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Journal of Algebra 319 (2008) 2668–2722

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Eigenvalues of unipotent elements in cross-characteristic representations of finite classical groups

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Received 10 October 2006

Available online 1 February 2008

Communicated by Aner Shalev

Abstract

Let H be a finite classical group, g be a unipotent element of H of order s and θ be an irreducible representation of H with $\dim\theta > 1$ over an algebraically closed field of characteristic coprime to s. We show that almost always all the s-roots of unity occur as eigenvalues of $\theta(g)$, and classify all the triples (H, g, θ) for which this does not hold. In particular, we list the triples for which 1 is not an eigenvalue of $\theta(g)$. We also give estimates of the asymptotic behavior of eigenvalue multiplicities when the rank of H grows and s is fixed.

Keywords: Finite classical groups; Cross-characteristic representations; Unipotent elements; Eigenvalue multiplicities

1. Introduction

In this paper we study the eigenvalues of unipotent elements in cross-characteristic representations of finite classical groups. Let H be a finite classical group and let g be a unipotent element of H of order s. We show that for almost every irreducible representation θ of H all the s-roots of unity occur as eigenvalues of $\theta(g)$, and we classify all the triples (H, g, θ) for which some

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.12.024

s-root of unity does not occur as an eigenvalue of $\theta(g)$. This is part of a broader project intended to study minimum polynomials of elements in group representations. In a previous paper [DM-Z] we solved a similar problem for semisimple elements of prime power order belonging to some parabolic subgroup of H. Other relevant papers are [Z86], where the work was started, and more recently [Z99,GMST,K-Z,Z06] and some papers in preparation.

We also study the asymptotic behavior of the eigenvalue multiplicities when the rank of H grows and $s = |g|$ is fixed. Not much is known about the asymptotic behavior of the eigenvalue multiplicities of matrices in group representations. Results of Gordeev [Go], Hall, Liebeck and Seitz [H-L-S], and Shalev [Sha] produce upper bounds for the multiplicity of a single eigenvalue in terms of the dimension of an irreducible representation. Results of Gluck [Gl1,Gl2,Gl3] and Gluck and Magaard [G-M1,G-M2] enable to obtain a lower bound for the eigenvalue multiplicities of a finite Chevalley group $H = H(q)$ in terms of the field parameter q. However, if q is bounded, no result was yet available, whatever large the order of G. In this paper we obtain lower bounds for the eigenvalue multiplicities of unipotent elements in cross-characteristic irreducible representations of finite quasi-simple classical groups in terms of the rank of H and the order of g (including the characteristic zero case). Bounds of a similar shape were worked out in [DM-Z] for semisimple elements of prime power order belonging to some parabolic subgroup of H. As a unipotent element belongs to a parabolic, this paper completes the analysis for elements of prime power order belonging to parabolic subgroups of classical groups. One may compare our results with those of Landazuri and Seitz [L-S], where lower bounds are obtained for the dimensions of irreducible non-trivial representations of quasi-simple Chevalley groups. Indeed, one can view these bounds as those for the eigenvalue 1 of the identity element of H.

In order to state our results, we introduce some notation, which will also be used throughout the paper. $\mathbb{F}_q$ denotes a finite field of order q, where q is a power of a prime p. V denotes a non-degenerate orthogonal, symplectic or unitary space of dimension $m > 1$ over a finite field F, and $I(V)$ denotes the group of the isometries of V. We assume that $F = \mathbb{F}_{q^2}$ if V is a unitary space and $F = \mathbb{F}_q$ otherwise. We denote by f the sesquilinear form defining the relevant structure of V (except when $p = 2$ and V is an orthogonal space defined by a quadratic form Q, in which case f denotes the bilinear form associated with Q). Our notation for classical groups is standard, namely $GL(m,q)$, $SL(m,q)$, $Sp(m,q)$ denote the general linear group, the special linear group and the symplectic group of degree m over $\mathbb{F}_q$, respectively, whereas $U(m,q)$ denotes the unitary group of degree m over $\mathbb{F}_{q^2}$. $Spin(m,q)$ for m odd and $Spin^{\pm}(m,q)$ for m even denote the spinor quasi-simple groups over $\mathbb{F}_q$, $\Omega(m,q)$ and $\Omega^{\pm}(m,q)$ being the subgroups of the relevant orthogonal groups consisting of the elements with spinor norm 1. Our main results will be stated under the assumption that the commutator subgroup $I(V)'$ is quasi-simple (which only excludes a few groups of low rank ($m < 5$)). Moreover, in view of well-known isomorphisms between simple classical groups, it will also be assumed that $m > 6$ in the orthogonal case, unless stated otherwise.

P denotes an algebraically closed field of characteristic prime to q, and $\mathrm{Irr}_P\, H$ denotes a set of representatives for the equivalence classes of the irreducible representations of H over P (or of the set of isomorphism classes of the irreducible PH-submodules depending on context).

For a square matrix M, we denote by $\deg M$ the degree of the minimum polynomial of M, and by $\mathrm{Spec}\, M$ the spectrum of M, respectively. Similarly we denote by $\mathrm{Spec}\, f$ the spectrum of a vector space endomorphism f. Note that in this paper the spectrum is defined as the *set* of all eigenvalues, *disregarding multiplicities*. For a matrix M, we denote by $\mathrm{Jord}\, M$ the Jordan canonical form of M; a Jordan block of size h is denoted by J_h.

The main aim of this paper is to prove the following results:

Theorem 1.1. *Let p be a prime and q be a power of p. Let H be one of the following groups: $Sp(m,q)$, $m>2$ and $(m,q)\neq(4,2)$; $SU(m,q)$, $m>2$ and $(m,q)\neq(3,2)$; $Spin(m,q)$, m odd, $m>5$; $Spin^{\pm}(m,q)$, m even, $m>6$. Let $g\in H$ be an element of order $s=p^{\alpha}$, $\alpha>1$, and set $t=g^{p^{\alpha-1}}$. Let $\theta\in \mathrm{Irr}_P H$ with $\dim\theta>1$. Then $\mathrm{Spec}\,\theta(g)$ contains all the s-roots of 1, unless one of the following holds*:

(1) *$H=Sp(m,p)$, with p odd, t is a transvection and θ is a Weil representation*;
(2) $H=Sp(4,3)$ *and* $\dim\theta=6, 10$ *or* 20;
(3) $H=Sp(4,9)$ *and* $\dim\theta=40$;
(4) $H=Sp(6,3)$ *and* $\dim\theta=78$;
(5) $H=Sp(8,3)$, $|g|=9$, $\mathrm{Jord}\,g=\mathrm{diag}\{J_4,J_4\}$ *and* $\dim\theta=40$;
(6) $H=Sp(6,2)$ *and* $\dim\theta=7$;
(7) $H=SU(4,3)$ *and* $\dim\theta=20$;
(8) $H=SU(m,2)$, $\mathrm{Jord}\,g=\mathrm{diag}\{J_k,\mathrm{Id}_{m-k}\}$ *with* $k=3$ *or* 5, *and θ is a Weil representation*;
(9) $H=SU(m,2)$, $\mathrm{Jord}\,g=\mathrm{diag}\{J_{m-2},J_2\}$ *and either* $m=5$ *and* $\dim\theta=10$, *or* $m=7$ *and* $\dim\theta=42$.

Remarks.

(1) We recall that the so-called Weil P-representations of $Sp(m,q)$, with $m=2n$ and q odd, are characterized by their dimensions, which are $(q^n\pm1)/2$ if $\mathrm{char}\,P\neq2$, $(q^n-1)/2$ and 1 if $\mathrm{char}\,P=2$. Similarly, the Weil P-representations of $SU(m,q)$ are characterized by their dimensions. These are $(q^m+(-1)^m q)/(q+1)$ and $(q^m-(-1)^m)/(q+1)$ if $(\mathrm{char}\,P,q+1)=1$, whereas one of the dimensions may be 1 and the greater dimension may not occur if $\mathrm{char}\,P$ divides $q+1$. We shall be especially concerned with the case $q=2$. Then $\mathrm{char}\,P=3$ and the greater dimension actually does not occur. For further details on Weil representations, see Section 6.
(2) Observe that we assume $\alpha>1$ in Theorem 1.1, as the case when the unipotent element g has order p is already known (for arbitrary Chevalley groups). The outcome is summarized in the following proposition, which is based on [Z86] and [Z88], except for the claims on dimensions. The latter can be found in [T-Z1] for $\mathrm{char}\,P=0$ and [GMST] for $\mathrm{char}\,P>0$, together with the additional fact that the representations involved are Weil.

For p odd, let us denote by $\Delta_1(p)$ (respectively: $\Delta_2(p)$) the set $1\cup\{\varepsilon^j\}$, where $1\neq\varepsilon\in P$, $\varepsilon^p=1$ and j runs over the non-squares (respectively: the squares) of $\mathbb{Z}$ modulo p. Then the following holds:

Proposition 1.2. *(Cf. [Z86] and [Z88].) Let H be a quasi-simple group of Lie type in characteristic p, such that $(p,|Z(H)|)=1$, and let $g\in H$ be an element of order p. Let θ be a faithful irreducible P-representation of H and suppose that $1<|\mathrm{Spec}\,\theta(g)|<p$. Then p is odd and one of the following holds*:

(1) $H=PSU(3,p)$, $\dim\theta=p(p-1)$ *and g is a transvection*;
(2) $H=SL(2,p^2)$, $\dim\theta=(p^2-1)/2$;
(3) $H=Sp(4,p)$, $\dim\theta=(p^2-1)/2$, $\deg\theta(g)=p-1$ *and g is not a transvection*;
(4) $H=PSp(4,p)$, $\dim\theta=p(p-1)^2/2$ *and g is a transvection*;

(5) *$H = Sp(2n, p)$ or $PSp(2n, p)$, $n > 1$, $\dim\theta = (p^n \pm 1)/2$, g is a transvection and* $\mathrm{Spec}\,\theta(g) = \Delta_1(p)$ *or* $\Delta_2(p)$. *If* char $P = 2$, *then* $H = PSp(2n, p)$ *and only the minus sign has to be taken in the expression for* $\dim\theta$. *If* char $P \neq 2$, *then* $H = Sp(2n, p)$ *if* $\dim\theta$ *is even, while* $H = PSp(2n, p)$ *if* $\dim\theta$ *is odd*;
(6) *$H = SL(2, p)$ or $PSL(2, p)$, and either* $\dim\theta = (p + 1)/2$ *with* $\mathrm{Spec}\,\theta(g) = \Delta_1(p)$ *or* $\Delta_2(p)$, *or* $\dim\theta = (p - 1)/2$ *with* $\mathrm{Spec}\,\theta(g) = \Delta_1(p) \setminus \{1\}$ *or* $\Delta_2(p) \setminus \{1\}$. *If* char $P = 2$, *then* $H = PSL(2, p)$ *and only* $(p - 1)/2$ *has to be taken for* $\dim\theta$;
(7) *$H = PSL(2, p)$ and* $\dim\theta = p - 1$.

$\mathrm{Spec}\,\theta(g)$ *consists of all the non-trivial p-roots of* 1 *except in cases* (5) *and* (6). *In case* (2) *the eigenvalue* 1 *does not occur for g belonging to one of the two unipotent conjugacy classes of H.*

The spectra $\theta(g)$ in the exceptional cases of Theorem 1.1 are known as well. Most (though not all) exceptions occur in Weil representations. In the latter case, the relevant information concerning cases (1) and (8) of Theorem 1.1 is collected in Theorems 1.3 and 1.4 below. In Theorem 1.3, η is a 3-root of ε, where ε is a primitive 3-root of unity in P. In Theorem 1.4, ζ is a primitive 8-root of unity in P. As above, for p odd, $\Delta_1(p)$ (respectively: $\Delta_2(p)$) denotes the set $1 \cup \{\varepsilon^j\}$, where j runs over the non-squares (respectively: the squares) of $\mathbb{Z}$ modulo p.

Theorem 1.3. *Let $H = Sp(m, p)$, with p odd. Let $g \in H$ be an element of order $s = p^\alpha$, $\alpha > 1$, such that $t = g^{p^{\alpha-1}}$ is a transvection, and let θ be a Weil representation of H. Then* $\mathrm{Spec}\,\theta(g)$ *contains all the $p^{\alpha-1}$-roots of the elements of $\Delta_1(p)$ or $\Delta_2(p)$, unless $p = 3$, $|g| = 9$ and one of the following holds*:

(1) *$m > 4$,* Jord $g = \mathrm{diag}\{J_4, \mathrm{Id}_{m-4}\}$, *and* $\mathrm{Spec}\,\theta(g) = \{1, \eta^3, \eta, \eta^4, \eta^7\}$ *or* $\{1, \eta^6, \eta^2, \eta^5, \eta^8\}$;
(2) *$m = 6$,* Jord $g = \mathrm{diag}\{J_4, J_2\}$, $\dim\theta = 13$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta^i \mid i \in \{1, 4, 7, 3, 6\}$ *or* $i \in \{2, 5, 8, 3, 6\}\}$;
(3) *$m = 4$, and either* $\dim\theta = 4$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta, \eta^4, \eta^7, \eta^6\}$ *or* $\{\eta^2, \eta^5, \eta^8, \eta^3\}$; *or* char $P \neq 2$, $\dim\theta = 5$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta, \eta^4, \eta^7, \eta^6, 1\}$ *or* $\{\eta^2, \eta^5, \eta^8, \eta^3, 1\}$.

Theorem 1.4. *Let $H = SU(m, 2)$, $m > 3$. Let $g \in H$ be an element of order $s = 2^\alpha$, $\alpha > 1$, such that $t = g^{2^{\alpha-1}}$ is a transvection, and let θ be a Weil representation of H. Then* $\mathrm{Spec}\,\theta(g)$ *contains all the s-roots of unity, unless one of the following holds*:

(1) Jord $g = \mathrm{diag}\{J_3, \mathrm{Id}_{m-3}\}$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon i = 0, 2, 6\}$;
(2) *$m > 5$,* Jord $g = \mathrm{diag}\{J_5, \mathrm{Id}_{m-5}\}$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon i \neq 4,\ 0 \leqslant i < 8\}$;
(3) *$m = 5$,* Jord $g = J_5$ *and either* $\dim\theta = 10$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon i \neq 4,\ 0 < i < 8\}$, *or* char $P \neq 3$, $\dim\theta = 11$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon i \neq 4, 0 \leqslant i < 8\}$;
(4) *$m = 5$,* Jord $g = \mathrm{diag}\{J_3, J_2\}$, $\dim\theta = 10$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon i = 2, 4, 6\}$;
(5) *$m = 7$,* Jord $g = \mathrm{diag}\{J_5, J_2\}$, $\dim\theta = 42$ *and* $\mathrm{Spec}\,\theta(g) = \{\zeta^i\colon 0 < i < 8\}$.

The exceptional cases listed in Theorem 1.1 which are not covered by Theorems 1.3 and 1.4 are the following: (2) with $H = Sp(4, 3)$, (3) with $H = Sp(4, 9)$, (4) with $H = Sp(6, 3)$, (5) with $H = Sp(8, 3)$, (6) with $H = Sp(6, 2)$, and (7) with $H = SU(4, 3)$. In all these cases $\dim\theta$ is provided in Theorem 1.1, and in fact for each representation of any of these dimensions there is a unipotent element g such that $\mathrm{Spec}\,\theta(g)$ contains less than $|g|$ elements. Complete information on these cases can be read off from Lemmas 6.14, 6.16, 6.18, 6.22, 5.7 and 6.15, respectively. The case $H = Sp(4,2) \simeq S_6$ is not considered, as S_6 is not quasi-simple.

Finally, we recall that the case $H = SL(m,q)$ was examined in [Z90]. For $m > 2$, if θ is a non-trivial representation of $SL(m,q)$, then every unipotent element g has exactly $|g|$ distinct eigenvalues except when $H = SL(3,2)$, $|g| = 4$ and $\dim\theta = 3$. The case $m = 2$ is contained in Proposition 1.2.

The detailed analysis carried out in the paper, in order to determine the exceptional cases listed in Theorem 1.1, provides in particular, as a byproduct, a list of the cases in which $\theta(g)$ acts fixed-point freely on the relevant representation space. Namely:

Theorem 1.5. *Under the assumptions of Theorem* 1.1, *assume that* 1 *is not an eigenvalue of* $\theta(g)$. *Then one of the following holds*:

(1) $G = Sp(4,3)$ *and* $\dim\theta \in \{4, 6, 10, 20\}$, *unless* $\operatorname{char} P = 2$, *in which case the value* 10 *must be discarded*;
(2) $G = Sp(6,3)$, $\operatorname{Jord} g = \operatorname{diag}\{J_4, J_2\}$ *and* $\dim\theta = 13$;
(3) $G = Sp(6,3)$, $\operatorname{Jord} g = \operatorname{diag}\{J_4, J_1, J_1\}$ *and* $\dim\theta = 78$;
(4) $G = Sp(4,9)$ *and* $\dim\theta = 40$;
(5) $G = Sp(8,3)$, $\operatorname{Jord} g = \operatorname{diag}\{J_4, J_4\}$ *and* $\dim\theta = 40$;
(6) $G = SU(4,3)$ *and* $\dim\theta = 20$;
(7) $G = SU(5,2)$, $\operatorname{Jord} g = \operatorname{diag}\{J_3, J_2\}$ *or* $\operatorname{Jord} g = J_5$, *and* $\dim\theta = 10$;
(8) $G = SU(7,2)$, $\operatorname{Jord} g = \operatorname{diag}\{J_5, J_2\}$ *and* $\dim\theta = 42$.

The next theorem produces the lower bounds for the eigenvalue multiplicities of unipotent elements in cross-characteristic irreducible representations of finite quasi-simple classical groups, announced at the beginning of the Introduction.

Theorem 1.6. *Let p be a prime and q be a power of p. Let H be one of the following groups*: $SL(m,q)$; $Sp(m,q)$; $SU(m,q)$; $Spin(m,q)$, *m odd*; $Spin^{\pm}(m,q)$, *m even. Let $g \in H$ be an element of order $s = p^{\alpha}$ and let $\theta \in \operatorname{Irr}_P H$ with* $\dim\theta > 1$. *Suppose that $m > 2p^{\alpha-1} + 4$. Then the multiplicity of every eigenvalue of $\theta(g)$ is at least* $q^{(\frac{m-6}{2} - s^2)}$.

Note. In contrast with Theorem 1.1, in whose proof all the exceptional cases usually occurring for small m or q are examined, the bounds obtained in Theorem 1.6 are not sharp. In fact, we only intend to show that eigenvalue multiplicities tend to the infinity when the order of g is bounded but the rank of the group tends to the infinity. A more accurate analysis, within the frame of the methods used in the paper, may lead to better lower bounds for eigenvalue multiplicities. Also observe that for some of the above groups better specific bounds are obtained even in the present paper. For details, we refer to the theorems and lemmas quoted in the proof of Theorem 1.6 (Section 8).

In characteristic zero Gluck [Gl1,Gl2] proves that, if $H = H(q)$ is a quasi-simple finite group of Lie type (where q is the field parameter), g belongs to $H \setminus Z(H)$ and χ is a non-trivial irreducible character of H, then there exists a non-negative real-valued function $\lambda(q)$ such that $\lambda(q)$ tends to 0 as q tends to the infinity and $|\chi(g)| \leqslant \lambda(q) \cdot \chi(1)$ (see also [G-M1] for further information on the function λ, specifically for unipotent elements in classical groups). From this one can easily deduce the following:

Let $\alpha \in \mathbb{N}$. There exists an increasing function $f_\alpha : \mathbb{N} \to \mathbb{N}$ such that, whenever: (a) $H = H(q)$ *is a finite quasi-simple group of Lie type and $g \in H$ has order α* mod $Z(H)$; (b) $n \in \mathbb{N}$, *and* $q > f_\alpha(n)$; (c) *θ is a non-trivial complex irreducible representation of H; then $\theta(g)$ has exactly α distinct eigenvalues and every eigenvalue has multiplicity at least n.*

It is an open problem whether such a function λ exists for Brauer characters in characteristic coprime to q, even for g unipotent. The reader may consult [G-M2] for some comments on this problem.

2. Preliminary results and machinery

In this section we collect a number of results which will play a crucial rôle in the sequel. They mainly concern finite groups containing an extraspecial normal subgroup and their representations.

Most facts about extraspecial groups quoted without explicit references are to be found in [Hu, Chapter III] and [H-B, Chapter IX]. Recall that an extraspecial group is a p-group $\mathcal{E}$ such that $|Z(\mathcal{E})| = p$ and $Z(\mathcal{E}) = \mathcal{E}' = \Phi(\mathcal{E})$, where $\Phi(\mathcal{E})$ stands for the Frattini subgroup of $\mathcal{E}$.

Let $W = \mathcal{E}/Z(\mathcal{E})$. Clearly W is an elementary abelian p-group, and thus can be viewed as a vector space over the prime field $\mathbb{F}_p$. For $a, b \in \mathcal{E}$, set $\bar{a} = aZ(\mathcal{E})$, $\bar{b} = bZ(\mathcal{E})$. Denoting by $[a, b]$ the commutator $aba^{-1}b^{-1}$ of a and b, and identifying $Z(\mathcal{E})$ with the additive group of $\mathbb{F}_p$, the bracket $(\bar{a}, \bar{b}) \to [a, b]$ defines a non-degenerate bilinear alternating form (,) on the space W. Thus, W has the structure of a symplectic space over $\mathbb{F}_p$. Let $\dim(W) = 2n$: then $\mathcal{E}$ has order p^{2n+1}, and in order to make the order of $\mathcal{E}$ explicit, we will write $\mathcal{E}_n$ for $\mathcal{E}$ of order p^{2n+1}. Let A be the group of all the automorphisms of $\mathcal{E}$ which induce the identity on $Z(\mathcal{E})$. Then there is a natural homomorphism $\varepsilon : A \to Sp(2n, p)$, whose kernel is $Inn(\mathcal{E})$. If p is odd, ε is surjective, whereas if $p = 2$ the image of ε is one of the orthogonal groups $O^+(W)$, $O^-(W)$, depending on the isomorphism type of $\mathcal{E}$. For any subset B of A, we will denote by $\bar{B}$ the image of B in $Sp(2n, p)$ under ε.

Lemma 2.1. *Let $\pi : \mathcal{E} \to W$ be the natural projection, and let X be a subgroup of $\mathcal{E}$. Then the following conditions are equivalent*:

(a) *X is extraspecial*;
(b) $Z(X) = Z(\mathcal{E}) \neq X$;
(c) *$\pi(X) \neq \{0\}$ is a non-degenerate subspace of W.*

Moreover, two subgroups Y_1, Y_2 of $\mathcal{E}$ commute elementwise if and only if $\pi(Y_1)$, $\pi(Y_2)$ are mutually orthogonal subspaces of W.

Lemma 2.2. *Let P be an algebraically closed field of characteristic coprime to p.*

(a) *Every faithful* (*equivalently*: *non-trivial on $Z(\mathcal{E}_n)$*) *irreducible P-representation φ of $\mathcal{E}_n$ has degree p^n.*
(b) *There is a bijection between such representations φ of $\mathcal{E}_n$ and the non-trivial characters $\zeta \in \mathrm{Irr}_P\, Z(\mathcal{E}_n)$, given by $\varphi|_{Z(\mathcal{E}_n)} = \zeta \cdot \mathrm{Id}$.*
(c) *Let $\mathcal{E}_k$ be a subgroup of $\mathcal{E}_n$ and φ be as in* (a). *Then $\varphi|_{\mathcal{E}_k}$ is a direct sum of p^{n-k} pairwise equivalent faithful irreducible representations of $\mathcal{E}_k$.*

Lemma 2.3. *Let B be a finite group containing an extraspecial normal subgroup $\mathcal{E}_n$ such that $B = \langle b, \mathcal{E}_n\rangle$, for some $b \in B \setminus \mathcal{E}_n$. Assume that $[b, Z(\mathcal{E}_n)] = 1$ (equivalently, $Z(\mathcal{E}_n) \subseteq Z(B)$). Then*:

(a) *every irreducible P-representation φ of B non-trivial on $Z(\mathcal{E}_n)$ has degree p^n, and $\varphi|_{\mathcal{E}_n}$ is irreducible*;
(b) *if ψ is an irreducible P-representation of B, such that $\varphi|_{Z(\mathcal{E}_n)} = \psi|_{Z(\mathcal{E}_n)}$, then $\varphi = \psi \otimes \eta$, where η is a 1-dimensional representation of B.*

The next two lemmas collect known facts about the Jordan form of unipotent elements in classical groups:

Lemma 2.4. *(E.g., see [Sp, pp. 19–20].) Let F be a field and suppose that u is a unipotent element of $GL(m, \bar{F})$, where $\bar{F}$ denotes the algebraic closure of F. Denote by $c_i(u)$ the number of blocks of size i in the Jordan normal form of u.*

The following holds:

(a) *Let m be even. Then u is conjugate to an element of $Sp(m, \bar{F})$ if and only if $c_i(u)$ is even whenever i is odd.*
(b) *Let* char $F > 2$. *Then u is conjugate to an element of $O(m, \bar{F})$ if and only if $c_i(u)$ is even whenever i is even. In particular, if m is even and $u \in O(m, \bar{F})$, then the Jordan form of u contains at least two blocks.*
(c) *Let* char $F = 2$ *and m be even. If $u \in Sp(m, \bar{F})$, then u is conjugate to an element of $O(m, \bar{F})$. Furthermore, u is conjugate to an element of $\Omega(m, \bar{F})$ if and only if the total number of Jordan blocks of u (that is, $\sum_i c_i(u)$) is even.*

Lemma 2.5.

(i) *Let $I(V)$ be the group of isometries of a non-degenerate symplectic or orthogonal space V over the field F, and let $G = \langle g\rangle \subset I(V)$, where g is unipotent. Then $V = V_1 \oplus \cdots \oplus V_k$, where the V_i's ($i = 1, \ldots, k$) are mutually orthogonal non-degenerate G-submodules and for each i the Jordan form of $g|_{V_i}$ consists either of a single block or of two blocks of equal size. In the latter case, G preserves two disjoint maximal totally isotropic subspaces of V_i (except possibly when* char $F = 2$).
(ii) *Let $I(V)$ be the group of isometries of a non-degenerate unitary space V, and let $G = \langle g\rangle \subset I(V)$, where g is unipotent. Then $V = V_1 \oplus \cdots \oplus V_k$, where the V_i's ($i = 1, \ldots, k$) are mutually orthogonal non-degenerate G-submodules and for each i the Jordan form of $g|_{V_i}$ consists of a single block (that is, V_i is indecomposable as a G-module).*

Proof. See [Zas, Lemma 2]. □

An easy consequence of Lemmas 2.1 and 2.5 is the following:

Lemma 2.6. *Let B be a finite group containing an extraspecial normal subgroup $\mathcal{E}_n$, and let b be an element of B centralizing $Z(\mathcal{E}_n)$ and inducing an automorphism of order l on $\mathcal{E}_n$. Then $\mathcal{E}_n$ is the central product of at least $[\frac{n}{l}]$ elementwise commuting extraspecial subgroups $\mathcal{E}_{n_i}$, such that $n_i \leqslant l$ and $b\mathcal{E}_{n_i}b^{-1} = \mathcal{E}_{n_i}$.*

As in the Introduction, if p is an odd prime we define $\Delta_1(p)$ (respectively: $\Delta_2(p)$) to be the set $1 \cup \{\xi^j\}$, where $1 \neq \xi \in P$, $\xi^p = 1$ and j runs over the non-squares (respectively: the squares) of $\mathbb{Z}$ modulo p.

Lemma 2.7. *Let P be as above and $\mathcal{F}_n \subset GL(p^n, P)$ be an irreducible p-subgroup isomorphic to $\mathcal{E}_n$. Let $b \in GL(p^n, P)$ be a p-element normalizing but not centralizing $\mathcal{F}_n$ and set $B = \langle b, \mathcal{F}_n\rangle$. Let p^α be the order of b modulo $Z(B)$ and let $\delta = \deg b$ (the degree of the minimum polynomial of b). Then the following holds*:

(a) *[Z85,Z88] Let $\alpha = 1$. Denote by $\bar{b}$ the element of $Sp(2n, p)$ induced by conjugation via b on the symplectic space $W = \mathcal{F}_n/Z(\mathcal{F}_n)$. Then $\delta = p$, unless $\bar{b}$ is a transvection in $Sp(2n, p)$, $p > 2$ and $|\mathcal{F}_n : C_{\mathcal{F}_n}(b)| = p$. In the latter case, $\delta = (p+1)/2$ and* $\mathrm{Spec}(b)$ *is either $\Delta_1(p)$ or $\Delta_2(p)$ up to a common multiplier.*

(b) *[Be-Z] Let $\alpha > 1$, $p > 2$ and $b_1 = b^{p^{\alpha-1}}$. Then $\delta = p^\alpha$, unless $b_1 \notin \mathcal{F}_n$ and $|\mathcal{F}_n : C_{\mathcal{F}_n}(b_1)| = p$. In the latter case, either*
   1. *$\delta = p^{\alpha-1}(p+1)/2$ and* $\mathrm{Spec}(b)$ *is the set of all the $p^{\alpha-1}$-roots of the elements of* $\mathrm{Spec}(b_1)$, *or*
   2. *$p = 3$, $\alpha = 2$, $|\mathcal{F}_n : C_{\mathcal{F}_n}(b)| = 3^3$ and* $\mathrm{Spec}(b)$, *up to a common multiplier, is $\{1, \varepsilon, \eta, \eta\varepsilon, \eta\varepsilon^2\}$, where $\eta^3 = \varepsilon \neq 1$ and $\varepsilon^3 = 1$.*

Remark. If $\bar{b}$ is a transvection, then the condition $|\mathcal{F}_n : C_{\mathcal{F}_n}(b_1)| = p$ in (b) is equivalent to $\langle b_1^p\rangle \cap \mathcal{E}_n \subset Z(\mathcal{E}_n)$.

Notation. If X is any square matrix over a field F, in the following lemma we denote by $\mu(X)$ the lowest multiplicity of an eigenvalue of X (in the algebraic closure $\bar{F}$ of F).

