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MV-algebras, multiple bets and subjective states

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Abstract

In this paper MV-algebras, the algebras of Łukasiewicz infinite-valued logics, are interpreted in a structure of bets, and a subjective interpretation of finitely additive measures on MV-algebras is given. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

MV-algebras were introduced by Chang [3] as the algebraic counterpart of Łukasiewicz logic [13]. They are an extension of boolean algebras just as Łukasiewicz logic is an extension of classical logic: boolean algebras coincide with idempotent MV-algebras.

Łukasiewicz logic and MV-algebras have been often used to deal with uncertain information. For example, in [4,11], a correspondence between MV-algebras and Ulam games is established, and Ulam games are easily translated in terms of error-correcting codes (see also [1,12]).

In this paper we shall use MV-algebra operations to describe *multiple bets*. Two players, A and B, agree on a finite set S of elementary events. A subset $X \subseteq S$ will be called an event. They also fix an integer $k > 0$. Player A buys from Player B (the bank) a sequence of events $u = X_1, \ldots, X_n$ (that we will call *multiple bet*), for a price, say $s(u)$\$, fixed by Player B. Then an elementary event

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$x \in S$ is extracted and Player B pays 1$ to Player A for each distinct X_i among $X_1, \ldots, X_n$ containing x. Further, we suppose that Player B cannot give to Player A more then k\$. By a suitable normalization we can suppose that the maximum winning is 1$ and that Player A wins (h/k)\$ if x is belongs to h many distinct elements of $\{X_1, \ldots, X_n\}$.

In general, different sequences of events can be considered equivalent whenever they lead to the same winnings for Player A.

In the following sections formal definitions of multiple bets and of space of events will be given, together with an equivalence relation enabling us to identify bets. The same equivalence will also be given in an algorithmic way, by means of a rewriting system (for an overview of rewriting systems see [6]). The resulting structure of multiple bets will be shown to be an MV-algebra, isomorphic to the boolean power of the MV-chain of k elements.

In Section 6 is described how the relation between prices and multiple bets is connected to the notion of *state* ([11]), i.e., finitely additive measure on MV-algebras.

2. Basic notions: MV-algebras

An *MV-algebra* is a structure $A = (A, \oplus, \neg, 0, 1)$ satisfying the following equations:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$
$$x \oplus y = y \oplus x,$$
$$x \oplus 0 = x,$$
$$x \oplus 1 = 1,$$
$$\neg 0 = 1,$$
$$\neg 1 = 0,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

As proved by Chang, boolean algebras coincide with MV-algebras satisfying the additional equation $x \oplus x = x$ (idempotency). Each MV-algebra contains as a subalgebra the two-element boolean algebra $\{0, 1\}$. The set $B(A)$ of all idempotent elements of an MV-algebra A is the largest boolean algebra contained in A and is called the *boolean skeleton* of A.

In any MV-algebra one defines the $\odot$ operation as follows:

$$x \odot y = \neg(\neg x \oplus \neg y).$$

The monoids $(A, \oplus, 0)$ and $(A, \odot, 1)$ are isomorphic via the map

$$\neg : x \mapsto \neg x.$$

Further any MV-algebra A is equipped with the order relation

$$x \leqslant y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Then A becomes a distributive lattice, and

$$x \wedge y = \inf\{x, y\} = \neg(\neg x \odot y) \odot y,$$
$$x \vee y = \sup\{x, y\} = \neg(\neg x \wedge \neg y).$$

For each $k = 1, 2, \ldots,$ we are interested in the following finite linearly ordered MV-algebra (also called *MV-chain*):

$$Ł_{k+1} = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}$$

equipped with the operations

$$x \oplus y = \min\{1, x + y\}, \quad x \odot y = \max\{0, x + y - 1\}, \quad \neg x = 1 - x.$$

Let L be a finite MV-chain. For any set S we denote by L^S the set of all functions $d : S \to L$ and we call *L-subsets* of S the elements of L^S. L^S inherits from L the structure of an MV-algebra: operations are obtained by pointwise application of the above operations and are called *Łukasiewicz union*, *intersection* and *complement*. Identifying subsets of S with their characteristic functions, the powerset 2^S of S then coincides with the boolean skeleton of L^S.

