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Note on the Markus–Yamabe conjecture for gradient dynamical systems

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Abstract

Let $v:\mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field which has a singular point O and its linearization is asymptotically stable at every point of $\mathbb{R}^n$. We say that the vector field v satisfies the Markus–Yamabe conjecture if the critical point O is a global attractor of the dynamical system $\dot{x} = v(x)$. In this note we prove that if v is a gradient vector field, i.e. $v = \nabla f$ $(f \in C^2)$, then the basin of attraction of the critical point O is the whole $\mathbb{R}^n$, thus implying the Markus–Yamabe conjecture for this class of vector fields. An analogous result for discrete dynamical systems of the form $x_{m+1} = \nabla f(x_m)$ is proved.

Keywords: Global attractor; Markus–Yamabe conjecture; Gradient dynamical system

1. Introduction

This paper is related to the problem of providing sufficient conditions in order that a critical point O (the origin of coordinates in what follows) of a C^1 vector field $v:\mathbb{R}^n \to \mathbb{R}^n$ $(v(O) = 0)$ be a global attractor (i.e. the ω-limit of any solution of the equation $\dot{x} = v(x)$, $x \in \mathbb{R}^n$, is the

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point O). In this line Markus and Yamabe proved in [1] the following theorem. In order to state it, we need to introduce some notation. In what follows ^{t}, Tr and Det stand for the transpose, the trace and the determinant of a matrix, respectively.

Theorem 1. *If the matrix $M(x) := Dv(x) + Dv(x)^t$ is negative definite ($Dv(x)$ is the Jacobian matrix of $v(x)$) and verifies that $|\mathrm{Tr}\, M(x)| < c_1$ and $|\mathrm{Det}\, M(x)| > c_2$ for some positive real constants c_1 and c_2 then O is a global attractor.*

In the same paper Markus and Yamabe stated the following conjecture:

Conjecture 1 *(Markus–Yamabe conjecture). If $Dv(x)$ is negative definite at every point of $\mathbb{R}^n$ (i.e. the linearization of $v(x)$ is asymptotically stable for any $x \in \mathbb{R}^n$) then O is a global attractor (in particular it is the only critical point).*

Apart from the interest of providing an easily verifiable condition to ensure that O is a global attractor, the importance of this conjecture lies in that it would imply (if true) the solution to the celebrated Jacobian problem [2]. In fact it would be enough to verify the Markus–Yamabe conjecture for vector fields of the type $v(x) = c - x + H(x)$, c being a real constant and $H(x)$ being a homogeneous vector polynomial of degree 3 with a nilpotent Jacobian matrix, in order to solve the Jacobian problem [2] (the so-called weak Markus–Yamabe problem).

Conjecture 1 remained open for many years until it was proved in $\mathbb{R}^2$ by Gutierrez [3], Fessler [4], and Glutsyuk [5]. In dimension greater than two the conjecture was proved to be wrong: analytic counterexamples in $\mathbb{R}^n$ ($n > 3$) were constructed by Bernat and Llibre [6] while polynomial counterexamples in $\mathbb{R}^n$ ($n > 2$) were found by Cima and coworkers [7].

A discrete version of the Markus–Yamabe conjecture was stated by La Salle in [8].

Conjecture 2 *(Discrete Markus–Yamabe conjecture). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that $F(O) = O$, and for any $x \in \mathbb{R}^n$, all the eigenvalues of $DF(x)$ have their modulus less than one. Then O is a global attractor for the discrete dynamical system generated by F.*

A positive answer to this problem for two-dimensional polynomial maps was found in [9]. The fact that the discrete Markus–Yamabe conjecture is generally false in $\mathbb{R}^2$ was shown in the same paper by providing a rational counterexample. In $\mathbb{R}^n$ ($n > 2$) polynomial counterexamples were constructed by Cima and coworkers [7]. The following related but weaker problem was proposed in [9].