Lemma 2.8. *Let n, k be natural numbers, with $n > k$, and let $\{X_i \mid 1 \leqslant i \leqslant n\}$ be a set of square matrices of size l_i over a field F, such that X_i^k is a non-zero scalar for every i. Let $M = \min(l_{i_1} \cdots l_{i_{n-k}})$, where the minimum is taken over all $(n-k)$-tuples $(i_1, \ldots, i_{n-k})$. Then $\mu(X_1 \otimes \cdots \otimes X_n) \geqslant M$. In particular, if $l = \min_i l_i$, then $\mu(X_1 \otimes \cdots \otimes X_n) \geqslant l^{n-k}$.*

Proof. First we observe that $|\mathrm{Spec}(\lambda X)| = |\mathrm{Spec}(X)|$ and $\mu(\lambda X) = \mu(X)$ for any matrix X and any $0 \neq \lambda \in \bar{F}$. This allows us to assume that 1 is an eigenvalue of each X_i $(1 \leqslant i \leqslant n)$. In particular, we may further assume that no X_i $(1 \leqslant i \leqslant n)$ is scalar. Now, set $Y_i = X_1 \otimes \cdots \otimes X_i$ $(1 \leqslant i \leqslant n)$, and reorder the X_i's in such a way that $|\mathrm{Spec}(Y_1)| < |\mathrm{Spec}(Y_2)| < \cdots < |\mathrm{Spec}(Y_j)| = |\mathrm{Spec}(Y_{j+1})| = \cdots = |\mathrm{Spec}(Y_n)|$. [Note that such an ordering always exists. For this, it suffices to prove that if $|\mathrm{Spec}(Y_j)| = |\mathrm{Spec}(Y_j \otimes X_k)|$ for some j and for all $k > j$, then $\mathrm{Spec}(Y_j) = \mathrm{Spec}(Y_k)$ for all $k > j$. Let $\mathrm{Spec}(Y_j) = \{\delta_1, \ldots, \delta_r\}$ and, for any $0 \neq \lambda \in \bar{F}$, write $\{\delta_1\lambda, \ldots, \delta_r\lambda\} = \mathrm{Spec}(Y_j) \cdot \lambda$. Clearly, if $\mathrm{Spec}(Y_j) \cdot \alpha = \mathrm{Spec}(Y_j) \cdot \beta$ for all pairs α, β of eigenvalues of X_k for all $k > j$, we are done. On the other hand, suppose that $\mathrm{Spec}(Y_j) \cdot \alpha \neq \mathrm{Spec}(Y_j) \cdot \beta$ for two eigenvalues α, β of some X_k with $k > j$. Then $|\mathrm{Spec}(Y_j)| = |\mathrm{Spec}(Y_j) \cdot \alpha| < |(\mathrm{Spec}(Y_j) \cdot \alpha) \cup (\mathrm{Spec}(Y_j) \cdot \beta)|$. Relabeling this X_k by X_{j+1} we obtain $|\mathrm{Spec}(Y_j)| < |\mathrm{Spec}(Y_{j+1})|$.] Since, for every i, the eigenvalues of Y_i are k-roots of 1, it is clear that $j \leqslant k$. Let $|\mathrm{Spec}(Y_j)| = a$, $M_0 = 1$, and $M_s = \min(l_{i_1} \cdots l_{i_s})$ taken over all s-tuples $(l_{i_1}, \ldots, l_{i_s})$ for $s = 1, \ldots, n-k$. We prove by induction on s that $\mu(Y_{j+s}) \geqslant M_s$, starting with $s = 0$, in which case the assertion is trivial. So, assume $\mu(Y_{j+s}) = r \geqslant M_s$. Let $\mathrm{Spec}(Y_{j+s}) = \{\varepsilon_1, \ldots, \varepsilon_a\}$, and let γ be any eigenvalue of X_{j+s+1}. As $|\mathrm{Spec}(Y_{j+s} \otimes X_{j+s+1})| = |\mathrm{Spec}(Y_{j+s+1})| = a$, the set

$\{\varepsilon_1\gamma, \dots, \varepsilon_a\gamma\}$ does not depend on the choice of γ. Hence $\varepsilon_1\gamma, \dots, \varepsilon_a\gamma$ are eigenvalues of Y_{j+s+1} of multiplicity at least $\mu(Y_{j+s})l_{j+s+1} \geqslant M_s l_{j+s+1} \geqslant M_{s+1}$. The lemma follows. □

We will also need the following elementary lemma:

Lemma 2.9. *Let $\varepsilon_1, \dots, \varepsilon_r$ be r (not necessarily distinct) k-roots of 1. If $r \geqslant k$, there exists a subset J of $\{1, \dots, r\}$ such that $1 \leqslant |J| \leqslant k$ and $\prod_{j\in J} \varepsilon_j = 1$.*

Proof. Let $\delta_i = \varepsilon_1 \cdots \varepsilon_i$, $1 \leqslant i \leqslant r$. Since the set $\{\delta_i\}$ has cardinality at most k, then either $\delta_i = \delta_j$ for some $i < j \leqslant k$, in which case $1 = \delta_i^{-1}\delta_j = \varepsilon_{i+1} \cdots \varepsilon_j$, or $r = k$ and the δ_i's are all distinct, so that $\delta_i = \varepsilon_1 \cdots \varepsilon_i = 1$ for some i. □

Lemma 2.10. *Let $\mathcal{F}_n \subset GL(p^n, P)$ be an irreducible p-subgroup isomorphic to $\mathcal{E}_n$. Set $B = \langle b, \mathcal{F}_n\rangle$, where b is a p-element of $GL(p^n, P)$ normalizing but not centralizing $\mathcal{F}_n$. Assume furthermore that $[b, Z(\mathcal{F}_n)] = 1$. Let l be the order of b modulo $Z(B)$, and assume $n > l^2$. Then the multiplicity of every eigenvalue of b is at least p^{n-l^2}.*

Proof. By Lemma 2.6, $\mathcal{F}_n$ is the central product of $r \geqslant [\frac{n}{l}]$ element-wise commuting extraspecial p-subgroups $\mathcal{E}_{n_i}$ $(1 \leqslant i \leqslant r)$ such that $b\mathcal{E}_{n_i}b^{-1} = \mathcal{E}_{n_i}$ and $n_i \leqslant l$. Let σ_i be the automorphism of $\mathcal{E}_n$ defined by $\sigma_i(x) = bxb^{-1}$ for $x \in \mathcal{E}_{n_i}$ and $\sigma_i(x) = x$ for $x \in \mathcal{E}_{n_j}$ $(j \neq i)$, and τ_i be the representation of $\mathcal{F}_n$ given by $x \to \sigma_i(x)$ $(x \in \mathcal{F}_n)$. Clearly τ_i is faithful and irreducible; moreover, as τ_i is the identity on $Z(\mathcal{F}_n)$, τ_i is equivalent to the identity representation $\mathrm{Id}_{\mathcal{F}_n}$ (cf. Lemma 2.2). Hence there exists $b_i \in GL(p^n, P)$ such that $b_i x b_i^{-1} = \sigma_i(x)$ for every $x \in \mathcal{F}_n$. Notice that b_i can be chosen to be of finite p-power order. (Indeed, let β_i be the order of b_i as an element of $\mathrm{Aut}(\mathcal{F}_n)$. Then $b_i^{\beta_i} = \lambda_i \cdot E$ for some $\lambda_i \in P$. Choose $\mu_i \in P$ such that $\mu_i^{\beta_i} = \lambda_i^{-1}$. Then $(b_i\mu_i)^{\beta_i} = E$, and we may replace b_i with $b_i\mu_i$.) Now, for each $i = 1, \dots, r-1$, we choose b_i arbitrarily (subject to the above conditions) and take $b_r = b_1 b_2 \cdots b_{r-1} b$. We claim that $b_i b_j = b_j b_i$ for $1 \leqslant i, j \leqslant r$. Let R_i, $R_{i'}$ denote the enveloping algebras of $\mathcal{E}_{n_i}$ and of all the $\mathcal{E}_{n_j}$'s with $j \neq i$, respectively. As the coset representatives of $\mathcal{F}_n/Z(\mathcal{F}_n)$ are linearly independent in $R =: \mathrm{Mat}(p^n, P)$, by dimension reasons we must have $\dim R_i = p^{2n_i}$ and $\dim R_{i'} = p^{2(n/n_i)}$. Clearly, R_i and $R_{i'}$ commute element-wise and R_i is simple, as $\mathcal{E}_{n_i}$ is homogeneous. Therefore, $R_i \simeq \mathrm{Mat}(p^{n_i}, P)$ and $R_{i'}$ coincides with the centralizer of R_i in R (by dimension reasons). Thus R_i is the centralizer of $R_{i'}$ in R. As b_i centralizes $R_{i'}$, it follows that $b_i \in R_i$. Since this holds for each i, we conclude that all the b_i's commute with each other. Set $B_i = \langle b_i, \mathcal{E}_{n_i}\rangle$ $(1 \leqslant i \leqslant r)$ and $D = B_1 \cdots B_r$, so that $B \subseteq D$. Observe that the B_i's are finite, commute elementwise, and have a common center $Z(\mathcal{E}_n) = Z(D)$. Consider the abstract group $D_1 = B_1 \times \cdots \times B_r$ (a direct product). Then D is the image of a representation τ of D_1. Clearly, $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ where $\tau_i \in \mathrm{Irr}\, B_i$ for $i = 1, \dots, r$ and, by Lemma 2.3, $\dim \tau_i = p^{n_i}$. In particular, there are elements $b_i' \in B_i$ such that $b = \tau_1(b_1') \otimes \cdots \otimes \tau_r(b_r')$. Set $g_i = \tau_i(b_i')$ for $i = 1, \dots, r$. Then $b = g_1 \otimes \cdots \otimes g_r$. By Lemma 2.8, the multiplicity of every eigenvalue of b is at least p^d, where $d = \min(n_{i_1} + n_{i_2} + \cdots + n_{i_{r-l}}) = n - \max(n_{j_1} + \cdots + n_{j_l}) \geqslant n - l^2$. □

The last item in this section is the following version of the so-called Higman's lemma, which will serve our purposes in the sequel:

Lemma 2.11. *(See [H-B, Chapter IX, Theorem 1.10].) Let $g \in GL(m, P)$ be an element of prime power order normalizing a (finite) abelian subgroup A of order coprime to* $\operatorname{char} P$. *Let $|g| = p^a$ and $[g^{p^{a-1}}, A] \neq 1$. Then the degree of the minimum polynomial of g equals p^a.*

3. The group $SL(m, q)$

Let p be a prime, $q = p^a$ for some integer $a > 0$, and let $H = SL(m, q)$ be the special linear group of degree m over F_q. Let $\theta \in \operatorname{Irr}_P H$ with $\dim \theta > 1$. The aim of this section is to provide information on the multiplicities of the eigenvalues of $\theta(g)$, when g is a p-element of H.

Lemma 3.1. *Let $g \in SL(m, q)$ be an element of order p^α, for some $\alpha > 0$. Set $t = g^{p^{\alpha-1}}$ and $G = \langle g\rangle$. Let V be the natural $SL(m, q)$-module and denote by V^t the space of fixed vectors of t. Then a vector $v \in V$ lies in a regular G-orbit if and only if $v \in V \setminus V^t$. Thus the number of vectors of V belonging to regular G-orbits equals $q^m - q^c$, where $c = \dim(V^t)$. (Observe that c equals the number of blocks in the Jordan normal form of t.)*

Proof. It is clear that $v \in V$ belongs to an orbit of length p^α if and only if it is not fixed by t. Since obviously $|V \setminus V^t| = q^m - q^c$, the result follows. □

Lemma 3.2. *Let $g \in SL(m, q)$ be an element of order p^α, for some $\alpha > 0$. Set $G = \langle g\rangle$ and denote by V the natural $SL(m, q)$-module. If* $\operatorname{Jord} g = J_m$, *suppose additionally that $m > p^{\alpha-1} + 1$. Then there exists a 1-dimensional subspace R of V such that $g(R) = R$ and G acts faithfully on V/R.*

Proof. Let $V = V_1 \oplus \cdots \oplus V_k$ the decomposition of V as a direct sum of G-stable indecomposable subspaces corresponding to the Jordan normal form of g. Furthermore, suppose that $\dim(V_1) \leqslant \cdots \leqslant \dim(V_k)$. Pick $0 \neq v \in V_1$ such that $g(v) = v$ and set $R = \langle v\rangle$. Now recall that a unipotent Jordan block J_r of size r has order p^γ such that $p^{\gamma-1} < r \leqslant p^\gamma$, and moreover $r \leqslant s$ implies $|J_r| \leqslant |J_s|$. Our claim readily follows. □

Proposition 3.3. *Let $H \in SL(m, q)$ with $m > 2$, and let g be an element of H of order p^α, for some $\alpha > 0$. Set $G = \langle g\rangle$, $t = g^{p^{\alpha-1}}$ and let c be the number of blocks in the Jordan normal form of t. Let θ be a non-trivial irreducible P-representation of $SL(m, q)$. Then the following holds:*

(i) *$\theta|_G$ contains at least one regular constituent, unless $m = 3$, $q = 2$ and* $\dim \theta = 3$.
(ii) *If $m > p^{\alpha-1} + 1$, then $\theta|_G$ contains at least* $\max\{1, (q^{m-1} - q^{c-1})/p^\alpha\}$ *regular constituents.*

Proof. Part (i) of the statement was proven in [Z90] (in the case $m = 3$, $q = 2$ and $\dim \theta = 3$, it is readily seen that $-1 \notin \operatorname{Spec} \theta(g)$). Next, suppose that $m > p^{\alpha-1} + 1$. Then the assumptions of Lemma 3.2 are fulfilled. Choose R as in Lemma 3.2, and let $U = \{x \in SL(m, q) \mid (x - \mathrm{Id})V \subseteq R\}$. U is an elementary abelian group of order q^{m-1} which can be viewed as a faithful $\mathbb{F}_q G$-module. Indeed, view U as a row $\mathbb{F}_q$-space, and let $\bar g$ be the projection of g onto V/R. Then $gxg^{-1} = x\bar g^{-1}$ for any $x \in U$; in other words, U is the dual of the natural $\langle \bar g\rangle$-module V/R. Let K be the group of P-characters of U. Let M denote the $SL(m, q)$-module afforded by θ. Then, we can write $M|_U = M_0 \oplus \sum_{\kappa \in O} M_\kappa$, where $O = K \setminus \{1_U\}$ and $M_\kappa = \{v \in M \mid uv = \kappa(u)v,\ \forall u \in U\}$. Clearly, since the action of G on U is contragredient to the action on the subspaces M_κ, we may apply Lemma 3.1 to K. Namely, K, and hence O, contains at least $q^{m-1} - q^{c-1}$ points belonging to regular orbits of G. As every regular orbit leads to a regular submodule of $M|_G$, the number of regular submodules of $M|_G$ is at least $(q^{m-1} - q^{c-1})/p^\alpha$, as desired. □

Corollary 3.4. *Under the assumptions of Proposition* 3.3*, the multiplicity of every eigenvalue of* g *is at least* $\max\{1, (q^{m-1} - q^{m-2})/p^\alpha\}$.

Proof. The statement follows from the observation that $c \leqslant m-1$. □

The following lemma deals with eigenvalue multiplicities in the case of $SL(2,q)$.

Lemma 3.5. *Let* $H = SL(2,q)$*, where* $q = p^a > p$ *and* p *is an odd prime, and let* g *be a non-trivial unipotent element of* H*. Let* θ *be a non-trivial irreducible* P*-representation of* H*. Then the following holds*:

(1) *If* $\dim\theta \in \{q, q \pm 1\}$*, then each* p*-root of* 1 *occurs as an eigenvalue of* $\theta(g)$ *with multiplicity at least* $\frac{q}{p} - 1$.
(2) *If* $\dim\theta = (q \pm 1)/2$ *and* a *is odd, then each* p*-root of* 1 *occurs as an eigenvalue of* $\theta(g)$ *with multiplicity at least* $(q/p - \sqrt{q/p})/2$.
(3) *If* $\dim\theta = (q \pm 1)/2$ *and* $a = 2r > 2$*, then each* p*-root of* 1 *occurs as an eigenvalue of* $\theta(g)$ *with multiplicity at least* $p^r - p$*. If* $a = 2$*, then each* p*-root of* 1 *occurs as an eigenvalue of* $\theta(g)$*, unless* $\dim\theta = (q-1)/2$*, in which case the eigenvalue* 1 *does not occur for* g *belonging to one of the two non-trivial unipotent conjugacy classes of* H.

Proof. It is well known that every irreducible P-representation of H lifts to a complex representation; so it suffices to deal with the case $P = \mathbb{C}$. Let χ be the character of θ. Recall that $\chi(1) \in \{q, q \pm 1, (q \pm 1)/2\}$. Set $G = \langle g \rangle$, and let 1_G and ρ_G denote the trivial and the regular character of G, respectively. Consider first the cases $\chi(1) \in \{q, q \pm 1\}$. Then $\chi(1) = q + k$, where $k = 0, 1$ or -1. Checking the character table of H, one observes that $\chi(g) = k$. It follows that $\chi|_G = \frac{q}{p}\rho_G + k \cdot 1_G$, so ρ_G occurs at least $\frac{q}{p} - 1$ times and the result follows.

Next, assume that $\chi(1) = (q \pm 1)/2$. There are two non-equivalent representations of each degree. Define $c = (-1)^{(q-1)/2}$. Then $\chi(g) = (-1 \pm \sqrt{cq})/2$ if $\chi(1) = (q-1)/2$ and $1 + (-1 \pm \sqrt{cq})/2$ if $\chi(1) = (q+1)/2$. It is convenient to denote by $\chi^\pm$ the two characters of degree $(q-1)/2$ and by $\chi_1^\pm$ the two characters of degree $(q+1)/2$, with signs chosen accordingly to their values listed above. It follows that $\chi_1^\pm(x) = 1 + \chi^\pm(x)$ for every $x \in G$. So it suffices to estimate $\chi^\pm|_G$. Furthermore, without loss of generality we may assume that $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Suppose first that a is odd. Then $c = (-1)^{p-1/2}$ and $\chi^\pm(g) = (-1 \pm \sqrt{cq})/2$. Denote by $\tau^\pm$ the characters of $SL(2,p)$ of degree $(p-1)/2$, with $\pm$ chosen accordingly. Then $(\chi^\pm - \sqrt{q/p}\,\tau^\pm)(g) = (\sqrt{q/p} - 1)/2$, whence

$$\chi^\pm|_G = \sqrt{q/p}\,\tau^\pm + \frac{\sqrt{q/p} - 1}{2} \cdot 1_G + \frac{q/p - \sqrt{q/p}}{2} \cdot \rho_G$$

(by comparison of the values of both sides at every $x \in G$). Therefore, the minimum eigenvalue multiplicity is $(q/p - \sqrt{q/p})/2$ in this case.

Let a be even, so that $c=1$. Then $\chi^{\pm}(g)=(-1\pm\sqrt{q})/2$, which is an integer. As $\frac{q-1}{2}=(-1\pm\sqrt{q})/2+(q\mp\sqrt{q})/2$, we have

$$\chi^{\pm}|_G=\frac{q\mp\sqrt{q}}{2p}\cdot\rho_G+\frac{-1\pm\sqrt{q}}{2}\cdot 1_G.$$

It follows that the eigenvalue multiplicity is minimal for the eigenvalue 1, for which it is equal to $(\sqrt{q}-p)(\sqrt{q}+1)/2p$. This is zero only when $q=p^2$, an exceptional case recorded in Proposition 1.2(2). □

4. The classical groups: preliminaries

In this and the following sections we deal with the classical groups mentioned in the Introduction, to which the reader is referred for the basic nomenclature and notation. Recall that, unless specified otherwise, V is a non-degenerate orthogonal, symplectic or unitary space of dimension $m>1$ over a finite field F of characteristic p, and $I(V)$ is the group of the isometries of V. Moreover, we denote by τ the Galois automorphism of $F=\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ in the unitary case, and the trivial automorphism of $F=\mathbb{F}_q$ in the symplectic and orthogonal cases. We also set $F_0=\{f\in F\mid \tau(f)=f\}$, the fixed field of τ.

Lemma 4.1. *Let $i(V)$ denote the number of non-zero isotropic (singular) vectors in V. Then:*

- $i(V)=|F|^m-1$ *if V is symplectic;*
- $i(V)=|F|^{m-1}-1$ *if V is orthogonal and m is odd;*
- $i(V)=|F|^{m-1}+|F|^{m/2}-|F|^{(m/2)-1}-1$ *if m is even and V is orthogonal of index $m/2$;*
- $i(V)=|F|^{m-1}-|F|^{m/2}+|F|^{(m/2)-1}-1$ *if m is even and V is orthogonal of index $\frac{m}{2}-1$;*
- $i(V)=(|F|^{m/2}-(-1)^m)(|F|^{(m-1)/2}-(-1)^{m-1})$ *if V is unitary.*
- *In particular: $i(V)\geqslant|F|^{m-2}-1$ in the orthogonal case, while $i(V)\geqslant|F|^{m/2}-1$ in the unitary case.*

Proof. $i(V)$ equals the index $|I(V):S_1|$, where S_1 denotes the stabilizer of an isotropic (singular) vector. Both values are well known. □

Lemma 4.2. *Let $\Omega=I(V)'$ be the commutator subgroup of $I(V)$. If $m=\dim(V)>3$, then Ω is transitive on every $I(V)$-orbit in V. (If V is unitary, the statement holds for $m>2$.)*

Proof. E.g., see [K-L, Lemma 2.10.5]. □

Lemma 4.3. *Let g be a unipotent element of $H=I(V)'$ and set $G=\langle g\rangle$. For $0\neq v\in V$ let $O=Hv$ be the orbit of v under H. Then the number of vectors $o\in O$ that belong to a regular G-orbit is at least $|F|^m-|F|^{m-1}$, $|F|^{m-4}$, and $|F|^{(m-2)/2}$, respectively in the symplectic, orthogonal and unitary case. In particular: the permutation H-module associated to O contains regular G-submodules.*

Proof. As in Lemma 3.1, let $|g|=p^{\alpha}$ and $t=g^{p^{\alpha-1}}$. Set $X=V\setminus V^t$, where V^t denotes the subspace of fixed vectors of t. Then $|Gx|=|G|$ for every $x\in X$. If V is symplectic, our claim follows immediately from Lemma 3.1. So, let V be orthogonal or unitary. Observe that $O\cap X\neq\emptyset$, as O spans V, unless V is orthogonal and $m \leqslant 2$ (e.g., see [K-L, Proposition 2.10.6]). Therefore, we may assume that $v \in X$. Denote by $\nu(u)$ the norm of a vector u in V ($\nu(u)$ is defined to be $Q(u)$ if V is orthogonal, $f(u,u)$ if V is unitary). If $w \in v^\perp$ and $\nu(w) = 0$, then $\nu(v+w) = \nu(v)$. Hence, by Lemma 4.2, $v + w \in O$ unless $m = 2$ and V is unitary, or $m \leqslant 3$ and V is orthogonal. Clearly, $v^\perp$ contains a non-degenerate subspace W of dimension at least $m-2$. Hence, by Lemma 4.1, the number of isotropic (singular) vectors in W is at least $|F|^{m-4}$, $|F|^{(m-2)/2}$ in the orthogonal and unitary case, respectively. (Recall that the zero vector is not counted in Lemma 4.1.) □

The following basic fact is well known:

Lemma 4.4. *Let g be a unipotent element of $I(V)$. Then g fixes an isotropic* (*singular*) *vector $v \in V$, unless V is orthogonal,* $\dim(V) = 2$ *and* $p = 2$.

Let g be a unipotent element of $I(V)$, let $v \in V$ be an isotropic (singular) vector fixed by g, and set $W = \langle v \rangle$. Let W_1 be a complement of W in $W^\perp$. It is clear that W_1 is non-degenerate. Thus $W_1^\perp$ is also non-degenerate and contains W. We choose a basis $B = \{b_1, \ldots, b_m\}$ such that $b_1 \in W$, $b_2, \ldots, b_{m-1} \in W_1$ and $b_m \in W_2$, where $W_1^\perp = W \oplus W_2$. With respect to B the Gram matrix of f is

$$\Gamma_f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \Phi & 0 \\ \varepsilon & 0 & 0 \end{bmatrix},$$

where Φ is the Gram matrix of the restriction of f to W_1, and $\varepsilon = 1$ unless q is odd and V is a symplectic space, in which case $\varepsilon = -1$. Clearly, $\tau(\Phi^t) = \varepsilon\Phi$, where t denotes the transpose. If $\tau \neq 1$, then Φ can be chosen to be Id_{m-2}. In particular, $\tau(\Phi) = \Phi$; we will always assume the latter in the sequel.

It is clear that the matrix of g with respect to the basis B has shape

$$\begin{bmatrix} 1 & * & * \\ 0 & h & * \\ 0 & 0 & 1 \end{bmatrix},$$

where $h \in I(W_1, f|_{W_1})$.

Set $S = Stab_H(W)$, $S_1 = Stab_H(v)$ and denote by U the unipotent radical of S, that is: $U = O_p(S)$, the largest normal p-subgroup of S. With respect to B, the elements of S have shape

$$\begin{bmatrix} \alpha & a & b \\ 0 & y & c \\ 0 & 0 & \alpha^* \end{bmatrix},$$

where $0 \neq \alpha \in F$, $y \in I(W_1, f|_{W_1})$ and $\alpha^* = (\alpha^{-1})^\tau$. The subgroup $Q = \{s \in S \mid s(W_i) = W_i,\ i = 1, 2\}$, consisting of all block-diagonal matrices $\mathrm{diag}(\alpha, y, \alpha^*)$ is called the (standard) Levi subgroup of S. It is well known (and readily seen) that $S = UQ$ (semidirect product). Furthermore, observe that, by our assumptions on H, Q contains no normal non-trivial p-subgroups.

For our purposes, it is also convenient to introduce one more subgroup related to Q. Namely, we denote by Y the subgroup of H consisting of all the matrices of shape $\mathrm{diag}(1, y, 1)$, so that $y \in I(W_1)$. In other words: $Y = \{M \in Q \mid M|_W = \mathrm{Id}\}$. It follows that $S_1 = UY$.

We observe explicitly that $g^i \in U$ if and only if $h^i = \mathrm{Id}$, that is, if and only if $(g^i - \mathrm{Id})W^\perp \subseteq W$. In particular, if $|g| = p^\alpha$ and $t = g^{p^{\alpha-1}}$, then $t \notin U$ if and only if $|g| = |h|$.

We also recall the following properties of the unipotent radical:

Lemma 4.5. *$U' \neq 1$, unless V is an orthogonal space (in any characteristic) or V is a symplectic space in characteristic 2. U is a group of exponent p, unless $p = 2$ and V is unitary, in which case U has exponent 4.*

Proof. Direct computation. E.g., see [DM-Z, pp. 240–241]. □

Lemma 4.6. *Let V be a unitary space of dimension $m > 2$, $g \in I(V)$ be unipotent of order $p^\alpha > 1$, and set $t = g^{p^{\alpha-1}}$. Then there exists an isotropic 1-dimensional subspace W of V such that $g(W) = W$ and $t \notin U$, except when one of the following holds:*

(i) $m = p^{\alpha-1} + 1$ *and* Jord g *consists of a single block*;
(ii) $m = p^{\alpha-1} + 2$ *and* Jord g *consists of a single block*;
(iii) $m = p^{\alpha-1} + 2$ *and* Jord g *consists of two blocks of sizes* 1 *and* $m - 1$, *respectively*;
(iv) $m = p^{\alpha-1} + 3$ *and* Jord g *consists of two blocks of sizes* 1 *and* $m - 1$, *respectively.*

Proof. By Lemma 2.5, we can write $V = V_0 \oplus V_1 \oplus \cdots \oplus V_r$, where the subspaces $V_1, \ldots, V_r$ are mutually orthogonal non-degenerate $\langle g\rangle$-submodules such that $g|_{V_0} = \mathrm{Id}$ and for each $i > 0$ the Jordan form of $g_i = g|_{V_i}$ consists of a single block of size > 1. We may also assume that the dimensions of the V_i's are non-decreasing for $i = 1, \ldots, r$. If $\dim(V_0) > 1$, we pick $0 \neq v \in V_0$ to be isotropic. Otherwise, we pick $v \in V_1$ to be isotropic with $gv = v$. Set $W = \langle v\rangle$. If $r > 1$ or $r = 1$ and $v \in V_0$, then $V_r \subseteq W^\perp$, $V_r \cap W = 0$ and $t|_{V_r} \neq \mathrm{Id}$ imply $(t - \mathrm{Id})W^\perp \nsubseteq W$, so $t \notin U$. We are left with the cases when $r = 1$ and $V_0 = 0$ or $\dim(V_0) = 1$. If $\dim(V_1) > p^{\alpha-1} + 2$, then $t \notin U$. So the lemma follows. (Observe that if $m = 3$, then t always belongs to U, and therefore g is 'exceptional'.) □

Lemma 4.7. *Let V be a symplectic or orthogonal space with* $\dim(V) > 4$. *Let $g \in I(V)$ be unipotent of order $p^\alpha > 1$, and set $t = g^{p^{\alpha-1}}$. Then there exists an isotropic (singular) 1-dimensional subspace W of V such that $g(W) = W$ and $t \notin U$, except when one of the following holds:*

(i) $p > 2$, $m = p^{\alpha-1} + 2$, V *is orthogonal and* Jord $g = J_m$.
(ii) $p > 2$, $m = p^{\alpha-1} + 1$, V *is symplectic*, t *is a transvection and* Jord $g = J_m$.
(iii) $p > 2$, $m = 2(p^{\alpha-1} + 1)$ *and* Jord g *consists of two blocks of size* $p^{\alpha-1} + 1$.
(iv) $p = 2$, $m = 2^{\alpha-1} + 2$ *with* $\alpha > 1$ *and* Jord $g = J_m$.
(v) $p = 2$, $m = 2(2^{\alpha-1} + 1)$ *and* Jord g *consists of two blocks of size* $2^{\alpha-1} + 1$.
(vi) $p > 2$, $m = 2(p^{\alpha-1} + 1) + 1$, V *is orthogonal and*

$$\mathrm{Jord}\, g = \mathrm{diag}\{J_1, J_{(m-1)/2}, J_{(m-1)/2}\}.$$

(vii) $p > 2$, $m = p^{\alpha-1} + 3$, V *is orthogonal and* Jord $g = \mathrm{diag}\{J_1, J_{m-1}\}$.
(viii) $p > 2$, $m = p^{\alpha-1} + 4$, V *is orthogonal and* Jord $g = \mathrm{diag}\{J_1, J_1, J_{m-2}\}$.
(ix) $p > 2$, $m = 2(p^{\alpha-1} + 1) + 2$, V *is orthogonal and*

$$\mathrm{Jord}\, g = \mathrm{diag}\{J_1, J_1, J_{(m-2)/2}, J_{(m-2)/2}\}.$$

(x) $p = 2$, $m = 2^{\alpha-1} + 4$ *with* $\alpha > 1$, V *is orthogonal and either* $\mathrm{Jord}\, g = \mathrm{diag}\{J_1, J_1, J_{m-2}\}$ *or* $\mathrm{Jord}\, g = \mathrm{diag}\{J_2, J_{m-2}\}$.

(xi) $p = 2$, $m = 2(2^{\alpha-1} + 1) + 2$, V *is orthogonal and either*

$$\mathrm{Jord}\, g = \mathrm{diag}\{J_1, J_1, J_{(m-2)/2}, J_{(m-2)/2}\} \quad \textit{or} \quad \mathrm{Jord}\, g = \mathrm{diag}\{J_2, J_{(m-2)/2}, J_{(m-2)/2}\}.$$

In particular, if g *belongs to one of the above exceptional cases and* t *is a transvection, then* (ii) *holds.*

Proof. By Lemma 2.5, we may write $V = V_1 \oplus \cdots \oplus V_r$, where the V_i's ($i = 1, \ldots, r$) are mutually orthogonal non-degenerate $\langle g \rangle$-submodules such that the Jordan form of each $g_i = g|_{V_i}$ consists of all blocks of a given size appearing in the Jordan form of g. If $r > 1$, by reordering the V_i's we may assume that the size of the Jordan blocks of g_{i+1} is greater than that of g_i, for $i = 1, \ldots, r - 1$. Thus $|g_1| \leqslant \cdots \leqslant |g_r|$. We claim that either:

(a_1) g fixes an isotropic (singular) vector $v \in V_1$; or:

(a_2) V is orthogonal, and either g fixes a singular vector $v \in V_2$, or $p = 2$, $\dim(V_1) = \dim(V_2) = 2$ and g fixes a singular vector $v \in V_1 \oplus V_2$.

Indeed, suppose that (a_1) does not hold. Then V is orthogonal and either $\dim(V_1) = 1$, in which case $p > 2$ and $g_1 = \mathrm{Id}$, or, by Lemma 4.4, $\dim(V_1) = 2$ and V_1 is anisotropic. As $m > 2$, $V_2 \neq 0$. If $\dim(V_1) = 1$, then, as $\dim(V_2) \geqslant 2$ and p is odd, g fixes a singular vector $v \in V_2$. Therefore (a_2) holds. Next, suppose that $\dim(V_1) = 2$ and V_1 is anisotropic. Then, by Lemma 2.4, either $\dim(V_2) \geqslant 3$, or $p = 2$, $g_1 = \mathrm{Id}$ and $\dim(V_2) = 2$. In the former case g fixes a singular vector $v \in V_2$ by Lemma 4.4; otherwise, it is easy to see that g fixes a singular vector $v \in V_1 \oplus V_2$. Thus (a_2) holds.

Set $W = \langle v \rangle$ and let B be as above. Recall that the claim that $(t - \mathrm{Id})W^{\perp} \nsubseteq W$ amounts to saying that $h^{p^{\alpha-1}} \neq \mathrm{Id}$, or equivalently $|g| = |h|$. We distinguish the following cases:

Case (1). (a_j) holds (for $j = 1$ or 2) and $r > j$. In this case $W \cap V_r = 0$, hence the projection $\lambda : W^{\perp} \to W^{\perp}/W$ is a G-module homomorphism injective on V_r. It follows that $|g| = |g_r| = |h|$.

Case (2). $r = j = 1$ (so $g = g_1$ and (a_1) holds). As $\ker(S_1 \to I(W^{\perp}/W)) = U$, either $|g| = |h|$ or there is $i \in \mathbb{N}$ such that $\mathrm{Id} \neq g^{p^i} \in U$. In the latter instance $1 \leqslant \dim(g^{p^i} - \mathrm{Id})V \leqslant 2$, since $\dim(u - \mathrm{Id})V \leqslant 2$ for any $u \in U$. If $\dim(g^{p^i} - \mathrm{Id})V = 1$, then $\mathrm{Jord}\, g$ consists of a single block of size $p^{\alpha-1} + 1$. If $\dim(g^{p^i} - \mathrm{Id})V = 2$, then $\mathrm{Jord}\, g$ consists either of a single block or of two blocks of equal size. In the former case, the size of the Jordan block must equal $p^i + 2$. Hence $i = \alpha - 1$, as U has exponent p. Suppose first that m is odd. Then V is orthogonal and $p > 2$. Hence $\dim(g^{p^i} - \mathrm{Id})V = 2$ (otherwise g^{p^i} would be a transvection) and g has a single Jordan block of size $p^{\alpha-1} + 2$. So we get (i). Now suppose that m is even. If p is odd, then either $\mathrm{Jord}\, g$ consists of a single block of size $m = p^{\alpha-1} + 1$, in which case by Lemma 2.4 V is symplectic and we get (ii), or $\mathrm{Jord}\, g$ consists of two blocks of equal size and $m = 2(p^{\alpha-1} + 1)$, and we get (iii). Let $p = 2$. If $\mathrm{Jord}\, g$ consists of a single block, then $m = p^{\alpha-1} + 2$ with $\alpha > 1$, yielding (iv). If $\mathrm{Jord}\, g$ consists of two blocks, then $m = 2(2^{\alpha-1} + 1)$ and we get (v).

Case (3). $r = j = 2$. Here (a_2) holds, and moreover $v \in V_2$, since by assumption $m > 4$ and $r = 2$. Arguing as above, we are reduced to the case when there exists $i \in \mathbb{N}$ such that $g_2^{p^i} - \mathrm{Id}$ is non-zero and has rank 1 or 2. First, suppose that $\dim(g^{p^i} - \mathrm{Id})V_2 = 1$. Then g_2 consists of a single block of size $p^{\alpha-1} + 1$, and therefore p must be odd. Assume first that $\dim(V_1) = 1$. Then $p^{\alpha-1} + 1 = m - 1$ is even, contradicting Lemma 2.4. Next, assume that $\dim(V_1) = 2$, that is, V_1 is an anisotropic plane. As p is odd, $m - 2 = p^{\alpha-1} + 1$ is even, contradicting once again Lemma 2.4. Now, suppose that $\dim(g^{p^i} - \mathrm{Id})V_2 = 2$. Assume that $\dim(V_1) = 1$. If m is odd, then $p > 2$, $\mathrm{Jord}\, g_2 = \mathrm{diag}\{J_{(m-1)/2}, J_{(m-1)/2}\}$ by Lemma 2.4, and we obtain (vi). If m is even, then again $p > 2$, $\mathrm{Jord}\, g_2 = J_{m-1}$ and $m - 1 = p^{\alpha-1} + 2$, and we obtain (vii). Next, assume that $\dim(V_1) = 2$. If m is odd, then $p > 2$, $g_1 = \mathrm{Id}$, $\mathrm{Jord}\, g_2 = J_{m-2}$ and $m - 2 = p^{\alpha-1} + 2$. This yields (viii). If m is even and $p > 2$, by Lemma 2.4 we cannot have $\mathrm{Jord}\, g_2 = J_{m-2}$; thus $\mathrm{Jord}\, g_2$ consists of two blocks of size $(m - 2)/2 = p^{\alpha-1} + 1$, and we get (ix). If m is even and $p = 2$, then either $g_1 = \mathrm{Id}$ or $g_1 = J_2$. This yields case (x) with $\alpha > 1$, and case (xi).

