-1	-1	1	i	$-i$	$-i$	i
χ_4	1	-1	-1	1	$-i$	i	i	$-i$
χ_5	2	0	0	-2	0	0	0	0

Table 2
Automorphisms of H_8.

	1	x	y	z
$\tau_1 = \mathrm{Id}$	1	x	y	z
τ_2	1	x	y	xyz
τ_3	1	y	x	$\frac{1}{2}(z + xz + yz - xyz)$
τ_4	1	y	x	$\frac{1}{2}(-z + xz + yz + xyz)$

4.2. *The second twisted Frobenius–Schur indicators for H_8*

The smallest semisimple Hopf algebra which is neither commutative nor cocommutative has dimension 8. We denote it by H_8. As an algebra, H_8 is generated by elements x, y and z, with relations:

$$x^2 = y^2 = 1, \qquad z^2 = \frac{1}{2}(1 + x + y - xy), \qquad xy = yx, \qquad xz = zy, \quad \text{and} \quad yz = zx.$$

The coalgebra structure of H_8 is given by the following:

$$\begin{gathered} \Delta(x) = x \otimes x, \qquad \varepsilon(x) = 1, \quad \text{and} \quad S(x) = x, \\ \Delta(y) = y \otimes y, \qquad \varepsilon(y) = 1, \quad \text{and} \quad S(y) = y, \\ \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \\ \varepsilon(z) = 1, \quad \text{and} \quad S(z) = z. \end{gathered}$$

The normalized integral is given by

$$\Lambda = \frac{1}{8}(1 + x + y + xy + z + xz + yz + xyz).$$

This Hopf algebra was first introduced by Kac and Paljutkin [KP66] and revisited later by Masuoka [Mas95].

The Hopf algebra H_8 has 4 one-dimensional representations and a single two-dimensional simple module. The characters for the irreducible representations of H_8 are listed in Table 1.

The automorphism group of H_8 is the Klein four-group. These automorphisms are given in Table 2. All four automorphisms satisfy $\tau^2 = \mathrm{Id}$, so the second twisted Frobenius–Schur indicator is defined for all of them. These indicators are given in Table 3.

5. The regular representation

We now return to the general case. In this section, we realize the twisted Frobenius–Schur indicators of the regular representation as the trace of an explicit linear endomorphism of H. Let χ_R denote the character of the left regular representation.

Table 3
Twisted Frobenius–Schur indicators for H_8.

	χ_1	χ_2	χ_3	χ_4	χ_5
$\nu_2(\chi,\tau_1)=\nu_2(\chi)$	1	1	1	1	1
$\nu_2(\chi,\tau_2)$	1	1	1	1	1
$\nu_2(\chi,\tau_3)$	1	1	0	0	1
$\nu_2(\chi,\tau_4)$	1	1	0	0	−1

Let $\Omega_m^\tau : H \to H$ be the linear map defined by

$$\Omega_m^\tau(h) = \sum_{(h)} S\big(\tau^{m-1}(h_1)\tau^{m-2}(h_2)\cdots\tau^2(h_{m-2})\tau(h_{m-1})\big).$$

Theorem 5.1. *The mth twisted Frobenius–Schur indicator of the regular representation satisfies*

$$\tilde{\nu}_m(\chi_R) = \operatorname{tr}\big(\Omega_m^\tau\big).$$

We will need two lemmas.

Lemma 5.2. *For any $h^1,\dots,h^{m-1}\in H$,*

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2)h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1})h^{m-1} \otimes \tau^{m-1}(\Lambda_m)$$
$$= \sum_{(\Lambda)} \Lambda_1 \otimes \tau\big(\Lambda_2 S\big(h^1_{m-1}\big)\big)h^2 \otimes \cdots \otimes \tau^{m-2}\big(\Lambda_{m-1}S\big(h^1_2\big)\big)h^{m-1} \otimes \tau^{m-1}\big(\Lambda_m S\big(h^1_1\big)\big).$$

Proof. By [LR88, Lemma 1.2(b)], we have

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \Lambda_2 = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 S\big(h^1\big).$$

Applying $\mathrm{Id}\otimes\Delta^{m-1}$ to both sides, we get

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m-1} \otimes \Lambda_m$$
$$= \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 S\big(h^1_{m-1}\big) \otimes \Lambda_2 S\big(h^1_{m-2}\big) \otimes \cdots \otimes \Lambda_{m-1} S\big(h^1_2\big) \otimes \Lambda_m S\big(h^1_1\big).$$

