On nonsingular sign regular matrices

J.M. Peña

Departamento de Matemática Aplicada, Universidad de Zaragoza, 50009 Zaragoza, Spain

Received 6 November 2001; accepted 16 April 2002

Submitted by M. Fiedler

Abstract

An $m \times n$ matrix $A$ is sign regular if, for each $k (1 \leq k \leq \min\{m, n\})$, all $k \times k$ submatrices of $A$ have determinant with the same nonstrict sign. The zero pattern of nonsingular sign regular matrices is analyzed. It is proved that the number of zero entries which can appear in a nonsingular sign regular matrix depends on its signature. A matrix is totally nonpositive if all its minors are nonpositive. A test for recognizing nonsingular totally nonpositive matrices is also provided.

Keywords: Sign regular matrices; Zero pattern; Totally nonpositive matrices; $N$-matrices; Neville elimination

1. Introduction

Let $1 \leq k \leq \min\{m, n\}$ and fix a $k$-vector of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, k$, which is called a signature. An $m \times n$ matrix $A$ is sign regular of order $k$ with signature $\varepsilon$ if, for each $j = 1, \ldots, k$, the sign of all minors of order $j$ coincides with $\varepsilon_j$. When $k = \min\{m, n\}$, a sign regular matrix of order $k$ is simply called sign regular matrix. The interest of nonsingular sign regular matrices comes from their characterizations as variation-diminishing linear maps: the number of sign changes in the consecutive components of the image of a vector is bounded above by the number of sign changes in the consecutive components of the vector (cf. [1, Theorems 5.3 and 5.6]). Many applications of sign regular matrices can be found in...
and, more recently, in [1,3,12,13]. In Section 2, we analyze the zero pattern of nonsingular sign regular matrices of order 3 (see Theorem 2.1). The zero pattern of nonsingular sign regular matrices is also described which depends on their signature.

A very important subclass of the sign-regular matrices is formed by the totally nonnegative matrices. A matrix is totally nonnegative (respectively, totally positive) if all its minors are nonnegative (respectively, positive). Totally nonnegative and totally positive matrices also have been called in the literature as totally positive and strictly totally positive matrices, respectively. On the other hand, a matrix is totally nonpositive (respectively, totally negative) if all its minors are nonpositive (respectively, negative). In [8] one can find a characterization of totally negative matrices, which were called strictly totally negative. In [5], several aspects of totally negative matrices were studied. From the results of Section 2 we see that, among all nonsingular sign regular matrices, the zero pattern of nonsingular totally nonpositive matrices is opposite to that of nonsingular totally nonnegative matrices. In Section 3, we include characterizations of nonsingular totally nonpositive matrices and provide a test of $O(n^3)$ operations to check if an $n \times n$ matrix is nonsingular totally nonpositive.

On the other hand, matrices with all principal minors nonpositive or negative have been considered and applied in the literature to several problems. An $N$-matrix is a matrix with all principal minors negative (they were called in [9] partially negative matrices). An $N_0$-matrix is a matrix with all principal minors nonpositive (they were called in [9] partially nonpositive matrices). Finally, a nonsingular $N_0$-matrix is called in [11] a weak $N$-matrix. Applications of these matrices can also be found in [14] and [2]. The principal minors of a nonsingular totally nonnegative matrix are positive. A similar result does not hold for a nonsingular totally nonpositive matrix. However, we see in Section 3 that, if $A$ is a nonsingular totally nonpositive matrix with $a_{11} < 0$ and $a_{nn} < 0$, then it is an $N$-matrix.

2. Zero patterns

Given $k, n \in \mathbb{N}$, $1 \leq k \leq n$, $Q_{k,n}$ will denote the set of all increasing sequences of $k$ natural numbers less than or equal to $n$. If $\alpha \in Q_{k,n}$, the complement $\alpha' \in Q_{n-k,n}$ is the increasingly rearranged sequence $\{1, 2, \ldots, n\} \setminus \alpha$. Let $A$ be a real $m \times n$ matrix. For $k \leq m$, $l \leq n$, and for any $\alpha \in Q_{k,m}$ and $\beta \in Q_{l,n}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$. The principal submatrices will be written in the form $A[\alpha] := A[\alpha|\alpha]$. A matrix is called nonnegative (nonpositive) if it has nonnegative (nonpositive) entries. Given any matrix $A = (a_{ij})_{i,j=1,\ldots,n}$, we define its conversion $A^\# := (a_{n-i+1,n-j+1})_{1 \leq i,j \leq n}$. Clearly, the conversion of a sign regular matrix is also a sign regular matrix with the same signature.