(Observe that if $m = 3$ or 4, then t always belongs to U, and therefore g is 'exceptional'.) □

Corollary 4.8. *Let $g \in H$ be unipotent of order $p^\alpha > 1$, and set $t = g^{p^{\alpha-1}}$. Suppose that $m > p^{\alpha-1} + 3$ if V is unitary, and $m > 2p^{\alpha-1} + 4$ if V is symplectic or orthogonal. Then there exists a singular 1-dimensional subspace W of V such that $g(W) = W$ and $(t - \mathrm{Id})W^\perp \nsubseteq W$.*

Proof. The statement follows immediately from Lemmas 4.6 and 4.7. □

Lemma 4.9. *Let V be a vector space over $\mathbb{F}_q$ and let $\chi : (V, +) \to P$ be a non-trivial character of the additive group $(V, +)$. Set $K = \ker \chi$. Then the following holds*:

(1) *K contains a unique hyperplane V_1 of V.*
(2) *If $q = p^a > p$ and $v \in V \setminus K$, then $\lambda v \in K$ for some $0 \neq \lambda \in \mathbb{F}_q$.*

Proof. (1) is proven in [L-S, Lemma 2.3]. As for (2), let $x \in K \setminus V_1$. Then, as $\dim(V/V_1) = 1$, both $V_1 + x$ and $V_1 + v$ generate V/V_1. Thus $\lambda(V_1 + v) = V_1 + x$ for some $0 \neq \lambda \in \mathbb{F}_q$, whence $\lambda v - x \in V_1$. It follows that $\lambda v \in K$. □

5. Orthogonal groups and symplectic groups of characteristic 2

As usual, if V is a (possibly degenerate) orthogonal space, we denote by $\mathrm{Rad}\, V$ the subspace of all vectors orthogonal to the whole of V. Further, we denote by $R_0(V)$ the set of all vectors $x \in \mathrm{Rad}\, V$ such that $Q(x) = 0$. Clearly, $R_0(V)$ is a subspace of codimension at most 1 in $\mathrm{Rad}\, V$.

Lemma 5.1. *Let V be an orthogonal space* (*possibly degenerate*) *over $\mathbb{F}_q$ such that $V/R_0(V)$ is not anisotropic. Then V is spanned by its singular vectors. Moreover, if $q = 2^a > 2$ and X is a subgroup of index 2 of the additive group of V, then V is spanned by the singular vectors belonging to X.*

Proof. By our assumption, $\mathrm{Rad}\, V \subset V$. Furthermore, without loss of generality we may assume that $R_0(V) = 0$. If $\mathrm{Rad}\, V = 0$, then our first claim follows readily from the classification and geometry of finite non-degenerate orthogonal spaces. Otherwise, $V = \mathrm{Rad}\, V \perp Y$, where Y is a non-anisotropic non-degenerate subspace of V (observe that V is not anisotropic; if $v \in V$ is singular, then $v \notin \mathrm{Rad}\, V$ and we may assume that $v \in Y$). Let $0 \neq x \in \mathrm{Rad}\, V$, so that, by our current assumptions, $Q(x) \neq 0$. Since $\dim(Y) > 1$ (otherwise Y would be totally singular), Y is spanned by its singular vectors and there exists $y \in Y$ such that $Q(y) = Q(x)$. Then $Q(x + y) = 0$. Since $\dim(\mathrm{Rad}\, V) = 1$, the claim follows.

Next, suppose that $q = 2^a > 2$ and X is a subgroup of index 2 of the additive group of V. Obviously, we can view X as the kernel of a suitable non-trivial character χ of $(V, +)$; thus, by the previous lemma, for any $v \in V \setminus X$ there exists $0 \neq \lambda \in \mathbb{F}_q$ such that $\lambda v \in X$. Since, if v is singular, so is λv, the second part of the statement follows. □

Lemma 5.2. *Let V be an orthogonal space (possibly degenerate) over $\mathbb{F}_q$ defined by a quadratic form Q such that $Q(V) \neq 0$. Let $0 \neq \lambda \in \mathbb{F}_q$. Then one of the following holds:*

(1) *V is spanned by the vectors v such that $Q(v) = \lambda$;*
(2) $\mathrm{Rad}\, V = R_0(V)$, *$V/\mathrm{Rad}\, V$ has dimension 2 and Witt index 1, and $q = 2$ or 3;*
(3) *q is odd and* $\dim(V/\mathrm{Rad}\, V) = 1$.

Proof. We first observe that Q is surjective on $\mathbb{F}_q$ unless q is odd and $V/\mathrm{Rad}\, V$ has dimension 1, that is unless (3) occurs. If $\mathrm{Rad}\, V = 0$, the statement of the lemma is well known. Indeed (e.g., see [K-L, Proposition 2.10.6]), $I(V)$ is irreducible, except when $\dim(V) = 2$, V is not anisotropic and $q = 2, 3$. Since the subspace generated by vectors of a given norm λ is $I(V)$-stable, the result follows. Assume $\mathrm{Rad}\, V \neq 0$. Then $V = \mathrm{Rad}\, V \oplus Y$ and $\mathrm{Rad}\, V = R_0(V) \oplus V_0$, where Y is non-degenerate (or $Y = 0$) and $Q(V_0) \neq 0$ if $V_0 \neq 0$. Set $N = \{x \in V \mid Q(x) = \lambda\}$. If $N = \emptyset$, then Q is not surjective on $\mathbb{F}_q$. By the above, we get (3). If $N \neq \emptyset$, let M denote the subspace spanned by N. By way of contradiction, suppose that $M \neq V$. Let $x \in N$. If $v \in R_0(V)$, then $Q(x + v) = Q(x) = \lambda$, hence $x + v \in N$ and $v \in M$. Therefore $R_0(V) \subseteq M$. If $V_0 \neq 0$, then q is even, $\dim(V_0) = 1$ and $Q(V_0) \neq 0$. Therefore $Q(V_0) = \mathbb{F}_q$, and hence $V_0 \subseteq M$. It follows that $\mathrm{Rad}\, V \subseteq M$. If $Y = 0$, we are done. Otherwise, $Y \cap M \neq Y$. If $Y \cap M = 0$, then $M = \mathrm{Rad}\, V$. In this case, $\lambda \neq 0$ and $N \subseteq \mathrm{Rad}\, V$ force q even. Then $Q(Y) = \mathbb{F}_q$, and therefore there must be $y \in Y$ such that $Q(y) = \lambda$, a contradiction. Since $Y \cap M$ is $I(V)$-stable, it follows from above that $\dim(Y) = 2$, Y is not anisotropic and $q = 2, 3$. If $V_0 \neq 0$, then $q = 2$, $\lambda = 1$ and one can easily check that $Y + V_0$ is spanned by its non-singular vectors. This forces $Y \subseteq M$, a contradiction. Thus $\mathrm{Rad}\, V = R_0(V)$, and $V/R_0(V) \simeq Y$ satisfies the requirements of (2). (In this case, $\dim(M) = \dim(V) - 1$.) □

Remark. Case (2) in the above lemma provides real exceptions. For, assume $\mathrm{Rad}\, V = 0$, so that $\dim(V) = 2$. Let $q = 2$. Then we may choose in V a basis (b, c) such that $Q(b) = 0$, $Q(c) = 1$ and $f_Q(b, c) = 1$. It follows that c is the only non-singular vector in V. Next, let $q = 3$. Then we may choose in V a basis (b, c) such that $Q(b) = 1 = -Q(c)$ and $f_Q(b, c) = 0$. It follows that $\pm b$ are the only vectors $x \in V$ such that $Q(x) = 1$.

Lemma 5.3. *Let V be an orthogonal space (possibly degenerate) over $\mathbb{F}_q$ defined by a non-zero quadratic form Q. Suppose that the codimension of* $\mathrm{Rad}\, V$ *in V is greater than* 1 *and let V_1 be a subspace of V of codimension* 1. *Set $J = \{x \in V \setminus V_1 \mid Q(x) \neq 0\}$ and $L = \langle J \rangle$. Then one of the following holds:*

(1) $L = V$ (*this includes the case when* $V/\operatorname{Rad} V$ *is either an anisotropic plane, or a hyperbolic plane with* $q > 3$).
(2) $q = 2$ *or* 3, $\operatorname{Rad} V = R_0(V)$, *and* $V/\operatorname{Rad} V$ *is a hyperbolic plane.* (*Observe that* $L = 0$ *iff this case holds with* $q = 2$ *and* $Q(V_1) \neq 0$; *whereas* $0 \neq L \neq V$ *iff* $q = 3$ *and* $Q(V_1) \neq 0$.)
(3) $q = 2$, $Q(\operatorname{Rad} V) \neq 0$ *and* $V/\operatorname{Rad} V$ *has dimension* 2.
(4) $q = 2$, $V/\operatorname{Rad} V$ *has dimension* 4 *and* V_1 *contains an anisotropic plane.* (*Additionally,* $Q(\operatorname{Rad} V_1) \neq 0$.)

Proof. The proof is based on induction on $\dim(V)$. By Lemma 5.2 $J \neq \emptyset$, hence $L \neq 0$, except possibly when case (2) of Lemma 5.2 holds [case (3) is ruled out by assumption]. Suppose the latter happens. Set $V/\operatorname{Rad} V = \langle b, c\rangle$. Then b and c may be chosen as in the remark above. By abuse of language, we identify b and c with elements of V. Let $q = 2$ and $x = r + \lambda b + \mu c$, with $r \in \operatorname{Rad} V$. Since $Q(x) = \mu^2 + \lambda\mu = \mu(\lambda + \mu)$, $Q(x) \neq 0$ iff $\mu = 1$ and $\lambda = 0$, i.e. iff $x = r + c$. It follows that $L = 0$ iff $\operatorname{Rad} V \subseteq V_1$ and $Q(V_1) \neq 0$. Now let $q = 3$, $x = r + \lambda b + \mu c$, with $r \in \operatorname{Rad} V$. Since $Q(x) = \lambda^2 - \mu^2 = (\lambda + \mu)(\lambda - \mu)$, $Q(x) \neq 0$ iff either $\lambda = 0$, $\mu = \pm 1$ or $\lambda = \pm 1$, $\mu = 0$. It follows $L \neq 0$. It is also easy to check that $L \neq V$ iff $Q(V_1) \neq 0$.

Suppose that neither case (1) nor case (2) holds. Then $0 \neq L \neq V$. Set $L_1 = L \cap V_1$. Clearly $L_1 \neq V_1$ (otherwise we would have $L = V_1$, whence $J \subseteq V_1$, a contradiction). Set $N = V_1 \setminus L_1$. Let $x = j + \alpha y$, where $j \in J$, $y \in N$ and $0 \neq \alpha \in \mathbb{F}_q$. Then $Q(x) = 0$, otherwise $x \in J$ and hence $y \in L$, which is a contradiction. Now observe that $0 = Q(x) = Q(j) + \alpha^2 Q(y) + \alpha f_Q(j, y)$. If $Q(y) = 0$, then $f_Q(j, y) \neq 0$ and $q = 2$ (indeed, if $q > 2$ we can always pick $\alpha \neq 0$ such that $Q(j + \alpha y) \neq 0$). If $Q(y) \neq 0$, then $f_Q(j, y) = 0$ and $q = 2$. Indeed, if $f_Q(j, y) \neq 0$, then $Q(j + \alpha y) = Q(j) \neq 0$ for $\alpha = -f_Q(j, y)/Q(y)$. Now, assume that $q > 2$. Then Lemma 5.2 implies that there exists some $y \in N$ such that $Q(j) + Q(y) = Q(j + y) \neq 0$ (which is a contradiction) unless possibly when (i): q is odd, $\dim(V_1/\operatorname{Rad} V_1) = 1$ and $Q(V_1) \neq \mathbb{F}_q$; (ii): $q = 3$ and $V_1/R_0(V_1)$ is a hyperbolic plane. Suppose that (i) holds. Assume first that $\operatorname{Rad} V \nsubseteq V_1$. Then $V = V_1 \oplus \langle x\rangle$ for some $x \in \operatorname{Rad} V$. This means that $\operatorname{Rad} V_1$ is properly contained in $\operatorname{Rad} V$, which in turn implies that $\operatorname{Rad} V$ has codimension at most 1 in V, against our assumptions. So, assume that $\operatorname{Rad} V \subseteq V_1$, and hence $\operatorname{Rad} V \subseteq \operatorname{Rad} V_1$. Set $\bar{V} = V/\operatorname{Rad} V$, $\overline{V_1} = V_1/\operatorname{Rad} V$. Then $\overline{V_1}$ has codimension 1 in $\bar{V}$; hence $\dim(\overline{V_1}^{\perp}) = 1$. It follows that $\dim(\operatorname{Rad} V_1/\operatorname{Rad} V) \leqslant 1$. If $\operatorname{Rad} V_1 = \operatorname{Rad} V$, then $\bar{V}$ is a plane. It is easy to check that, if $\bar{V}$ is anisotropic or hyperbolic with $q > 3$, then $L = V$ and we fall under case (1). Otherwise $q = 3$ and we fall under case (2). So we may assume that $\dim(\operatorname{Rad} V_1/\operatorname{Rad} V) = 1$. In this case $\dim(\bar{V}) = 3$, and therefore $\operatorname{Rad} V_1/\operatorname{Rad} V$ is a maximal totally singular subspace of $\bar{V}$. Thus, each vector in $V \setminus \operatorname{Rad} V_1$ is anisotropic. Without loss of generality, we may assume that $\operatorname{Rad} V = 0$. Let $r \in \operatorname{Rad} V_1$, $v_1 \in V_1 \setminus \operatorname{Rad} V_1$, $x \in V \setminus V_1$. Then $\langle r, v_1, x\rangle = \langle r + x, v_1 + x, x\rangle = V$. Since $r + x, v_1 + x, x \notin V_1$, it follows that $L = V$ and we are back to case (1). So, suppose that (ii) holds. Then, by Lemma 5.1, $V_1/R_0(V_1)$ is generated by its singular vectors. It follows that N contains an element y' such that $Q(y') = 0$. By the above, this would imply $q = 2$, a contradiction.

To sum up, at this stage we may assume that $q = 2$. Furthermore, we know that, for $j \in J$, $y \in N$, $f_Q(j, y) \neq 0$ if $Q(y) = 0$, and $f_Q(j, y) = 0$ if $Q(y) \neq 0$. Suppose first that V contains a totally singular subspace V_2 of codimension 1. Then, by our assumptions, $\operatorname{Rad} V$ is (properly) contained in V_2. In particular, $R_0(V) = \operatorname{Rad} V$ and therefore $V/\operatorname{Rad} V$ is a non-degenerate orthogonal space containing a totally singular subspace of codimension 1. It follows that $V/\operatorname{Rad} V$ is a plane and we fall under cases (1) or (2). Thus, from now on, we assume that

(∗) V does not contain any totally singular subspace of codimension 1.

In particular, $Q(V_1) \neq 0$ and $V_1 \nsubseteq \operatorname{Rad} V$ (the latter is clear, since $\operatorname{Rad} V$ has codimension at least 2 in V). Suppose first that $V_1 = \operatorname{Rad} V_1$. Then $\operatorname{Rad} V \nsubseteq \operatorname{Rad} V_1$ would imply $\operatorname{Rad} V + \operatorname{Rad} V_1 = V$, whence $V_1 = \operatorname{Rad} V_1 \subset \operatorname{Rad} V$, a contradiction. It follows that $\operatorname{Rad} V \subset \operatorname{Rad} V_1$, and hence, as seen above, $\dim(\operatorname{Rad} V_1/\operatorname{Rad} V) = 1$. This means that $V/\operatorname{Rad} V$ has dimension 2, and we fall into case (3). So we may assume that $\dim(V_1/\operatorname{Rad} V_1)$ is greater than 1 and therefore, by induction, that the statement of our lemma is true for V_1 (with V_1 replaced by a subspace L_2 of codimension 1 in V_1 and containing L_1). So, we proceed to evaluate all options (1)–(4) case-by-case.

Case (1a). Here V_1 is spanned by the non-singular elements belonging to N. Hence $L \subseteq V_1^{\perp}$. If $\operatorname{Rad} V \subseteq V_1$, then $L \subseteq V_1^{\perp} \subseteq V_1 + \operatorname{Rad} V = V_1$, which is a contradiction. Therefore $\operatorname{Rad} V \nsubseteq V_1$, and hence $\operatorname{Rad} V + V_1 = V$. It follows that $f_Q(V, L) = f_Q(V_1, L) = 0$, whence $L \subseteq \operatorname{Rad} V$. Now, write $\operatorname{Rad} V = R_0(V) \oplus V_0$. Clearly $L \subseteq \operatorname{Rad} V$ forces $V_0 \neq 0$. Suppose first that $R_0(V) \nsubseteq V_1$. Then, for any $r \in R_0(V) \setminus V_1$ and for any $y \in N$ with $Q(y) \neq 0$, we have $r + y \notin V_1$. As $Q(r + y) = Q(y) \neq 0$, we get $r + y \in J \subseteq L \subseteq \operatorname{Rad} V$, and hence $y \in \operatorname{Rad} V$. As the y's span V_1, we get $V_1 \subseteq \operatorname{Rad} V$, contrary to our assumptions. Thus $R_0(V) \subseteq V_1$, and hence $R_0(V) = V_1 \cap \operatorname{Rad} V = R_0(V_1)$. Moreover, as $\operatorname{Rad} V_1 \subseteq \operatorname{Rad} V$, $R_0(V_1) = \operatorname{Rad} V_1$. Now observe that $V_1 \cap V_0 = \{0\}$ and $V_0 \setminus \{0\} \subseteq J$. Let $v_1 \in V_1$ with $Q(v_1) = 0$, $0 \neq v_0 \in V_0$. Then $Q(v_1 + v_0) = Q(v_0) \neq 0$ and $v_1 + v_0 \notin V_1$. Thus $v_1 + v_0 \in L$, whence $v_1 \in L$. It follows that V_1 is not spanned by its singular vectors, and hence, by Lemma 5.1, $V_1/R_0(V_1)$ is an orthogonal (anisotropic) plane. As $V_1/R_0(V_1) = V_1/(V_1 \cap \operatorname{Rad} V) \simeq (V_1 + \operatorname{Rad} V)/\operatorname{Rad} V = V/\operatorname{Rad} V$, we fall into case (3).

Case (2a). Here $R_0(V_1) = \operatorname{Rad} V_1$ and $V_1/R_0(V_1)$ is a hyperbolic plane. Thus V_1 contains a totally singular subspace V_2 of codimension 1. In particular, V_2 has codimension 2 in V, and hence by $(*)$ is a maximal totally singular subspace of V. Assume first that $\operatorname{Rad} V \nsubseteq V_2$. Note that $\operatorname{Rad} V + V_2 \neq V$, otherwise $V_2 \subseteq \operatorname{Rad} V$, whence $\operatorname{Rad} V = V$, against our assumptions. Thus $\operatorname{Rad} V + V_2$ is a subspace of codimension 1 in V. It follows that $(\operatorname{Rad} V + V_2)/\operatorname{Rad} V$ is a totally singular subspace of codimension 1 in $V/\operatorname{Rad} V$. Hence $V/\operatorname{Rad} V$ is a plane. Now observe that $R_0(V) \neq \operatorname{Rad} V$ (otherwise $\operatorname{Rad} V + V_2$ would be a totally singular subspace of codimension 1 in V). In particular, $\operatorname{Rad} V \nsubseteq R_0(V_1) = \operatorname{Rad} V_1$, and hence $\operatorname{Rad} V \nsubseteq V_1$. It follows that $\operatorname{Rad} V + V_1 = V$, whence $R_0(V_1) \subseteq \operatorname{Rad} V$, which in turn implies that $R_0(V_1) = R_0(V) = V_2 \cap \operatorname{Rad} V$, by dimension reasons. We claim that these conditions force $L = V$. Clearly, we may assume $R_0(V_1) = R_0(V) = 0$. Set $\operatorname{Rad} V = \langle r \rangle$, $V_2 = \langle x \rangle$, $W = \langle r, x \rangle = \operatorname{Rad} V \oplus V_2$, a subspace of codimension 1 in V. Clearly $Q(r) = Q(r + x) = 1$ and neither r nor $r + x$ lie in V_1. Choose $v_1 \in V_1 \setminus V_2$. Then $r + v_1 \notin W$, and hence $f_Q(v_1, x) = 1$ (as $V_2^{\perp} = W$). Also, we may assume that $Q(v_1) = 0$. [For, suppose $Q(v_1) = 1$. Then $(v_1 + x) \in V_1 \setminus V_2$, $Q(v_1 + x) = Q(v_1) + f_Q(v_1, x) = 0$ and we replace v_1 with $v_1 + x$.] Thus $Q(r + v_1) = Q(r) + Q(v_1) = 1$. As $r + v_1 \notin V_1$, we conclude that $L = \langle r, r + x, r + v_1 \rangle = V$. Next, suppose that $\operatorname{Rad} V = V_2$. Then $V/\operatorname{Rad} V$ is an anisotropic plane and it is easily seen that this leads again to $L = V$, a contradiction. (Indeed, let $V = \langle \operatorname{Rad} V, b, c \rangle$, $V_1 = \langle \operatorname{Rad} V, b \rangle$, where $Q(b) = Q(c) = Q(b + c) \neq 0$. Then $c, b + c \in J$ forces $b \in L$, whence $L = V$.) So, we are reduced to the instance $\operatorname{Rad} V \subsetneq V_2$. In this case $R_0(V) = \operatorname{Rad} V$ and $V/\operatorname{Rad} V$ has dimension 4, since it is non-degenerate and contains a totally singular subspace of codimension 2. W.l.o.g. we may assume $\operatorname{Rad} V = 0$, so that V is a hyperbolic 4-dimensional space. Set $V_2 = \langle a, c \rangle \subset V_1$, $V = \langle a, b \rangle \perp \langle c, d \rangle$, where (a, b) and (c, d) are hyperbolic pairs. It is then easy to compute that, once again, $L = V$.

Case (3a). Here $Q(\operatorname{Rad} V_1) \neq 0$ and $V_1/\operatorname{Rad} V_1$ is a plane. If $\operatorname{Rad} V \nsubseteq V_1$, then $V = V_1 + \operatorname{Rad} V$, and hence $\operatorname{Rad} V_1$ is properly contained in $\operatorname{Rad} V$. It follows that $V/\operatorname{Rad} V$ is a plane. Since $R_0(V) \neq \operatorname{Rad} V$, we fall into case (3). If $\operatorname{Rad} V \subseteq V_1$, then clearly $V/\operatorname{Rad} V$ has dimension $\geqslant 4$. Set $\bar{V} = V/\operatorname{Rad} V$, $\overline{V_1} = V_1/\operatorname{Rad} V$, $\overline{R_0(V_1)} = (R_0(V_1) + \operatorname{Rad} V)/\operatorname{Rad} V$. Clearly $\overline{V_1}$ has codimension 1 in $\bar{V}$. As $\bar{V}$ is non-degenerate, $\dim(\overline{V_1}^{\perp}) \leqslant 1$. Hence, $\dim(\operatorname{Rad} \overline{V_1}) \leqslant 1$. As $q = 2$, $\dim(\operatorname{Rad} \overline{V_1})$ is exactly 1 (otherwise $\bar{V}$ would have odd dimension). As $\overline{R_0(V_1)} \subseteq \operatorname{Rad} \overline{V_1}$ and $|V : (R_0(V_1) + \operatorname{Rad} V)| \leqslant 4$, we conclude that $V/\operatorname{Rad} V$ has dimension 4. Thus V falls into case (4).

Case (4a). Let $V_1 = \operatorname{Rad} V_1 \oplus W$, so that W is non-degenerate and $\dim(W) = 4$. Observe that W is spanned by its singular vectors as well as by its non-singular vectors. Moreover, W has 9 or 5 non-zero singular vectors and 6 or 10 non-singular vectors, depending on the Witt index of W. As W is non-degenerate, $V = W \oplus W^{\perp}$, $\operatorname{Rad} V_1 \subset W^{\perp}$ and $W^{\perp} \nsubseteq V_1$. Hence $W^{\perp} \cap V_1$ has codimension 1 in $W^{\perp}$. Assume first that there is $l \in W^{\perp}$ such that $l \notin V_1$ and $Q(l) = 1$. (By Lemma 5.2, this is always possible unless: (**) $\operatorname{Rad} W^{\perp} = R_0(W^{\perp})$, $\dim(W^{\perp}/\operatorname{Rad} W^{\perp}) = 2$ and $W^{\perp}/\operatorname{Rad} W^{\perp}$ has Witt index 1.) Let us consider the vectors $l + w$ such that $w \in W$ and $Q(w) = 0$. Then $Q(l + w) = Q(l) = 1$, and hence $l + w \in L$. As W is spanned by its singular vectors, L contains $\langle l, W\rangle$. Suppose that V is non-degenerate. As $\operatorname{Rad} V = 0$, $\dim(\operatorname{Rad} V_1) = 1$, hence $\dim(W^{\perp}) = 2$. Set $\operatorname{Rad} V_1 = \langle r\rangle$. Then $W^{\perp} = \langle l, r\rangle$ is non-degenerate and hence $f_Q(l, r) = 1$. It follows that $Q(l + r) = Q(l) + f_Q(l, r) + Q(r) = Q(r)$. If $Q(r) = 1$, then $l + r \in L$; so $\operatorname{Rad} V_1 \subset L$ and $L = V$. Hence we assume $Q(r) = 0$. Then $Q(l + w + r) = 1 + Q(w) + f_Q(l, r) = Q(w)$. If $Q(w) = 1$, then $l + w + r \in L$. As $L \supseteq \langle l, W\rangle$, again we conclude that $L = V$.

Now assume that $\operatorname{Rad} V \neq 0$. By factoring out $R_0(V)$, we can assume with no loss of generality that $\dim(\operatorname{Rad} V) = 1$ and $Q(x) = 1$ for $0 \neq x \in \operatorname{Rad} V$. Now $Q(l + w + x) = Q(w)$. It follows that $Q(l + w + x) = 1$ provided we pick w such that $Q(w) = 1$. If $x \in V_1$, then $l + w + x \in L$, whence $L = V$, as $L \supseteq \langle l, W\rangle$. Suppose $x \notin V_1$. If $\operatorname{Rad} V_1 = 0$, then $V = W \oplus \langle l\rangle = L$. Otherwise, pick $0 \neq y \in \operatorname{Rad} V_1$. Then $y + w + x \notin V_1$ and $Q(y + w + x) = Q(y) + Q(w) + 1$. If we choose $w \in W$ such that $Q(y) + Q(w) = 0$, then $y + w + x \in L$, whence $y + w \in L$, $V_1 \subseteq L$, and hence $L = V$.

So, we are left to consider case (**): here we have $Q(l) = 0$ for all $l \in W^{\perp} \setminus (W^{\perp} \cap V_1)$. Assume first that $\operatorname{Rad} V = 0$. Then $\dim(V) = 6$ forces $\dim(\operatorname{Rad} V_1) = 1$. Also, $\dim(W^{\perp}) = 2$ and $W^{\perp}$ is not anisotropic. Moreover, $Q(l) = 0$ for all $l \in W^{\perp} \setminus (W^{\perp} \cap V_1)$ implies $Q(\operatorname{Rad} V_1) \neq 0$. For, let $\operatorname{Rad} V_1 = \langle r\rangle$. Then $r^{\perp} = V_1$. Hence $Q(r + l) = Q(r) + Q(l)$. If $Q(r) = 0$, then $Q(r + l) = 1$, with $r + l \in W^{\perp} \setminus (W^{\perp} \cap V_1)$: a contradiction. Let us consider the vectors $l + w$, where w runs over the non-singular vectors of W. As $Q(l + w) = Q(w) = 1$, all such vectors belong to L. Let $W_L = L \cap W$. As the 4-dimensional space W is spanned by its non-singular vectors, and $(l + w) - (l + w') = w - w' \in W_L$ whenever w and w' are non-singular, it follows that $\dim(W_L) \geqslant 3$. From this it also follows that W_L contains a non-singular vector w''. Then $l + w'' \in L$ forces $l \in L$ and hence $w \in L$ for all non-singular vectors $w \in W$. It follows that $\langle l, W\rangle \subseteq L$. Now, observe that $Q(r) = 1 = f_Q(l, r)$, as $W^{\perp}$ is non-degenerate of dimension 2. Hence $Q(l + w + r) = Q(l + w) = Q(w)$. Picking w non-singular, $l + w \in L$ forces $r \in L$, whence $L = V$.

Finally, assume that $\operatorname{Rad} V \neq 0$. As $\operatorname{Rad} V = R_0(V) = \operatorname{Rad} W^{\perp}$, we may write $V = R_0(V) \oplus U$, where U is non-degenerate and contains W. As $\dim(W^{\perp}/\operatorname{Rad} W^{\perp}) = 2$, it follows that $\dim(U) = 6$. Suppose first that $R_0(V) \subseteq V_1$ and let $U_1 = U \cap V_1$. If $U \subseteq V_1$, $R_0(V) \subseteq V_1$

forces $V \subseteq V_1$, a contradiction. So, U_1 has codimension 1 in U. Let $J_U = \{l \in (U \setminus U_1) \mid Q(l) = 1\}$, $L_U = \langle J_U \rangle$. Since $\dim(U) > 4$, it follows by induction that $L_U = U \subseteq L$. As $Q(l+r) = 1$ and $l + r \notin V_1$ for any $l \in J_U$, $r \in R_0(V)$, it follows that $L = V$. Now, we are left with the case where $R_0(V) \nsubseteq V_1$. Set $\widehat{R}_0 = R_0(V) \setminus (R_0(V) \cap V_1)$ and consider $L_1 = L \cap V_1$. Let M be the set of all the non-singular vectors in V_1. As $\dim(V_1/\operatorname{Rad} V_1) = 4$, by Lemma 5.2 V_1 is spanned by M. Let $v_1 \in M$, $r \in \widehat{R}_0$: then $Q(v_1 + r) = 1$, hence $v_1 + r \in L$. It follows that, for $v_1, v_2 \in M$, $(v_1 + r) + (v_2 + r) = v_1 + v_2 \in L_1$. In particular, if $(v_1, v_2, \ldots, v_h)$ is a basis of V_1 contained in M, then the vectors $v_1 - v_j$ $(2 \leqslant j \leqslant h)$ are independent, and hence $\dim(V_1/L_1) \leqslant 1$. As V_1 contains the non-degenerate subspace W, $Q(L_1) \neq 0$. Now, pick $l_1 \in L_1$ with $Q(l_1) = 1$. As both l_1 and $l_1 + r$ belong to L, r also belongs to L. Thus $v_1 \in L$, for all $v_1 \in M$. We conclude that $V = \langle V_1, \widehat{R}_0 \rangle = L$. □

Remark. Observe that case (2) is afforded by the examples given in the remark preceding Lemma 5.3. Case (3) is afforded by the following example. Let $q = 2$ and define V of dimension 3 via a basis (b, c, r) such that $\langle r \rangle = \operatorname{Rad} V$, $Q(b) = Q(c) = Q(r) = 1$ and $f_Q(b, c) = 1$. Set $V_1 = \langle b, c \rangle$. Then V_1 is anisotropic and $L = \langle r \rangle$. Additionally, if one chooses $V_1 = \langle c, r \rangle$, then $L = \langle b, b + c \rangle$. Case (4) also actually arises. To see this, define V to be the orthogonal sum of two anisotropic planes over $\mathbb{F}_2$, say $V = P_1 \perp P_2$, and pick $V_1 = \langle P_2, d \rangle$, where $0 \neq d \in P_1$. Let (p, d) be a basis for P_1. Then it is easy to see that $J = \{p, p + d\}$. Hence $L = \langle J \rangle$ has dimension 2. Moreover, $0 = R_0(V_1) \neq \operatorname{Rad} V_1 = \langle d \rangle$.