Conjecture 3 *(Fixed point conjecture). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map such that $F(O) = O$, and for any $x \in \mathbb{R}^n$, $DF(x)$ has all its eigenvalues with modulus less than one. Then O is the unique fixed point of F.*

Cima and collaborators proved in [9] that this conjecture holds if and only if the Jacobian conjecture is true.

Since both Markus–Yamabe claims (continuous and discrete) are wrong in general, even for polynomial dynamical systems, this suggests to test the validity of Conjectures 1 and 2 in $\mathbb{R}^n$ ($n > 2$) for more restricted families of vector fields or maps, for example, homogeneous polynomial vectors, general polynomial vectors of degree lower or equal to 3 or gradient dynamical systems. In this note we prove that both Markus–Yamabe conjectures hold when $v = \nabla f$ ($F = \nabla f$ in the discrete case), f being a C^2 function.

Theorem 2 *(Markus–Yamabe conjecture for gradient vector fields). Let $v = \nabla f$ ($f \in C^2$) be a vector field such that its linearization is asymptotically stable at every point of $\mathbb{R}^n$. Then if O is a critical point of v, it is a global attractor (in particular, there are no other zeroes of v).*

Theorem 3 *(Discrete Markus–Yamabe conjecture for gradient maps). Let $F = \nabla f$ ($f \in C^2$) be a map such that its linearization has eigenvalues with modulus less than one on the whole $\mathbb{R}^n$. Then if O is a fixed point of F it is a global attractor (in particular, there are no other fixed points of F).*

The organization of this paper is as follows. In Section 2 we prove Theorem 2 in a remarkably simple manner (we show that the function $\frac{1}{2}(x_1^2 + \cdots + x_n^2)$ is globally Lyapunov for the system $\dot{x} = \nabla f(x)$). In Section 3 we prove that the function f itself is globally Lyapunov thus providing another, more involved, proof of Theorem 2. As a consequence of this result we show that the level sets of f on $\mathbb{R}^n - O$ are diffeomorphic to S^{n-1}. Finally, in Section 4 we prove the discrete version of the previous results (Theorem 3). The authors hope that this note will motivate other researchers to investigate the Markus–Yamabe problem for different types of dynamical systems (homogeneous polynomial vectors, polynomial vectors of degree $\leqslant 3, \ldots$). The fact that the weak Markus–Yamabe conjecture remains open suggests that this topic is far from trivial and could give rise to interesting mathematical results.

2. Markus–Yamabe problem for gradient vector fields I

First of all let us recall the concept of a global Lyapunov function.

Definition 1. The C^1 function $V(x)$ is globally Lyapunov for the system $\dot{x} = v(x)$ if

- $V(O) = 0$ and $V(P) > 0\,(< 0)$ if $P \neq O$;
- $\lim_{\|x\| \to \infty} V(x) = +\infty\,(-\infty)$;
- $\nabla V(x) \cdot v(x) < 0\ (> 0)$ on $\mathbb{R}^n - O$.

This definition implies that O is a global minimum (a global maximum) of V. The existence of a global Lyapunov function is very useful because it implies (Lyapunov's theorem [10]) that the critical point O is a global attractor. The following lemma will be useful in the proof of Theorem 2.

Lemma 1. $\nabla f(x) = [\int_0^1 D^2 f(sx)\,\mathrm{d}s]x$.

Proof. If we make the change of variables $y = sx$ ($\Rightarrow \mathrm{d}y = x\,\mathrm{d}s$) we get

$$\left[\int_0^1 D^2 f(sx)\,\mathrm{d}s\right]x = \int_0^x D^2 f(y)\,\mathrm{d}y = \int_0^x \frac{\mathrm{d}(\nabla f(y))}{\mathrm{d}y}\,\mathrm{d}y = \nabla f(x)$$

as we desired to prove. Note that we are assuming that 0 is a critical point of $\nabla f(x)$, i.e. $\nabla f(0) = 0$. $\square$

In the case of gradient vector fields the property that Dv is negative definite implies that the Hessian matrix $D^2 f$ is negative definite. This trivial observation and Lemma 1 are enough to prove that the function

$$V(x) = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)$$

is globally Lyapunov for the system $\dot{x} = \nabla f(x)$.