We then apply $\mathrm{Id}\otimes\tau\otimes\tau^2\otimes\cdots\otimes\tau^{m-1}$ to get

$$\sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) \otimes \tau^{m-1}(\Lambda_m)$$
$$= \sum_{(\Lambda)} \Lambda_1 \otimes \tau\big(\Lambda_2 S\big(h^1_{m-1}\big)\big) \otimes \cdots \otimes \tau^{m-2}\big(\Lambda_{m-1} S\big(h^1_2\big)\big) \otimes \tau^{m-1}\big(\Lambda_m S\big(h^1_1\big)\big).$$

The lemma follows by right multiplying this equation by $h^1\otimes h^2\otimes\cdots\otimes h^{m-1}\otimes 1$. □

Next, define a linear map $\psi : \widetilde{H}^{\otimes(m-1)} \to \widetilde{H}^{\otimes(m-1)}$ by

$$\begin{aligned}
&\psi\big(h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}\big) \\
&\quad = \sum_{(h^1)} \tau\big(S\big(h^1_{m-1}\big)\big)h^2 \otimes \tau^2\big(S\big(h^1_{m-2}\big)\big)h^3 \otimes \cdots \otimes \tau^{m-2}\big(S\big(h^1_2\big)\big)h^{m-1} \otimes \tau^{m-1}\big(S\big(h^1_1\big)\big).
\end{aligned}$$

Lemma 5.3.

$$\operatorname{tr}(\psi) = \operatorname{tr}\big(\alpha|_{\widetilde{(V^{\otimes m})^H}}\big).$$

Proof. To prove the lemma, it suffices to find a linear isomorphism

$$\varphi : \widetilde{H}^{\otimes(m-1)} \to \big(H \otimes \widetilde{H}^{\otimes(m-1)}\big)^H$$

making the diagram

$$\begin{tikzcd}
\widetilde{H}^{\otimes(m-1)} \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \widetilde{H}^{\otimes(m-1)} \arrow[d, "\varphi"] \\
\big(H \otimes \widetilde{H}^{\otimes(m-1)}\big)^H \arrow[r, "\alpha"'] & \big(H \otimes \widetilde{H}^{\otimes(m-1)}\big)^H
\end{tikzcd}$$

commute. Recall that for any H-module W, there is a linear isomorphism $W \to (H \otimes W)^H$ given by $w \mapsto \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 w$. Let φ be this isomorphism for $W = \widetilde{H}^{\otimes(m-1)}$.

Calculating gives

$$\begin{aligned}
&(\alpha \circ \varphi)\big(h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}\big) \\
&\quad = \sum_{(\Lambda)} \tau(\Lambda_2)h^1 \otimes \tau^2(\Lambda_3)h^2 \otimes \cdots \otimes \tau^{m-1}(\Lambda_m)h^{m-1} \otimes \Lambda_1 \\
&\quad = \sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2)h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1})h^{m-1} \otimes \tau^{m-1}(\Lambda_m) \\
&\quad = \sum_{(\Lambda)} \Lambda_1 \otimes \tau\big(\Lambda_2 S\big(h^1_{m-1}\big)\big)h^2 \otimes \cdots \otimes \tau^{m-2}\big(\Lambda_{m-1} S\big(h^1_2\big)\big)h^{m-1} \otimes \tau^{m-1}\big(\Lambda_m S\big(h^1_1\big)\big) \\
&\quad = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2)\tau\big(S\big(h^1_{m-1}\big)\big)h^2 \otimes \cdots \\
&\qquad \otimes \tau^{m-2}(\Lambda_{m-1})\tau^{m-2}\big(S\big(h^1_2\big)\big)h^{m-1} \otimes \tau^{m-1}(\Lambda_m)\tau^{m-1}\big(S\big(h^1_1\big)\big) \\
&\quad = (\varphi \circ \psi)\big(h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}\big).
\end{aligned}$$

Here, the second and third equalities use Lemmas 3.3 and 5.2 respectively. $\square$

Proof of Theorem 5.1. By the previous lemma, we need only show that $\operatorname{tr}(\psi) = \operatorname{tr}(\Omega^\tau_m)$. Choose a basis $b^1, \dots, b^n \in H$ with dual basis $b^*_1, \dots, b^*_n \in H^*$. Writing out $\operatorname{tr}(\psi)$ in terms of the induced basis on $H^{\otimes m}$, we obtain

Table 4
The linear maps Ω_2^τ for H_8.