Lemma 5.4. *Let x be a unipotent element of $GL(m, q)$ of order p^α. Suppose that* $\operatorname{Jord} x = J_m$ *and* $x^{p^\beta} = y \neq \mathrm{Id}$, *for some* $\beta > 0$. *Then* $\operatorname{Jord} y$ *contains at least two non-trivial blocks of equal size, unless*:

(i) *p is odd, $m = p^{\alpha-1} + 1$ and $y = x^{p^{\alpha-1}}$ is a transvection*;
(ii) *$p = 2$ and m is odd.*

In case (ii) $\operatorname{Jord} x^2 = \operatorname{diag}\{J_{h+1}, J_h\}$; *furthermore, any other non-identity* 2*-power of x has at least two non-trivial blocks of equal size, unless $h = 2^\gamma$ for some γ and $\beta = \gamma + 1$, in which case y is a transvection.*

Proof. Recall that if a unipotent Jordan block J_m of size m has order p^α, then $p^{\alpha-1} < m \leqslant p^\alpha$. An easy computation shows that the blocks of $\operatorname{Jord} x^p$ have sizes $\lceil \frac{m}{p} \rceil$, $\lceil \frac{m-1}{p} \rceil$, $\lceil \frac{m-2}{p} \rceil, \ldots$ (where $\lceil x \rceil$ denotes the least integer not less than x). Thus, if $m = ph$, then x^p has p blocks of size h. If $m = ph + r$ $(0 < r < p)$, then x^p has r blocks of maximal size $h + 1$ and $p - r$ blocks of size h. It follows that every non-identity p-power of x has at least two non-trivial blocks of equal size, provided $m \neq 1 \bmod p$. Suppose that $r = 1$, so that x^p has $p - 1$ blocks of size h. Observe that two blocks of sizes $h + 1$ and h, respectively, have the same order, unless $h = p^\gamma$ for some γ, in which case $|J_h| = p^\gamma$, whereas $|J_{h+1}| = p^{\gamma+1}$. Thus, if $p > 2$ and $h \neq p^\gamma$, we are done. If $p > 2$ and $h = p^\gamma$, then $\gamma = \alpha - 1$ and (i) holds. If $p = 2$ and $h \neq 2^\gamma$, then either h or $h + 1$ is even, and therefore every non-identity 2-power of the corresponding block (hence of x^2) has at least two non-trivial blocks of equal size. Next, suppose that $p = 2$ and $h = 2^\gamma$. Then each power $(x^2)^{2s}$, $s < \gamma$, has at least two non-trivial blocks of equal size, whereas $\operatorname{Jord}(x^2)^{2^\gamma} = \operatorname{diag}\{(J_{2^\gamma+1})^{2^\gamma}, \mathrm{Id}_{2^\gamma}\} = \operatorname{diag}\{J_2, \mathrm{Id}_{2^{\gamma+1}-1}\}$, hence $(x^2)^{2^\gamma}$ is a transvection. □

Lemma 5.5. *Let V be either a symplectic space or a (non-degenerate) orthogonal space of dimension $m \geqslant 5$ over $\mathbb{F}_q$. Let $\chi : (V, +) \to P$ be a non-trivial character of $(V, +)$, $\varepsilon \in P$ be a non-trivial p-root of 1, and $0 \neq u \in V$. Then for each $i \in \{0, \dots, p-1\}$ there exists $x_i \in I(V)'$ such that $\chi(x_i(u)) = \varepsilon^i$. Additionally, the same holds if $\dim(V) = 4$ and V is either symplectic or orthogonal with $q > p = 2$.*

Proof. Let $0 \neq v_i \in V$ be a vector such that $\chi(v_i) = \varepsilon^i$. If V is symplectic, then obviously u and v_i lie in the same $I(V)'$-orbit and the result follows. So, assume that V is orthogonal with defining quadratic form Q. By Lemma 2.3 in [L-S] (cf. Lemma 4.9), V contains a unique hyperplane V_1 such that $\chi(V_1) = 1$. Suppose first that $i = 0$ and $v_0 \in V_1$. As V is non-degenerate, $V_1 = \operatorname{Rad} V_1 \oplus W$, where $\dim(\operatorname{Rad} V_1) \leqslant 1$ and W is non-degenerate. As $\dim(W) \geqslant 3$, W contains a non-zero vector w such that $Q(w) = Q(u)$ and $\chi(w) = 1$. By Lemma 4.2, u and w lie in the same $I(V)'$-orbit, and we conclude that the statement of the lemma is true for $i = 0$. Next, suppose that $i \neq 0$ and p is odd. Assume first that V_1 is degenerate, and let $0 \neq r \in \operatorname{Rad} V_1$. Then $v_i \notin r^{\perp}$, and hence, without loss we can assume that $f_Q(v_i, r) = 1$. As $Q(v_i + ar) = Q(v_i) + 2a$ for any $a \in \mathbb{F}_q$, we can choose a such that $Q(v_i + ar) = Q(u)$. As $\chi(v_i + ar) = \chi(v_i)$, we are done, again by Lemma 4.2. Now, suppose that V_1 is non-degenerate. Set $L = V_1 \cap v_i^{\perp}$ and let $w \in L$. Then $\chi(v_i + w) = \chi(v_i) = \varepsilon^i$ and $Q(v_i + w) = Q(v_i) + Q(w)$. We need to show that w can be chosen in such a way that $Q(u) = Q(v_i) + Q(w)$ and $v_i + w \neq 0$. To this purpose, we have to check that $Q|_L$ is surjective. If $L = V_1$, we are done provided $m > 2$. Otherwise, $\dim(L) = m - 2$ forces $\dim(\operatorname{Rad} L) \leqslant 2$, and hence $\dim(L/\operatorname{Rad} L) \geqslant 2$ provided $m \geqslant 6$. It follows that the quadratic form induced on $L/\operatorname{Rad} L$ by Q is surjective on $\mathbb{F}_q$, and we are done. On the other hand, if $m = 5$ then $\dim(L) = 3$. Hence, $L = \operatorname{Rad} L \oplus X$, where X is a 2-dimensional non-degenerate space. This implies that $Q|_L$ is surjective also when $m = 5$.

We are left with the case when $p = 2$. Clearly, if $Q(\operatorname{Rad} L) \neq 0$ we are done. So, we may assume that $\operatorname{Rad} L = R_0(L)$. This implies that $L/\operatorname{Rad} L$ inherits an orthogonal structure from L. Observe that $\dim(L) \geqslant m - 2$ forces $\dim(\operatorname{Rad} L) \leqslant 2$. Thus, the non-degenerate space $L/\operatorname{Rad} L$ has dimension at least $m - 4$. It follows that the quadratic form induced on $L/\operatorname{Rad} L$ by Q is surjective on $\mathbb{F}_q$ provided $m \geqslant 6$, and we are done.

Finally, suppose that $m = 4$ (and $q > p = 2$). Then $V_1 = Y \oplus \operatorname{Rad} V_1$, where $\dim(\operatorname{Rad} V_1) = 1$ and Y is a non-degenerate subspace of dimension 2. As V_1 contains a non-zero vector of any norm, the case $i = 0$ is done by Lemma 4.2. Let $i \neq 0$. Then $V = \langle V_1, v_i \rangle$ and $V_1 \neq v_i^{\perp}$ (as v_i is isotropic). Let $\langle r \rangle = \operatorname{Rad} V_1$. As above, we may assume that $f_Q(v_i, r) = 1$. If $Q(\operatorname{Rad} V_1) = 0$, $Q(v_i + ar) = Q(v_i) + a = Q(u)$ for some $a \in \mathbb{F}_q$, and we are done. So suppose that $Q(\operatorname{Rad} V_1) \neq 0$. Then, as $f_Q(v_i, r) \neq 0$, $r \notin L = V_1 \cap v_i^{\perp}$. Hence $V_1 = \operatorname{Rad} V_1 \oplus L$. If L is not anisotropic, then we are done. If L is anisotropic and $Q(u) = 0$, then we cannot find in L a non-zero vector with the same norm as u. However, if $q > 2$, then $\ker\chi \supsetneq V_1$. Consider $\ker\chi \cap v_i^{\perp}$. Then $\ker\chi \cap v_i^{\perp} \supsetneq L$ (indeed, $|V : L| = q^2$, while $|V : \ker\chi \cap v_i^{\perp}| = 2q$). As $v_i^{\perp}$ has dimension 3, it contains a non-zero vector x such that $Q(x) = 0$. As $\ker\chi$ has index 2 in the additive group of V, a suitable non-zero multiple of x lies in $\ker\chi$, by Lemma 4.9(2). (Indeed, if $x \notin \ker\chi$ and also $\mu x \notin \ker\chi$ $(\mu \neq 0, 1)$, then $x + \mu x = (1 + \mu)x \in \ker\chi$. Thus the vector $(1 + \mu)x$ will do.) □

At this stage, we are able to prove the following result. (Observe that the statement addresses to a central extension of H, in order to include in our treatment the spinor group in the orthogonal case.)

Theorem 5.6. *Let F be a finite field of characteristic p and order p^a. Suppose that either V is a non-degenerate orthogonal space over F, or $p = 2$ and V is a non-degenerate symplectic space. Moreover, assume $m = \dim(V) > 4$ ($m > 6$ if $q = 2$). Let $I(V)' \subseteq H \subseteq I(V)$, and let $\widetilde{H}$ be a central extension of H such that $(|Z(\widetilde{H})|, q) = 1$. Let g be an element of $\widetilde{H}$ of order $s = p^\alpha > 1$, and $\theta \in \mathrm{Irr}_P \widetilde{H}$ with $\dim\theta > 1$. Then $|\operatorname{Spec}\theta(g)| = s$. Furthermore, if $m > 2p^{\alpha-1} + 4$, then the multiplicity of every eigenvalue of $\theta(g)$ is at least $\max\{1, p^{a(m-6)-\alpha}\}$.*

Proof. We first observe that, for m and q even, every unipotent element of $Sp(V)$ is conjugate to an element of $O^+(V)$ or $O^-(V)$ (see [S-Se, Lemma 4.1]). Therefore, we may restrict ourselves to the case when V is orthogonal. Also, since $I(V)/I(V)'$ has exponent 2, we may assume that $g \in \widetilde{H}'$ if $p > 2$. If $p = 2$, then $g^2 \in \widetilde{H}'$, but in this case $\widetilde{H}$ splits to $Z(\widetilde{H}) \times H$.

For a subgroup X of H, we denote by $\widetilde{X}$ the preimage of X in $\widetilde{H}$. Observe that $\widetilde{H}$ acts on V via the homomorphism $\widetilde{H} \to H$. Let $W = \langle v\rangle$ and W_1 be defined as above. Set $G = \langle g\rangle$, $\widetilde{S}_1 = Stab_{\widetilde{H}}(v)$, $\widetilde{U} = O_p(\widetilde{S}_1)$. Then $\widetilde{U} \simeq U$ is an elementary abelian group of order q^{m-2} (cf. Lemma 4.5). For this reason, we shall write U for $\widetilde{U}$. Let K denote the group of characters of U, and let ϕ be an irreducible constituent of $\theta|_{\widetilde{S}_1}$ which is non-trivial on U (such a ϕ certainly exists, since H' is quasi-simple, hence $\ker\theta$ has order coprime to p). Let T be the $P\widetilde{S}_1$-module afforded by ϕ. Then $T|_U$ decomposes into homogeneous components T_κ, namely $T|_U = \bigoplus_{\kappa\in K} T_\kappa$, where $T_\kappa = \{x \in T:\ ux = \kappa(u)x\}$ and the summation runs over an $\widetilde{S}_1$-orbit O of non-trivial elements of K. Obviously, U lies in the kernel of this action, so in fact K is acted upon by $\widetilde{S}_1/U \simeq \widetilde{Y}$. Observe that U can be endowed in an obvious way with the structure of F-vector space, and viewed as an $F\widetilde{Y}$-module isomorphic to W_1. Since W_1 is self-dual, K is isomorphic to W_1 as $F\widetilde{Y}$-modules. This isomorphism turns K into a non-degenerate orthogonal space with quadratic form Q, say, and $\widetilde{S}_1/U$ preserves Q. It follows that O is permutationally isomorphic to an orbit of $\widetilde{Y}$ on W_1.

As above, let h denote the projection of g into Y. If $|g| = |h|$, then each h-orbit on O is also a g-orbit, and in particular the number of regular g-orbits coincides with the number of regular h-orbits. Therefore, we can use Lemma 4.3 to estimate the number of the regular g-orbits on O: this is at least $p^{an-\alpha}$, where $n = m - 6$. It follows that the underlying space of ϕ contains a direct sum of at least $p^{a(m-6)-\alpha}$ copies of the regular FG-module. If $m > 2p^{\alpha-1} + 4$, by Corollary 4.8 v can be chosen such that $|g| = |h|$, and the result follows.

By the above, we may now assume that $|g| > |h|$. We have to show that also in this case $|\operatorname{Spec}\theta(g)| = s$, that is, every $|g|$-root of 1 occurs as an eigenvalue of $\theta(g)$. Set $t = g^{p^{\alpha-1}}$: then $t \in U$. It follows that one of the exceptional cases listed in Lemma 4.7 holds (for, otherwise, we can switch to a conjugate $\tilde{g}$ of g such that $\tilde{g}^{p^{\alpha-1}} \notin U$). Furthermore, by Proposition 1.2, we may assume that $g \neq t$. Let $K_t = \{\kappa \in K:\ \kappa(t) = 1\}$. Then $|K : K_t| = p$ and $gK_t = K_t$. As noted above, we can use the additive notation for K, and view K as an FG-module dual to, hence isomorphic to W_1. Observe that the map χ_t sending κ to $\kappa(t)$ is a character of K, and $K_t = \ker\chi_t$. Thus, by Lemma 2.2 in [L-S], K_t contains a unique F-subspace K' of codimension 1 in K, which is therefore g-stable as well.

Let ε be a non-trivial p-root of 1. Set $K_i = \{\kappa \in K \mid T_\kappa \neq 0,\ \kappa(t) = \varepsilon^i\}$ and $T^{(i)} = \bigoplus_{\kappa\in K_i} T_\kappa$ for $i = 0, \dots, p-1$ (thus $T^{(i)}$ is the t-eigenspace for the eigenvalue ε^i). Due to our assumptions on $\dim(V)$, Lemma 5.5 applied to $V = U \simeq W_1$, $u = t$ and $I(V) = I(W_1) \simeq S_1/U$ ensures that K_i is non-empty for each i. Obviously $gT^{(i)} = T^{(i)}$, hence $gK_i = K_i$ for each i. We claim that, for each i, there exists $\kappa_i \in K_i$ such that $|G\kappa_i| = p^{\alpha-1}$. For this, it suffices to show that

$g_1 = g^{p^{\alpha-2}}$ does not act trivially on K_i for each i. Suppose the contrary: then there is some i such that g_1 acts trivially on the subspace $\langle K_i \rangle$. We claim that this leads to a contradiction.

First, we observe that $(*)$: $g_1|_K \neq \mathrm{Id}$, otherwise $g_1|_{W_1} = \mathrm{Id}$, and the latter implies that $g_1 = g^{p^{\alpha-2}} \in U$, which is not the case as U has exponent 2.

Next, we show that $(**)$: if $(m, p) \neq (6, 2)$, then g_1 does not act trivially on any subspace X of codimension 1 in K.

Indeed, assume $(**)$ is false. Then $\dim(g_1 - \mathrm{Id})K = 1$. This implies $\dim(g_1 - \mathrm{Id})W_1 = 1$, whence $\dim(g_1 - \mathrm{Id})V \leqslant 3$. As $|g_1| = p^2$, $\mathrm{Jord}\, g_1$ has a block of size at least $p + 1$; hence $\dim(g_1 - \mathrm{Id})V \geqslant p$. It follows that $p \leqslant 3$. Assume first that $p = 3$. Then, in view of the limitation on $\dim(g_1 - \mathrm{Id})V$, $\mathrm{Jord}\, g_1 = \mathrm{diag}\{J_4, \mathrm{Id}_{m-4}\}$. However, this contradicts Lemma 2.4(b). So $p = 2$. In view of the above, $\mathrm{Jord}\, g_1$ has blocks of size at least 3, but cannot have blocks of size greater than 4. By Lemma 2.4(a), $\mathrm{Jord}\, g_1$ cannot contain a single block of size 3. It follows that $\mathrm{Jord}\, g_1$ has no blocks of size 3, whence $\mathrm{Jord}\, g_1 = \mathrm{diag}\{J_4, \mathrm{Id}_{m-4}\}$. In this case, $g = g_1$ by Lemma 5.4. This implies (case (x) of Lemma 4.7) that $m = 6$, against our current assumptions.

Set $M = K' \cap \kappa_i^{\perp}$ and assume first that $m > 6$. We distinguish two cases.

Case (1). Suppose first that $M = K'$. In this case, denote by J the set of all singular vectors of M. Clearly, $J \neq 0$, as $\dim(M) = m - 3$ and $m > 6$. Since $Q(\kappa_i + \beta) = Q(\kappa_i)$ for every $\beta \in J$, by Lemma 4.2 the orbit $Y\kappa_i$ contains all vectors $\kappa_i + \beta$, where $\beta \in J$: hence $T_{\kappa_i+\beta} \neq 0$ and $\kappa_i + \beta \in K_i$. Then $g_1(\kappa_i + \beta) = \kappa_i + \beta$ implies $g_1(\beta) = \beta$ for each $\beta \in J$. Since $|K : M| = 1$ and K is a non-degenerate space of dimension at least 5 (at least 6 if $p = 2$), $\dim(\mathrm{Rad}\, M) \leqslant 1$. Hence $\dim(M/R_0(M))$ is at least 3 if p is odd, and at least 4 if $p = 2$. In both cases, by Lemma 5.1 M is generated by J. It follows that $g_1|_{K'} = \mathrm{Id}$, which contradicts $(**)$.

Case (2). Here $M = K' \cap \kappa_i^{\perp}$ has codimension 1 in K'. Observe first that $Q(K') \neq 0$, as $|K : K'| = 1$ and K is a non-degenerate space of dimension at least 5. Also, $\dim(\mathrm{Rad}\, K') \leqslant 1$ forces $\dim(K'/\mathrm{Rad}\, K') \geqslant 3$. Let us denote by J the set of all vectors $\beta \in (K' \setminus M)$ such that $Q(\beta) \neq 0$. Moreover, if $q = 2$, let us make the additional assumption that $\dim(K'/\mathrm{Rad}\, K') > 4$. Then K' satisfies the assumptions of Lemma 5.3 but does not fulfill the conditions stated in cases (2)–(4) of the same lemma. Hence J spans K'. Since $\beta \notin \kappa_i^{\perp}$ for each $\beta \in J$, setting $\nu = -f_Q(\kappa_i, \beta)/Q(\beta)$ we obtain $Q(\kappa_i + \nu\beta) = Q(\kappa_i) + \nu f_Q(\kappa_i, \beta) + \nu^2 Q(\beta) = Q(\kappa_i)$. Replacing every β with $\nu\beta$, where ν is chosen as above, we conclude that K' is spanned by the set J_1 of all $\beta \in J$ such that $Q(\kappa_i + \beta) = Q(\kappa_i)$. As in case (1), we only need to show that $\kappa_i + J_1$ contains a vector x with $|Gx| = p^{\alpha-1}$. If this is not so, $g_1|_{K'} = \mathrm{Id}$, which contradicts $(**)$. Finally, it remains to consider the case when $q = 2$ and $\dim(K'/\mathrm{Rad}\, K') = 4$. In this instance, $\dim(\mathrm{Rad}\, K') \leqslant 1$ implies that $\dim(K) \leqslant 6$, and hence $m \leqslant 8$. Therefore, we are left with the groups $O^{\pm}(8, 2)$, which can be handled scrutinizing the Atlas [Atl], together with the Modular Atlas [MAtl]. In conclusion, we have proved that, for each i, there exists $\kappa_i \in K_i$ such that $|G\kappa_i| = p^{\alpha-1}$. Considering the G-submodule $\bigoplus_{\kappa \in Y\kappa_i} T_\kappa$ of $T^{(i)}$, it now follows from [DM-Z, Lemma 2.14], that $\mathrm{Spec}\, \phi(g)$ contains all $p^{\alpha-1}$-roots of ε^i. As i is arbitrary, we deduce that $|\mathrm{Spec}\, \theta(g)| = s$.

Finally, we deal with the case where $m = 6$, $q > p = 2$. Keeping the notation introduced above, we distinguish two cases:

Case (1a). $M = K'$. Then $K' \subseteq k_i^{\perp}$, hence $K' = k_i^{\perp}$ as both K' and $k_i^{\perp}$ are of codimension 1 in K. As $p = 2$, $k_i \in k_i^{\perp} = K'$, which implies $i = 0$. If $k_0^{\perp}$ is spanned by its singular vectors, the argument developed for case (1) still works. So, let us suppose that $k_0^\perp$ is not spanned by its singular vectors. Then, by Lemma 5.1, $k_0^\perp/R_0(k_0^\perp)$ is anisotropic. In particular, as $\dim(K')=3$, $\mathrm{Rad}(k_0^\perp)=R_0(k_0^\perp)$, whence $Q(k_0)=0$ (as $k_0\in\mathrm{Rad}(k_0^\perp)$). Also, for any $\lambda\in F$ $\chi(\lambda k_0)=1$, as $\lambda k_0\in k_0^\perp=K'\subset K_t$. By Lemma 5.3 (with $V_1=k_0^\perp$), we know that K is spanned by the non-singular vectors lying in $K\setminus k_0^\perp$. It follows, by Lemma 4.9(2), that K is also spanned by the set J_t of the non-singular vectors lying in $K_t\setminus k_0^\perp$. For any $x\in J_t$, $\lambda\in F$, one has $\chi_t(\lambda k_0+x)=\chi_t(\lambda k_0)\chi_t(x)=1$ (as $\lambda k_0\in K_t$) and $Q(\lambda k_0+x)=\lambda f_Q(k_0,x)+Q(x)$. Thus, we can choose $\lambda=Q(x)/f_Q(k_0,x)$ to obtain $Q(\lambda k_0+x)=0$. It follows that the vectors k_0 and λk_0+x (with $x\in J_t$ and λ such that $Q(\lambda k_0+x)=0$) span the whole of K. All these vectors lie in K_0. So g_1 acts non-trivially on K_0, as otherwise $g_1|_K=\mathrm{Id}$, which is false.

Case (2a). Here $M=K'\cap k_i^\perp\neq K'$, so $\dim(M)=2$. Let J' denote the set of non-singular vectors in $K'\setminus M$. As $\dim(K')=3$, $K'=\langle J'\rangle$ by Lemma 5.3. For $x\in J'$, set $\lambda=f_Q(k_i,x)/Q(x)$. Then $Q(k_i+\lambda x)=Q(k_i)+\lambda(f_Q(k_i,x)+\lambda Q(x))=Q(k_i)$ and $\chi_t(k_i+\lambda x)=\chi_t(k_i)$, as $\lambda x\in K'\subseteq K_t$. Therefore $k_i+\lambda x\in K_i$. It follows that $\langle K_i\rangle$ contains k_i and $\langle J'\rangle=K'$, hence $\langle K_i\rangle\supseteq\langle k_i\rangle+K'$. Assume first $i\neq 0$. Then $k_i\notin K'$, so $K=\langle k_i\rangle+K'$, and hence $\langle K_i\rangle=K$, contradicting (∗). Now let $i=0$. The same contradiction holds if $k_0\notin K'$. Hence we may assume that $k_0\in K'$. It follows $K'\subseteq\langle K_0\rangle\neq K$, whence $K'\subseteq\langle K_0\rangle$. In addition, $\lambda k_0\in K'$ for every $\lambda\in F$. If $Q(k_0)\neq 0$, $k_0^\perp$ is spanned by its singular vectors, hence there is $x\in k_0^\perp\setminus M$ such that $Q(x)=0$. By Lemma 4.9(2), a multiple of x lies in K_t, hence we can assume $x\in K_t$.

Thus $Q(k_0+x)=Q(k_0)$, and moreover, as $k_0\in K'\subseteq K_t$ and $x\in K_t$, $\chi(k_0+x)=1$. So $k_0+x\in K_0$, whence $x\in K_0$ and $K=\langle x,K'\rangle\subseteq K_0$.

So now we assume that $Q(\kappa_0)=0$. Observe that $q^4=|K|\neq|K'\cup\kappa_0^\perp|\leqslant 2q^3$. Pick $x\notin(K\setminus(K'\cup\kappa_0^\perp))$. Then every non-zero scalar multiple of x is not in $K\setminus(K'\cup\kappa_0^\perp)$. By Lemma 4.9(2), we can pick $x\in K_t$ such that $x\notin(K'\cup\kappa_0^\perp)$. Then $\chi(\lambda\kappa_0+x)=1$ for any $\lambda\in F$. In addition, $Q(\lambda\kappa_0+x)=\lambda f_Q(\kappa_0,x)+Q(x)$, so $Q(\lambda\kappa_0+x)=0=Q(\kappa_0)$ for a suitable $\lambda\in F$. Therefore, $\lambda\kappa_0+x\in K_0$ for such λ, whence $x\in\langle K_0\rangle=K'$. This is a contradiction. □

The previous theorem leaves us to examine the groups $H=Sp(4,q)$ and $H=Sp(6,2)$, whenever one of the exceptional cases listed in Lemma 4.7 applies to the unipotent element g.

The group $Sp(6,2)$ provides a true exception, as shown by the following lemma (where the notation of [Atl] for conjugacy classes is used).

Lemma 5.7. *Let $H=Sp(6,2)$ and let g be a 2-element of H. Then g has $|g|$ distinct eigenvalues in every non-trivial irreducible representation θ of H, unless $\dim\theta=7$. In the latter instance, one of the following holds*:

(i) $|g|=4$, $g\in(4A)$, $\mathrm{Jord}\,g=\mathrm{diag}\{J_4,J_1,J_1\}$, $\deg\theta(g)=3$ *and* $\mathrm{Spec}\,\theta(g)=\{\pm\sqrt{-1},1\}$;
(ii) $|g|=8$, $g\in(8A)$, $\mathrm{Jord}\,g=J_6$, $\deg\theta(g)=6$ *and* $\pm\sqrt{-1}\notin\mathrm{Spec}\,\theta(g)$;
(iii) $|g|=8$, $g\in(8B)$, $\mathrm{Jord}\,g=J_6$, $\deg\theta(g)=7$ *and* $-1\notin\mathrm{Spec}\,\theta(g)$.

Proof. We may either inspect the Brauer characters in [MAtl], or make use of the package GAP (see [GAP]). The details for θ of dimension 7 are as follows. If $g\in 4A$, then $g^2\in 2B$ and $\chi(g)=3$, $\chi(g^2)=-1$. It follows that $\mathrm{Jord}(\theta(g))=\mathrm{diag}(\sqrt{-1},\sqrt{-1},-\sqrt{-1},-\sqrt{-1},1,1,1)$. In the case of the other elements of order 4, $|\mathrm{Spec}\,\theta(g)|=4$. Let $|g|=8$. If $g\in 8A$, then $g^2 \in 4A$, $g^4 \in 2B$ and $\chi(g) = -1$, $\chi(g^2) = 3$, $\chi(g^2) = -1$. Hence $\mathrm{Jord}(\theta(g)) = \mathrm{diag}(\varepsilon, \varepsilon^3, \varepsilon^5, \varepsilon^7, -1, -1, 1)$ (here $\varepsilon^2 = \sqrt{-1}$). Similarly, if $g \in 8B$, then $\mathrm{Jord}(\theta(g)) = \mathrm{diag}(\varepsilon, \varepsilon^3, \varepsilon^5, \varepsilon^7, \sqrt{-1}, -\sqrt{-1}, 1)$. The other elements of order 8 have 8 distinct eigenvalues. □

The next lemma deals with the group $Sp(4,q)$ (in which case $|g| \leqslant 4$).

Lemma 5.8. *Let $H = Sp(4,q)$, where $q > 2$ is even. Let $g \in H$ with $|g| = 4$. Then $\theta(g)$ has 4 distinct eigenvalues in every non-trivial P-representation θ.*

Proof. Let us start assuming that $P = \mathbb{C}$. Observe that g is conjugate to g^{-1} in H (e.g., cf. [T-Z2, Theorem 1.8]), hence $\mathrm{Spec}\,\theta(g)$ contains $\pm\sqrt{-1}$. By [S-Se, Lemma 4.1], g is conjugate to an element of either $H_1 =: O^+(4,q)$ or $H_2 =: O^-(4,q)$. As $q > 2$, the group H_i' is perfect and of index 2 in H_i. It is also well known that $H_1' \cong SL(2,q) \times SL(2,q)$ and $H_2' \cong SL(2,q^2)$, e.g. see [D, Ch. II, §10].

(1) Suppose first that $g \in H_2$ and let T denote a Sylow 2-subgroup of H_2'. We can assume that g normalizes T, so that $\langle g, T\rangle$ is a non-abelian 2-group and $1 \neq g^2 \in T$. Observe that g acts by conjugation as an outer automorphism of H_2'. [Indeed, suppose the contrary. Then there exists $h \in H_2'$ such that $g^{-1}Tg = h^{-1}Th = T$. As h induces on T an automorphism of order 2, and $N_{H_2'}(T) = T.\mathbb{Z}_{q^2-1}$ where q^2-1 is odd, it follows that h must belong to T. But then h centralizes T, a contradiction.] We may identify T with $(\mathbb{F}_{q^2}, +)$. As the only involutory outer automorphism of $SL(2,q^2)$ is the field automorphism associated to the Galois automorphism γ of $\mathbb{F}_{q^2}/\mathbb{F}_q$, the commutator $[x,g]$ for $x \in T$ corresponds to trace $x + \gamma(x)$, where $x \in \mathbb{F}_{q^2}$. Since the trace form is surjective and $q > 2$, it follows that the quotient group $\langle g, T\rangle/\langle g^2\rangle$ is non-abelian. In particular, g acts non-trivially on the group $T/\langle g^2\rangle$. Let M be the $\mathbb{C}H_2'$-module afforded by a non-trivial irreducible constituent ϕ of $\theta|_{H_2'}$. As q is even, every non-trivial irreducible complex character ϕ of $SL(2,q^2)$ is of degree q^2 or $q^2 \pm 1$. From the character table of $H_2' \cong SL(2,q^2)$ one can also observe that $\phi|_T$ is equal to $\rho_T - 1_T$, ρ_T, $\rho_T + 1_T$, respectively when $\phi(1) = q^2-1, q^2, q^2+1$. Let $t \in T$ and M^t be the 1-eigenspace for t in M. Take $t = g^2$ and consider the module $M_1 = M^t + gM^t \subset M + gM$. From the previous remark on the values of $\phi|_T$, it readily follows that $T/\langle t\rangle$ acts faithfully on M^t, and hence on M_1. Clearly M_1 is g-stable (in other words, M_1 is a $\langle g, T\rangle$-module). As g acts non-trivially on $T/\langle t\rangle$, g acts on M_1 as a non-scalar element of order 2. It follows that $g|_{M_1}$ has eigenvalues ± 1, and we are done.

(2) Next, suppose that $g \in H_1$. Let R denote a Sylow 2-subgroup of $H_1' = X_1 \times X_2$, where $X_1 \simeq X_2 \simeq SL(2,q)$, and set $r = g^2$. Again, $r \neq 1$ and $r \in H_1'$. Moreover, r belongs to none of the two direct factors of H_1'. [For, suppose the contrary: say, $r \in X_1$. Then $C_{H_1'}(r) = C_{X_1}(r) \times X_2$. As $C_{X_1}(r)$ is abelian, it follows that $(C_{H_1'}(r))' = X_2$. Thus g normalizes X_2 (and hence also X_1). Observe that we may take a basis of the natural $O^+(4,q)$-module with respect to which $X_2 \simeq SL(2,q)$ consists of matrices of shape $\left[\begin{smallmatrix} S & 0\\ 0 & S\end{smallmatrix}\right]$, $S \in SL(2,q)$. Consider the enveloping algebra of X_2: $\left[\begin{smallmatrix} A & 0\\ 0 & A\end{smallmatrix}\right]$, $A \in \mathrm{Mat}(2,q)$. Since g acts on X_2 by conjugation, it follows that g acts on $\mathrm{Mat}(2,q)$ preserving the scalars. Thus, by the Skolem–Noether theorem, g acts as an inner automorphism of $\mathrm{Mat}(2,q)$. However this is impossible, since, by the same argument used in (1), g must act on X_2 as an outer automorphism.] As in case (1), let ϕ be a non-trivial irreducible constituent of $\theta|_{H_1'}$, and let M be the module afforded by ϕ. As $q > 2$, g acts non-trivially on $R/\langle r\rangle$. [Indeed, let $R = E_q^1 \times E_q^2$, where $E_q^i \subset X_i$ $(i = 1,2)$. Set $X = \langle g, R\rangle$ and observe that g permutes E_q^1

and E_q^2. Suppose that g acts trivially on $R/\langle r\rangle$. Then $\langle g, R\rangle' = \langle r\rangle$. Consider the map $f : X \to \langle r\rangle$ defined by $x \to [x, g]$. As, for any $x_1, x_2 \in X$, $[x_1x_2, g] = [x_1, g][x_2, g]$, f is homomorphism of X onto $\langle r\rangle$. Thus, $|X : C_X(g)| = 2$. Set $C_X(g) = Y$, $Y \cap E_q^1 = Y_1$, $Y \cap E_q^2 = Y_2$. As $q > 2$, $|E_q^1| = |E_q^2| \geqslant 4$. On the other hand, $|E_q^i : Y_i| \leqslant 2$; hence $|Y_1| \geqslant 2$. But $gY_1g^{-1} \subseteq X_2$, a contradiction.] In addition, via tedious but elementary computations involving the values of $\phi|_R$, one can show as above that $R/\langle r\rangle$ acts faithfully on M^r. The result follows as in (1).

To establish the lemma for a field P of odd characteristic, observe that every irreducible P-representation of $SL(2, F)$ lifts to a complex representation of the same degree. The facts about the restriction of the representation to the Sylow 2-subgroups of H_1' and H_2' remain true, so the lemma follows. □

6. Symplectic and unitary groups of odd characteristic

From now on, we assume that the space V is neither orthogonal nor symplectic of even characteristic. Set $H = I(V)'$. As above, let $v \in V$ be isotropic and set $S_1 = Stab_H(v)$, $U = O_p(S_1)$. Then (see Lemma 4.5) U is non-abelian and consists of the $(m \times m)$-matrices u satisfying the condition $u^t \Gamma_f u^\tau = \Gamma_f$ (note that U is completely determined by this condition as a subgroup of the upper unitriangular group). Thus U consists of the matrices of shape

$$u = \begin{bmatrix} 1 & -\varepsilon(c^t\Phi)^\tau & b \\ 0 & \mathrm{Id}_{m-2} & c \\ 0 & 0 & 1 \end{bmatrix},$$

where c is any $(m-2) \times 1$-matrix and b satisfies the condition $\varepsilon b + b^\tau + (c^\tau)^t \Phi^\tau c = 0$. Computation shows that $Z(U)$ consists of the matrices of U such that $c = 0$, and hence such that $\varepsilon b + b^\tau = 0$. It follows that $Z(U)$ may be identified with the additive group $(\mathbb{F}_q, +)$ of F in the symplectic case, and with the additive group of the fixed field F_0 ($\simeq \mathbb{F}_q$) in the unitary case (more precisely, $Z(U)$ can be viewed as a 1-dimensional space over F_0). Also, one observes that $Z(U) = U'$, the commutator subgroup of U (see [DM-Z, Lemma 3.1]). Drawing further data from the analysis carried out in [DM-Z] (specialized to the case $d = 1$), we record the following facts:

(1) The group $U_0 = U/Z(U)$ is elementary abelian of order $|F|^{m-2}$. U_0 has a natural structure of vector space over F, and hence can be viewed in a natural way as an FS_1-module. Namely, the conjugation action of S_1 on U induces a module action on U_0. Recall that $S_1 = U : Y$, where Y is the subgroup of H consisting of all matrices of shape $\mathrm{diag}(1, y, 1)$ (so that $y \in I(W_1)'$). Restricting to the subgroup Y, we obtain the action $c \to yc$. Viewing the column vector c as an element of W_1 and setting $Y_1 = \{y \mid \mathrm{diag}(1, y, 1) \in Y\}$, we conclude that $Y_1 \equiv I(W_1)'$ and the conjugation action of Y on U turns U_0 into an FY-module isomorphic to the natural Y_1-module W_1.