In the sequel, the cardinality of a set T will be denoted by $\mathrm{card}(T)$.

3. Identifying bets

Let $\mathscr{B} = (B, \vee, \wedge, 0, 1)$ be a boolean algebra. We will denote by B^+ the free semigroup on the domain B of $\mathscr{B}$. In other words, B^+ is the set of words $X_1 \cdots X_n$ with $X_i \in B$, equipped with the operation of juxtaposition, such that, if ϵ is the neutral element with respect to the juxtaposition, $B^* = B^+ \cup \{\epsilon\}$ is the free monoid over B. In the sequel we shall not make distinctions between $\mathscr{B}$ and B.

For any boolean algebra B and for any n-tupla $u = X_1 \cdots X_n \in B^+$, let us denote by B_u the (finite) boolean subalgebra of B generated by $X_1, \ldots, X_n$. The set of atoms of B_u will be denoted by $at(B_u)$.

Let $k > 0$ be a natural number. For every $u = X_1 \cdots X_n \in B^+$ and $X \in B$ the quantity

$$c_k(X, u) = \frac{\min\{k,\ \mathrm{card}(\{i \mid X \leqslant X_i\})\}}{k}$$

is called the *frequency* (*up to k*) of X in u. Note that it is an element of the MV-algebra $Ł_{k+1}$.

Lemma 3.1. *Let* $u, v \in B^+$, $X \in at(B_{uv})$ *and* $Z \in at(B_u)$, $Z' \in at(B_v)$ *such that* $Z \wedge Z' \neq 0$. *Then*:
(i) $c_k(X, uv) = c_k(X, u) \oplus c_k(X, v)$.
(ii) $c_k(Z \wedge Z', uv) = c_k(Z \wedge Z', u) \oplus c_k(Z \wedge Z', v) = c_k(Z, u) \oplus c_k(Z', v)$.
(iii) $c_k(Y, u) = c_k(Z, u)$ *for every* $Y \in B$ *such that* $Y \leqslant Z$.
(iv) $at(B_{uv}) = \{\overline{Z} \wedge \overline{Z}' \mid \overline{Z} \in at(B_u), \overline{Z}' \in at(B_v), \overline{Z} \wedge \overline{Z}' \neq 0\}$. □

Definition 3.2. Given $u = X_1 \cdots X_n$ and $v = Y_1 \cdots Y_m$ in B^+ we set $u \preceq_k v$ if and only if, for every $X \in at(B_{uv})$,

$$c_k(X, u) \leqslant c_k(X, v).$$

The relation $\preceq_k$ is a pre-order, i.e., a reflexive and transitive relation.

Definition 3.3. Two elements $u, v \in B^+$ are *k-equivalent*, and we write $u \equiv_k v$, if $u \preceq_k v$ and $v \preceq_k u$.

It is easy to see that the relation $\leqslant_k$ given by

$$[u] \leqslant_k [v] \iff u \preceq_k v$$

is a partial order relation over $B^+/\equiv_k$.

Since B^+ is freely generated over B, then $\equiv_k$ is a congruence in the semigroup B^+. We can then consider the operation $\oplus$ in $B^+/\equiv$ induced by the operation in B^+ as follows:

$$[X_1 \cdots X_n] \oplus [Y_1 \cdots Y_m] = [X_1 \cdots X_n Y_1 \cdots Y_m].$$

Further let us denote by $\mathbf{0}$ the element $[0^k] = [0^{k-1}] = \cdots = [0]$. The resulting structure $(B^+/\equiv, \oplus, \mathbf{0})$ is a monoid. We shall denote by $\mathbf{1}$ the element $[1^k]$, where 1 is the unit element of B.