Let $T$ be an $n \times n$ lower (resp. upper) triangular matrix. The minors $T[\alpha|\beta]$ with $\alpha_k \geq \beta_k$ (resp. $\alpha_k \leq \beta_k$ $\forall k$) are called nontrivial minors of $T$. Then a matrix $T$ is called $\Delta$STP if the nontrivial minors of $T$ are all positive.
For an $n \times n$ matrix $C$ with $C[\gamma]$ invertible, the Schur complement of $C[\gamma]$ in $C$, denoted by $C/C[\gamma]$, is defined as
\[
C/C[\gamma] = C[\gamma'] - C[\gamma'][C[\gamma]]^{-1}C[\gamma][\gamma'].
\]
Then
\[
\det C[\gamma] = \frac{\det C}{\det(C/C[\gamma])}. \tag{2.1}
\]

For nonsingular matrices $A$, Gaussian elimination consists of a succession of $n - 1$ steps resulting in a sequence of matrices as follows:
\[A = A^{(1)} \rightarrow A^{(2)} \rightarrow \ldots \rightarrow A^{(n)} = U,\]
where $U$ is an upper triangular matrix. At the end of step $t - 1$, the matrix $A^{(t)} = (a_{ij}^{(t)})_{1 \leq i, j \leq n}$ will have been constructed, having zeros below its main diagonal in its $t - 1$ first columns. To obtain $A^{(t+1)}$ from $A^{(t)}$ we produce zeros in column $t$ below the pivot element $a_{ij}^{(t)}$ by subtracting multiples of row $t$ from the rows beneath it. Since we have assumed that $A$ is nonsingular, it is well-known (use (2.1)) that, if no row exchanges are needed, one has, for $i \geq t$, $j \geq t$,
\[a_{ij}^{(t)} = \frac{\det A[1, 2, \ldots, t-1, i|1, 2, \ldots, t-1, j]}{\det A[1, 2, \ldots, t-1]}.
\]

The following result shows the possible zero patterns of sign regular matrices of order 3 with a given signature. The proof will use the matrix $P$, obtained by reversing the order of the rows of the identity matrix $I$.

**Theorem 2.1.** Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a nonsingular sign regular matrix of order 3 with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

(i) If $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_2 = -1$, then $a_{ij} \neq 0$ whenever $(i, j) \notin \{(1, 1), (n, n)\}$ and the remaining elements can be zero.

(ii) If $\varepsilon_1 \neq \varepsilon_3$ and $\varepsilon_2 = -1$, then $a_{ij} \neq 0$ whenever $(i, j) \notin \{(1, n), (n, 1)\}$ and the remaining elements can be zero.

(iii) If $\varepsilon_1 \neq \varepsilon_3$ and $\varepsilon_2 = -1$, then $a_{n-i+1,i} \neq 0$ for all $i = 1, \ldots, n$ and the remaining elements can be zero. Besides, if $a_{ij} = 0$, then $a_{kl} = 0 \forall k \leq i, l \leq j$, if $j < n - i + 1$ and $a_{kl} = 0 \forall k \geq i, l \geq j$ if $j > n - i + 1$.

(iv) If $\varepsilon_1 = \varepsilon_3$ and $\varepsilon_2 = 1$, then $a_{ij} \neq 0$ for all $i = 1, \ldots, n$ and the remaining elements can be zero. Besides, if $a_{ij} = 0$, then $a_{kl} = 0 \forall k \geq i, l \leq j$, if $i > j$ and $a_{kl} = 0 \forall k \leq i, l \geq j$ if $i < j$.

**Proof.** (i) Assume that $\varepsilon_i = -1$ for $i = 1, 2, 3$. Let us first check that $a_{1n} < 0$. Otherwise $a_{1n} = 0$ and there exist $i \neq 1, j \neq n$, such that $a_{1n}$ and $a_{1j}$ are negative because $A$ is nonsingular. Then $\det A[1, i|j, n] = a_{1j}a_{1n} > 0$, which is a contradiction.