(2) Let us view $U_0 = U/Z(U)$ as an F_0-space. Then the commutator map $(u, v) \to [u, v]$, for $u, v \in U$, induces a non-degenerate alternating F_0-bilinear form on U_0. For $u \in U$, let $\pi(u)$ denote the projection of u into U_0. We observe explicitly that $u_1, u_2 \in U$ commute if and only if $\pi(u_1), \pi(u_2)$ are orthogonal with respect to the above alternating form.

(3) Let λ be an irreducible P-representation of U, non-trivial on $Z(U)$. It was shown in [DM-Z, Lemma 3.12], that the composition of λ with the canonical projection $\pi : \lambda(U) \to \lambda(U)/Z(\lambda(U))$ induces a group isomorphism ξ of U_0 onto $\lambda(U)/Z(\lambda(U))$. It follows from this that $\lambda(U) = Z(\lambda(U)) \cdot \mathcal{E}$, where $\mathcal{E}$ is an extraspecial p-group of order $p \cdot |F|^{m-2}$ and

$\mathcal{E} \cap Z(\lambda(U)) = Z(\mathcal{E})$ (see [DM-Z, Lemma 3.13]). In fact, under our current restrictions it turns out that $\lambda(U)$ is indeed extraspecial. More precisely, the following holds:

Lemma 6.1. *Let λ be an irreducible P-representation of U, non-trivial on $Z(U)$. Then $Z(\lambda(U)) = \lambda(Z(U))$ has order p, and hence $\lambda(U) = \mathcal{E}$. If furthermore $q = p$, then $\lambda(U) \simeq U$.*

Proof. The bilinear form induced by the commutator map on U_0 is non-degenerate. Hence, for every $u \in U \setminus Z(U)$ and for every $1 \neq z \in Z(U)$ there exists $u_1 \in U$ such that $[u, u_1] = z$. Suppose that $Z(\lambda(U))$ properly contains $\lambda(Z(U))$. Then, there exists $u \in U \setminus Z(U)$ such that $\lambda(u)$ commutes with $\lambda(v)$ for every $v \in U$; that is, $[\lambda(u), \lambda(v)] = \lambda([u, v]) = 1$. Choose $v = u_1$. Then $\lambda([u, u_1]) = \lambda(z) = 1$. This contradicts the assumption that λ is non-trivial on $Z(U)$. So $Z(\lambda(U)) = \lambda(Z(U))$. As $Z(U)$ has exponent p and $\lambda(Z(U))$ is cyclic (by the irreducibility of λ), we have $|Z(\lambda(U))| = |\lambda(Z(U))| = p$. As $\mathcal{E} \cap Z(\lambda(U)) = Z(\mathcal{E})$, it follows that $\lambda(U) = \mathcal{E}$. Finally, suppose that $q = p$. Then $\ker \lambda$ must be trivial; hence $\lambda(U) \simeq U$. □

(4) Let $|F| = p^a$. The commutator map on $\lambda(U)$ induces on $\lambda(U)/Z(\lambda(U)) \simeq \mathcal{E}/Z(\mathcal{E})$ the structure of a symplectic space of dimension $a(m-2)$ over the prime field $\mathbb{F}_p$. The symplectic structure of U_0 over F_0 considered in (2) is related to the symplectic structure of $\lambda(U)/Z(\lambda(U))$ over $\mathbb{F}_p$ via the isomorphism ξ defined in (3). In particular, ξ allows to translate the action of Y on U_0 (defined in (1)) into a symplectic action of Y on the space $\lambda(U)/Z(\lambda(U))$: in other words, ξ induces a faithful embedding $\varepsilon : Y \to Sp(a(m-2), \mathbb{F}_p)$.

(5) (See [DM-Z, Lemma 3.14(iii)].) No element of Y acts on the symplectic $\mathbb{F}_p$-space $\lambda(U)/Z(\lambda(U))$ as a transvection, unless p is odd, $F = \mathbb{F}_p$, and $I(V) = Sp(m, \mathbb{F}_p)$. Furthermore, in the latter case transvections of Y map to transvections of $\varepsilon(Y)$.

We start with some observations and preliminary results on representations of S_1.

Lemma 6.2. *(See [Gé, Theorems 2.4 and 3.3].) Let $H = Sp(m, q)$ with $m > 2$ and q odd, or $H = SU(m, q)$ with $m \geqslant 3$. Let $S_1 = Stab_H(v)$, where v is an isotropic vector of V, $U = O_p(S_1)$ and $Z = Z(S_1)$. Then $Z = Z(U) \cong (\mathbb{F}_q, +)$, and for every non-trivial character $\zeta : Z \to P$ there exists a representation $\tau : S_1 \to GL(|F|^{\frac{m-2}{2}}, P)$ such that $\tau|_U$ is irreducible and $\tau(z) = \zeta(z) \cdot \mathrm{Id}$ for $z \in Z$. In addition: if S_1 is perfect, then τ is unique.*

Remarks.

(1) In [Gé] P is the field of complex numbers. However, one can use the Brauer reduction of τ modulo every prime r distinct from p to obtain a representation over P. The latter is irreducible, as $|U|$ is coprime to r.
(2) The last claim in Lemma 6.2 can be justified as follows. As $\tau|_U$ is irreducible, τ is unique as a projective representation of S_1. Two ordinary representations that coincide as projective representations only differ by scalars, that is, one is obtained from the other by tensoring with a one-dimensional representation. If S_1 is perfect, then the only one-dimensional representation is the trivial one, and so τ is unique.

Under the assumptions of Lemma 6.2, $S_1 = YU$, where $Y \simeq Sp(m-2, q)$ or $SU(m-2, q)$, respectively. The restriction $\tau|_Y$ is a so-called *generic Weil representation* of Y. It is reducible, and its irreducible constituents are also called Weil representations. It obviously depends on the choice of ζ, which is however irrelevant in the unitary case. In the symplectic case, two generic

Weil representations $\tau|_Y(\zeta)$ and $\tau|_Y(\zeta')$ are equivalent if and only if ζ and ζ' belong to the same S-orbit, where $S = Stab_H(\langle v \rangle)$. As the non-trivial characters of Z are parametrized by the elements of $\mathbb{F}_q^*$, we can think of ζ and ζ' as elements of $\mathbb{F}_q^*$. Then ζ and ζ' belong to the same S-orbit if and only if they belong to the same coset of $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$, where $(\mathbb{F}_q^*)^2$ denotes the group of squares in $\mathbb{F}_q^*$.

The Weil representations of symplectic and unitary groups have been intensively studied in the recent years. They have many nice properties, which often characterize the representations themselves. Most of them are described in [GMST]. Here we mention the following, for later use:

(A) Let $H = Sp(m, q)$, where $m = 2n$ and q is odd. Then H has exactly two generic Weil representations (of dimension q^n). If char $P \neq 2$, each of them decomposes into two irreducible constituents of dimensions $(q^n + 1)/2$ and $(q^n - 1)/2$ respectively, thus producing exactly four distinct irreducible Weil representations. In addition, these are trivial on $Z(H)$ if and only if their dimension is odd. If char $P = 2$, then a generic Weil representation of H is not completely reducible, and its composition series contains two isomorphic irreducible constituents of dimension $(q^n - 1)/2$, plus the trivial one. Conversely, every non-trivial irreducible representation of H of the previous dimensions (according to char P) is a Weil representation of H.

Two irreducible Weil representations (as well as their characters or Brauer characters) are said to be associated if they occur as constituents of a single generic Weil representation of H. The characters of Weil representations of equal dimension coincide on semisimple elements of H. If χ_1 and χ_2 are the characters of two associated Weil representations with $\chi_1(1) < \chi_2(1)$ and $g \in H$ has odd order, then $\chi_1(g) + 1 = \chi_2(g)$.

For $m' < m$, let $\alpha : Sp(m', q) \to Sp(m, q)$ be a standard embedding. If ψ is a Weil representation of H (or a generic Weil representation of H), then the irreducible constituents of $\psi \circ \alpha$ are associated Weil representations of $Sp(m', q)$ (e.g., cf. [Z85, Theorem 2]). The following converse result will be of particular relevance to us:

Lemma 6.3. *(Cf. [GMST, Theorem 2.3 and Corollary 2.4].) Let $H = Sp(m, q)$, with $m > 4$ and q odd. Let ψ be a non-trivial irreducible representation of H such that, for some m' with $2 < m' < m$, the non-trivial irreducible constituents of the restriction of ψ to a standard subgroup $Sp(m', q)$ are Weil representations of $Sp(m', q)$. Then ψ is a Weil representation of H.* [*The same also holds for $m' = 2$, provided all the non-trivial irreducible constituents of the restriction of ψ to $SL(2, q)$ are associated Weil representations.*]

(B) Let $H = SU(m, q)$, with $m > 2$. Then a generic Weil representation of H has one irreducible constituent of dimension $(q^m + (-1)^m q)/(q + 1)$ and q irreducible constituents of dimension $(q^m - (-1)^m)/(q + 1)$, unless char P divides $q + 1$, in which case one of the dimensions can be 1 and the greater dimension may not occur (see [H-M, Proposition 9], for a precise information). Conversely, if a non-trivial irreducible representation of H is of the above dimension (and it exists, depending on char P), then it is a Weil representation of H.

For $m' < m$, let $\alpha : SU(m', q) \to SU(m, q)$ be a standard embedding. As in the symplectic case, if ψ is a Weil representation of H (or a generic Weil representation of H), then the irreducible constituents of $\psi \circ \alpha$ are Weil representations of $SU(m', q)$. The following converse result will be of particular importance to us:

Lemma 6.4. *(Cf. [GMST, Theorem 2.5].) Let $H = SU(m, q)$, with $m > 3$. Let ψ be a non-trivial irreducible representation of H such that, for some m' with $2 < m' < m$, the non-trivial* *irreducible constituents of the restriction of ψ to a standard subgroup $SU(m', q)$ are Weil representations of $SU(m', q)$. Then ψ is a Weil representation of H.*

Observe that $Sp(m, q)$ (respectively, $SU(m, q)$ for $m > 2$) has no non-trivial P-representation of degree less that $(q^{m/2} - 1)/2$ (respectively, $(q^m - q)/(q + 1)$ if m is odd, $(q^m - 1)/(q + 1)$ if m is even), see [Se, Theorem 1]. Finally, in connection to the representations τ of S_1, the following holds:

Lemma 6.5. *Let S_1 be as in Lemma 6.2. Let $\phi \in \mathrm{Irr}_P\, S_1$ and assume that $\phi(Z) \neq \mathrm{Id}$. Then $\phi = \tau \otimes \lambda$ where $\tau, \lambda \in \mathrm{Irr}_P\, S_1$, $\tau|_U$ is irreducible of dimension $|F|^{\frac{m-2}{2}}$ and $\lambda(U) = \mathrm{Id}$. In addition: if S_1 is perfect and $\dim\phi = |F|^{\frac{m-2}{2}}$, then $\phi|_Y$ is a generic Weil representation of Y.*

Proof. Suppose that $\sigma \in \mathrm{Irr}_P\, U$ and $\sigma(Z) \neq \mathrm{Id}$. Then (see Lemma 6.1 above) $\sigma(U) \simeq \mathcal{E}$, where $\mathcal{E}$ is an extraspecial p-group of order $p \cdot |F|^{m-2}$. By Lemma 2.2, $\dim\sigma = |F|^{(m-2)/2}$ and σ is equivalent to $\sigma' \in \mathrm{Irr}_P\, U$ if and only if $\sigma|_Z$ is equivalent to (hence coincides with) $\sigma'|_Z$. As $\phi|_Z$ is scalar, it follows that $\phi|_U$ is homogeneous. Hence one can think of $\phi|_U$ as $\phi' \otimes \mathrm{Id}_n$, where $\phi' \in \mathrm{Irr}_P\, U$ and $n = (\dim\phi)/|F|^{(m-2)/2}$. Let $\tau : S_1 \to GL(|F|^{(m-2)/2}, P)$ be such that $\tau|_U = \phi'$. By Lemma 6.2, such a τ exists. For $x \in S_1$, set $\lambda'(x) = \phi(x) \cdot (\tau(x^{-1}) \otimes \mathrm{Id}_n)$. Then, it is easily seen that $\lambda'(U) = \mathrm{Id}$ and $\lambda'(x)\phi(u) = \phi(u)\lambda'(x)$ for every $x \in S_1$, $u \in U$ [indeed, the latter equality can be translated into $\phi(u^x) = \tau(u^x) \otimes \mathrm{Id}_n$]. Therefore, $\lambda'(S_1)$ belongs to the centralizer C of $\phi(U)$ in $GL(d, P)$, where $d = \dim\phi$. Clearly, $C \cong GL(n, P)$. So $\lambda' \cong \mathrm{Id}_{|F|^{(m-2)/2}} \otimes \lambda$, where $\lambda : S_1 \to GL(n, P)$ and $\lambda(U) = \mathrm{Id}$. By Burnside's theorem, $\tau(S_1) \subseteq \langle\tau(U)\rangle$, so $\tau(S_1) \otimes \mathrm{Id}_n$ centralizes $\lambda'(S_1)$. This implies that λ' is a representation. Clearly, λ' is irreducible as so is ϕ. This completes the proof of the main claim. The additional claim follows from Lemma 6.2 and the remark (2) following it. □

Lemma 6.6. *Let V be a symplectic space of odd characteristic and let $L = \{M \in \mathrm{Mat}(m, F)$: $\Gamma_f M = -M^t \Gamma_f\}$ (that is, L is the Lie algebra $\mathfrak{sp}(V)$ associated to V). Let W be a 1-dimensional subspace of V. Set $L_W = \{\ell \in L \mid \ell W = 0,\ \ell W^\perp \subseteq W\}$. Then the following holds:*

(1) *L_W consists of all the matrices*

$$L_{x,y} = \begin{bmatrix} 0 & x\Phi & y \\ 0 & 0 & x^t \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where } \Gamma_f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \Phi & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

(2) *Both $W^\perp$ and L_W are FS_1-modules (with respect to the natural action of S_1 on V and the conjugation action of S_1 on $\mathrm{Mat}(m, F)$, respectively), and the mapping $\alpha : W^\perp \to L_W$ defined by $\alpha\left(\left[\begin{smallmatrix} y \\ x^t \end{smallmatrix}\right]\right) = L_{\frac{1}{2}x, y}$ realizes an S_1-module isomorphism between $W^\perp$ and L_W.*

Proof. Let $W = \langle v\rangle$ and choose a basis B of V as above, so that

$$\Gamma_f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \Phi & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then L_W consists of all the matrices

$$L_{x,y} = \begin{bmatrix} 0 & x & y \\ 0 & 0 & -\Phi^{-1}x^t \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular $L_W(V) = W^{\perp}$, and (1) is proven. Let

$$s = \begin{bmatrix} 1 & z^t\Phi Y & u \\ 0 & Y & z \\ 0 & 0 & 1 \end{bmatrix} \in S_1.$$

Direct computation shows that $\alpha\big(s \cdot \big[\begin{smallmatrix} y \\ x^t \end{smallmatrix}\big]\big) = s \cdot \alpha\big(\big[\begin{smallmatrix} y \\ x^t \end{smallmatrix}\big]\big)$, thus proving (2). □

Lemma 6.7. *Let V be as in the previous lemma and define a map $\lambda : L_W \to U$ setting $\lambda(\ell) = \mathrm{Id} + \ell + \ell^2/2$ for $\ell \in L_W$. Then*:

(1) *λ is a bijection*;
(2) *$\lambda(C_{L_W}(s)) = C_U(s)$ for every $s \in S_1$*;
(3) *assume $q = p$. Then a subset U_1 of U is a subgroup if and only if $\lambda^{-1}(U_1)$ is a subspace of L_W. In particular, if $|U : C_U(s)| = p$, then $C_{L_W}(s)$ is of codimension 1 in L_W. In addition, $C_{W^{\perp}}(s)$ is of codimension 1 in $W^{\perp}$ and $C_V(s)$ is of codimension $\leqslant 2$ in V.*

Proof. Observe that $\ell^3 = 0$ as $\ell W = 0$, and hence $\ell V \subseteq W^{\perp}$. So λ is just the exponential map $\ell \to \exp(\ell) =: \sum_{i=0}^{p-1} \frac{1}{i!}\ell^i$. It is well known that if $\ell^p = 0$ for all $\ell \in L_W$, then the image of the exponential map is a subgroup of $GL(V)$, and its inverse is provided by the logarithmic map $u \to \sum_{i=1}^{p-1}(-1)^i \frac{(u-1)^i}{i}$. It follows from the definition of U that $\exp(L_W) = U$. This justifies (1). (2) is obvious. (3) is easy. □

Lemma 6.8. *Let $H = Sp(m,q)$, where $m > 2$ and q is odd, or $H = SU(m,q)$, where $m > 2$. Let S_1, Y and U be defined as above (so that $Y_1 \simeq Sp(m-2,q)$ or $SU(m-2,q)$, respectively), and let $t \in S_1$ be of order p. Suppose that $|U : C_U(t)| = p$. Then $q = p$, and either $t \in U$ or $H = Sp(m,p)$, the projection of t into $Y_1 \simeq Sp(m-2,p)$ is a transvection and* $\dim(t - \mathrm{Id})V \leqslant 2$.

Proof. Set $U_1 = \{u \in U \mid [t,u] \in Z(U)\}$. Since $C_U(t) \subseteq U_1$, the assumption that $|U : C_U(t)| = p$ implies either $U = U_1$ or $C_U(t) = U_1$. Suppose first $U \neq U_1$. Then t acts non-trivially on $U_0 = U/Z(U)$, and by the above this action is a linear transformation of U_0 viewed as an FS_1-module. Hence $|U : C_U(t)| = p$ implies that $|F| = p$ and the fixed point subspace of t on U_0 is of codimension 1. The latter means that t projects to a transvection in $Sp(m-2,p)$. The claim that $\dim(t - \mathrm{Id})V \leqslant 2$ follows from Lemma 6.7(3). Next, suppose that $U = U_1$. Then the mapping from U_0 to $Z(U)$ defined by $uZ(U) \to [t,u]$ is F_0-linear. Therefore, its kernel is an F_0-subspace of U_0, whence $|F_0| = p$. So $q = p$. As observed above, the commutator map $(u,u') \to [u,u']$, for $u, u' \in U$, induces a non-degenerate F_0-bilinear form on U_0. Therefore, there exists $u' \in U$ such that $[u',u] = [t,u]$ for all $u \in U$. Whence $t^{-1}u' \in C_H(U)$. By the well-known Borel–Tits theorem $C_H(U) \subseteq U \cdot Z(H)$, and hence $C_H(U) = Z(U) \cdot Z(H)$. As $(t^{-1}u')^p = 1$, this implies $t^{-1}u' \in Z(U)$, and the result follows. □

Lemma 6.9. *Let $H = Sp(m,q)$, where $m > 2$, q is odd and $(m,q) \neq (4,3)$, or $H = SU(m,q)$, $m > 3$. Let S_1, Y and U be defined as above. Let $y \in S_1 \setminus Z(S_1)$ be of order p, and let ϕ be an irreducible P-representation of S_1 non-trivial on $Z(U)$. Then $|\operatorname{Spec}\phi(y)| = p$, unless one of the following holds*:

- (i) *$H = Sp(4,p)$, $\dim\phi = p(p-1)/2$ and $|\operatorname{Spec}\phi(y)| = p-1$.*
- (ii) *$H = Sp(m,p)$, the projection of y into $Sp(m-2,p)$ is a transvection and* $\operatorname{Spec}\phi(y) = \Delta_1(p)$ *or $\Delta_2(p)$, up to a common multiplier. ($\Delta_1(p)$, $\Delta_2(p)$ are defined before Lemma* 2.7.) *Furthermore, if y itself is a transvection, then* $\operatorname{Spec}\phi(y) = \Delta_1(p)$ *or $\Delta_2(p)$.*

Proof. By Lemma 6.5, $\phi = \tau \otimes \lambda$ where $\tau, \lambda \in \operatorname{Irr}_P S_1$, $\tau(U)$ is irreducible and $\lambda(U) = \mathrm{Id}$. In particular, λ can be viewed as a representation of Y_1.

Then $\phi(y) = \tau(y) \otimes \lambda(y)$ and $\operatorname{Spec}\phi(y)$ is the product elementwise of $\operatorname{Spec}\tau(y)$ and $\operatorname{Spec}\lambda(y)$. Obviously, if $|\operatorname{Spec}\lambda(y)| = p$ or $p-1$, then $|\operatorname{Spec}\phi(y)| = p$ (this is because $\tau(y)$ is not a scalar). Thus, we may assume that $|\operatorname{Spec}\lambda(y)| \leqslant p-2$.

Suppose first that λ is non-trivial. Set $y = y_1 u$, where $y_1 \in Y$, $u \in U$. If $(m,q) \neq (4,p)$, then by Proposition 1.2 $Y \simeq Sp(m-2,p)$, y_1 is a transvection and $\operatorname{Spec}\lambda(y) = \Delta_1(p)$ or $\Delta_2(p)$ (whence $p > 3$). Thus $H = Sp(m,p)$. By Lemma 2.7 (applied to $b = \phi(y)$, $\mathcal{F} = \phi(U)$, $B = \langle b, \mathcal{F}\rangle$), either $|\operatorname{Spec}\tau(y)| = p$ or $\operatorname{Spec}\tau(y) = \Delta_1(p)$ or $\Delta_2(p)$, up to a common multiplier. Observing that $\Delta_i(p) \times \Delta_j(p) = \{1, \varepsilon, \dots, \varepsilon^{p-1}\}$, we obtain that $|\operatorname{Spec}\phi(y)| = p$. If $(m,q) = (4,p)$, then $\operatorname{Spec}\lambda(y) = \Delta_i(p) \setminus \{1\}$ if $\dim\lambda = (p-1)/2$. This leads to (i), as $\Delta_i(p) \times (\Delta_j(p) \setminus \{1\})$ equals either $\{1, \varepsilon, \dots, \varepsilon^{p-1}\}$ or $\{\varepsilon, \dots, \varepsilon^{p-1}\}$ (recall that $p > 3$).

Now suppose that λ is trivial. Then $\phi = \tau$, so $\phi(U)$ is irreducible. Therefore, by Lemma 6.1, $\phi(U)$ is extraspecial of order $p|F|^{m-2}$. As observed above (cf. (4)), the conjugation action of Y on U embeds Y into $Sp(a(m-2),p)$. Again by Lemma 2.7, either $|\operatorname{Spec}\phi(y)| = p$ or $|\mathcal{F}_n : C_{\mathcal{F}_n}(b)| = p$. The latter is equivalent to $|U : C_U(y)| = p$. As $y \notin U$, by the previous lemma $q = p$ and the projection of y into $Sp(m-2,p)$ is a transvection. Thus we get the first part of (ii).

Finally, suppose that y itself is a transvection. Then y is conjugate under S_1 to an element of Y, and the result follows from [Z87, Proposition 2]. □

Lemma 6.10. *Let $H = Sp(m,p)$, where $m > 2$ and p is odd, and let $g \in H$ be an element of order p^α such that $t = g^{p^{\alpha-1}}$ is a transvection. Let $1_H \neq \theta \in \operatorname{Irr}_P H$ and $1 \neq \varepsilon \in \operatorname{Spec}\theta(t)$. Then the multiplicity of ε as an eigenvalue of $\theta(t)$ is at least $p^{(m-2)/2}$. If $\alpha > 1$, then* $\operatorname{Spec}\theta(g)$ *contains all the $p^{\alpha-1}$-roots of ε and the multiplicity of each $p^{\alpha-1}$-root of ε as an eigenvalue of $\theta(g)$ is at least* $\max\{1, p^{(m-2/2)-(p^{\alpha-1})^2}\}$.

Proof. Let V be the natural $\mathbb{F}_p H$-module, and let v, S_1 and U be defined as at the beginning of this section. Then $U \cong \mathcal{E}_{(m-2)/2}$. Observe that, without loss of generality, we may assume that $v \in (t - \mathrm{Id})V$ and $Z(U) = \langle t\rangle$. It is clear that there exists an irreducible constituent ϕ of $\theta|_S$ such that $\phi(t) = \varepsilon \cdot \mathrm{Id}$. Furthermore, as $|Z(U)| = p$, $\phi(U) \simeq U$. By Lemma 2.2, $\dim\phi \geqslant p^{(m-2)/2}$. As the multiplicity of ε in $\theta(t)$ is at least $\dim\phi$, the assertion about ε follows.

Now assume that $\alpha > 1$ and set $g_1 = g^{p^{\alpha-2}}$, $b = \phi(g_1)$, $\mathcal{F} = \phi(U)$, $B = \langle b, \mathcal{F}\rangle$. Since b has order p modulo $Z(B)$ $(= Z(\mathcal{F}))$, we can apply Lemma 2.7(a) to B, obtaining that $|\operatorname{Spec}\phi(g_1)| = p$ (that is, $\operatorname{Spec}\phi(g_1)$ consists of all the p-roots of ε), except when $|U : C_U(g_1)| = p$ and the image $\bar{g}_1$ of g_1 in $Sp(U/Z(U))$ is a transvection. We proceed to show that the latter exceptional case does not occur. First, we recall that $U/Z(U)$ is isomorphic, as a Y-module, to W_1. It follows that $\dim(\mathrm{Id} - g_1)V \leqslant 3$. As $|g_1| = p^2$, we deduce that $p = 3$ and $\mathrm{Jord}\, g_1$ contains a unique non-trivial block, which has size 4. By Lemma 2.5(i), $V = V_1 \oplus V_2$ where V_1, V_2 are non-degenerate mutually orthogonal g-submodules, $\dim(V_1) = 4$ and g_1 acts trivially on V_2. Moreover, by Lemma 5.4, $g = g_1$. Let g' be the projection of g_1 to $Sp(V_1) = Sp(4,3)$. As g' has order 9, one sees (e.g., cf. [Atl, p. 27]) that $|C_{Sp(V_1)}(g')| = 2880 = 2^5 \cdot 5 \cdot 9$. Let us denote by $\widetilde{Y} \simeq Sp(V_1)$ the subgroup of H consisting of all elements acting trivially on V_2. As v is fixed by g, we may assume that $v \in V_1$. It follows readily that $U_1 = U \cap \widetilde{Y}$ has order 27. As $\exp U = 3$, $g' \notin U_1$. Therefore $C_{U_1}(g_1)$ has order at most 3. It follows that $|U_1 : C_{U_1}(g_1)| > 3$, whence also $|U : C_U(g_1)| > 3$. We conclude that $|U : C_U(g_1)| > p$, and therefore $|\mathrm{Spec}\,\phi(g_1)| = p$, in all cases. Furthermore, we may apply Lemma 2.7(b) to $B_1 = \langle \phi(g), \mathcal{F}\rangle$. As $\phi(g)$ has order $p^{\alpha-1}$ mod $Z(B_1)$, we obtain that $|\mathrm{Spec}\,\phi(g)| = p^{\alpha-1}$; hence $\mathrm{Spec}\,\phi(g)$ consists of all the $p^{\alpha-2}$-roots of the elements of $\mathrm{Spec}\,\phi(g_1)$, that is, all the $p^{\alpha-1}$-roots of ε. This proves the assertion about $\mathrm{Spec}\,\theta(g)$. Finally, by Lemma 2.10, the multiplicity of every eigenvalue of $\phi(g)$ is at least $\max\{1, p^{(m-2/2)-(p^{\alpha-1})^2}\}$, as claimed. □

Lemma 6.11. *Let $H = Sp(m,q)$, where $m = 2n$ and q is odd. Then the following holds*:

(1) *If n is odd, H has a single conjugacy class of unipotent elements whose Jordan form consists of two blocks of size n. If n is even H contains exactly two conjugacy classes of such elements.*
(2) *Let $g \in H$ be a unipotent element whose Jordan form consists of two blocks of size n. Then g belongs to a subgroup isomorphic either to $GL(n,q)$ or to $U(n,q)$. Additionally, if n is even, then g also belongs to a subgroup isomorphic to $Sp(n,q^2)$.*
(3) *If $g \in H$ is a unipotent element whose Jordan form consists of two blocks of size n, then g is rational* (*that is, g is conjugate in $Sp(m,q)$ to g^j for all j's coprime to $|g|$*).

Proof. (1) The claim follows from the general theory of algebraic groups. Indeed, there is a single class of elements whose Jordan form is $\mathrm{diag}(J_n, J_n)$ in $Sp(m, \overline{\mathbb{F}_p})$, where $\overline{\mathbb{F}_p}$ denotes the algebraic closure of the prime field $\mathbb{F}_p$ (cf. [T-Z2, Lemma 4.1]). It follows (see [T-Z2, Lemmas 4.7 and 4.10]) that the number of classes of such elements in H is as indicated in (1).

(2) Let x be an element with Jordan form J_n in $GL(n,q)$, $U(n,q)$ and $Sp(n,q^2)$, respectively. Observe that there are two distinct classes of such elements in $Sp(n,q^2)$. Let π be a standard embedding of each of these groups into H. Then $\pi(x)$ has Jordan form $\mathrm{diag}(J_n, J_n)$. Let χ denote the complex (generic) Weil character of H. Then $\chi(\pi(x))$ equals q in the $GL(n,q)$ case, $(-1)^{n+1}q$ in the unitary case (cf. [Gé]). If n is even, these values are distinct, and therefore $\pi(x)$ gives rise to two distinct conjugacy classes of H. Additionally, if n is even it also follows from [Gé] that $\chi(\pi(x))$ equals $\pm(-1)^{\frac{q^2-1}{4}}q$ in the symplectic case, where the choice of $+$ and $-$ corresponds to the two conjugacy classes of x in $Sp(n,q^2)$, which yield two distinct conjugacy classes of H.

(3) Let x be as in (2). Then the rationality of g follows from the rationality of x in $GL(n,q)$ and $U(n,q)$ (e.g., cf. [T-Z2, Theorem 1.9]). □

Lemma 6.12. *Let $H = Sp(m,p)$, where p is odd, and let ϕ be an irreducible representation of S_1, non-trivial on $Z(U)$. Let $g \in S_1$ be an element of order $p^\alpha > p$ and set $t = g^{p^{\alpha-1}}$. Then* $\mathrm{Spec}\,\phi(g)$ *consists of all the $p^{\alpha-1}$-roots of* $\mathrm{Spec}\,\phi(t)$, *unless $|g| = 9$ and $|U : C_U(t)| = 3$. In the exceptional case,* $\mathrm{Spec}\,\phi(g)$ *contains all the* 3*-roots of at least one non-trivial* 3*-root of* 1.

Proof. Set $A = \langle g, U\rangle$. As already mentioned above, $Z(U) = Z(S_1)$ forces $\phi(U)$ to be homogeneous. Hence, by Lemma 6.1, $U \simeq \phi(U) \simeq \mathcal{E}_n$. Observe that $(\ker\phi) \cap U = 1$ implies $[\ker\phi, U] = 1$. A direct computation shows that $C_{S_1}(U) = Z(U)$. This forces $\ker\phi = 1$. Thus ϕ is faithful on S_1. Let τ be any irreducible constituent of $\phi|_A$. The above argument can also be applied to τ, showing that τ is faithful on A and hence $|\tau(g)| = |g|$. Set $b = \tau(g)$, $b_1 = \tau(t)$ and $B = \tau(A)$. Then Lemma 2.7 applies to B. Hence, by 2.7(b), $\mathrm{Spec}\,\tau(g)$ consists of all $p^{\alpha-1}$-roots of $\mathrm{Spec}\,\tau(t)$, unless possibly when $|g| = 9$ and $|U : C_U(t)| = 3$. In the latter case, $\mathrm{Spec}\,\tau(g)$ contains elements $a, a\varepsilon, a\eta, a\eta\varepsilon, a\eta\varepsilon^2$, where $a \in P$ is some 9-root of 1, $\eta^9 = 1$ and $\eta^3 = \varepsilon \neq 1$ (cf. Lemma 2.7). The claim follows. □

Lemma 6.13. *Let $H = Sp(m, q)$ with q odd and $m > 2$, or $SU(m, q)$ with $m > 3$, and let T be an irreducible PH-module affording a representation θ with* $\dim\theta > 1$. *Let v, S_1, Y, U be as above, and $1 \neq t \in Z(U)$. Let T_1 the 1-eigenspace of t on T. Then Y acts non-trivially on T_1, unless possibly when $H = Sp(4, p)$. In addition, if $H = Sp(m, q)$ and $m > 4$, then* $\dim(T_1) \geqslant (q^{m-2} - 1)/2$.

Proof. It is easy to observe that Y contains an element t' conjugate to t in H. In addition, t is a transvection as well as t'. It is well known that Y is generated by the transvections conjugate to t'. As Y centralizes $Z(U)$, $YT_1 = T_1$. Suppose that Y acts trivially on T_1. Then t' acts trivially on T_1. As t and t' are conjugate, their 1-eigenspaces have the same dimension. Therefore, T_1 is the 1-eigenspace for t' as well. It follows that t' acts fixed point-freely on T/T_1, as well as on every irreducible constituent τ of $Y|_{T/T_1}$. If q is odd this contradicts Lemma 6.9, unless possibly when $H = Sp(4, p)$. If q is even, then t' would act as $-\mathrm{Id}$ on T/T_1, which is clearly impossible. The additional claim follows, as the minimum dimension of a non-trivial representation of $Y \simeq Sp(m-2, q)$ equals $(q^{m-2} - 1)/2$. □

Lemma 6.14. *Let $H = Sp(4, 3)$ and let g be an element of H of order* 9. *Let $\theta \in \mathrm{Irr}_P H$ with* $\dim\theta > 1$. *Suppose that* $|\mathrm{Spec}\,\theta(g)| < 9$. *Then one of the following holds*:

(1) $\dim\theta = 4$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta, \eta^4, \eta^7, \eta^6\}$ *or* $\{\eta^2, \eta^5, \eta^8, \eta^3\}$, *where η is a primitive 9-root of* 1.
(2) $\mathrm{char}\,P \neq 2$, $\dim\theta = 5$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta, \eta^4, \eta^7, \eta^6, 1\}$ *or* $\{\eta^2, \eta^5, \eta^8, \eta^3, 1\}$.
(3) $\dim\theta = 6$ *and* $\mathrm{Spec}\,\theta(g) = \{\eta, \eta^2, \eta^4, \eta^5, \eta^7, \eta^8\}$.
(4) $\mathrm{char}\,P \neq 2$, $\dim\theta = 10$ *and*

$$\mathrm{Spec}\,\theta(g) = \{\eta, \eta^2, \eta^4, \eta^5, \eta^6, \eta^7, \eta^8\} \quad \text{or} \quad \{\eta, \eta^2, \eta^3, \eta^4, \eta^5, \eta^7, \eta^8\}.$$

(5) $\dim\theta = 20$, $|\mathrm{Spec}\,\theta(g)| = 8$ *and* $1 \notin \mathrm{Spec}\,\theta(g)$. (*Primitive* 9*-roots of* 1 *occur with multiplicity* 3, *primitive* 3*-roots of* 1 *occur with multiplicity* 1.)