Proposition 3.4. *If* $X, Y \in B$ *then*
(i) $XY \equiv_k YX$,
(ii) $XY \equiv_k (X \vee Y)(X \wedge Y)$,
(iii) $X^{k+1} \equiv_k X^k$,
(iv) $X0 \equiv_k X$,
(v) $XY \equiv_k (X - Y)(Y - X)(X \wedge Y)^2$.

Proof. We will prove (ii). Proofs of the other conditions are similar.

We must compare the words $u = XY$ and $v = (X \vee Y)(X \wedge Y)$. If $k = 1, 2$ then the result trivially holds. Suppose that $k > 2$. For every $Z \in at(B_{uv})$ possible cases are:

- $c_k(Z, u) = 0$; then $Z \nleqslant X$ and $Z \nleqslant Y$, so $Z \nleqslant X \vee Y$, $Z \nleqslant X \wedge Y$ and $c_k(Z, v) = 0$;
- $c_k(Z, u) = 1$; then we can assume that $Z \leqslant X$ and $Z \nleqslant Y$ so that $Z \leqslant X \vee Y$, $Z \nleqslant X \wedge Y$ and $c_k(Z, v) = 1$;

- $c_k(Z,u) = 2$; then $Z \leqslant X$ and $Z \leqslant Y$, so that $Z \leqslant X \wedge Y$, $Z \leqslant X \vee Y$ and $c_k(Z,v) = 2$.

So $c_k(Z,u) = c_k(Z,v)$ and $u \equiv_k v$. □

Proposition 3.4 suggests a rewriting system that enables us to choose in an effective way a particular element as the representative of an equivalence class.

Definition 3.5. A word on B can be transformed into another element of B^+ applying the following *rewriting rules*:
(a) if $X \wedge Y \neq 0$ then $X^nY^m \to (X-Y)^n(Y-X)^m(X \wedge Y)^{n+m}$;
(b) if $X \wedge Y = 0$ then $X^nY^n \to (X \vee Y)^n$;
(c) if $n \leqslant m$ then $Y^mX^n \to X^nY^m$;
(d) if $h > k$ then $X^h \to X^k$.

This rule system is *terminating*, i.e., after a finite number of applications of rules to a word w over B it is not possible to apply other rules. The expression resulting from such *derivation* is called *normal form* of w and will be denoted by $N(w)$.

Proposition 3.6. *A word w is in normal form if and only if it has the form $X_1^{m_1} \cdots X_n^{m_n}$ where X_i are non-zero pairwise disjoint elements of B and $(m_i)_{i=1,\dots,n}$ is a strictly increasing sequence of positive integers $\leqslant k$.*

Proposition 3.7. *For every $w \in B^+$, $N(w)$ is unique.*

Proof. We give here only a sketch of the proof. Using the above notations, we have to prove that our rule system is *locally confluent*, i.e, if two expressions w_1 and w_2 are deducible from the expression w, then w_1 and w_2 have the same normal form. Since the system is terminating, this property assures the uniqueness of the normal form (See [6]). We will prove that if w_1 and w_2 are different words deducible from w, then a finite number of application of rules to w_1 and w_2 yields to the same word (i.e., there is w' such that $w_1 \to^* w'$ and $w_2 \to^* w'$ where $\to^*$ is the transitive closure of $\to$).

- If w_1 and w_2 are obtained from w applying rules to two different disjoint sub-words of w, then applying the rules again in the opposite order, we obtain the same word w' and then the same normal form.
- Otherwise, suppose that w_1 and w_2 are obtained from w using respectively rules h and k (where $h,k \in \{(a),(b),(c),(d)\}$), applied to the same sub-word X^nY^m of w. Then surely $\{h,k\} \neq \{(a),(b)\}$ and applying h and k, respectively to w_2 and w_1, we obtain the same word w'. In symbols:

$$w \xrightarrow{h} w_2 \xrightarrow{k} w'$$

$$w \xrightarrow{k} w_1 \xrightarrow{h} w'. \qquad \square$$

Proposition 3.8. *If u is obtained from v using rules* (a), (b), (c), (d), *then* $u \equiv_k v$.