Proof of Theorem 2. $\dot{V}(x) = \nabla V(x) \cdot \nabla f(x) = x \cdot \nabla f(x)$. Now apply Lemma 1 to get $\dot{V}(x) = x[\int_0^1 D^2 f(sx)\,\mathrm{d}s]x \leqslant 0$. In fact, since $D^2 f(x)$ is negative definite on the whole $\mathbb{R}^n$ we conclude that $\dot{V}(x) < 0$ on $\mathbb{R}^n - O$. Since $V(x)$ fulfills all the conditions in Definition 1 it is a global Lyapunov function for $\dot{x} = \nabla f(x)$, thus implying, via Lyapunov's theorem, that O is a global attractor. □

It is interesting to observe that our proof still works if we assume that $D^2 f(x)$ is negative definite in $\mathbb{R}^n$ except for a discrete set of points. The details are left to the reader.

Note that if $v(x)$ is an arbitrary vector field then the identity

$$v(x) = \left[\int_0^1 Dv(sx)\,\mathrm{d}s\right]x$$

also holds. Therefore the proof of Theorem 2 fails for the general case because the property that $Dv(x)$ is negative definite does not imply $xDv(x)x \leqslant 0$, since $Dv(x)$ is a non-symmetric matrix.

3. Markus–Yamabe problem for gradient vector fields II

In this section we prove that $f(x)$ is a global Lyapunov function for $\dot{x} = \nabla f(x)$. This will not only provide another proof of Theorem 2 but an interesting topological characterization of the level sets of the function f.

Lemma 2. *$\dot{f} > 0$ on $\mathbb{R}^n - O$.*

Proof. $\dot{f}(x) = \nabla f(x) \cdot \nabla f(x)$. On account of Lemma 1 we can write

$$\dot{f}(x) = x\left[\int_0^1 D^2 f(sx)\,\mathrm{d}s\right]\left[\int_0^1 D^2 f(s'x)\,\mathrm{d}s'\right]x.$$

Since $D^2 f(x)$ is negative definite on $\mathbb{R}^n$ and the product of two negative definite matrixes is positive definite we reach that $\dot{f}(x) > 0$ except at the critical point O ($\nabla f(O) = 0 \Rightarrow \dot{f}(O) = 0$). □

This lemma proves that the only critical point of f is O. If we assume (without loss of generality) that $f(O) = 0$, then Lemma 2 implies that $f < 0$ on a neighborhood of O except for O, thus showing that f is a local Lyapunov function (O is a local maximum of f). The level sets of f near O are therefore topological spheres (S^{n-1}) [11]. Let us now prove that f is, in fact, a global Lyapunov function.

Lemma 3. $\lim_{\|x\|\to\infty} f(x) = -\infty$.

Proof. Let $v \in \mathbb{R}^n$ be a unitary vector ($\|v\| = 1$). Define now the function $f_v : \mathbb{R} \to \mathbb{R}$ as $f_v(t) = f(tv)$. Since $f_v''(t) = vD^2 f v < 0$ then $f_v(t)$ is a concave function and hence $t = 0$ is a global maximum ($f_v(t) < 0$ and $f_v'(t) < 0$ if $t > 0$). The concavity of $f_v(t)$ implies that

$$f_v(t) \leqslant f_v(1) + f_v'(1)(t-1) < f_v'(1)t - f_v'(1)$$

if $t > 0$. The set $\|v\| = 1$ is a sphere of dimension $n-1$ in $\mathbb{R}^n$ and hence $f_v'(1)$ attains maximum and minimum values, which are strictly negative (call M the maximum and m the minimum of $f_v'(1)$). Summarizing we have that $f_v(t) < Mt - m$ ($t > 0$). If we apply this inequality to the values $t = \|x\|$ and $v = x/\|x\|$, it is obtained that $f(x) < M\|x\| - m$ ($M, m < 0$) thus implying that $\lim_{\|x\|\to\infty} f(x) = -\infty$. □

Taking into account Lemmas 2 and 3 we conclude that f is a global Lyapunov function (see Definition 1) and therefore O is a global attractor. Now it is not difficult to characterize the topology of the level sets of f.