Let us now prove that $a_{12} < 0$. Otherwise, since $A$ is nonsingular, we could find $j > 1$ such that $a_{j2} < 0$. Then $0 \geq \det A[1, j|1, 2] = a_{11}a_{j2} \geq 0$, which implies
that \( a_{11} = 0 \). Also because \( A \) is nonsingular there must exist \( i,j > 1 \) such that \( \det A[i,j][1,2] < 0 \). Now it follows that the \( 3 \times 3 \) submatrix \( A[1,i,j][1,2,n] \) has a positive determinant, which is a contradiction. Applying the same arguments to \( A^T, A^\# \) and \((A^\#)^T\) we derive \( a_{21}, a_{n,n-1}, a_{n-1,n} \neq 0 \).

Let us now assume that \( a_{ij} = 0 \) for some \( i,j \notin \{1,1\} \). Since \( A \) is nonsingular, we could find \( t \neq i \) such that \( a_{ij} \neq 0 \). Let us suppose that \( t > i \). Then \( a_{ih} = 0 \) for all \( h < j \) because otherwise \( \det A[i,t][h,j] = a_{ih}a_{ij} > 0 \). In consequence, \( i > 2 \) since \( a_{12}, a_{21} \neq 0 \), and there exists \( h > j \) such that \( a_{ih} \neq 0 \). Then \( \det A[2,i][h] = a_{21}a_{ih} > 0 \), a contradiction. Analogously, suppose that \( t < i \). Then \( a_{ih} = 0 \) for all \( h > j \) because otherwise \( \det A[t,i][j,h] = a_{ij}a_{ih} > 0 \). Therefore, \( i < n - 1 \) since \( a_{n,n-1}, a_{n-1,n} \neq 0 \), and there exists \( h < j \) such that \( a_{ih} \neq 0 \). Then \( \det A[i,n-1][h,n] = a_{ih}a_{n-1,n} > 0 \), the final contradiction.

In order to prove that the entries \((1,1)\) and \((n,n)\) can be zero, take any \( n \times n \) totally negative matrix \( A \) (its existence is guaranteed in [5, Section 3] for any \( n \)). Let \( B = (b_{ij})_{1 \leq i,j \leq n} \) be the matrix given by \( b_{ij} := a_{ij} \) for \( (i,j) \neq (1,1) \) and \( b_{11} := 0 \). Clearly, the matrix \( B \) is nonsingular totally nonpositive. Analogously, we can form a matrix \( C = (c_{ij})_{1 \leq i,j \leq n} \) given by \( c_{ij} := b_{ij} \) for \( (i,j) \neq (n,n) \) and \( c_{nn} := 0 \), and then \( C \) is nonsingular totally nonpositive. Therefore \( C \) is a nonsingular sign regular matrix of order 3 with \( c_{11} = c_{nn} = 0 \) and signature \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \), \( \varepsilon_i = -1 \) for all \( i = 1,2,3 \).

The case \( \varepsilon_1 = 1 = \varepsilon_3 \) and \( \varepsilon_2 = -1 \) can be derived from the previous one by observing that \(-A \) will be sign regular with \( \varepsilon_i = -1 \) for all \( i = 1,2,3 \).

(ii) It is a consequence of (i) because \( PA \) will be sign regular with signature corresponding to the case (i).

(iii) Assume that \( \varepsilon_1 = \varepsilon_2 = -1 \) and \( \varepsilon_3 = 1 \). Suppose that \( a_{ij} = 0 \) for some \( i,j \). Since \( A \) is nonsingular, \( a_{ij} \neq 0 \) for some \( t \neq i \). If \( t > i \) then \( a_{it} = 0 \) for all \( l < j \) because otherwise \( \det A[i,t][l,j] = a_{il}a_{ij} > 0 \), which is a contradiction. Again by the nonsingularity of \( A, a_{ih} \neq 0 \) for some \( h > j \). Then \( a_{rs} = 0 \) for all \( r < i, s \leq j \) because otherwise \( \det A[r,i][s,h] = a_{rs}a_{ih} > 0 \), which is a contradiction, and we have proved that \( a_{kl} = 0 \) for some \( k \leq i \). Since \( A \) is nonsingular, \( j < n - i + 1 \). Analogously, it can be proved that, if \( t < i \), then \( a_{kl} = 0 \) for all \( k \geq i, l \geq j \) and, since \( A \) is nonsingular, \( j > n - i + 1 \). As a consequence, \( a_{n-i+1,i} \neq 0 \) for all \( i = 1, \ldots, n \). The matrix \(-P\) shows that the remaining elements can be zero.