An immediate consequence is that every word over B is equivalent to its normal form.

We will refer to an element $\alpha = [X_1^{m_1} \cdots X_n^{m_n}]$ written in normal form as the generic element of the quotient $B^+/\equiv_k$.

Using the operation $\oplus$ and the order relation $\leqslant_k$, we can introduce the complement $\neg[u]$ of an element $[u]$ of $B^+/\equiv_k$ as the least element $[z] \in B^+/\equiv_k$ such that $[u] \oplus [z] = [1^k]$.

Taking normal forms as representatives of equivalence classes, we are able to describe the complement of an element of $B^+/\equiv_k$ in a simple way and, at the same time, to prove its existence. Indeed if $[w]$ is in normal form, say $[w] = [X_1^{m_1} \cdots X_n^{m_n}]$, it is easy to prove that

$$\neg[w] = \neg[X_1^{m_1} \cdots X_n^{m_n}] = \begin{cases} [X_0^k X_1^{k-m_1} \cdots X_n^{k-m_n}], & \text{if } m_n \neq k, \\ [X_0^k X_1^{k-m_1} \cdots X_{n-1}^{k-m_{n-1}}], & \text{if } m_n = k, \end{cases} \tag{1}$$

where $X_0 = 1 - \bigvee_{i=1}^{n} X_i$.

Lemma 3.9. *For any $[u] \in B^+/\equiv_k$, the identity $[v] = \neg[u]$ holds if and only if for every $X \in at(B_u)$, $c_k(X, v) = \neg c_k(X, u)$.*

Within the above algebraic context, we can formalize multiple bets described in Section 1. To this purpose let us fix a finite set S (space of events) and let

$$\mathscr{B}_{(k)} = ((2^S)^+/\equiv_k, \oplus, \neg, [\emptyset^k], [S^k]).$$

Then an element of $(2^S)^+/\equiv_k$ will be called *multiple bet*. If Player A buys from Player B the multiple bet $\alpha = [u] = [X_1 \cdots X_n]$ paying $s(\alpha)$, the *winning* given by elementary event $x \in S$ is the frequency $c_k(x, u)$ of x in u. The total *gain* is given by the difference $c_k(x, u) - s(\alpha)$.

The relation $\preceq_k$ over $(2^S)^+$ is such that $X_1 \cdots X_n \preceq_k Y_1 \cdots Y_m$ if and only if for every $x \in S$,

$$\min\{k,\ \mathrm{card}\{i \mid x \in X_i\}\} \leqslant \min\{k,\ \mathrm{card}\{j \mid x \in Y_j\}\}.$$

We say that a multiple bet $X_1 \cdots X_n$ is smaller than $Y_1 \cdots Y_m$ provided that, whenever the elementary event x happens, the winning resulting from the first is less than the winning resulting from the second. Consequently, two multiple bets are equivalent if they led to the same winning.

A normal form for an element of $(2^S)^+$ has the form $X_1^{m_1} \cdots X_n^{m_n}$ with $X_i \subseteq S$ and it represents a multiple bet such that if an elementary event $x \in S$ happens then Player A wins m_i if $x \in X_i$, for a suitable i, otherwise he wins 0. The complement of a multiple bet α is the least bet β such that if a player plays on α and β then the sum of the winnings is exactly 1.

More generally, for any boolean algebra $(B, \vee, \wedge, \neg, 0, 1)$, the structure $\mathscr{B}_{(k)} = (B^+/\equiv_k, \oplus, \neg, \mathbf{0}, \mathbf{1})$ will be called the *algebra of k-bets*. We will denote by $B_{(k)} = B^+/\equiv_k$ the underlying set of $\mathscr{B}_{(k)}$. The boolean algebra $\mathscr{B} = (B, \vee, \wedge, \neg, 0, 1)$ can be easily embedded in such a structure by considering the function

$$i : X \in B \to [X^k] \in B^+/\equiv_k .$$

Moreover for $k = 1$, $\mathscr{B}_{(k)}$ is isomorphic to $\mathscr{B}$.