Proof. Direct computation, using the data provided by the complex and modular character tables of H (from [Atl] and [MAtl]). □

Remarks.

(1) The above lemma does not contradict Lemma 2.7(b), as there the spectrum is the η^3-multiple of $\{\eta, \eta^4, \eta^7, \eta^6, 1\}$.

(2) In (5) only one representation of degree 20 has to be chosen, namely the one with character value -7 at g^3.

Lemma 6.15. *Let $H = SU(4,3)$ and $\theta \in \mathrm{Irr}_P\, H$ with $\dim\theta > 1$. Let $g \in H$ be of order 9. Then $\mathrm{Spec}\,\theta(g)$ contains all the 9-roots of 1 unless $\dim\theta = 20$, in which case $\mathrm{Spec}\,\theta(g)$ contains all the non-trivial 9-roots of 1.*

Proof. Inspection of the complex and Brauer character tables in [Atl] and [MAtl]. (There are three 20-dimensional representations, of which two are faithful. In characteristic 2 there is only one 20-dimensional representation, faithful for $PSU(4,3)$.) □

Lemma 6.16. *Let $H = Sp(4,9)$ and $\theta \in \mathrm{Irr}_P\, H$ with $\dim\theta > 1$. Let $g \in H$ be of order 9. Then $\mathrm{Spec}\,\theta(g)$ contains all the 9-roots of 1, unless $\dim\theta = 40$. H has exactly two irreducible representations θ_1, θ_2 of dimension 40 and either $|\mathrm{Spec}\,\theta_1(g)| = 8$ and $|\mathrm{Spec}\,\theta_2(g)| = 9$, or conversely. If $|\mathrm{Spec}\,\theta_i(g)| = 8$, then $1 \notin \mathrm{Spec}\,\theta_i(g)$ for this i.*

Proof. Assume first that P is of characteristic 0. Then it is known (e.g., cf. [T-Z2, Theorem 1.7]) that g is rational, that is, g is conjugated to all its powers g^i, where i is coprime to 9. It follows that, for every non-trivial $\theta \in \mathrm{Irr}_P\, H$, all the primitive 9-roots of 1 are eigenvalues of $\theta(g)$ with the same multiplicity, say t. Similarly, the non-trivial 3-roots of 1 appear as eigenvalues of $\theta(g)$ with the same multiplicity, say u. Let v be the multiplicity of the eigenvalue 1 in $\theta(g)$ and let χ be the character afforded by θ. Then it is readily seen that $\chi(g) = -u + v$ and $\chi(g^3) = -3t + 2u + v$. As $\chi(1) = 6t + 2u + v$, it follows that $9t = \chi(1) - \chi(g^3)$, $6u = \chi(1) + 2\chi(g^3) - 3\chi(g)$ and $9v = \chi(1) + 2\chi(g^3) + 6\chi(g)$.

We refer to [Sri] for the character table, as well as the labeling of classes and characters of H. There, the two classes of elements of order 9 are labeled A_{41} and A_{42}, respectively. In both cases, direct computation based on inspection of the character values at g and g^3 shows that $(\chi|_{\langle g\rangle}, \lambda) > 0$ for every non-trivial character χ of H and every irreducible character λ of $\langle g\rangle$, unless $\chi(1) = 40$. In the notation of [Sri], the characters of degree 40 are labeled θ_7 and θ_8. If g belongs to A_{41}, then $\theta_7(g) = -2$ and $\theta_7(g^3) = -14$, while $\theta_8(g) = 1$ and $\theta_8(g^3) = 13$. It follows that for $\theta_7(g)$ $t = 6$ and $u = 2$, while $v = 0$. Similarly, one sees that in $\theta_8(g)$ $t = 3$, $u = 7$ and $v = 8$. If g belongs to A_{42} then one gets the same result swapping θ_7 with θ_8. So the result follows.

Next, suppose that $\mathrm{char}\, P = r > 0$. We only have to inspect the cases $r = 2$, 5 or 41. The r-decomposition matrices for $Sp(4,q)$ are known (see [Wh1,Wh2,Wh3,Wh4] and [O-W]). If $r = 5$, then all the Brauer characters are liftable, except two characters $\varphi_{11} = \theta_{11} - 1_H$, $\varphi_{12} = \theta_{12} - 1_H$ in the principal block (cf. [Wh2]). As $\theta_{11}(1) = \theta_{12}(1) = 369$, $\theta_{11}(g) = \theta_{12}(g) = 0$, $\theta_{11}(g^3) = -36$ and $\theta_{12}(g^3) = 45$ (regardless of the class of g), we get $t = 45$, $u = v = 33$ for $\theta_{11}(g)$, and $t = 36$, $u = v = 51$ for $\theta_{12}(g)$. Thus $|\mathrm{Spec}\,\varphi_j(g)| = 9$, for $j = 11, 12$, and we are done.

If $r = 2$, then all the Brauer characters are liftable (cf. [Wh1]), except for: (a) a single character λ belonging to the block $b_{III}(r)$ in the notation of [Wh1] and expressible as $\xi_{42} - \xi_3$ on the $2'$-classes; (b) two characters in the principal block: $\phi_1 = \Phi_3 - \theta_7 - \theta_{10}$, $\phi_2 = \Phi_4 - \theta_8 - \theta_{10}$, $\phi_3 = \theta_{12} - 1_H$. (The remaining characters in the block, labeled ϕ_4, ϕ_5 and ϕ_6, lift to the Weil characters θ_7, θ_8 and to the character θ_{10} of degree 288, respectively. Furthermore, the undetermined parameter x in [Wh1] has value 1 in our case. Indeed, as $\Phi_4 = \phi_2 + \theta_8 + x\theta_{10}$, we get $\phi_2(g) = x - 2$ for $g \in A_{41}$. But from $\Phi_7 = 1_H + \phi_2 + \theta_7 + 2\theta_8 + (x+1)\theta_{10} + \theta_{12}$ and $\theta_{10}(g) = 0$

we also get $\phi_2(g) = -1$; whence $x = 1$.) In the case of λ, arguing as above one obtains the following data: for ξ_3, $t = 81$, $u = 111$, $v = 112$ at both A_{41} and A_{42}; for ξ_{42}, $t = 465$, $u = 437$, $v = 436$ at A_{41} and $t = 435$, $u = 496$, $v = 498$ at A_{42}. Whence $|\operatorname{Spec}\lambda(g)| = 9$. Considering $\phi_1 = \Phi_3 - \theta_7 - \theta_{10}$, one obtains the following data: for Φ_3, $t = 342$, $u = v = 300$ at A_{41} and $t = 315$, $u = v = 354$ at A_{42}; for θ_{10}, $t = 36$, $u = v = 24$. Again, it follows that $|\operatorname{Spec}\phi_1(g)| = 9$. As for ϕ_2, this character is conjugate to ϕ_1 under an automorphism of H, and one gets the same results by swapping the data at A_{41} and A_{42}; finally, ϕ_3 has already been dealt with above.

If $r = 41$, there are only two blocks to consider (cf. [Wh3]): (a) a block containing 4 Brauer characters, two of which lift to θ_7 and θ_8, whereas the remaining two are expressible as $\phi_5 = \theta_5 - \theta_7$, $\phi_6 = \theta_6 - \theta_8$; (b) the principal block, containing 3 non-trivial Brauer characters, one of which lifts to θ_{10}, whereas the remaining two are expressible as $\phi_9 = \theta_9 - 1_H$, $\phi_{13} = \theta_{13} - \theta_9 + 1_H$. In the case of ϕ_5, the data for θ_5 are as follows: $t = 378$, $u = v = 324$ at A_{41}, and $t = 351$, $u = v = 378$ at A_{42}. It follows that $|\operatorname{Spec}\phi_5(g)| = 9$. As for ϕ_6, one gets the same results by swapping the data at A_{41} and A_{42}. Considering θ_9, one gets the following data: $t = 45$, $u = v = 60$ at both A_{41} and A_{42}. This yields the desired result for ϕ_9. Finally, to deal with ϕ_{13}, we observe that for θ_{13} we have $t = u = v = 729$ regardless of the class of g, whence $|\operatorname{Spec}\phi_{13}(g)| = 9$. □

Lemma 6.17. *Let $H = Sp(m, q)$ or $SU(m, q)$, where $m > 2$, q is odd and $(m, q) \neq (4, 3)$, and let $\theta \in \operatorname{Irr}_P H$ with $\dim\theta > 1$. Let $g \in H$ be an element of order $s = p^\alpha > 1$ and set $t = g^{s/p}$. Then one of the following holds*:

(i) $\operatorname{Spec}\theta(g)$ *contains all the s-roots of* 1.
(ii) *$H = Sp(m, p)$ and t is a transvection.*
(iii) *$H = Sp(4, 9)$ and t is a transvection.*
(iv) *$H = Sp(8, 3)$, $|g| = 9$ and* $\operatorname{rank}(t - \mathrm{Id}) = 2$.
(v) *$g = t$, and either $H = Sp(4, p)$ and t is not a transvection, or $H = SU(3, p)$ and t is a transvection.*

Suppose that $m > \max\{8, \frac{s}{p} + 3\}$. If case (ii) *does not hold, then every eigenvalue of $\theta(g)$ is of multiplicity at least $\max\{1, p^{n-s^2}\}$, where $n = a(m - 2)/2$ and $p^a = |F|$.*

Proof. We shall say that g is generic if there exists an isotropic 1-dimensional subspace W of V such that $g(W) = W$ and $t \notin Z(U)$; otherwise, we shall say that g is exceptional.

If g is generic, choose $W = \langle v\rangle$ according to the condition stated above. Otherwise, let $W = \langle v\rangle$ to be any isotropic 1-dimensional subspace fixed by g. Let $S_1 = Stab_H(v)$, $U = O_p(S_1)$, and denote by h the projection of g into $I(W_1)$, where $W_1 \simeq W^\perp/W$.

Let ϕ be an irreducible constituent of $\theta|_{S_1}$ non-trivial on $Z(U)$. As $Z(U) = Z(S_1)$, $\phi(Z(U))$ is scalar, and hence $\phi(U)$ is homogeneous. Thus $\phi(U) \simeq \mathcal{E}_n$. Set $A = \langle g, U\rangle$ and let τ be an irreducible constituent of $\phi|_A$. Set $b = \tau(g)$, $b_1 = \tau(t)$ and $B = \tau(A)$. Note that U is non-abelian and the order of g modulo $Z(U)$ is either s or s/p, the former happening if and only if $t \notin Z(U)$. We claim that the same holds after applying τ. First, recall that $Z(\tau(U)) = \tau(Z(U))$ (cf. Lemma 6.1). Next, observe that $t \in Z(U)$ iff $\tau(t) \in \tau(Z(U))$. [Indeed, suppose that $t \notin Z(U)$. Then, for every $z \in Z(U)$ there exists $u_1 \in U$ such that $[u_1, t] = z$. As τ is non-trivial on $Z(U)$, this implies that $t \notin Z(\tau(U)) = \tau(Z(U))$.] It follows that the order of $b = \tau(g)$ modulo $Z(\tau(U))$ is either s or s/p, the former happening if and only if $b_1 = \tau(t) \notin \tau(Z(U))$. As $Z(B) = Z(\tau(U))$, the same happens for the order of b modulo $Z(B)$.

Step 1. If g is generic, then (i) and the claim on eigenvalue multiplicities hold, except possibly when $(*)$ $H = Sp(m, p)$ and t is a transvection.

Suppose that $(*)$ does not hold. By Lemma 2.7 applied to $B = \tau(A)$, we observe that $\operatorname{Spec}\tau(g)$ is the set of all the s-roots of 1, unless possibly when $\tau(t) \notin \tau(U)$ and $|\tau(U) : C_{\tau(U)}(\tau(t))| = p$. The latter is equivalent to $t \notin U$ and $|U : C_U(t)| = p$, which implies by Lemma 6.8 that $(*)$ holds. (Case $(*)$ for $m > 4$ will be considered in Lemma 6.23 below. The case $m = 4$ is covered by Proposition 1.2.) As $\operatorname{Spec}\tau(g)$ contains all the s-roots of 1, the claim on multiplicities follows from Lemma 2.10. Also, observe that the assumption $m > \max\{8, \frac{s}{p} + 3\}$ forces g to be generic (see Lemmas 4.6 and 4.7). Thus, the stated claim on multiplicities follows.

Step 2. The lemma is true if g is exceptional and $H \neq Sp(m, p)$, $Sp(m, p^2)$ or $SU(m, p)$.

As g is exceptional, $t \in Z(U)$. Hence t is a transvection and therefore $\operatorname{rank}(t - \mathrm{Id}) = 1$. It follows that either case (i) or (iii) of Lemma 4.6, or case (ii) of Lemma 4.7, hold for g. Write $\phi(t) = \varepsilon \cdot \mathrm{Id}$ (the case $\varepsilon = 1$ is not excluded). Observe that $Z(U)$ can be viewed as a vector space of dimension 1 over $\mathbb{F}_q$ or of dimension a over $\mathbb{F}_p$. Moreover, $S = N_H(Z(U))$ and the conjugation action of S on $Z(U)$ can be described as follows: as S/S_1 is isomorphic to $GL(1, q)$ in the symplectic case and to $GL(1, q^2)$ in the unitary case, the action in question is permutationally equivalent to $b \to aa^\tau b$ for $b \in \mathbb{F}_q$ and $a \in \mathbb{F}_{q^2}$ in the unitary case, and to $b \to a^2 b$ for $a, b \in \mathbb{F}_q$ in the symplectic case. Let χ be the character of $Z(U)$ such that $\phi(z) = \chi(z) \cdot \mathrm{Id}$ for $z \in Z(U)$, and let $K = \ker\chi$.

Assume first that H is unitary. As the norm map $\mathbb{F}_{q^2} \to \mathbb{F}_q$ is surjective, t is conjugate to t^i for each i coprime to p. In addition, if $q > p$ then K contains a conjugate of t. As ϕ is non-trivial on $Z(U)$, it follows that, given any p-root ε' of 1, there is a conjugate t' of t in S such that $\phi(t') = \varepsilon' \cdot \mathrm{Id}$. If $q = p$, then $K = 1$, $\varepsilon \neq 1$ and the above is only true for $\varepsilon' \neq 1$. This shows that $|\operatorname{Spec}\theta(t)| = p$ unless, possibly, $H = SU(m, p)$. The order of g modulo $Z(U)$ is s/p (as U is of exponent p); nevertheless, Lemmas 2.7 and 6.8 tell us that $\operatorname{Spec}\phi(g)$ contains all the (s/p)-roots of ε'. Hence $\operatorname{Spec}\theta(g)$ contains all the s-roots of 1, unless $q = p$ and $\varepsilon' = 1$.

Let H be symplectic. Then there are two conjugacy classes of transvections in H, and the number of elements of $Z(U)$ in each class equals $(q - 1)/2$. As $|Z(U) : K| = p$, one observes that any transvection is conjugate to an element of K as long as $q > p^2$. Suppose that $q = p^2$ (so that K has order p). Up to conjugacy, we may think of the transvections in $Z(U)$ as of rational elements of $SL(2, p^2)$. Thus any transvection in $Z(U)$ is conjugate (under H) to all its non-identity powers, and henceforth the transvections in K are conjugate to each other. Let z_1, z_2 be transvections in $Z(U)$, and suppose that $z_1 \in K$. Then there is $z_1' \in Z(U) \setminus K$ which is conjugate to z_1. If $q > p^2$, then an analogue of the argument used in the unitary case can be exploited to show that $\operatorname{Spec}\theta(g)$ contains all the s-roots of 1. If $q = p^2$ such an argument works only if t is conjugate to z_1. Suppose that t is not conjugate to z_1. If $1 \neq \varepsilon \in \operatorname{Spec}\theta(t)$, then we can choose ϕ so that $\phi(t) = \varepsilon \cdot \mathrm{Id}$, and $\phi(g)$ contains all the $p^{\alpha-1}$-roots of ε by Lemma 2.7. So we are left to deal with the case where $\varepsilon = 1 \in \operatorname{Spec}\theta(t)$. Furthermore, observe that we may assume that $Z(U)$ acts trivially on the subspace of t-fixed points. Indeed, let T be the underlying space of θ, and let E_1 be the subspace of z_1-fixed points (that is, E_1 is the 1-eigenspace of $\theta(z_1)$). Clearly S_1 acts on E_1. If z_2 acts non-trivially on E_1, let ϕ_1 be an irreducible constituent of $E_1|_{S_1}$ such that $\phi_1(z_2) \neq \mathrm{Id}$. Then we are in the same situation as above, with z_2 playing the role of z_1. So we are done, unless z_2, and hence the whole of $Z(U)$, acts trivially on E_1. In conclusion, we have shown that $\operatorname{Spec}\theta(g)$ contains all the s-roots of 1, except possibly when t is a transvection, $H = Sp(m, p)$ or $Sp(m, p^2)$ (and in the latter case $Z(U)$ acts trivially on the subspace of t-fixed points).

Step 3. The lemma is true if $H = SU(m, p)$ or $Sp(m, p^2)$.

By Step 1, we only have to examine the case where $t \in Z(U)$ and g is exceptional. Let T be the underlying space of θ, and let E_1 be the subspace of t-fixed points. By Proposition 1.2

$E_1 \neq 0$ unless $m = 3$ and $q = p$. In this case $H = SU(3, p)$ and $g = t$, as recorded in (v). So assume $m > 3$. Suppose first that $U|_{E_1} \neq \mathrm{Id}$ and let T_1 be an irreducible PS_1-submodule of $E_1|_{S_1}$ such that $U|_{T_1} \neq \mathrm{Id}$. Let us consider the elementary abelian group $U_0 = U/Z(U)$. As shown in Step 2, we may assume that $Z(U)$ acts trivially on T_1, so that T_1 is in fact acted upon by U_0. Observe that $\langle g\rangle/\langle t\rangle$ also acts on T_1. As $\langle g\rangle/\langle t\rangle$ acts faithfully on U_0 by conjugation, we may apply Lemma 4.1 to the group $\langle\langle g\rangle/\langle t\rangle, U_0\rangle \subseteq GL(T_1)$ and obtain (i).

So we are left with the case where $U|_{E_1} = \mathrm{Id}$. In this case, E_1 is acted upon by $S_1/U \simeq Y$. By Lemma 6.13, Y acts non-trivially on E_1, unless possibly when $H = Sp(4, p)$ or $SU(3, p)$. However both these cases are ruled out by our current assumptions.

So, we may assume that Y acts non-trivially on E_1, and the action of g on E_1 is realized by the action of h. We wish to apply what we have already proven to this situation, in order to obtain that h has s/p distinct eigenvalues on E_1. By Step 1, if $\langle g\rangle$ contains no transvection, then $\mathrm{Spec}\,\theta(g)$ contains all the s-roots of 1. Using this, by taking $Sp(m-2, p^2)$ or $SU(m-2, p)$ for H and h for g, we conclude by the above that $h|_{E_1}$ has s/p distinct eigenvalues, unless possibly when $t_1 = h^{s/p^2}$ is a transvection (observe that this is necessarily so if $m = 4$).

In order to examine the case when t_1 is a transvection, set $t_2 = g^{s/p^2}$, so that t_1 is the projection of t_2 to Y. Then $\dim(t_2 - \mathrm{Id})V \leqslant 3$ and $t_2^p = t$. Hence $1 = \dim(t - \mathrm{Id})V = \dim(t_2^p - \mathrm{Id})V = \dim(t_2 - \mathrm{Id})^p V$. As the right-hand side is equal to 0 for $p > 3$, we conclude that $p = 3$ and $\dim(t_2 - \mathrm{Id})V = 3$. Since the minimum polynomial of t_2 has shape $(x-1)^i$ for some i, we deduce that $i = 4$, and hence $\mathrm{Jord}\,t_2 = \mathrm{diag}(J_4, \mathrm{Id}_{m-4})$. It follows from Lemma 5.4 that $g = t_2$. Recall that g is exceptional. Thus, if H is symplectic we are in case (ii) of Lemma 4.7, whence $H = Sp(4, 9)$, as recorded in (iii). If H is unitary, then by Lemma 4.6 $m = 4$ or 5, that is, either $H = SU(4, 3)$ (which is excluded by our assumptions) or $H = SU(5, 3)$.

To rule out the case $SU(5, 3)$, let $X \simeq SU(4, 3)$ be the stabilizer of a non-isotropic vector of V in H, and consider $\theta|_X$. Notice that g^3 is a transvection, hence g is conjugate to an element of X. (Alternatively: notice that $g = t_2$. As $\mathrm{Jord}\,t_2 = \mathrm{diag}(J_4, 1)$, it follows that g is conjugate to an element of X.) By Lemma 6.15, we may assume that the non-trivial irreducible constituents of $\theta|_X$ are 20-dimensional, so all of them are Weil representations of X. By Lemma 6.4, it follows that θ is also a Weil representation. Hence θ lifts to characteristic zero, that is, there exists an irreducible complex representation $\bar\theta$ of H whose character on elements of order coprime to char P coincides with the Brauer character of θ. It is well known that $\bar\theta|_X$ contains an irreducible constituent λ, say, of degree 21. By Lemma 6.15, $\mathrm{Spec}\,\lambda(g)$ contains nine elements. Hence $\mathrm{Spec}\,\bar\theta(g)$ contains nine elements, and so does θ.

Step 4. The lemma is true if $H = Sp(m, p)$ and t is not a transvection.

Suppose the contrary. Let $V = V_1 \oplus \cdots \oplus V_r$, as in the proof of Lemma 4.7, and set $V'' = V_2 \oplus \cdots \oplus V_r$ (possibly, $V'' = 0$). Thus $V = V_1 \oplus V''$ and $W \subseteq V_1$. By Step 1 and Lemma 6.8, we may assume that $|U : C_U(t)| = p$ and the projection $\bar t$ of t to Y is a transvection in $Sp(m-2, p)$. Set $\bar W = W^\perp/W$. Clearly, $W^\perp$ contains V''. Since $W \cap V'' = 0$, it follows that $\bar W = W' \oplus W''$, where $W'' \simeq V''$. Thus, as $U \simeq \tau(U) \simeq \mathcal{E}_n$, the condition that $\bar t$ is a transvection is equivalent to the condition that $h^{p^{\alpha-1}}$ is a transvection. By Lemma 6.7, $\dim(t - \mathrm{Id})V \leqslant 2$; so, as t is not a transvection, $\dim(t - \mathrm{Id})V = 2$. It follows that $\mathrm{Jord}\,g$ has exactly two blocks of size $p^{\alpha-1} + 1$, and possibly other blocks of lower sizes. Therefore, either $\mathrm{Jord}\,g$ consists of exactly two blocks of size $p^{\alpha-1} + 1$, or the two blocks occur in $\mathrm{Jord}\,g|_{V_2}$. However, in the latter case $\bar t$ cannot be a transvection.

So $\mathrm{Jord}\,g = \{J_{p^{\alpha-1}+1}, J_{p^{\alpha-1}+1}\}$ and $m = 2(p^{\alpha-1} + 1)$. By Lemma 6.11, g belongs to a subgroup K isomorphic to $Sp(m/2, p^2)$. Suppose first that $m > 4$. Since, by Step 3, the lemma holds for $Sp(m/2, p^2)$ except when $K = Sp(4, 9)$, we are left with the case $m = 8$ and $H = Sp(8, 3)$, which is recorded in (iv). Finally, if $m = 4$, by Proposition 1.2 we are lead to the first part of (v). □

Remark. As already observed (cf. Proposition 1.2) the above lemma cannot be extended to the case $H = Sp(2, q)$ with $q = p^2$, as there are irreducible representations of $Sp(2, p^2)$ of dimension $(p^2 - 1)/2$ in which an element of order p does not have eigenvalue 1.

The next two lemmas deal with exceptional cases that need to be examined in order to work out in detail case (ii) of Lemma 6.17 (see Lemma 6.23 below).

Lemma 6.18. *Let $H = Sp(6, 3)$, $g \in H$ be of order 9 and $1_H \neq \theta \in \mathrm{Irr}_P\, H$.*

(A) *Suppose that* $\mathrm{rank}(g - \mathrm{Id}) = 4$. *Then the following holds*:
   (A1) *If* $\dim\theta > 13$, *then* $\mathrm{Spec}\,\theta(g)$ *consists of all* 3*-roots of the elements in* $\mathrm{Spec}\,\theta(g^3)$. *More precisely,* $|\mathrm{Spec}\,\theta(g)| = 9$ *if* θ *is not of dimension* 14, *whereas* $\mathrm{Spec}\,\theta(g) = \{\eta^i\}$ *or* $\{\eta^{-i}\}$, *where* $i \in \{0, 3, 6, 1, 4, 7\}$, *if* θ *is of dimension* 14.
   (A2) *If* $\dim\theta = 13$, *then* $|\mathrm{Spec}\,\theta(g)| = 5$ *and either* $\mathrm{Spec}\,\theta(g) = \{\varepsilon, \varepsilon^2, \eta, \eta\varepsilon, \eta\varepsilon^2\}$ *or* $\mathrm{Spec}\,\theta(g) = \{\varepsilon, \varepsilon^2, \eta^2, \eta^2\varepsilon, \eta^2\varepsilon^2\}$, *where* $\eta^3 = \varepsilon \neq 1$ *and* $\varepsilon^3 = 1$. *In particular,* $1 \notin \mathrm{Spec}\,\theta(g)$.

(B) *Suppose that* $\mathrm{rank}(g - \mathrm{Id}) = 3$. *Then the following holds*:
   (B1) *If* $n \neq 13, 14, 78$, *then* $|\mathrm{Spec}\,\theta(g)| = 9$.
   (B2) *If* $\dim\theta = 78$, *then* $|\mathrm{Spec}\,\theta(g)| = 8$ *and* $1 \notin \mathrm{Spec}\,\theta(g)$.
   (B3) *If* $\dim\theta = 13$ *or* 14, *then* $|\mathrm{Spec}\,\theta(g)| = 5$ *and*

$$\text{either}\quad \mathrm{Spec}\,\theta(g) = \{1, \eta^3, \eta, \eta^4, \eta^7\} \quad\text{or}\quad \mathrm{Spec}\,\theta(g) = \{1, \eta^6, \eta^2, \eta^5, \eta^8\}.$$

Proof. If $\mathrm{char}\,P = 0$, one can inspect the character table of H from [Atl]. If $\mathrm{char}\,P > 0$, the character table of H is not available explicitly but it is easily recovered from the decomposition matrices available on the [MAtl] website. Let $\mathrm{char}\,P = 2$. Then all the irreducible characters are trivial on $Z(H)$. According to [MAtl], there are 7 irreducible 2-modular characters that do not lift to ordinary characters. They are denoted by ϕ_i, with $i = 6, 7, 12, 13, 14, 15, 16$. Let the ordinary characters of H be labeled χ_j, as in [Atl]. Then $\phi_6 = \chi_{11} - \phi_3$, $\phi_7 = \chi_{10} - \phi_2$, $\phi_{12} = \chi_{91} - \phi_4$, $\phi_{13} = \chi_{90} - \phi_4$, $\phi_{14} = \chi_{27} - \phi_4$, $\phi_{15} = \chi_{46} - \phi_6 - \phi_{13}$ and $\phi_{16} = \chi_{47} - \phi_7 - \phi_{12}$, where the equalities above hold for the character values at elements of odd order. From this one can easily deduce the lemma (observe that in characteristic 2 there are no irreducible Brauer characters of degree 14). In a similar way, one obtains the stated result inspecting the cases when $\mathrm{char}\,P > 3$. □

Lemma 6.19. *Let $H = Sp(8, 3)$ and $\theta \in \mathrm{Irr}_P\, H$ with $\dim\theta > 1$. Let $g \in H$ be an element of order 9 such that* $\mathrm{rank}(g - \mathrm{Id}) = 3$. *Then one of the following holds*:

(1) $|\mathrm{Spec}\,\theta(g)| = 9$;
(2) θ *is a Weil representation of H and either* $\mathrm{Spec}\,\theta(g) = \{1, \eta^3, \eta, \eta^4, \eta^7\}$ *or* $\mathrm{Spec}\,\theta(g) = \{1, \eta^6, \eta^2, \eta^5, \eta^8\}$, *where η is a primitive* 9*-root of* 1.

Proof. Set $t = g^3$. Then t is a transvection and $\operatorname{Spec}\theta(t)$ is either $\{1, \varepsilon\}$ or $\{1, \varepsilon^2\}$ or $\{1, \varepsilon, \varepsilon^2\}$, where ε is a non-trivial 3-root of 1. Set $0 \neq w \in W = (g^3 - \mathrm{Id})V$ and $S_1 = Stab_H(w)$. Furthermore, let U and Y be as above and h be the projection of g into Y. Then $|h| = 3$ and h is a transvection as well. Observe that there exists $g' \in Y$ which is an H-conjugate of g. Let ϕ be an irreducible constituent of $\theta|_{S_1}$ non-trivial on $Z(U)$. By Lemma 6.5, $\phi = \tau \otimes \lambda$, where $\tau, \lambda \in \operatorname{Irr}_P S_1$, $\tau(U)$ is irreducible and $\lambda(U) = \mathrm{Id}$. By Lemma 6.9, $1 \in \operatorname{Spec}(\tau(g^3))$ and by Lemma 2.7 $\operatorname{Spec}\tau(g') = \alpha\{1, \eta^3, \eta, \eta^4, \eta^7\}$ for some 9-root α of 1. As λ can be viewed as a representation of $Y \cong Sp(6,3)$, we observe from Lemma 6.18 that $\operatorname{Spec}\lambda(g')$ contains $\beta\{1, \eta^3, \eta, \eta^4, \eta^7\}$ for some 9-root β of 1. Then it is an easy matter to check that $\operatorname{Spec}\tau(g') \cdot \operatorname{Spec}\lambda(g')$ contains all the 9-roots of 1. Therefore, if the lemma is false then λ is trivial. Hence $\phi|_Y$ is a Weil representation of Y. By Lemma 6.18(B3), $\operatorname{Spec}\phi(g')$ contains either $\{1, \eta^3, \eta, \eta^4, \eta^7\}$ or $\{1, \eta^6, \eta^2, \eta^5, \eta^8\}$. In particular, $1 \in \operatorname{Spec}\theta(g)$. It then follows from Lemma 6.18, that the non-trivial irreducible constituents of $\theta|_Y$ are all of dimension 13 or 14, and so they are Weil representations of Y. By [GMST, Theorem 2.3], θ is a Weil representation of H. Thus $\operatorname{Spec}\theta(t)$ consists of two elements. We conclude that only one of the two options recorded in Lemma 6.18 is realized for the constituents of $\theta|_Y$ and the result follows. $\square$

Next, in order to dispose of the exceptional case arisen in Lemma 6.17(iv), we need two auxiliary results. The first of these is concerned with the Weil representations of $Sp(4,3)$.

Lemma 6.20. *Let $H = Sp(4,3)$ and let $SL(2,9) \hookrightarrow M$ be a standard embedding of $SL(2,9)$ into H. Let $1_H \neq \theta \in \operatorname{Irr}_P H$. Suppose that the irreducible constituents of $\theta|_M$ are either trivial or associated Weil representations of M. Then either θ is a Weil representation of H, or θ is a unique representation of dimension* 6.

Proof. Suppose first that $\operatorname{char} P = 0$ and let χ be the character of θ. Let χ_1, χ_2 be the characters of two associated Weil representations of M, ordered so that $\chi_1(1) = 4$ and $\chi_2(1) = 5$. Observe that $Z(M) = Z(H)$. Hence, there exists two integers k and l with $k > 0$, such that $\chi|_M = k\chi_2 + l \cdot 1_M$ if $\theta(Z(M))$ is the identity, and $\chi|_M = k\chi_1$ otherwise. Let $g \in M$ be of order 5 and let $h \in M$ be of order 8 (so that h projects to an element of $M/Z(M)$ of order 4). Then $\chi_1(g) = -1$ and $\chi_2(h) = -1$ (the same is true for the characters of the other pair of associated Weil representations of M, as their values at elements of order 5 and 8 in Weil representations of the same degree coincide). In particular, viewed as an element of $H/Z(H)$, h belongs to the class $4B$ in [Atl]. Suppose first that $\theta(Z(M)) \neq \mathrm{Id}$. Then $\chi(g) = -k$, and hence $k = 1$, as $\chi(g) \in \{-1, 0, 1\}$ (see [Atl] for the character table of H). Thus $\chi(1) = \chi_1(1) = 4$, as required. Next, let $\theta(Z(M)) = \mathrm{Id}$. Then $\chi|_M = k\chi_2 + l \cdot 1_M$. As $\chi_2(g) = 0$, we have $l \leqslant 1$ (as $\chi(g) \in \{-1, 0, 1\}$). In addition, as $\chi_2(h) = -1$, $\chi(h) = -k + l$. As $\chi(h) \in \{-1, 0, 1\}$, we conclude that $k \leqslant 2$, and hence $\chi(1) = 5, 6$, or 10. Taking into account the values of χ at h, one rules out $\chi(1) = 10$. We conclude that either $\chi(1) = 5$, in which case θ is a Weil representation, or $\chi(1) = 6$, in which case $\theta|_M = \chi_2 + 1_M$. This yields the result.