The case \( \varepsilon_2 = -1 = \varepsilon_3 \) and \( \varepsilon_1 = 1 \) can be derived from the previous one by observing that the signature of \(-A\) will be \( \varepsilon_1 = \varepsilon_2 = -1 \) and \( \varepsilon_3 = 1 \).

(iv) It is a consequence of (iii) because the signature of \( PA \) will correspond to case (iii). □

Suppose that the signature sequence \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) of an \( n \times n \) nonsingular sign regular matrix \( A \) satisfies one of the following four cases:

\[
\varepsilon_i = 1 \quad \forall i = 1, \ldots, n, \tag{2.2}
\]

\[
\varepsilon_i = (-1)^i \quad \forall i = 1, \ldots, n. \tag{2.3}
\]
\[ \varepsilon_1 = 1, \quad \varepsilon_i = -1 \quad \forall i = 2, \ldots, n, \]  \hspace{1cm} (2.4)\\
\[ \varepsilon_1 = -1, \quad \varepsilon_i = (-1)^{i+1} \quad \forall i = 2, \ldots, n. \]  \hspace{1cm} (2.5)

These cases appear when \( A, -A, PA \) or \( -PA \), respectively, is totally nonnegative. Then the maximal number of zero entries can be achieved, as show the matrices \( I, -I, P \) or \( -P \), respectively. If, in cases (iii) and (iv) of Theorem 2.1 (that is, when \( \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1 \)), we impose a higher order of sign regularity different from cases (2.2) to (2.5), hence restrictions on zero elements can appear, depending on the signature \( \varepsilon \), as the following result shows.

**Theorem 2.2.** Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) such that (2.2)–(2.5) do not hold. Then there exists a positive integer \( k \geq 2 \) such that one of the following possibilities holds:

\[ \varepsilon_i = 1 \quad \forall i < k, \quad \varepsilon_k = -1, \]  \hspace{1cm} (2.6)\\
\[ \varepsilon_i = (-1)^i \quad \forall i < k, \quad \varepsilon_k = (-1)^{k-1}, \]  \hspace{1cm} (2.7)\\
\[ \varepsilon_1 = 1, \quad \varepsilon_i = -1 \quad \forall i \in [2, \ldots, k-1], \quad \varepsilon_k = 1, \]  \hspace{1cm} (2.8)\\
\[ \varepsilon_1 = -1, \quad \varepsilon_i = (-1)^{i+1} \quad \forall i \in [2, \ldots, k-1], \quad \varepsilon_k = (-1)^k. \]  \hspace{1cm} (2.9)

If either (2.6) or (2.7) holds, then \( a_{ij} \neq 0 \) whenever \( |i - j| < n - k + 2 \). If either (2.8) or (2.9) holds, then \( a_{ij} \neq 0 \) whenever \( k \leq |i + j| \leq 2n - k + 2 \).

**Proof.** The existence of \( k \) is obvious. Let us assume that (2.6) holds and let us prove the result by induction on \( k \geq 3 \). The case \( k = 3 \) follows from Theorem 2.1(ii). Let us assume that the result holds for \( k - 1 \) and let us prove it for \( k > 3 \).