Definition 3.10. For every $u, v \in B^+$ the conjunction $\odot$ is defined by

$$[u] \odot [v] = \neg(\neg[u] \oplus \neg[v]).$$

From Lemmas 3.1(i) and 3.9 it follows that if $[u] \odot [v] = [w]$ then for every $X \in B_{uv}$,

$$c_k(X, w) = c_k(X, u) \odot c_k(X, v). \tag{2}$$

In the next section, using boolean powers, we will demonstrate that $\mathscr{B}_{(k)}$ is an MV-algebra.

4. Boolean powers

Let us recall the definition of boolean power of an MV-algebra (see also [2,8] and references therein):

Definition 4.1. Let B be a boolean algebra and A a finite MV-algebra. The *boolean power* $\mathbf{A}[\mathbf{B}] = (A[B], \oplus, \neg, \mathbf{0}, \mathbf{1})$ is defined in the following way:

$$A[B] = \{f \in B^A \mid f(a_1) \wedge f(a_2) = 0 \quad \text{if } a_1 \neq a_2 \quad \text{and} \quad \bigvee_{a \in A} f(a) = 1\},$$
$$(f \oplus g)(x) = \bigvee_{h \oplus k = x} f(h) \wedge g(k),$$
$$\neg f(x) = f(\neg x)$$

and where $\mathbf{0}$ is the characteristic function of $\{0\}$ and $\mathbf{1}$ is the characteristic function of $\{1\}$, i.e.,

$$\mathbf{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \quad \mathbf{1}(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases}$$

The boolean power of an MV-algebra is an MV-algebra. Further, if the boolean algebra is supposed to be complete then it is possible to define boolean powers for infinite MV-algebras.

Using the following theorem we will show that every algebra $\mathscr{B}_{(k)}$ of k-bets is an MV-algebra and that every boolean power of the form $\mathbf{L_{k+1}}[\mathbf{B}]$ where $L_{k+1} = \{0, 1/k, \ldots, (k-1)/k, 1\}$, can be interpreted as an algebra of k-bets.

Theorem 4.2. *The algebra of k-bets* $\mathscr{B}_{(k)}$ *is isomorphic to the MV-algebra* $(L_{k+1}[B], \oplus, \neg, \mathbf{0}, \mathbf{1})$. *Thus, in particular,* $\mathscr{B}_{(k)}$ *is an MV-algebra.*

Proof. We will construct an isomorphism

$$F : \mathscr{B}_{(k)} \to \mathbf{L_{k+1}}[\mathbf{B}]$$

starting from a homomorphism of semigroups $G : B^+ \to L_{k+1}[B]$. For every $u \in B^+$ we define the function $\psi_u \in L_{k+1}[B]$ such that for every $r \in L_{k+1}$:

$$\psi_u(r) = \bigvee\{X \in at(B_u) \mid c_k(X, u) = r\}.$$

In case $u = X_1^{m_1} \cdots X_n^{m_n}$ is in normal form, then ψ_u becomes

$$\psi_u(r) = \begin{cases} X_i & \text{if } r = \frac{m_i}{k}, \\ X_0 = 1 - \bigvee_{i=1}^n X_i & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the interpretation of multiple bets, $\psi_u(r)$ is the set of elementary events for which Player A wins r.

Fact 1. $\psi_u \in L_{k+1}[B]$.