Next, suppose that $r = \operatorname{char} P > 0$. We only have to inspect the cases $r = 2$ and $r = 5$. The decomposition numbers for M are known, and one can easily deduce from them that for $r = 2$ or 5, if τ is a non-trivial irreducible representation of M over the complex numbers and the composition factors of $\tau \pmod r$ are either trivial or associated Weil representations, then τ itself is a Weil representation. [Here, one needs to recall that in characteristic 2 the non-trivial Weil representations of M have dimension 4. Furthermore, if τ_1, τ_2 are the two distinct Brauer characters of M of degree 4, then τ_1 and τ_2 are not associated.] Therefore, for the representations of H

that either lift to characteristic 0, or occur in a decomposition of a characteristic 0 representation of H, the other terms being only trivial, the result follows from the above.

Let $r = 2$. There is only one Brauer character of H, namely ϕ_5 in [MAtl], that does not lift to characteristic 0, and $\phi_5(1) = 14$. Moreover, $\phi_5(x) = \chi_7(x) - 1_H$ for all $x \in H$ of odd order, where χ_7 is an ordinary irreducible character. By the observation above χ_7 should either be a Weil representation, or have dimension 6, which is false.

Let $r = 5$. There are only two Brauer characters of H that do not lift to characteristic 0. In the [MAtl] notation these are ϕ_{10} and ϕ_{18}, which occur as constituents on the $5'$-classes of the ordinary irreducible characters $\chi_{10} = \phi_{10} + 1_H$ and $\chi_{19} = \phi_{18} + \phi_4$. Here $\phi_{10}(1) = 23$, $\phi_4(1) = 6$ and $\phi_{18}(1) = 58$. As $\chi_{10} = \phi_{10} + 1_H$, the observation above also applies to χ_{10}, yielding a contradiction. Next, observe that χ_{19} is trivial on $Z(H)$, and hence the irreducible constituents of $\phi_{18|M}$ are of dimension 1 or 5. Let τ be the Brauer character of a Weil representation of H of dimension 5; then $\tau(h) = -1$. Therefore, h belongs to the class $4B$ in [Atl]. Thus $\phi_{18|M} = k\tau + l \cdot 1_H$ and $\phi_{18}(h) = -k + l$. As ϕ_4 lifts to characteristic 0, $\phi_4(h) = 0$. Therefore, as $\chi_{19}(h) = 0$, we obtain that $k = l$. On the other hand, $\chi_{19}(h^2) = 0$, $\phi_4(h^2) = 2$ and $\tau(h^2) = 1$, so we get $0 = \chi_{19}(h^2) = k + l + 2$, which is false.

Finally, observe that the 6-dimensional exception still lives when $r = 2$ (in which case $\theta|_M = \tau + 2 \cdot 1_M$, τ being a Weil representation of dimension 4) or $r = 5$ (in which case $\theta|_M = \tau + 1_M$, with $\dim \tau = 5$). □

Lemma 6.21. *Let $H = Sp(8,3)$ and let $Sp(4,9) \hookrightarrow N$ be a standard embedding of $Sp(4,9)$ into H. Let $\operatorname{char} P \neq 2$ and $\theta \in \operatorname{Irr}_P H$ with $\dim\theta > 1$. Suppose that the irreducible constituents of $\theta|_N$ are associated Weil representations of N. Then θ is a Weil representation of H.*

Proof. Let $Sp(2,9) \hookrightarrow N_1$ be a standard embedding of $Sp(2,9)$ into N. As the irreducible constituents of $\theta|_N$ are Weil representations of N associated to each other, the irreducible constituents of $\theta|_{N_1}$ are Weil representations of N_1 associated to each other. Observe that N_1 is contained in a subgroup H_1 of H isomorphic to $Sp(4,3)$ standardly embedded into H. By Lemma 6.20 the non-trivial irreducible constituents of $\theta|_{H_1}$ either are Weil representations of H_1, or have dimension 6. As $\operatorname{char} P \neq 2$, $\theta|_{N_1}$ does not contain the trivial representation. It follows that $\theta|_{H_1}$ does not contain 6-dimensional subrepresentations (otherwise 1_{N_1} would appear in $\theta|_{N_1}$). Hence, by Lemma 6.3, θ is a Weil representation of H. □

Now we are ready to deal with the $Sp(8,3)$ case in Lemma 6.17(iv).

Lemma 6.22. *Let $H = Sp(8,3)$ and $\theta \in \operatorname{Irr}_P H$ with $\dim\theta > 1$. Let $g \in H$ be an element of order 9 such that $\operatorname{Jord} g = \operatorname{diag}\{J_4, J_4\}$. Then one of the following holds:*

(1) $\operatorname{Spec}\theta(g)$ *contains all the* 9*-roots of* 1.
(2) $\dim\theta = 40$ *and up to conjugacy there is exactly one element $g \in H$ with the above Jordan form such that $|\operatorname{Spec}\theta(g)| = 8$. In this case $1 \notin \operatorname{Spec}\theta(g)$.*

Proof. Suppose first that $\operatorname{char} P \neq 2$. By Lemma 6.11 we may assume that g is contained in a subgroup $H_1 \cong Sp(4,9)$. Suppose that (1) is false. Then $|\operatorname{Spec}\phi(g)| < 9$ for every non-trivial irreducible constituent ϕ of $\theta|_{H_1}$. Applying Lemma 6.16 to these ϕ's, we conclude that they either are trivial or have dimension 40. In fact, the trivial ones cannot occur as otherwise $|\operatorname{Spec}\theta(g)| = 9$. For the same reason, the constituents of dimension 40 are all associated to each other. As char $P \neq 2$, by Lemma 6.21, θ is a Weil representation of H. It is well known that the Weil representations of $Sp(2n, q)$ remain irreducible under restriction to $Sp(n, q^2)$. Therefore $\dim\theta = 40$ and the result follows from Lemma 6.16.

Next, assume that char $P = 2$. Recall that in this case, by our definitions (cf. the discussion following Lemma 6.2) the trivial representation is considered to be Weil (unlike in [GMST]). Suppose that (1) is false. Again by Lemma 6.11, we may assume that g is contained in a subgroup isomorphic either to $GL(4, 3)$ or to $SU(4, 3)$. By [Z90] we may rule out the first option (cf. the result quoted in the Introduction, following Theorem 1.4). So, we assume that $g \in K_1 \simeq SU(4, 3)$ and consider the restriction $\theta|_{K_1}$. By Lemma 6.15, the irreducible constituents of $\theta|_{K_1}$ are 20-dimensional Weil representations. It follows that, restricting further from K_1 to a subgroup K_2 isomorphic to $SU(3, 3)$, the irreducible constituents of θ on such a subgroup are also Weil representations. Now, let H_1 be a standard subgroup of H isomorphic to $Sp(6, 3)$ and containing K_2. Direct computations using complex character tables in [Atl] and 2-modular decomposition matrices available on the [MAtl] website show that if an irreducible representation ϕ of $Sp(6, 3)$, when restricted to $SU(3, 3)$, has irreducible constituents which are all Weil for $SU(3, 3)$, then ϕ itself is a Weil representation. We conclude, by [GMST, Theorem 2.3] and Lemma 6.16 that θ is a 40-dimensional Weil representation of H and (2) holds. □

The following lemma completes the analysis of case (ii) in Lemma 6.17.

Lemma 6.23. *Let $H = Sp(m, p)$, with p odd, $m > 4$ and $(m, p) \neq (6, 3)$, and let $\theta \in \mathrm{Irr}_P\, H$ with $\dim\theta > 1$. Let $g \in H$ be an element of order $s = p^\alpha$ such that $t = g^{p^{\alpha-1}}$ is a transvection. Let $\varepsilon \in \mathrm{Spec}\,\theta(t)$. Then one of following holds*:

- (i) $\mathrm{Spec}\,\theta(g)$ *contains all the* $p^{\alpha-1}$*-roots of* ε;
- (ii) $\varepsilon = 1$, $|g| = 9$, $\mathrm{rank}(g - \mathrm{Id}) = 3$, *and θ is a Weil representation of $H = Sp(m, 3)$. In this case* $|\mathrm{Spec}\,\theta(g)| = 5$ *and* $1 \in \mathrm{Spec}\,\theta(g)$.

Furthermore, the multiplicity of every eigenvalue of $\theta(g)$ is at least

$$\max\{1, p^{n-2-p^{2\alpha-2}}\}, \quad \text{where } n = (m-2)/2.$$

Proof. Suppose first that $g = t$. In this case the content of the lemma reduces to the claim about multiplicities. Let $\varepsilon \in \mathrm{Spec}\,\theta(t)$. If $\varepsilon \neq 1$, then the multiplicity of ε is at least $p^{(m-2)/2}$ by Lemma 6.10. If $\varepsilon = 1$, then the multiplicity of ε is at least $(p^{(m-2)/2} - 1)/2$ by Lemma 6.13. As $(p^{(m-2)/2} - 1)/2 \geqslant p^{n-1}$, the lemma is true in this case.

So we assume that $g \neq t$. Furthermore, $m = 4$ and $g \neq t$ forces $p = 3$, which case has been dealt with in Lemma 6.14. In addition, Lemma 6.10 settles the case where $\varepsilon \neq 1$.

In order to deal with the case $\varepsilon = 1$, we set $W = (t - \mathrm{Id})V$, so that $t \in Z(U)$. As usual, let h be the projection of g into $Y \cong Sp(m-2, p)$. Then $|h| < |g|$.

Step 1. $h^{p^{\alpha-2}}$ is not a transvection, except when $|g| = 9$ and $p = 3$.

Indeed, let $t_2 = g^{p^{\alpha-2}}$, so that $h^{p^{\alpha-2}}$ is the projection of t_2 into Y. Set $d := \dim(t_2 - \mathrm{Id})V$. Then $(t_2 - \mathrm{Id})^{d+1} = 0$. Suppose that $h^{p^{\alpha-2}}$ is a transvection. Then $d \leqslant 3$. Since $1 = \dim(t - \mathrm{Id})V = \dim(t_2^p - \mathrm{Id})V = \dim(t_2 - \mathrm{Id})^p V$, it follows that $p = d = 3$. This is only possible when $\mathrm{Jord}\, t_2 = \mathrm{diag}(J_4, \mathrm{Id}_{m-4})$. It follows from Lemma 5.4 that $g = t_2$.

Let T be the underlying space of θ, and let E_1 be the 1-eigenspace of $\theta(t)$. By Proposition 1.2 $E_1 \neq 0$. Moreover, by Lemma 6.13 S_1 acts non-trivially on E_1.

Step 2. The lemma is true if $U|_{E_1} \neq \mathrm{Id}$.

Let T_1 be an irreducible PS_1-submodule of $E_1|_{S_1}$ such that $U|_{T_1} \neq \mathrm{Id}$. Let us consider the elementary abelian group $U_0 = U/Z(U)$ and denote by K_0 its group of characters. As $Z(U)$ acts trivially on T_1, T_1 is acted upon by U_0: thus $T_1|_{U_0} = \bigoplus_{\kappa \in K_0} T_\kappa$, where $T_\kappa = \{x \in T_1\colon ux = \kappa(u)x$, for all $u \in U_0\}$ and the summation runs over a Y-orbit of non-trivial elements of K_0. As the natural $Sp(m-2, p)$-module is self-dual, K_0 is isomorphic to U_0 as $Sp(m-2, p)$-modules. It follows that Y is transitive on $K_0 \setminus \{1\}$. By Lemma 4.3, h has more than $p^{m-3-\alpha}$ regular orbits on $K_0 \setminus \{1\}$. It follows that $\langle h\rangle|_{T_1}$ contains a direct sum of more than $p^{m-3-\alpha}$ regular submodules, which justifies the claim on multiplicities for this case.

Step 3. The lemma is true if $U|_{E_1} = \mathrm{Id}$ and $h^{p^{\alpha-2}}$ is not a transvection.

By Lemma 6.17, applied to $Y \cong Sp(m-2, p)$ acting on E_1, the spectrum of $g|_{E_1} = h|_{E_1}$ contains all the $p^{\alpha-1}$-roots of 1 unless possibly when $Y \cong Sp(8, 3)$, hence $H = Sp(10, 3)$ and $\mathrm{rank}(h^3 - \mathrm{Id}) = 2$, or $Y \cong Sp(4, p)$, hence $H = Sp(6, p)$ and $\mathrm{rank}(h - \mathrm{Id}) = 2$ [notice that, as h is not a transvection, $Y \neq Sp(4, 3)$]. Suppose first that $H = Sp(6, p)$. As $|g| > p$, it follows that $p \leqslant 5$. If $p = 5$, then $\mathrm{Jord}\, g = J_6$. However, $\mathrm{rank}(h - \mathrm{Id}) = 2$ forces $\mathrm{rank}(g - \mathrm{Id}) \leqslant 4$, a contradiction. The case $H = Sp(6, 3)$ is excluded by assumption. So, let $H = Sp(10, 3)$. As $\mathrm{rank}(h^3 - \mathrm{Id}) = 2$, $\mathrm{Jord}\, g$ is not J_{10}. But then $|g| = 9$, which contradicts the assumption $|h| < |g|$. The claim on multiplicities follows from Lemma 6.17 applied to $Y|_{E_1}$. Indeed, Lemma 6.17 gives the bound $p^{(m-4)/2 - p^{2\alpha-2}} > p^{n-2-p^{2\alpha-2}}$.

Step 4. The lemma is true if $U|_{E_1} = \mathrm{Id}$ and h is a transvection.

By Step 1, this can only happen if $p = 3$ and $\mathrm{Jord}\, g = \mathrm{diag}(J_4, \mathrm{Id}_{m-4})$, so $|g| = 9$. If $m = 8$ the result follows from Lemma 6.19. Otherwise, g can be included in a subgroup X of H isomorphic to $Sp(8, 3)$. By the same lemma all the non-trivial irreducible constituents of $\theta|_X$ are Weil representations. Therefore, θ is a Weil representation of H by [GMST, Theorem 2.3]. As $Y|_{E_1} \neq \mathrm{Id}$ and h is a transvection, the argument preceding Step 1 applied to Y and h yields that the multiplicity of every eigenvalue of h on E_1 is at least $(3^{(m-4)/2} - 1)/2 > 3^{n-2}$, which will do. □

7. Unitary groups of characteristic 2

Unless stated otherwise, it is assumed in this section that $p = 2$ (hence $\mathrm{char}\, P \neq 2$) and V is a unitary space of dimension $m > 2$. As above, g is a unipotent element of H, $v \in V$ is an isotropic vector fixed by g, $W = \langle v\rangle$ and W_1 is a complement of W in $W^\perp$. Set $S = Stab_H(W)$, $S_1 = Stab_H(v)$ and $U = O_2(S)$ $(= O_2(S_1))$. Observe that $S = U : Q$ and $S_1 = U : Y$, where Q and Y are the groups defined after Lemma 4.4. Also observe that $Z(S_1) = Z(U) \cong (\mathbb{F}_q, +)$.

We begin with a lemma that refines Lemma 6.2.

Lemma 7.1. *Let $H = SU(m, q)$ and let g be as above. For a given non-trivial irreducible character $\zeta : Z(U) \to P$ let $\tau : S_1 \to GL(q^{m-2}, P)$ be an irreducible representation such that $\tau|_U$ is irreducible and $\tau(z) = \zeta(z) \cdot \mathrm{Id}$ for all $z \in Z(U)$. Then the following holds*:

- (i) *Let χ be the character afforded by τ. If g is not conjugate in S_1 to an element of $YZ(U)$, then $\chi(g) = 0$. If g is conjugate to an element of Y, then $\chi(g) = (-1)^m(-q)^{d(g)}$, where $d(g) = \dim\ker(h - \mathrm{Id})$ and h is the projection of g to Y. If $g \in Z(U)$, then $\chi(g) = q^{m-2}\zeta(g)$.*
- (ii) *Suppose that $\mathrm{Jord}\, g$ consists of a single block and $t = g^{2^{\alpha-1}}$ is a transvection. Then $\chi(g^i) = 0$ for all $i < 2^{\alpha-1}$, and $\chi(t) = q^{m-2}\zeta(t) = \pm q^{m-2}$. Furthermore*: *if $\zeta(t) = -1$,*

then $\operatorname{Spec}\tau(g)$ *consists of all the primitive* 2^α*-roots of* 1; *if* $\zeta(t)=1$, *then* $\operatorname{Spec}\tau(g)$ *consists of all the* $2^{\alpha-1}$*-roots of* 1.

(iii) *Suppose, as in* (ii), *that* $\operatorname{Jord} g$ *consists of a single block. Assume* $|g|=2^\alpha>2$ *and* $\operatorname{rank}(g^{2^{\alpha-1}}-\mathrm{Id})\geqslant 2$. *Then* $\chi(g^i)=0$ *for all* $i<2^\alpha$. *Furthermore,* $\operatorname{Spec}\tau(g)$ *contains all the* 2^α*-roots of* 1, *with equal multiplicity.*

Proof. As already observed in the remark following Lemma 6.2, the result for P-representations follows immediately from the analogue for complex representations. Indeed, the Brauer reduction of τ modulo any prime r distinct from p remains irreducible (as $|U|$ is coprime to r).

(i) The statement follows from [Gé, Theorems 4.5(b) and 4.9.2], except for the refinement to $g\in Z(U)$. As in the latter case $\tau(g)$ is scalar, the claim clearly follows.

(ii) Without loss of generality we may assume that $v\in(t-\mathrm{Id})V$. Let g_1 be a conjugate of g in S_1. As $t\in Z(S_1)$, we have $g_1^{2^{\alpha-1}}=t$. As $t\notin Y$, we also have that $g_1^{2^l}\notin Y$ for all $l\leqslant\alpha-1$. Moreover, $g_1^{2^l}\notin YZ(S_1)$ for all $l\leqslant\alpha-2$. Indeed, if $g_1^{2^l}\in YZ(S_1)$, then $g_1^{2^{l+1}}\in Y$. Whence $l+1\geqslant\alpha$, i.e. $l\geqslant\alpha-1$.

(iii) By (i), it suffices to show that $g^{2^{\alpha-1}}$ is not conjugate in S_1 to an element of $YZ(U)$. Let $B=\{b_1=v,\dots,b_m\}$ be the canonical basis defining S_1 and Y, so that $W=\langle b_1\rangle$ and Y fixes b_m. Observe that, if $x\in YZ(U)$, then $(x-\mathrm{Id})b_m\in W$. Let g_1 be a conjugate of g in S_1. Clearly, $(g-\mathrm{Id})^{m-1}W^\perp=0=(g_1-\mathrm{Id})^{m-1}W^\perp$. As $b_m\notin W^\perp$, it follows that $(g_1-\mathrm{Id})^{m-1}b_m\neq 0$. Hence $(g_1-\mathrm{Id})^{m-2}b_m\notin W$. If $g^{2^{\alpha-1}}\in YZ(U)$, then $(g^{2^{\alpha-1}}-\mathrm{Id})b_m=(g-\mathrm{Id})^{2^{\alpha-1}}b_m\in W$, whence $2^{\alpha-1}>m-2$. This is a contradiction, as the assumptions on $\operatorname{Jord} g$ and $\operatorname{rank}(g^{2^{\alpha-1}}-\mathrm{Id})$ force m to be odd, and hence $m>2^{\alpha-1}+1$.

In order to prove the second claim in (ii) and (iii), set $G=\langle g\rangle$ and let χ_i denote the character of G sending g to ε^i, where ε is a primitive $|g|$-root of 1. We have $(\chi,\chi_i)_G=\frac{1}{|g|}(\chi(1)+\chi(g^{2^{\alpha-1}})\cdot\varepsilon^{-i\cdot 2^{\alpha-1}})=\frac{1}{2^\alpha}(q^{m-2}+(-1)^i q^{m-2}\zeta(t))$, which is equal to 0 if and only if $(-1)^i\zeta(t)=-1$. If $\zeta(t)=-1$, then i is even, which means that ε^i is an eigenvalue of $\varphi(g)$ if and only if i is odd, that is, ε^i is primitive. If $\zeta(t)=1$, then i is odd, which means that all the ε^i's with i even occur as eigenvalues of $\varphi(g)$. Thus the claim in (ii) follows. As for (iii), $(\chi,\chi_i)_G=\frac{1}{2^\alpha}q^{m-2}\neq 0$. □

Next, we need to prove a series of lemmas in order to single out some exceptional low-dimensional cases and establish an inductive basis for general results.

Lemma 7.2. *Let* θ *be a non-trivial irreducible* P*-representation of* $H=SU(4,2)$ *and let* g *be an element of* H *of order* 4 *such that* $\operatorname{rank}(g-\mathrm{Id})=2$. *Then* $|\operatorname{Spec}\theta(g)|=4$, *except when* $\dim\theta=5$ *or* $\operatorname{char}P\neq 3$ *and* $\dim\theta=6$. *In the exceptional cases* $\operatorname{Spec}\theta(g)=\{1,\pm\sqrt{-1}\}$.

Proof. Observe that $\operatorname{Jord} g=\operatorname{diag}\{J_3,J_1\}$, so that g^2 is a transvection. In the [Atl] notation, g belongs to the class $4A$ and squares to the class $2A$. Direct computation based on inspection of ordinary and Brauer characters (cf. [Atl] and [MAtl]) shows that $|\operatorname{Spec}\theta(g)|=4$ provided $\dim\theta>6$. Furthermore, one sees that $\chi_\theta(g)=1$ and $\chi_\theta(g^2)=-3$ if $\dim\theta=5$, whereas $\chi_\theta(g)=2$ and $\chi_\theta(g^2)=-2$ if $\dim\theta=6$. In both cases the result follows. □

Lemma 7.3. *Let* θ *be a non-trivial irreducible* P*-representation of* $H=SU(m,2)$, $m>4$. *Suppose that* $g\in H$ *has order* 4 *and* $\operatorname{rank}(g-\mathrm{Id})=2$. *If* $|\operatorname{Spec}\theta(g)|<4$, *then* θ *is a Weil representation of* H *and* $\operatorname{Spec}\theta(g)=\{1,\pm\sqrt{-1}\}$.

Proof. Let R be a non-degenerate subspace of V of dimension $m-4$ and let $X \simeq SU(4,2)$ be the pointwise stabilizer of R in H. Clearly g can be assumed to be an element of X. Suppose that $|\operatorname{Spec}\theta(g)| < 4$. Then, by Lemma 7.2, the non-trivial irreducible constituents of $\theta|_X$ have dimension 5 or 6, hence they are Weil representations of $SU(4,2)$. Thus, by [GMST, Theorem 2.5], θ is a Weil representation of H. Finally, as the non-trivial irreducible constituents of $\theta|_X$ are Weil representations of $SU(4,2)$, the result follows, again by Lemma 7.2. □

Lemma 7.4. *Let θ be a non-trivial irreducible P-representation of $H = SU(5,2)$ and let g be an element of H of order* 8. *Let Σ_{-1} denote the set of all* 4*-roots of* -1. *The following holds*:

(1) *If* $\dim\theta > 11$, *then* $|\operatorname{Spec}\theta(g)| = 8$.
(2) *If* $\dim\theta = 10$, *then* $\operatorname{Spec}\theta(g) = \{\pm\sqrt{-1}, \Sigma_{-1}\}$.
(3) *If* $\dim\theta = 11$, *then* $\operatorname{Spec}\theta(g) = \{1, \pm\sqrt{-1}, \Sigma_{-1}\}$.
(4) *If* $\dim\theta = 10$, *then* $\operatorname{Spec}\theta(g^2) = \{-1, \pm\sqrt{-1}\}$.
(5) *If* $\dim\theta > 10$, *then* $|\operatorname{Spec}\theta(g^2)| = 4$.

Proof. Observe that, in the [Atl] notation, g belongs to the class $8A$ and g^2 belongs to the class $4B$; furthermore $\operatorname{Jord}(g^2) = \operatorname{diag}(J_3, J_2)$. The statement then follows from computations on ordinary and Brauer characters. □

Remark. Observe that, if char $P = 3$, then H has no irreducible representations of degree 11.

Lemma 7.5. *Let θ be a non-trivial irreducible P-representation of $H = SU(m,2)$, where* $3 < m \leqslant 6$. *Suppose that $g \in H$ has order* 4. *If* $|\operatorname{Spec}\theta(g)| < 4$, *then θ is a Weil representation of H and either* $\operatorname{Jord} g = \operatorname{diag}\{J_3, \mathrm{Id}_{m-3}\}$ *and* $\operatorname{Spec}\theta(g) = \{1, \pm\sqrt{-1}\}$, *or* $m = 5$, $\dim\theta = 10$, $\operatorname{Jord} g = \operatorname{diag}\{J_3, J_2\}$ *and* $\operatorname{Spec}\theta(g) = \{-1, \pm\sqrt{-1}\}$.

Proof. If $\operatorname{rank}(g - \mathrm{Id}) = 2$ the statement follows from Lemma 7.3. So, suppose that $\operatorname{rank}(g - \mathrm{Id}) > 2$. Let l be the highest dimension of an indecomposable g-submodule. Then $l = 3$ or 4 and there exists two non-degenerate g-submodules V_1, V_2 of V such that $g|_{V_1}$ is indecomposable, $\dim(V_1) = l$ and $V = V_1 \oplus V_2$. Let $X_1 = \{x \in H\colon xV_1 = V_1 \text{ and } x|_{V_2} = \mathrm{Id}\}$ and $X_2 = \{x \in H\colon xV_2 = V_2 \text{ and } x|_{V_1} = \mathrm{Id}\}$. Then $g = g_1g_2$, where $g_i \in X_i$ for $i = 1, 2$. Furthermore, if ϕ is an irreducible constituent of $\theta|_{X_1X_2}$ then $\phi = \phi_1 \otimes \phi_2$, where ϕ_i is an irreducible representation of X_i for $i = 1, 2$. We can assume that ϕ_1 is non-trivial.

Suppose first that $l = 4$. In the [Atl] notation, $g_1 \in SU(4,2)$ belongs to the class $4B$ and squares to the class $2B$. Computation based on inspection of ordinary and Brauer characters (cf. [Atl] and [MAtl]) shows that $\operatorname{Spec}\phi_1(g_1)$ consists of all the 4-roots of 1. Therefore, both $\operatorname{Spec}\phi(g)$ and $\operatorname{Spec}\theta(g)$ consist of all the 4-roots of 1.

Let $l = 3$. Then, $\operatorname{rank}(g - \mathrm{Id}) > 2$ forces $m > 4$. The case $m = 5$ has been considered in Lemma 7.4(4) and (5). Assume that $m = 6$. Then the Jordan form of g is either $\operatorname{diag}\{J_3, J_2, J_1\}$ or $\operatorname{diag}\{J_3, J_3\}$. In the first case g lies in a subgroup X of H isomorphic to $SU(5,2)$. If ϕ is a non-trivial irreducible constituent of $\theta|_X$, then by Lemma 7.4 $|\operatorname{Spec}\phi(g)| = 4$ (and hence $|\operatorname{Spec}\theta(g)| = 4$), unless $\dim\phi \leqslant 10$. If every irreducible constituent of $\theta|_X$ is of dimension at most 10, then by Lemma 6.4 θ is a Weil representation of H. In the latter case θ is of dimension 21 or 22, and hence $\theta|_X$ either contains irreducible constituents of dimension 11, or it contains irreducible constituents of dimension 1 and 10. In both cases, we deduce from Lemma 7.4 that $\operatorname{Spec}\theta(g)$ consists of all the 4-roots of 1. Next, suppose that $\operatorname{Jord} g = \operatorname{diag}\{J_3, J_3\}$. Then there is an element h of order 3 in $C_H(g)$ such that $|C_H(gh)| = 3 \cdot 2^4 = 48$. Therefore gh belongs to one of the classes $12F$, $12G$ or $12H$, whence g^3 belongs to one of the classes $4C$, $4D$ or $4E$. It follows that g^2 belongs to the class $2B$. With this data, it is easy to check that $\operatorname{Spec}\theta(g)$ consists of all the 4-roots of 1 provided $\operatorname{char} P = 0$. Tedious but elementary computations (using the decomposition matrices for r-modular representations ($r = 3, 5, 7, 11$) available on the [MAtl] website) show that the same holds when $\operatorname{char} P > 2$. □

Most of the contents of the following lemma can be extracted from the general discussion of Weil representations following Lemma 6.2, but we record them for the reader's convenience.

Lemma 7.6. *Let $H = SU(m, 2)$, where $m > 3$, and let θ be a (non-trivial) irreducible Weil representation of H. Let $X \cong SU(m-1, 2)$ be the stabilizer of an anisotropic vector, and let* $\operatorname{char} P = r$.

(1) *Suppose that $r = 0$. Then $\dim\theta = (2^m - (-1)^m)/3$ or $2(2^{m-1} + (-1)^m)/3$. The restriction $\theta|_X$ is the sum of two irreducible Weil representations of X; in addition, if m is even, then at least one of the two constituents is of dimension $(2^{m-1} + 1)/3$.*
(2) *Suppose that $r > 0$. If $r \neq 3$, then the claims in (1) remain true. If $r = 3$, then $\dim\theta = (2^m - 2)/3$ if m is odd, otherwise $\dim\theta = (2^m - 1)/3$. In both cases, θ lifts to characteristic zero. In addition, if m is even then 1_X is an irreducible constituent of $\theta|_X$.*

Proof. (1) The claim on dimensions is just the specialization of the general dimension formula at $q = 2$. It is easy to observe that $\theta|_X$ is the sum of irreducible Weil representations of X (for instance, see [T-Z1, Lemma 4.2]). By comparing the dimensions, one also obtains the last part of the statement.

(2) If θ is as in (1) and $r \neq 3$, then $\theta \pmod r$ remains irreducible (see [H-M, Proposition 9]). So the claim follows for $r \neq 3$. If $r = 3$, the first assertion follows again from Proposition 9 and other comments on p. 755 in [H-M]. The additional claim follows from the comparison of dimensions. □

Lemma 7.7. *Let θ be an irreducible Weil representation of $H = SU(m, 2)$, where $m > 5$. Suppose that $g \in H$ has order 8 and $\operatorname{rank}(g - \mathrm{Id}) = 4$. Let Σ_{-1} denote the set of all 4-roots of -1. Then* $\operatorname{Spec}\theta(g) = \{1, \pm\sqrt{-1}, \Sigma_{-1}\}$.

Proof. As $|g| = 8$, $\operatorname{Jord} g$ must contain a block of size $\geqslant 5$. As by assumption $\operatorname{rank}(g - \mathrm{Id}) = 4$, it follows that $\operatorname{Jord} g = \operatorname{diag}\{J_5, \mathrm{Id}_{m-5}\}$. Let R be a non-degenerate subspace of V of dimension $m - 5$ and let $X \simeq SU(5, 2)$ be the pointwise stabilizer of R in H. Clearly g can be assumed to be an element of X. The irreducible constituents of $\theta|_X$ are Weil representations of X, and furthermore Lemma 7.6 tells us that, if $m = 6$, then one of the irreducible constituents of $\theta|_X$ is of dimension 11, except when $r = 3$, in which case a constituent of dimension 10 occurs together with a trivial one. This, together with Lemma 7.4, yields the result. □

Lemma 7.8. *Let θ be a Weil irreducible representation of $H = SU(7, 2)$ of dimension 42. Suppose that $g \in H$ has order 8 and $\operatorname{Jord} g = \operatorname{diag}\{J_5, J_2\}$. Let Σ_{-1} denote the set of all 4-roots of -1. Then* $\operatorname{Spec}\theta(g) = \{-1, \pm\sqrt{-1}, \Sigma_{-1}\}$.

Proof. We can write $g = yz$, where z is a transvection and $\mathrm{Jord}\, y = \{J_5, \mathrm{Id}_2\}$. Set $W = (z - \mathrm{Id})V$ and let S_1 be the stabilizer in H of a non-zero vector of W. By Lemma 2.5 (considering the Jordan form of g) we may assume that $z \in Z(U)$ and $y \in Y$. It follows from [L-S, 4.4(a)], that $\theta|_{S_1} = \phi \oplus \tau$, where ϕ is an irreducible representation of S_1 non-trivial on $Z(U)$ (hence of dimension 2^5) and τ is an irreducible representation of S_1 trivial on U (hence of dimension 10). By Lemma 7.1(i), the character value of ϕ at y^i is equal to 2 if i is odd, -4 if $i = 2$ or 6 and -16 if $i = 4$. It follows that $\mathrm{Spec}\,\phi(y) = \{1, \pm\sqrt{-1}, \Sigma_{-1}\}$ and $\mathrm{Spec}\,\tau(y) = \{\pm\sqrt{-1}, \Sigma_{-1}\}$ (cf. Lemma 7.4). As $z \in Z(U)$ and $g = yz$, $\phi(z) = -\mathrm{Id}$ and $\phi(g) = -\phi(y)$. The result follows. □

Theorem 7.9. *Let g be a 2-element of $H = SU(m, q)$. Suppose that* $\mathrm{Jord}\, g = J_m$ *and* $t = g^{2^{\alpha-1}}$ *is not a transvection. If* $\theta \in \mathrm{Irr}_P\, H$ *with* $\dim\theta > 1$*, then* $|\mathrm{Spec}\,\theta(g)| = 2^\alpha$.