By Theorem 2.1(iv), \( a_{11} > 0 \), and so we can perform a step of Gaussian elimination to produce zeros in the first column of \( A \) below \( a_{11} \) and obtaining the matrix \( A^{(2)} = (a^{(2)}_{ij})_{1 \leq i, j \leq n} \). By (2.1), \( A^{(2)}[2, \ldots, n] \) is an \( (n-1) \times (n-1) \) nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_2, \ldots, \varepsilon_n) \) such that (2.6) holds replacing \( k \) by \( k - 1 \). Applying the induction hypothesis to \( A^{(2)}[2, \ldots, n] \), we deduce that \( a^{(2)}_{ij} \neq 0 \) for all \( i, j \geq 2 \) such that \( |i - j| < n - 1 - (k - 1) + 2 = n - k + 2 \). Since the matrix \( A^{(2)} \) is nonnegative and we obtain \( A \) from \( A^{(2)} \) by adding to each row a nonnegative multiple of the first row, the mentioned nonzero entries of \( A^{(2)} \) are also nonzero entries of \( A \). Since \( A^t \) is also a nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) such that (2.6) holds we get that entries \( (i, j) \) of \( A^t \) are nonzero for all \( i, j \geq 2 \) such that \( |i - j| < n - k + 2 \). In conclusion, \( a_{ij} \neq 0 \) whenever \( |i - j| < n - k + 2 \).

If we assume that \( A \) is a nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) such that (2.7) (2.8) or (2.9) respectively) holds, then \(-A (PA \) or \(-PA, \) respectively) is a nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) such that (2.6) holds and the result follows. \( \square \)
Let us observe that, among all possibilities of matrices satisfying the hypotheses of Theorem 2.2, the case of maximal number of possible zero elements appears when \( k = n \). The following example shows \( n \times n \) matrices with this maximal number of zero elements.

**Example 2.3.** The \( n \times n \) (\( n \geq 3 \)) matrix

\[
C_n = \begin{pmatrix}
\frac{n-1}{n} & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 2 & 1
\end{pmatrix}
\]

is a nonsingular sign regular matrix with signature \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) such that (2.6) holds for \( k = n \). It can be proved by induction on \( n \) taking into account that, if we perform a step of Gauss elimination and obtain \( C_{n}^{(2)} \), then \( C_{n}^{(2)}[2, \ldots, n] = C_{n-1} \).

On the other hand, \(-C_n, PC_n\) or \(-PC_n\) are sign regular matrices and their signature satisfies (2.7), (2.8) or (2.9), respectively.

## 3. Characterizations and tests for nonsingular totally nonpositive matrices

This section includes characterizations of nonsingular totally nonpositive matrices in terms of their factorizations and provides tests for recognizing if a given matrix satisfies the mentioned properties. A key tool for the tests presented in this section is an elimination procedure called Neville elimination (see more details in [7]). Roughly speaking, Neville elimination (NE) is a procedure to make zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one. For a nonsingular matrix \( A = (a_{ij})_{1 \leq i,j \leq n} \), it consists of \( n - 1 \) major steps resulting in a sequence of matrices as follows:

\[
A = A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n,
\]

where \( A_t = (a_{ij}^{(t)})_{1 \leq i,j \leq n} \) has zeros below its main diagonal in the \( t - 1 \) first columns. If no row exchanges are needed, the matrix \( A_{t+1} \) is obtained from \( A_t \) \((t = 1, \ldots, n)\) according to the formula:

\[
a_{ij}^{(t+1)} := \begin{cases}
    a_{ij}^{(t)} - (a_{it}^{(t)}/a_{i-1,t}^{(t)})a_{i-1,j}^{(t)} & \text{if } i \geq t + 1, j \geq t \text{ and } a_{i-1,t}^{(t)} \neq 0, \\
a_{ij}^{(t)} & \text{otherwise}.
\end{cases}
\]

(3.1)

The element

\[
p_{ij} := a_{ij}^{(t)} \quad 1 \leq j \leq n, \quad j \leq i \leq n,
\]

(3.2)
is called the \((i, j)\) pivot of the NE of \(A\). The pivots \(p_{ij}\) will be referred to as diagonal pivots. The element
\[
m_{ij} = p_{ij} / p_{i-1,j}, \quad 1 \leq j \leq n, \quad j < i \leq n,
\]
is called the \((i, j)\) multiplier of the NE of \(A\). The matrix \(U := A_n\) is upper triangular and has the diagonal pivots on its main diagonal. The complete Neville elimination (CNE) of a nonsingular matrix \(A\) consists in performing the NE of \(A\) until getting the upper triangular matrix \(U\) and, afterwards, proceeding with the NE of \(U^T\) (the transpose of \(U\)) until obtaining a diagonal matrix with the diagonal pivots on its main diagonal. When we say that the CNE of \(A\) is possible without row or column exchanges, we mean that there have not been any row exchanges in the NE of \(A\) or \(U^T\). Finally, the \((i, j)\) pivot of the CNE of \(A\) is the \((i, j)\) pivot of the NE of \(A\) if \(i \geq j\) and the \((j, i)\) pivot of the NE of \(U^T\) if \(i \leq j\). Analogously can be defined the multipliers of the CNE of \(A\).