Indeed, if $r \neq s \in L_{k+1}$ *we have*

$$\begin{aligned}&\psi_u(r) \wedge \psi_u(s)\\ &\quad = \bigvee\{X \in at(B_u) \mid c_k(X, u) = r\} \wedge \bigvee\{Y \in at(B_u) \mid c_k(Y, u) = s\}\\ &\quad = \bigvee(X \wedge Y \mid X, Y \in at(B_u), c_k(X, u) = r, c_k(Y, u) = s\}) = 0.\end{aligned}$$

since different atoms of B_u *are always disjoint.*

Further, $\bigvee_{r \in L_{k+1}} \psi_u(r) = 1$, *because*

$$\bigvee_{r\in L_{k+1}} \psi_u(r) = \bigvee_{r\in L_{k+1}} \Big(\bigvee\{X \in at(B_u) \mid c_k(X, u) = r\}\Big) = \bigvee_{X \in at(B_u)} X = 1.$$

Fact 2. *The function* $G : u \in B^+ \to \psi_u \in L_{k+1}[B]$ *is an epimorphism of semigroups.*

Indeed, if $u, v \in B^+$ *then for every* $r \in L_{k+1}$,

$$\begin{aligned}(\psi_u \oplus \psi_v)(r) &= \bigvee_{i \oplus j = r} (\psi_u(i) \wedge \psi_v(j))\\ &= \bigvee_{i\oplus j=r} \Big(\bigvee\{X \mid X \in at(B_u) \text{ and } c_k(X, u) = i\} \wedge,\\ &\quad \bigvee\{Y \mid Y \in at(B_v) \text{ and } c_k(Y, v) = j\}\Big)\\ &= \bigvee_{i\oplus j=r} \bigvee\{X \wedge Y \mid X \in at(B_u), Y \in at(B_v), c_k(X, u) = i, c_k(Y, v) = j\}.\end{aligned}$$

If i and j are integer numbers such that $i \oplus j = r$, *and* $X \in at(B_u)$ *and* $Y \in at(B_v)$ *are such that* $X \wedge Y \neq 0$, $c(X,u) = i$, $c(Y,v) = j$, *then* $Z = X \wedge Y$ *is an atom of* B_{uv}, *such that* (*by Lemma* 3.1(ii)), $c_k(Z,uv) = c_k(Z,u) \oplus c_k(Z,v) = r$.

Conversely, let Z be an atom in B_{uv}, *such that* $c_k(Z,uv) = r$. *Then* (*by Lemma* 3.1(iv)), *there exist* $X \in at(B_u)$, $Y \in at(B_v)$ *such that* $X \wedge Y = Z$. *By setting* $i = c(X,u)$ *and* $j = c(Y,v)$ *we have*

$$r = c_k(Z,uv) = c_k(Z,u) \oplus c_k(Z,v) = c_k(X,u) \oplus c_k(Y,v) = i \oplus j.$$

Thus,

$$(\psi_u \oplus \psi_v)(r) = \bigvee\{Z \in at(B_{uv}) \mid c_k(Z,uv) = r\} = \psi_{uv}(r).$$

In order to show that G is surjective, let $f : L_{k+1} \to B$ *be an element of the boolean power* $\mathbf{L_{k+1}}[\mathbf{B}]$. *Then the element*

$$u = f\left(\frac{1}{k}\right) f\left(\frac{2}{k}\right)^2 \ldots f\left(\frac{k-1}{k}\right)^{(k-1)} f(1)^k \in B^+$$

satisfies the identity $f = \psi_u = G(u)$.

Fact 3. *The congruence* $\equiv_G$ *associated to G, defined by* $u \equiv_G v$ *if and only if* $G(u) = G(v)$, *coincides with the congruence* $\equiv_k$.

Indeed, let $u \equiv_G v$, *i.e., for every* $r \in L_{k+1}$, $\psi_u(r) = \psi_v(r)$. *Then*

$$\bigvee\{X \in at(B_u) \mid c_k(X,u) = r\} = \bigvee\{Y \in at(B_v) \mid c_k(Y,v) = r\}. \tag{3}$$

Let Z be an atom of B_{uv} *such that* $c_k(Z,u) = r$. *Then there exists* $X \in at(B_u)$ *such that* $Z \leqslant X$ *and* $c_k(X,u) = r$. By (3)

$$Z \leqslant \bigvee\{Y \in at(B_v) \mid c_k(Y,v) = r\}$$

whence there exists $Y \in at(B_v)$ *such that* $Z \leqslant Y$ *and* $c_k(Y,v) = r$. *Consequently,* $c_k(Z,v) = r = c_k(Z,u)$ *and this proves that* $u \equiv_k v$.