Proof. Let S_1 be defined as above, and let ϕ be an irreducible constituent of $\theta|_{S_1}$ non-trivial on $Z(U)$. Then, by Lemma 6.5 $\phi = \tau \otimes \lambda$ where $\tau, \lambda \in \mathrm{Irr}_P\, S_1$, $\tau|_Y$ is a generic Weil representation, $\tau|_U$ is irreducible of dimension $|F|^{\frac{m-2}{2}}$ and $\lambda(U) = \mathrm{Id}$. As t is not a transvection, Lemma 7.1(iii) applies, hence $|\mathrm{Spec}\,\tau(g)| = |g|$. Thus, this is also true of $\phi(g)$, and hence of $\theta(g)$. □

Theorem 7.10. *Let g be a 2-element of $H = SU(m, q)$, where $(m, q) \neq (3, 2)$. Suppose that* $\mathrm{Jord}\, g = J_m$*, where $m = 2^{\alpha-1} + 1$ (so that $t = g^{2^{\alpha-1}}$ is a transvection). If* $\theta \in \mathrm{Irr}_P\, H$ *with* $\dim\theta > 1$*, then* $|\mathrm{Spec}\,\theta(g)| = 2^\alpha$ *unless $q = 2$, $m = 5$ and θ is a Weil representation of H.*

Proof. We have set above $S = \mathit{Stab}_H(W)$, $S_1 = \mathit{Stab}_H(v)$, where $v \in V$ is a non-zero isotropic vector fixed by g and $W = \langle v\rangle$; furthermore, $U = O_2(S)$. Thus $Z(U)$ is an elementary abelian normal subgroup of S of order q. Observe that without loss of generality we may assume that $v \in (t - \mathrm{Id})V$, and hence $t \in Z(U)$. Let K denote the group of characters of $Z(U)$. The action of S on $Z(U)$ induces an action of S on K. The group S acts transitively on the non-identity elements of $Z(U)$, and hence S has a single non-trivial orbit on K. On the other hand, S_1 acts trivially on $Z(U)$, and hence also on K. Let T be the PH-module afforded by θ. Then $T|_{Z(U)}$ decomposes into homogeneous components T_ζ, namely $T|_{Z(U)} = \bigoplus_{\zeta\in K} T_\zeta$, where $T_\zeta = \{x \in T\colon zx = \zeta(z)x$ for all $z \in Z(U)\}$ and ζ runs over K^*. Clearly, $\zeta(t) = \pm 1$, and $\zeta(t) = -1$ for some ζ. Let R be an irreducible constituent of S_1 on a component T_ζ such that $\zeta(t) = -1$, and let ρ be the corresponding representation of S_1. By Lemma 6.5 $\rho = \varphi \otimes \lambda$, where φ behaves as τ in Lemma 7.1 and λ is an irreducible representation of S_1 trivial on U. Therefore, $\mathrm{Spec}\,\rho(g) = \mathrm{Spec}\,\varphi(g) \times \mathrm{Spec}\,\lambda(g)$. Let Σ_{-1} denote the set of all $2^{\alpha-1}$-roots of -1 and Σ_1 the set of all $2^{\alpha-1}$-roots of 1. By Lemma 7.1(ii), $\mathrm{Spec}\,\varphi(g) = \Sigma_{-1}$. Obviously, $\mathrm{Spec}\,\lambda(g)$ is a subset of Σ_1 and $\Sigma_{-1} \times \Sigma_1 = \Sigma_{-1}$. Therefore, $\mathrm{Spec}\,\rho(g) = \Sigma_{-1}$. As ρ is a constituent of θ, we conclude that $\mathrm{Spec}\,\theta(g)$ contains Σ_{-1}.

We are left to show that $\mathrm{Spec}\,\theta(g)$ contains Σ_1. Suppose first that $q > 2$, so that $Z(U)$, and hence K, is not cyclic. As S acts transitively on $K - \{1\}$, there exists a component $T_{\zeta'}$ of $T|_{Z(U)}$ such that $\zeta' \neq 1$ and $\zeta'(t) = 1$. Let $\rho' = \varphi' \otimes \lambda'$ be an analogue of the representation ρ considered above, but corresponding to ζ'. Again by Lemma 7.1(ii), we conclude that $\mathrm{Spec}\,\varphi'(g)$ coincides with Σ_1, and hence $\mathrm{Spec}\,\rho'(g)$ contains Σ_1. As above, since ρ' is a constituent of θ we are done.

Next, suppose that $q = 2$. Then $Z(U) = \langle t\rangle$ has order 2, and $T|_{Z(U)} = T_1 \oplus T_{-1}$, where $T_1 = T_\zeta$ with $\zeta = 1$ and $T_{-1} = T_\zeta$ with $\zeta \neq 1$. It follows that $\mathrm{Spec}(g|_{T_{-1}}) = \Sigma_{-1}$. As $Z(U)$ acts trivially on T_1, T_1 is in fact acted upon by $S_1/Z(U)$. Set $d = g^{2^{\alpha-2}}$, so that $|d| = 4$ and $d^2 = t$. As $t|_{T_{-1}} = -\mathrm{Id}$, d has eigenvalues $\pm\sqrt{-1}$ on T_{-1}. We claim that, provided $m > 3$, the subgroup Y of S_1 (isomorphic to $SU(W_1)$) contains a conjugate of t which acts non-trivially on T_1. Indeed, as $m > 3$, Y does contain a conjugate t' of t under H and t' acts on both T_1 and T_{-1}. Suppose $t'|_{T_1} = \mathrm{Id}$. Then $t'|_{T_{-1}}$ is not scalar, as otherwise t' would centralize S_1: hence the eigenvalue 1 occurs in t' with multiplicity greater than $\dim(T_1)$, which is exactly the multiplicity of 1 in t. This is a contradiction, as t and t' are conjugate. Hence Y acts non-trivially on T_1 (cf. also Lemma 6.13). Observe that $d \notin U$, unless $m = 3$ and $g = d$. Indeed, if $d \in U$, then $\mathrm{rank}(d - \mathrm{Id}) = 2$. As $\mathrm{Jord}\, d = \mathrm{diag}(J_3, (2^{\alpha-2} - 1)J_2)$, it follows that $m = 3$ and $\alpha = 2$, that is $g = d$. However in this case $H = SU(3, 2)$, against our assumptions on H.

We now distinguish two cases:

(1) Suppose first that U acts non-trivially on T_1. This means that T_1 is acted upon non-trivially by the quotient $U_0 = U/Z(U)$. Let us consider S_1 acting on U_0 by conjugation. Then, as U acts trivially on U_0, setting $\bar{S}_1 = S_1/U \simeq Y_1 = SU(W_1)$, U_0 can be identified to the natural Y_1-module W_1. Denoting as usual by h the projection of g into Y, we also observe that, by Lemma 4.6, $|g| > |h|$. On the other hand, by the above, $d \notin U$, so $|h| = 2^{\alpha-1}$. This ensures that, considering the group $\langle\langle g\rangle/\langle t\rangle, U_0\rangle$ acting on T_1, the assumptions of Lemma 2.11 are fulfilled and therefore $g|_{T_1}$ has $o(h)$ distinct eigenvalues; as these are exactly the $2^{\alpha-1}$-roots of 1, we are done.

(2) Next, suppose that U acts trivially on T_1. Thus, T_1 is acted upon by $S_1/U \simeq Y_1 = SU(W_1)$. As seen above, $|h| = 2^{\alpha-1}$. In particular, since $\mathrm{rank}(g^{2^{\alpha-2}} - \mathrm{Id}) > 3$ if $m > 5$, $h^{2^{\alpha-2}}$ is not a transvection in Y_1 unless $m = 5$. Therefore, by Theorem 7.9, if $m > 5$ then $\mathrm{Spec}(h|_{T_1})$ consists of $2^{\alpha-1}$ elements. The result follows.

Finally, if $m = 5$ and $q = 2$, then θ is a Weil representation by Lemma 7.4. □

The following two lemmas concerning Weil representations are instrumental in the proof of the subsequent Theorems 7.13 and 7.15. (Observe that in both lemmas q need not be even.)

Lemma 7.11. *Let $H = SU(m, q)$. Then every Weil representation of H lifts to characteristic* 0.

Proof. The statement can be deduced from computations on the characters of Weil representation of H available in [D-T]. Indeed, let ζ_i for $i = 0, \dots, q$ be the characters of the Weil representations of H labeled as in [D-T]. Let r be a prime. If $(r, q + 1) = 1$, then every Weil representation of H remains irreducible under reduction modulo r by [D-T, Theorem 7.2]. This is also true for the Weil representations of degree x, where $x = (q^m - 1)/(q + 1)$ if m is even, and $x = (q^m - q)/(q + 1)$ if m is odd, as x is the least degree of a non-trivial representation of H. Assume $(r, q + 1) \neq 1$. According to the proof of Theorem 7.2(ii) in [D-T], $\zeta_i \pmod r = \zeta_j \pmod r + (-1)^m(\delta_{i,0} - \delta_{j,0})$ (where $\delta_{i,k}$ is the Kronecker symbol) whenever $i - j$ is divisible by ℓ, the r'-part of $q + 1$. Let m be even. Then $\zeta_j(1) = x$ for $j > 0$ and $\zeta_0 \pmod r = 1 + \zeta_\ell \pmod r$. So the lemma follows for m even. Let m be odd. If j is not an r-power then $\zeta_i \pmod r$ is irreducible (see [H-M, Proposition 9]). Otherwise, $\zeta_i \pmod r = 1 + \zeta_0$ by the above or alternatively by [H-M, proof of Proposition 9]. So the lemma follows. □

Lemma 7.12. *Let $H = SU(m, q)$ and let $V = V_1 \oplus V_2$, where V_1, V_2 are mutually orthogonal non-degenerate subspaces of V. Set $k = \dim(V_2)$ and $X = X_1X_2$, where $X_i = \{x \in H\colon xV_i = V_i,\ x|_{V_j} = \mathrm{Id}\}$, for $i, j \in \{1, 2\}$ and $i \neq j$. Let θ be a (non-trivial) Weil representation of H and let ϕ_1 be an irreducible constituent of $\theta|_{X_1}$. Then there exists an irreducible constituent ϕ of $\theta|_X$ and an irreducible constituent ϕ_2 of $\theta|_{X_2}$ such that $\dim\phi_2 \geqslant (q^k - q)/(q + 1)$ and $\phi = \phi_1 \otimes \phi_2$.*

Proof. Let τ be a complex irreducible representation of H such that θ is a constituent of $\tau \bmod r$, where $r = \operatorname{char} P$. If σ is an irreducible constituent of $\tau|_X$, then we may express σ as $\sigma = \sigma_1 \otimes \sigma_2$, where σ_i is an irreducible Weil representation of $X_i \simeq SU(V_i)$. Suppose that σ is chosen so that ϕ_1 is a constituent of $\sigma_1 \bmod r$. Recall that $\sigma_2 \bmod r$ has an irreducible constituent, say ϕ_2, of dimension at least $(q^k - q)/(q + 1)$ (cf. the comments preceding Lemma 6.4). It follows that $\sigma \bmod r$ has a constituent $\phi = \phi_1 \otimes \phi_2$, where $\dim \phi_2 \geqslant (q^k - q)/(q + 1)$. □

Theorem 7.13. *Let $H = SU(m, q)$, where $(m, q) \neq (3, 2)$, and let g be a non-trivial 2-element of H. If $\theta \in \operatorname{Irr}_P H$ with $\dim \theta > 1$, then $|\operatorname{Spec} \theta(g)| < |g|$ if and only if $q = 2$, θ is a Weil representation, and one of the following holds*:

(1) $m = 5$ *and* $\operatorname{Jord} g = J_m$;
(2) $\operatorname{Jord} g = \operatorname{diag}(J_l, \mathrm{Id}_{m-l})$, *where* $m > l$ *and* $l = 3$ *or* 5;
(3) $\operatorname{Jord} g = \operatorname{diag}(J_{m-2}, J_2)$ *and either* $m = 5$ *and* $\dim \theta = 10$, *or* $m = 7$ *and* $\dim \theta = 42$.

Proof. If g consists of a single Jordan block, then Theorems 7.9 and 7.10 yield case (1) of the statement. So, suppose that g has more than one block and let l be the maximum size of a Jordan block in $\operatorname{Jord} g$. If $l = 2$, then $|g| = 2$, in which case the theorem is trivial. Let $l > 2$. According to Lemma 2.5, we can write $V = V_1 \oplus V_2$, where $gV_i = V_i$ for $i = 1, 2$ and $\operatorname{Jord} g|_{V_1} = J_l$. Let $X = X_1 \times X_2$ be a subgroup of H such that $XV_i = V_i$, $X_i \cong SU(V_i)$ and X_i acts trivially on V_j for $j \neq i$. Then $g \in X$ and $g = g_1 g_2$, where $g_i \in X_i$ and $\operatorname{Jord} g_1|_{V_1} = J_l$. Let ϕ be an irreducible constituent of $\theta|_X$. Then $\phi = \phi_1 \otimes \phi_2$, where ϕ_i is an irreducible representation of X_i for $i = 1, 2$. In addition, $\phi(g) = \phi_1(g_1) \otimes \phi_2(g_2)$, and hence $\operatorname{Spec} \phi(g) = \operatorname{Spec} \phi_1(g_1) \cdot \operatorname{Spec} \phi_2(g_2)$. Suppose that $|\operatorname{Spec} \theta(g)| < |g|$. Then $|\operatorname{Spec} \phi(g)| < |g|$ and hence $|\operatorname{Spec} \phi_1(g_1)| < |g_1| = |g|$. Clearly, we can choose ϕ such that the kernel of ϕ_1 lies in $Z(X_1)$. If $l > 5$ or $q > 2$, or $l = 4$ and $q = 2$, it follows from Theorems 7.9 and 7.10 that $|\operatorname{Spec} \phi_1(g_1)| = |g_1|$, which is a contradiction.

So, we may suppose that $q = 2$ and $l = 3$ or 5. If $g_2 = \mathrm{Id}$, then $\operatorname{Jord} g = \operatorname{diag}(J_l, \mathrm{Id}_{m-l})$ and Lemma 7.3 together with Theorem 7.10 yields case (2). So assume $g_2 \neq \mathrm{Id}$.

Suppose first that $l = 5$. Then $m > 6$, as $g_2 \neq \mathrm{Id}$. Since $|\operatorname{Spec} \phi_1(g_1)| < |g_1|$, Theorem 7.10 tells us that, for every choice of ϕ, either ϕ_1 is trivial or ϕ_1 is a Weil representation of $SU(5, 2)$. By [GMST, Theorem 2.5], θ is a Weil representation of $SU(m, 2)$. By Lemma 7.11, every Weil representation lifts to characteristic zero. Therefore, we may assume that $r = 0$. Furthermore, recall that the irreducible constituents of $\theta|_{X_i}$ are Weil representations of X_i for $i = 1, 2$. Therefore, every irreducible constituent ϕ of $\theta|_X$ has shape $\phi = \phi_1 \otimes \phi_2$ where both ϕ_1 and ϕ_2 are Weil representations of X_1 and X_2, respectively. Now, $\theta|_{X_1}$ has an irreducible constituent of dimension 11 (by induction, as this is true for $m = 6$, see Lemma 7.6), and hence there exists ϕ such that $\dim \phi_1 = 11$. By Lemma 7.4, $|\operatorname{Spec} \phi_1(g_1)| = 7$. Thus $|\operatorname{Spec} \phi_1(g_1) \cdot \operatorname{Spec} \phi_2(g_2)| = 8$, unless $\phi_2(g_2)$ is scalar. This implies either $m - 5 = 3$ and $\dim \phi_2 = 2$ or $m - 5 = 2$ and $\dim \phi_2 = 1$. Suppose first that $m = 7$. Then either $\dim \theta = 42$ or $\dim \theta = 43$. The case where $\dim \theta = 42$ has been dealt with in Lemma 7.8. So, assume that $\dim \theta = 43$. In this case the reduction of θ modulo 3 has two irreducible constituents, one trivial and the other one of dimension 42 (cf. Lemma 7.6). It follows from this and Lemma 7.8 that $|\operatorname{Spec} \theta(g)| = 8$. This completes the proof in the case $m = 7$. Next, suppose that $m = 8$. Then $\dim(V_2) = 3$, and either $\operatorname{Jord} g|_{V_2} = \operatorname{diag}(J_2, \mathrm{Id}_1)$ or $\operatorname{Jord} g|_{V_2} = J_3$. In the former case g belongs to a subgroup $H_1 \cong SU(7, 2)$ and the restriction of θ to H_1 contains as a constituent a Weil representation of dimension 43 (cf. Lemma 7.6). So we are done in this case. Finally, suppose that $\operatorname{Jord} g|_{V_2} = J_3$. Then, we may choose $\phi = \phi_1 \otimes \phi_2$

so that $\dim\phi_2 = 3$, in which case $\phi_2(g_2)$ has 3 distinct eigenvalues. This and Lemma 7.4 easily imply that $|\operatorname{Spec}\phi_1(g_1)\cdot\operatorname{Spec}\phi_2(g_2)| = 8$.

Finally, suppose that $l = 3$, so that $|g| = 4$. In view of Lemma 7.5, we may assume that $m > 6$. Arguing as in the case $l = 5$, we see that θ is a Weil representation of $SU(m,2)$. For $v \in V_2$ set $S = Stab_H(v)$. Suppose that g has Jordan form distinct from $\operatorname{diag}(J_3, 1, \dots, 1)$. Then v can be chosen so that the projection h of g into $S/O_2(S) \cong SU(m-2,2)$ has Jordan form distinct from $\operatorname{diag}(J_3, 1, \dots, 1)$ and $|h| = 4$. Let η be an irreducible constituent of $\theta|_S$ non-trivial on $Z(S)$. It suffices to show that $\eta(h)$ has 4 distinct eigenvalues. Observe that η lifts to characteristic 0, so we may assume $r = 0$. Then Lemma 7.1 allows to compute the character of η. In particular, $\eta(1) = 2^{m-2}$ and the absolute value of $\eta(h^2)$ and $\eta(h) = \eta(h^{-1})$ does not exceed 2^{m-3} and 2^{m-5}, respectively. As $2^{m-2} - 2^{m-3} - 2\cdot 2^{m-5} > 0$, it follows that every irreducible character of $\langle h\rangle$ is a constituent of the restriction of η to $\langle h\rangle$. This completes the proof of the theorem. □

We need yet another useful fact concerning Weil representations:

Lemma 7.14. *(See [GMST, Corollary 12.4].) Let $H = SU(m,q)$ and let $1_H \neq \theta \in \operatorname{Irr}_P H$ be such that all the 1-dimensional constituents of $\theta|_U$ are trivial. Then θ is a Weil representation of H.*

Theorem 7.15. *Let $H = SU(m,q)$ and $1_H \neq \theta \in \operatorname{Irr}_P H$. Let g be a non-trivial 2-element of H of order s. Suppose that $m > \max\{s+3, 12\}$. Then the multiplicity of every eigenvalue of $\theta(g)$ is at least q^{m-2}/s, unless θ is a Weil representation of H, in which case the multiplicity of every eigenvalue of $\theta(g)$ is at least $(q^{m-s-3} - q)/(q+1)$.*

Proof. By Lemma 7.14, either θ is a Weil representation of H or the restriction $\theta|_{S_1}$ contains an irreducible constituent ϕ trivial on $Z(U)$ and non-trivial on U. Let T be the PS_1-module afforded by ϕ. As $\phi(U)$ is abelian, we can write $T = \bigoplus T_\alpha$, where α runs over a Y-orbit O of non-trivial elements of K_0, K_0 being the group of characters of $U_0 = U/Z(U)$. Set $t = g^{s/2}$, so that $t^2 = 1$. As $m > s+3$, by Lemma 4.6 we can choose U such that $t \notin U$. Let h be the projection of g into Y. Then $|g| = |h|$. Observe that K_0 can be obtained from U_0 as the τ-twist of the dual of U_0, where τ is the Galois automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Therefore U_0 and K_0 are isomorphic as $\mathbb{F}_{q^2}\langle h\rangle$-modules. Lemma 4.3 applied to $Y \simeq SU(m-2,q)$ tells us that $\langle h\rangle$ has at least q^{m-2}/s regular orbits in O. It follows that T as a $P\langle g\rangle$-module contains a direct sum of at least q^{m-2}/s regular submodules. So the result follows.

Next, suppose that θ is a Weil representation of H. As θ lifts to characteristic zero, by Lemma 7.11, and the eigenvalues of $\theta(g)$ are preserved under lifting, we may suppose that $\operatorname{char} P = 0$.

By Lemma 2.5(ii), we may write V as a direct sum of two mutually orthogonal non-degenerate $\langle g\rangle$-submodules $V = V_1 \oplus V_2$, and suppose that g acts faithfully on V_1 (not excluding the option $V_2 = 0$). Let $X = X_1 \times X_2$ be a subgroup of H such that $XV_i = V_i$, $X_i \cong SU(V_i)$ and X_i acts trivially on V_j for $j \neq i$. Then $g \in X$ and $g = g_1g_2$, where $g_i \in X_i$. As the irreducible constituents of $\theta|_{X_i}$ are Weil representations of X_i for $i = 1,2$, every irreducible constituent ϕ of $\theta|_X$ has shape $\phi = \phi_1 \otimes \phi_2$ where both ϕ_1 and ϕ_2 are Weil representations of X_1 and X_2, respectively. Thus $\phi(g) = \phi_1(g_1) \otimes \phi_2(g_2)$. (In the previous setting, it is understood that, if $V_2 = 0$, then $X_2 = \{1_H\}$, $g_2 = 1_H$ and ϕ_2 is trivial.)

We observe first that the theorem is true if $|\operatorname{Spec}\phi_1(g_1)| = |g_1| = |g|$ (that is, g_1 does not belong to one of the exceptional cases in Theorem 7.13) and furthermore $\dim(V_2) > 2$. Indeed, according to Lemma 7.12, we may choose ϕ in such a way that $\dim\phi_2 \geqslant (q^{\dim(V_2)} - q)/(q+1)$. It follows that each of the s-roots of 1 occurs as an eigenvalue of $\phi(g)$ (and hence of $\theta(g)$) with multiplicity at least $\max\{1, (q^{\dim(V_2)} - q)/(q+1)\}$. Now, we distinguish two cases.

Case (1). Either $q > 2$ or $q = 2$ and $|g| > 8$. In this case we choose V_1 to be indecomposable, hence of dimension $\leqslant s$. As $m > s+3$, we have $\dim(V_2) > 2$, and hence we can use the estimate above. We conclude, by Theorem 7.13, that each s-root of 1 occurs as an eigenvalue of $\theta(g)$ with multiplicity at least $(q^{m-s} - q)/(q+1)$.

Case (2). $q = 2$ and $|g| \leqslant 8$.

We need to choose V_1 to be of minimal dimension such that $|\operatorname{Spec}\phi_1(g_1)| = |\operatorname{Spec}\theta(g)|$. Suppose first that $|\operatorname{Spec}\theta(g)| < s$. Then $s > 2$, and by Theorem 7.13 either $\dim(V) < 8$ or $\operatorname{Jord} g = \operatorname{diag}\{J_l, \mathrm{Id}_{m-l}\}$, where $l \leqslant 5$. By our assumptions, we may ignore the first instance. In the second instance, we may choose V_1 of dimension at most 5, and again $m > s+3$ forces $\dim(V_2) > 2$.

Finally, suppose that $|\operatorname{Spec}\theta(g)| = s$. Then, by Theorem 7.13, the Jordan form of g is not of shape $\operatorname{diag}\{J_l, \mathrm{Id}_{m-l}\}$, where $l = 3, 5$. If g has a Jordan block of size $r = 6$ or 7, we choose V_1 such that $\operatorname{Jord} g|_{V_1} = J_r$ and then, by Theorem 7.13, $|\operatorname{Spec}\phi_1(g_1)| = s$. In this case, as above, we are done. Otherwise, each Jordan block of g has size at most 5. If $|g| = 8$, then g has at least one block of size 5. As $m > 12$, we can choose V_1 of dimension at most 10 such that $|\operatorname{Spec}\phi_1(g_1)| = s$. If $|g| = 4$ and $m > 8$, we can choose V_1 of dimension at most 6 such that $|\operatorname{Spec}\phi_1(g_1)| = 4$; if $|g| = 2$ and $m > 6$, we can choose V_1 of dimension at most 4. We conclude that, in all cases, every eigenvalue of $\theta(g)$ occurs with multiplicity at least $(q^{m-s-3} - q)/(q+1)$. □

8. Proofs of the main results

Proof of Theorem 1.1. Suppose first that $H = Sp(m,q)$, with q odd. The case where $(m,q) = (4,3)$ is dealt with in Lemma 6.14. Otherwise, by Lemma 6.17, $\operatorname{Spec}\theta(g)$ contains all the s-roots of unity unless one of the following holds: (a) $H = Sp(m,p)$ and t is a transvection; (b) $H = Sp(4,9)$ and t is a transvection; (c) $H = Sp(8,3)$, $|g| = 9$ and $\operatorname{rank}(t - \mathrm{Id}) = 2$. Case (b) is examined in Lemma 6.16, whereas case (c) is examined in Lemma 6.22. So, we can assume in what follows that (a) holds. Furthermore, as $m = 4$ and $|g| > p$ forces $p = 3$, we may suppose $m > 4$. By Proposition 1.2, $\operatorname{Spec}\theta(t)$ contains all the p-roots of unity, unless θ is a Weil representation of $H = Sp(m,p)$. It then follows from Lemma 6.23 that $\operatorname{Spec}\theta(g)$ contains all the s-roots of unity, unless θ is a Weil representation of H or $(m,q) = (6,3)$. The latter case is dealt with in Lemma 6.18. So, assume that θ is a Weil representation of H. Again by Lemma 6.23, $\operatorname{Spec}\theta(g)$ contains all the (s/p)-roots of every $\varepsilon \in \operatorname{Spec}\theta(t)$, unless $p = 3$, $|g| = 9$ and either $\operatorname{Jord} g = \operatorname{diag}(J_4, \mathrm{Id}_{m-4})$ or $m = 6$. In these cases $\operatorname{Spec}\theta(g)$ is described in detail in Theorem 1.3.

Next, suppose that $H = Sp(m,q)$, with q even. Then, by Theorem 5.6 and Lemma 5.7, $\operatorname{Spec}\theta(g)$ contains all the s-roots of unity unless $H = Sp(6,2)$ and $\dim\theta = 7$.

Now, let $H = SU(m,q)$. If q is odd, then the result follows from Lemma 6.17, except for the case $H = SU(4,3)$ which is examined in Lemma 6.15. Let q be even. By Theorem 7.13, $\operatorname{Spec}\theta(g)$ contains all the s-roots of unity, unless $q = 2$ and either (a) $\operatorname{Jord} g = \operatorname{diag}(J_l, \mathrm{Id}_{m-l})$ with $l = 3, 5$ or (b) $m = 5$ or 7 and $\operatorname{Jord} g = \operatorname{diag}(J_2, J_{m-2})$. Case (a) is settled in Lemma 7.3 for

$l = 3$, whereas for $l = 5$ is settled in Lemmas 7.7 ($m > 5$) and 7.4 ($m = 5$). Case (b) is dealt with in Lemma 7.8 for $m = 7$ and in Lemma 7.5 for $m = 5$.

Finally, suppose that $H = Spin(m, q)$ with m odd, or $Spin^{\pm}(m, q)$, with m even. Then, by Theorem 5.6, $\operatorname{Spec}\theta(g)$ contains all the s-roots of unity. (Additionally, it is worthwile to observe that the orthogonal groups examined in Theorem 5.6 cover a broader range than those recorded in Theorem 1.1.) □

Proof of Theorem 1.3. The data collected in the statement are drawn from the analysis carried out in Section 6. For case (1) see Lemma 6.18(B3) for $m = 6$, Lemmas 6.23(ii) and 6.19(2) for $m > 6$; for case (2) see Lemma 6.18(A2); for case (3) see Lemma 6.14(1) and (2). □

Proof of Theorem 1.4. The data collected in the statement are drawn from the analysis carried out in Section 7. For case (1) see Lemmas 7.2 and 7.3; for case (2) see Lemma 7.7; for case (3) see Lemma 7.4; for case (4) see Lemmas 7.4(4) and 7.5; for case (5) see Lemma 7.8. □

Proof of Theorem 1.5. The data collected in the statement are drawn from the analysis carried out in Sections 6 and 7. For case (1), see Lemma 6.14; for cases (2) and (3), see Lemma 6.18(A2) and (B2); for case (4), see Lemma 6.16; for case (5), see Lemma 6.22; for case (6), see Lemma 6.15. Case (7) arises from Lemma 7.4, whereas case (8) arises from Lemma 7.8. □

Proof of Theorem 1.6. The case when $H = SL(m, q)$ has been considered in Corollary 3.4, whereas the cases when $H = Sp(m, q)$ with q even or H is a spinor orthogonal group were done in Theorem 5.6. The case when $H = Sp(m, q)$ and either $q > p > 2$, or $q = p > 2$ and $\langle g\rangle$ contains no transvections, has been examined in Lemma 6.17. This lemma also covers the unitary groups in odd characteristic. The case when $H = Sp(m, q)$ with $q = p > 2$ and $\langle g\rangle$ contains a transvection, has been examined in Lemma 6.23. The unitary groups in characteristic 2 are dealt with in Theorem 7.15. Observe that if $s = 2$ or 4, then Theorem 7.15 remains valid as long as $m > 6$ and $m > 8$, respectively (cf. the last paragraph of the proof). Therefore, we can use Theorem 7.15 for all $m > 2p^{\alpha-1} + 4$, as required for Theorem 1.6. □

Acknowledgments

We would like to thank the referee for his careful reading of the original manuscript, and for providing useful suggestions in order to clarify and/or simplify some arguments in several places, with particular regard to the proof of Theorem 7.13.

Appendix A. Classical simple groups with exceptional Schur multiplier

In this appendix, for the sake of completeness, we deal with the spectra of unipotent elements in the case of the universal coverings of simple classical groups with exceptional Schur multiplier.

So, let H be a finite simple classical group of characteristic p and G be a universal central extension of H. Suppose that $H = G/Z(G)$ is such that $|Z(G)|$ is a multiple of p (that is, H has an exceptional Schur multiplier). Then H is one of the following groups: $PSL(2, 4)$, $PSL(2, 9)$, $PSL(3, 2)$, $PSL(3, 4)$, $PSL(4, 2)$, $PSU(4, 2)$, $PSp(6, 2)$, $PSU(4, 3)$, $PSU(6, 2)$, $\Omega(7, 3)$, $\Omega^+(8, 2)$. Let Z_0 be a Sylow p-subgroup of $Z(G)$. It is known that G/Z_0 is either a quasi-simple classical group or the spinor orthogonal group. Now, let g be an element of G such that the order $|g|$ is a p-power, and denote by $o(g)$ the order of g modulo $Z(G)$ (of course, it may happen that $o(g) = |g|$). We wish to give a list of all the irreducible representations θ of G over an algebraically closed field P of characteristic $r \neq p$, such that $\deg\theta(g) < o(g)$ and $\theta(Z_0) \neq 1$.

Observe that we need not to deal with the case $H = PSL(2,4)$, as $o(g) = 2$ in this case. If $H = PSL(2,9)$, then $o(g) = 3$. If $H = PSL(3,2)$, then $G \cong SL(2,7)$ and $o(g) = 4$. If $H = PSL(3,4)$, then $o(g) = 4$. If $H = PSL(4,2)$, then $G = \tilde{A}_8$ and $o(g) = 4$. If $H = PSU(4,2)$, then $G \simeq Sp(4,3)$ and $o(g) = 4$. If $H \in \{PSp(6,2), PSU(6,2), \Omega^+(8,2)\}$, then $o(g) = 4$ or 8. If $H \in \{PSU(4,3), \Omega(7,3)\}$, then $o(g) = 3$ or 9.

The table below gives the list of the P-representations of universal coverings G, providing exceptional spectra at unipotent elements. For each relevant simple group H the results were obtained from the ordinary and Brauer character tables, making use of packages available from [GAP]. We have denoted by μ, i, ω, λ, ν elements of P such that $\mu^4 = -1$, $i = \mu^2$, $\omega^3 = 1$ ($\omega \neq 1$), $\lambda \in \langle\omega\rangle$ and $\nu^3 = \omega$, respectively.

H	$\|Z_0\|$	r	H-classes	$\dim\theta$	$\operatorname{Spec}\theta(g)$
$PSL(3,2)$	2	7	4A	2	(μ, μ^{-1})
		any	4A	3	$(1, i, -i)$
$PSL(3,4)$	16	any	4A	6	$\pm(1,1,i,-i,i,-i)$
		3	4A	4	$\pm(1,1,i,-i)$, $\pm(1,-1,i,i)$
$PSU(4,2)$	2	any	4A	4	$\pm(1,1,i,-i)$
$PSp(6,2)$	2	any	4A	8	$\pm(1,1,1,1,i,-i,i,-i)$
		any	8A	8	$(1,1,\mu,\mu^3,\mu^4,\mu^4,\mu^5,\mu^7)$
		any	8B	8	$\pm(1,1,\mu,\mu^2,\mu^3,\mu^5,\mu^6,\mu^7)$
$\Omega^+(8,2)$	4	any	4A	8	$\pm(1,1,1,1,i,-i,i,-i)$
		any	8A	8	$(1,1,\mu,\mu^3,\mu^4,\mu^4,\mu^5,\mu^7)$
		any	8B	8	$\pm(1,1,\mu,\mu^2,\mu^3,\mu^5,\mu^6,\mu^7)$
$PSU(4,3)$	9	any	3A	6	$\lambda(1,1,1,\omega,\omega,\omega)$
		any	9A, 9B	6	$\lambda(\nu,\nu^3,\nu^3,\nu^4,\nu^7,1)$
		any	9C, 9D	6	$(\nu,\nu^2,\nu^4,\nu^5,\nu^7,\nu^8)$
		any	9A, 9B	15	$\lambda(1,1,\nu,3\nu^2,\nu^4,3\nu^5,\nu^6,\nu^7,3\nu^8)$
		any	9A, 9B	15	$\lambda(1,3\nu,\nu^2,3\nu^4,\nu^5,2\nu^6,3\nu^7,\nu^8)$

Remarks.

(1) In the above table we have only listed those representations of the universal covering G which do not contain Z_0 in their kernel. Furthermore, when considering reductions of characteristic zero representations modulo a prime $r > 0$, we have also allowed non-isomorphic reductions.
(2) The fourth column of the table lists, in the Atlas notation [Atl], the 'class-type' of the group H to which $g \bmod Z(G)$ belongs. Of course, several G-classes may correspond to a unique H-class: e.g. for $H = PSL(3,4)$, $\dim\theta = 4$, four distinct classes of elements of G of order 4 map to the single class 4A.
(3) In several cases the representation listed in the fifth column is not unique. For details about the number of such representations and their interrelationships, the reader is referred to [Atl, MAtl] as well as to the [GAP] package.

(4) In the case $H = PSU(4, 2)$, the representations of G of dimension 5 and 6 are not included in the table as Z_0 is in their kernel. These representations are dealt with in Lemma 7.2 of the present paper. We also take this opportunity to note that, as G is isomorphic to $Sp(4, 3)$, the above mentioned 6-dimensional representation of G provides an 'exceptional' representation of $Sp(4, 3)$ which was missed in [T-Z1], while it is correctly considered in [GMST]. In particular, the remark quoting [T-Z1] in [DM-Z, p. 230] is inaccurate.

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