Let us also recall that checking if an \(n \times n\) matrix is a \(P\)-matrix (i.e., it has positive principal minors) requires \(\mathcal{O}(2^n)\) elementary operations (see [15]). However, checking if a matrix is strictly sign regular requires \(\mathcal{O}(n^4)\) operations (see [9]) and checking if it is nonsingular totally nonnegative or totally positive has a complexity of \(\mathcal{O}(n^3)\) (see [6] or the reformulations in [7]). However, there are no similar tests to check if a nonsingular matrix is sign regular. Let us compare the totally nonnegative case with the totally nonpositive case. A nonsingular matrix is totally nonnegative if and only if we can perform its CNE without row or column exchanges, all multipliers are nonnegative and the diagonal pivots are positive. Roughly speaking, the proof uses the fact that the steps of CNE preserve the total nonnegativity and that the test is equivalent to express the matrix as a product of matrices which are bidiagonal nonnegative matrices and one of them is diagonal (obtained after the CNE). Since each of these matrices are totally nonnegative, the product is also totally nonnegative by [1, Theorem 3.1]. If the matrix is nonsingular totally nonpositive, by Theorem 2.1 (i) we cannot produce zeros preserving the total nonpositivity (except in entries \((1, 1)\) and \((n, n)\)). However, in the next theorem we provide a test of complexity \(\mathcal{O}(n^3)\) to check if a matrix is nonsingular totally nonpositive.

Let us define the bidiagonal matrix
\[
G = (g_{ij})_{1 \leq i,j \leq n}, \quad g_{ij} := \begin{cases} 
1 & \text{if } i = j, \\
1 & \text{if } i = 2 \text{ and } j = 1, \\
0 & \text{elsewhere.}
\end{cases}
\]
(3.4)

It can happen that a nonsingular totally nonpositive matrix does not have an \(LDU\) factorization: for instance, if \(a_{11} = 0\). This motivated us to distinguish several cases in order to obtain the corresponding characterization.

**Theorem 3.1.** Let \(A = (a_{ij})_{1 \leq i,j \leq n} \ (n \geq 2)\) be a matrix with \(a_{11} \leq 0\) and \(a_{nn} \leq 0\). Let \(G\) be the matrix defined in (3.4). Let us define the matrix
A minors of order greater than 1 of columns are nonpositive. This follows from the nonpositivity of \( A \) by [1, Theorem 2.1], it is sufficient to see that the minors of \( AG \) corresponding minors of \( AG \) are nonnegative. For the converse, take into account that \( AG \) is totally nonpositive if and only if \( AG \) has zeroes in the first column below 11. Therefore, the (1,1) entry of \( B \) is nonnegative (taking\( C := A[\alpha 1 \leq j \leq n] \)). Then the following properties are equivalent:

(i) \( A \) is nonsingular totally nonpositive.

(ii) The CNE of \( B \) can be performed without row or column exchanges, with nonnegative multipliers, and diagonal pivots

\[
p_{11} < 0, \quad p_{ii} > 0 \quad \forall i > 1. \tag{3.5}
\]

(iii) \( B \) can be decomposed as \( B = LDU \) with \( L \) (resp. \( U \)) a nonsingular lower (resp. upper) triangular totally nonnegative matrix and \( D \) a diagonal matrix whose entries \( p_{ii} \) on its main diagonal satisfy (3.5).