Vice-versa assume that for every atom Z of B_{uv} *we have* $c_k(Z,u) = c_k(Z,v)$. *Then,*

$$\begin{aligned}\bigvee\{X \in at(B_u) \mid c_k(X,u) = r\} &= \bigvee\{Z \in at(B_{uv}) \mid c_k(Z,u) = r\} \\ &= \bigvee\{Z \in at(B_{uv}) \mid c_k(Z,v) = r\} \\ &= \bigvee\{Y \in at(B_v) \mid c_k(Y,v) = r\}.\end{aligned}$$

This proves that $u \equiv_G v$.

Fact 4. *The map* $F : [w] \in B^+/\equiv_k \to G(w) \in L_{k+1}[B]$ *is an isomorphism of MV-algebras.*

Indeed, since $\equiv_k$ *and* $\equiv_G$ *coincide, it follows that F is an isomorphism of semigroups. We have* $F(\mathbf{0}) = G(0^k) = \psi_{0^k}$, *where*

$$\psi_{0^k}(r) = \bigvee\{X \in at(B_{0^k}) \mid c_k(X, 0^k) = r\}.$$

Since $B_{0^k} = \{0, 1\}$ *then* ψ_{0^k} *is the characteristic function of* 0.

Let us denote $\neg[w]$ *by* $[z]$. *We then have*

$$F(\neg[w]) = F([z]) = G(z) = \psi_z.$$

By Lemmas 3.9, *for every* $r \in L_{k+1}[B]$ *we get*

$$\psi_z(r) = \bigvee\{X \in at(B_z) \mid c(X, z) = r\} = \bigvee\{X \in at(B_w) \mid c(X, w) = \neg r\}$$
$$= \psi_w(\neg r). \qquad \square$$

From Definition 3.10 it follows that the operation $\odot$ is the Łukasiewicz conjunction in the MV-algebra $\mathscr{B}_{(k)}$. From (2) we have

$$[u] \odot [v] = 0 \iff c_k(X, u) \odot c_k(X, v) = 0. \tag{4}$$

for every $X \in at(B_{uv})$.

5. Subjective states

De Finetti [5] used the idea of *fair betting system* as a foundation for the theory of probability (see also [10]). A betting system is a set of events and rates fixed by the bank. A player bet over events and win in accordance with rates. The betting system is said to be unfair if, no matter which event occurs, the player always wins or always looses. If the distribution of rates satisfies the probability rules, then there does not exist any set of bets for which the player or the bank always wins (Dutch book theorem), and the game is fair.

In [12] the author defines finitely additive measures for MV-algebras, called *states*. Since MV-algebras are a generalization of boolean algebras, this kind of measure turns out to be an extension of classical probability.

Conditional states have been defined in [7] and their analogies with conditional probabilities have been examined in [9].

In this section we will analyze how multiple bets with limited winning furnish a subjective approach to the notion of state in an MV-algebra.

As in [12], by a *state* of an MV-algebra A we mean a function $s : A \to [0, 1]$ satisfying the following conditions:

(i) $s(0) = 0$,

(ii) $s(1) = 1$,

(iii) $s(a) + s(b) = s(a \oplus b)$ whenever $a, b \in A$ and $a \odot b = 0$.

Note that a state over an MV-algebra A that happens to be a boolean algebra, is a finitely additive probability measure on A.

Let us consider the MV-algebra of k-bets over the finite boolean algebra 2^S,

$$\mathscr{B}_{(k)} = (B_{(k)} = (2^S)^+/\equiv_k, \oplus, \neg, [\emptyset^k], [S^k]).$$

An element of $B_{(k)}$ has the form $[X_1^{m_1} \cdots X_n^{m_n}]$ with $X_i \subseteq S$ disjoint events and $(m_i)_{i\in I}$ strictly increasing sequence of positive integers $\leqslant k$.