Proposition 1. *The level sets* $\{f = c\}$, $c < 0$, *are diffeomorphic to* S^{n-1}.

Proof. Around O the claim is immediate [11] (O is a maximum of f). Since f is a global Lyapunov function it follows that its level sets on $\mathbb{R}^n - O$ are compact codimension 1 submanifolds. Note that f is a Morse function [12] and therefore if we had a bifurcation of the topology S^{n-1} to other (compact) topology there would be critical points different from O as a consequence of Morse's theory [12]. The fact that $\|\nabla f\| > 0$ on $\mathbb{R}^n - O$ completes the proof. Note that the level sets of f are submanifolds in $\mathbb{R}^n$ and therefore they inherit the standard differentiable structure of $\mathbb{R}^n$, no problems with exotic differentiable structures arise. □

In ending this section it is interesting to observe that the key to prove that f is a global Lyapunov function is that $D^2 f$ is negative definite. In fact it is not difficult to construct examples (even in $\mathbb{R}^2$) such that O is a global attractor (global maximum) of ∇f (of f) but f itself is not globally Lyapunov.

Example 1. Consider the analytic function

$$f(x, y) = e^{-x^2} - y^2 - 1,$$

$(x, y) \in \mathbb{R}^2$. The only critical point of $\nabla f(x, y)$ is $(0, 0)$ and it is immediate to verify that $f(x, y) < 0$ on $\mathbb{R}^2 - (0, 0)$. $(0, 0)$ is a global attractor for the orbits of $\nabla f(x, y)$ but contrary to the results in this section $f(x, y)$ is not globally Lyapunov because the limit of $f(x, y)$ at infinity does not exist. In particular, the level sets $f(x, y) = c$, which are topological circles when $c > -1$, become unbounded when $c \leqslant -1$. Note that $D^2 f(x, y)$ is not negative definite on the whole plane.

4. Markus–Yamabe problem for gradient maps

In this section we consider discrete dynamical systems generated by maps $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form $F = \nabla f$ ($f : \mathbb{R}^n \to \mathbb{R}$ a C^2 function). The origin of coordinates O is assumed to be a fixed point of F ($\Rightarrow \nabla f(O) = 0$). If the modulus of any eigenvalue of DF is less than 1 on $\mathbb{R}^n$

these gradient maps satisfy a remarkable property: the fixed point O is a global attractor for the dynamical system $x_{m+1} = F(x_m)$. This result is the discrete counterpart to the theorem proved in Sections 2 and 3 in the continuous case, that is the discrete Markus–Yamabe conjecture for gradient maps (Theorem 3).

The following two elementary lemmas will be useful in the proof of Theorem 3. The first one is a well-known property of any square symmetric matrix, so we state it without proof. The second one is an estimate of the norm $\|\int_0^1 A(s)\,\mathrm{d}s\|$ in terms of the norm $\|A(s)\|$, for a square matrix A depending on a parameter s.

Lemma 4. *If A is an $n \times n$ symmetric matrix then its norm is given by $\|A\| = \max_{\lambda\in\sigma} |\lambda|$, where σ stands for the set of eigenvalues of A.*

Lemma 5. *Let $A(s)$, $s \in [0,1]$, be an $n \times n$ matrix-valued function. If $A(s)$ is integrable and $\|A(s)\| < K$, $K \in \mathbb{R}$, for any $s \in [0,1]$, then $\|\int_0^1 A(s)\,\mathrm{d}s\| < K$.*