	$\Omega_2^{\tau_1}$	$\Omega_2^{\tau_2}$	$\Omega_2^{\tau_3}$	$\Omega_2^{\tau_4}$
1	1	1	1	1
x	x	x	y	y
y	y	y	x	x
xy	xy	xy	xy	xy
z	z	xyz	$\frac{1}{2}(z+xz+yz-xyz)$	$\frac{1}{2}(-z+xz+yz+xyz)$
xz	yz	xz	$\frac{1}{2}(z+xz-yz+xyz)$	$\frac{1}{2}(z-xz+yz+xyz)$
yz	xz	yz	$\frac{1}{2}(z-xz+yz+xyz)$	$\frac{1}{2}(z+xz-yz+xyz)$
xyz	xyz	z	$\frac{1}{2}(-z+xz+yz+xyz)$	$\frac{1}{2}(z+xz+yz-xyz)$

$$\begin{aligned}
\operatorname{tr}(\psi) &= \sum_{i_1,\dots,i_{m-1}=1}^{n} \left\langle b_{i_1}^* \otimes \cdots \otimes b_{i_{m-1}}^*, \psi\left(b^{i_1} \otimes \cdots \otimes b^{i_{m-1}}\right)\right\rangle \\
&= \sum_{i_1,\dots,i_{m-1}=1}^{n} b_{i_1}^*\left(\tau\left(S\left(b_{m-1}^{i_1}\right)\right)b^{i_2}\right) b_{i_2}^*\left(\tau^2\left(S\left(b_{m-2}^{i_1}\right)\right)b^{i_3}\right)\cdots \\
&\quad \times b_{i_{m-2}}^*\left(\tau^{m-2}\left(S\left(b_2^{i_1}\right)\right)b^{i_{m-1}}\right) b_{i_m-1}^*\left(\tau^{m-1}\left(S\left(b_1^{i_1}\right)\right)\right) \\
&= \sum_{i_1,\dots,i_{m-2}=1}^{n} b_{i_1}^*\left(\tau\left(S\left(b_{m-1}^{i_1}\right)\right)b^{i_2}\right) b_{i_2}^*\left(\tau^2\left(S\left(b_{m-2}^{i_1}\right)\right)b^{i_3}\right)\cdots \\
&\quad \times b_{i_{m-2}}^*\left(\tau^{m-2}\left(S\left(b_2^{i_1}\right)\right)\tau^{m-1}\left(S\left(b_1^{i_1}\right)\right)\right) \\
&= \cdots = \sum_{i_1=1}^{n} b_{i_1}^*\left(\tau\left(S\left(b_{m-1}^{i_1}\right)\right)\tau^2\left(S\left(b_{m-2}^{i_1}\right)\right)\cdots\tau^{m-2}\left(S\left(b_2^{i_1}\right)\right)\tau^{m-1}\left(S\left(b_1^{i_1}\right)\right)\right) \\
&= \sum_{i=1}^{n} b_i^*\left(\tau\left(S\left(b_{m-1}^{i}\right)\right)\tau^2\left(S\left(b_{m-2}^{i}\right)\right)\cdots\tau^{m-2}\left(S\left(b_2^{i}\right)\right)\tau^{m-1}\left(S\left(b_1^{i}\right)\right)\right) \\
&= \sum_{i=1}^{n} b_i^*\left(S\left(\tau\left(b_{m-1}^{i}\right)\right)S\left(\tau^2\left(b_{m-2}^{i}\right)\right)\cdots S\left(\tau^{m-2}\left(b_2^{i}\right)\right)S\left(\tau^{m-1}\left(b_1^{i}\right)\right)\right) \\
&= \sum_{i=1}^{n} b_i^*\left(S\left(\tau^{m-1}\left(b_1^{i}\right)\tau^{m-2}\left(b_2^{i}\right)\right)\cdots\tau^2\left(b_{m-2}^{i}\right)\tau\left(b_{m-1}^{i}\right)\right) \\
&= \operatorname{tr}\left(\Omega_m^\tau\right),
\end{aligned}$$

as desired. □

Example 5.4. We revisit the Hopf algebra H_8 described in Section 4.2. The linear maps Ω_2^τ from Theorem 5.1 are given in Table 4. Computing the traces, one obtains the twisted Frobenius–Schur indicators for the regular representation: $\nu_2(\chi_R, \tau_1) = 6$, $\nu_2(\chi_R, \tau_2) = 6$, $\nu_2(\chi_R, \tau_3) = 4$, and $\nu_2(\chi_R, \tau_4) = 0$. These can, of course, also be calculated from the information in Table 3.

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