Definition 5.1. A *subjective quotation* in a multiple bets game with space of events S, is a function over the MV-algebra $\mathscr{B}_{(k)}$ of k-bets

$$s : B_{(k)} \rightarrow [0, 1].$$

Subjective quotations can be interpreted as tables of prices for multiple bets established by Player B.

Definition 5.2. A favorable (resp., unfavorable) *Dutch book* for a subjective MV-quotation s, is a set of multiple bets T (that is, a subset of $B_{(k)}$) such that for every $x \in S$

$$\sum_{w\in T} s(w) - c_k(x, w) < 0 \text{ (resp., } > 0).$$

In other words, a favorable (resp., unfavorable) Dutch book T for a quotation s is a set of bets such that whatever elementary events x in S occurs, Player A wins more (resp., less) than he has paid.

A subjective quotation for which it is not possible to construct a Dutch book, will be called a *coherent quotation*.

Theorem 5.3. *Any coherent quotation satisfies axioms* (i), (ii) *and* (iii) *in the definition of states and is therefore a state.*

Proof. Let s be a coherent quotation. First of all note that if $s([\emptyset^k]) > 0$ then $\{[\emptyset^k]\}$ would be a favorable Dutch book for s, since

$$s([\emptyset^k]) - c_k(x, \emptyset^k) = s([\emptyset^k]) - 0 > 0.$$

So we have

$$s([\emptyset^k]) = 0.$$

Consider a multiple bet $\alpha = [u]$ and its complement $\neg\alpha = [w]$. If $s([u]) + s([w]) < 1$ then from Lemma 3.9 and from (1), we get, for every $x \in S$, $c_k(x, u) = 1 - c_k(x, w)$ and hence

$$s([u]) - c_k(x, u) + s([w]) - c_k(x, w) = s([u]) + s([w]) - 1 < 0.$$

Symmetrically, if $s([u]) + s([w]) > 1$ we are similarly led to an unfavorable Dutch book. So for a coherent quotation we have

$$s(\alpha) + s(\neg\alpha) = 1. \tag{5}$$

Further, $s(1) = s([S^k]) = s(\neg[\emptyset^k]) = 1$.

Let us consider the case of two disjoint bets $\alpha = [u]$ and $\beta = [v]$, and let $\alpha \oplus \beta = [w]$. By Definition 3.10, for every $x \in S$

$$c_k(x,u) \odot c_k(x,v) = c_k(x,\emptyset^k) = 0,$$

whence

$$c_k(x,u) \oplus c_k(x,v) = c_k(x,u) + c_k(x,v). \tag{6}$$

Suppose

$$s([u]) + s([v]) + s([w']) < 1.$$

From Eq. (6) and Lemmas 3.1(i), 3.9, for every $x \in S$ we have

$$c_k(x,w') = 1 - (c_k(x,u) \oplus c_k(x,w)) = 1 - c_k(x,u) - c_k(x,v)$$

hence

$$s([u]) + s([v]) + s([w']) - c_k(x,u) - c_k(x,v) - c_k(x,w') < 0.$$

So in case $s([u]) + s([v]) + s([w']) < 1$, $\{[u],[v],[w']\}$ would be a Dutch book.

Symmetrically we get

$$s(\alpha) + s(\beta) + s(\neg(\alpha \oplus \beta)) = 1.$$

Thus, using (5),

$$s(\alpha) + s(\beta) = s(\alpha \oplus \beta). \qquad \square$$

6. Conclusions

The game of multiple bets represents a new interpretation of MV-algebra operations. Actually, monoidal structure of multiple game bets is isomorphic to Ulam game monoidal structure, but multiple bets game allows an intuitive interpretation of order relation in terms of winnings. It gives also a clear context in which properties of a probability of uncertain events can be explained. A next step will be the investigation of a subjective conditional uncertain probability and of an algebraic structure of conditioned MV-events.

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