**Proof.** We shall prove the equivalence of (i)–(iii) in the case when \( a_{11} < 0 \) and \( a_{nn} < 0 \). In the case \( a_{11} = 0 \) and \( a_{nn} < 0 \), observe that the (1,1) entry of \( B \) is nonnegative (taking\( C := A[\alpha 1 \leq j \leq n] \)). Therefore, the NE of \( A \) is actually performing the NE of the totally nonnegative matrix \( A[2, \ldots, n] \). Then by [7, Theorem 4.1] all the remaining multipliers of the NE of \( A \) are nonnegative and the pivots \( p_{ij} > 0 \) for \( i > 1 \). After the NE of \( A \) we obtain the upper triangular matrix \( A_n = U = (u_{ij})_{1 \leq i,j \leq n} \) and, since \( A[2, \ldots, n] \) is totally nonnegative, \( U[2, \ldots, n] \) is also totally nonnegative (cf. [1, Theorem 3.5]).

\[
B := \begin{cases} 
A & \text{if } a_{11} < 0 \text{ and } a_{nn} < 0, \\
AG & \text{if } a_{11} = 0 \text{ and } a_{nn} < 0, \\
AG^# & \text{if } a_{11} < 0 \text{ and } a_{nn} = 0, \\
AGG^# & \text{if } a_{11} = 0 \text{ and } a_{nn} = 0.
\end{cases}
\]
any $k \times k$ minor $\det(U^T)_2[\alpha, 2, \ldots, k]$ using initial columns of the lower triangular matrix $(U^T)_2[2, \ldots, n]$ is nonnegative:

$$\det(U^T)_2[\alpha, 2, \ldots, k] = \det(U^T)[\alpha_1 - 1, \alpha[1, \ldots, k]]_{a_1, a_1 - 1} \geq 0.$$ 

Therefore by [1, Corollary 2.6] the nonsingular triangular matrix $(U^T)_2[2, \ldots, n]$ is totally nonnegative. As above, if we continue the NE of $A$ we perform the NE of the totally nonnegative matrix $(U^T)_2[2, \ldots, n]$ and, by [7, Theorem 4.1], all the remaining multipliers of the NE of $U^T$ are nonnegative.

(ii) $\Rightarrow$ (iii). The matrix interpretation of the CNE of $A$ provides the $LDU$ factorization of $A$, with $L$ and $U$ factorized as a product of bidiagonal nonnegative matrices since the multipliers are nonnegative (see more details in [7, Theorem 4.1]). Taking into account that a bidiagonal nonnegative matrix is totally nonnegative and the product of totally nonnegative matrices is totally nonnegative (cf. [1, Theorem 3.1]), (iii) follows.

(iii) $\Rightarrow$ (i). By (iii) $A$ is nonsingular. By [4, Theorem 5.2] each of the triangular matrices $L$ and $U$ can be written as a limit of $\Delta STP$ matrices: $L = \lim_{p \to \infty} L_p$ and $U = \lim_{p \to \infty} U_p$ with $L_p$, $U_p$ $\Delta STP$ for all $p$. For each positive integer $p$ let us define $A_p := L_p DU_p$. Observe that $\lim_{p \to \infty} A_p = LDU = A$. Since $a_{nn} < 0$, we can assume that the $(n, n)$ entry of the matrices $A_p$ is also negative and then we can apply [8, Remark 3.6] and conclude that each $A_p$ is totally negative (STN with the notation of that paper). Since the set of totally nonpositive matrices is closed, $\lim_{p \to \infty} A_p = A$ is totally nonpositive. □

Observe that part (ii) of the previous result provides an algorithm of $O(n^3)$ operations to check (in each of the four possibilities) if a nonsingular matrix is totally nonpositive.

By the proof of Theorem 2.1(i), there exist nonsingular totally nonpositive matrices with $a_{11} = 0$ and $a_{nn} = 0$. If both elements are negative then all principal minors are negative, as the following result shows.

**Theorem 3.2.** If $A = (a_{ij})_{1 \leq i, j \leq n}$ is a nonsingular totally nonpositive matrix with $a_{11} < 0$ and $a_{nn} < 0$, then $A$ is an $N$-matrix.

**Proof.** By Theorem 2.1(i), $A$ has all negative entries. Form the matrix $B := SA^{-1}S$, where $S = \text{diag}(1, -1, 1, -1, \ldots)$. It is an easy calculation via Jacobi’s identity (see [1, formula (1.32)]) that all $(n - 1) \times (n - 1)$ minors of $B$ are positive and all remaining proper minors of $B$ are nonnegative. Hence, by [1, Corollary 3.8], all proper principal submatrices of $B$ are $P$-matrices. From which it follows, again by [1, formula (1.32)], that $A$ is an $N$-matrix. □
References