Proof. If $A_i(s)$ is the ith row of the matrix $A(s)$ then it is immediate that $\|[\int_0^1 A(s)\,\mathrm{d}s]x\|^2 = \sum_{i=1}^n (\int_0^1 (A_i(s)\cdot x)\,\mathrm{d}s)^2$. Now, on account of the Cauchy–Schwartz inequality, we get

$$\sum_{i=1}^n \left(\int_0^1 \big(A_i(s)\cdot x\big)\,\mathrm{d}s\right)^2 \leqslant \sum_{i=1}^n \int_0^1 \big(A_i(s)\cdot x\big)^2\,\mathrm{d}s = \int_0^1 \big\|A(s)x\big\|^2\,\mathrm{d}s.$$

Since $\int_0^1 \|A(s)x\|^2\,\mathrm{d}s \leqslant \int_0^1 \|A(s)\|^2\|x\|^2\,\mathrm{d}s$ and $\|A(s)\|^2 < K^2$ we have that $\|[\int_0^1 A(s)\,\mathrm{d}s]x\|^2 < K^2\|x\|^2$ and therefore

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} < K,$$

as we desired to prove. □

Now we are ready to prove Theorem 3 by using Lemmas 1, 4 and 5.

Proof of Theorem 3. The dynamical system $x_{m+1} = F(x_m) = \nabla f(x_m)$ can be written as $x_{m+1} = [\int_0^1 D^2 f(sx_m)\,\mathrm{d}s]x_m$ (Lemma 1). Recall that the origin of coordinates O is a zero of ∇f thus implying that it is a fixed point of F. Since the eigenvalues of $D^2 f(x)$ have modulus less than 1 we have that $\|D^2 f(x)\| < 1$ for any $x \in \mathbb{R}^n$ (Lemma 4). In particular, if B is a compact ball (of arbitrary radius) centered at O then $\max_{x\in B} \|D^2 f(x)\| = K < 1$. By Lemma 5 we get that

$$\left\|\left[\int_0^1 D^2 f(sx)\,\mathrm{d}s\right]x\right\| = \big\|F(x)\big\| \leqslant K\|x\| < \|x\|$$

and hence $F(B) \subset B$. This implies that O is a global attractor for the dynamics induced by F, which is, in fact, the only fixed point. □

In ending this section let us illustrate this result by showing an example of a gradient map $F = \nabla f$ in $\mathbb{R}^2$ whose fixed point is a global attractor. Let us observe that the function f does not satisfy Proposition 1, contrary to the continuous case. Note that it is not clear how to obtain global stability in this example without invoking Theorem 3, thus showing its utility.

Example 2. Consider the analytic function

$$f(x,y)=\frac{1}{10}\arctan\bigl(1+x^2\bigr)e^{-y^2},$$

$(x,y)\in\mathbb{R}^2$. Note that $(0,0)$ is the only critical point of ∇f. After some computations the reader can check that the modulus of any eigenvalue of $D^2f(x,y)$ is bounded above by a constant lower than 1. Therefore, we conclude from Theorem 3 that the origin is a global attractor for the dynamical system $x_{m+1}=\nabla f(x_m)$. Note that the level sets of f are all unbounded and therefore f is not a Lyapunov function, unlike the continuous case where the level sets are topological spheres (see Section 3).

5. Final remarks

In this note we have proved the Markus–Yamabe conjecture for gradient dynamical systems in $\mathbb{R}^n$ (both continuous and discrete). Moreover, we have shown that the function $f(x)$ is globally Lyapunov for the system $\dot{x}=\nabla f(x)$ if O is a critical point and $D^2f(x)$ is negative definite. As a consequence of this result we have been able to characterize the topology of the level sets of f, i.e. the non-critical fibres $f^{-1}(c)$ are diffeomorphic to S^{n-1}.

It remains open to study other classes of vector fields for which the Markus–Yamabe conjecture holds. Apart from the family of vector fields related to the Jacobian problem we think that it would match with interest to find other families of vector fields for which the Markus–Yamabe conjecture is solved affirmatively.

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