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Exact λ-numbers of generalized Petersen graphs of certain higher-orders and on Möbius strips[☆]

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ABSTRACT

An $L(2,1)$-labeling of a graph G is an assignment f of nonnegative integers to the vertices of G such that if vertices x and y are adjacent, $|f(x)-f(y)| \geq 2$, and if x and y are at distance two, $|f(x)-f(y)| \geq 1$. The λ-number of G is the minimum span over all $L(2,1)$-labelings of G. A *generalized Petersen graph* (GPG) of order n consists of two disjoint copies of cycles on n vertices together with a perfect matching between the two vertex sets. By presenting and applying a novel algorithm for identifying GPG-specific isomorphisms, this paper provides exact values for the λ-numbers of all GPGs of orders 9, 10, 11, and 12. For all but three GPGs of these orders, the λ-numbers are 5 or 6, improving the recently obtained upper bound of 7 for GPGs of orders 9, 10, 11, and 12. We also provide the λ-numbers of several infinite subclasses of GPGs that have useful representations on Möbius strips.

1. Introduction

One well-known application of graph labelings is the channel assignment problem [10] in which frequencies are assigned to transmitters in a network in such a way that two transmitters receive sufficiently different frequencies to avoid interference due to transmitter proximity. In one model, adjacent vertices receive labels at least two apart, and vertices at distance two receive labels at least one apart. More formally, we say that an $L(2,1)$-*labeling* is a function f from the vertex set of a graph G to the non-negative integers such that $|f(x)-f(y)| \geq 2$ if vertices x and y are adjacent, and $|f(x)-f(y)| \geq 1$ if x and y are at distance two. $L(2,1)$-labelings were first suggested by Roberts in a personal communication to Griggs in 1988 and formally introduced by Griggs and Yeh in 1992 [9,20]. These labelings have been subsequently studied extensively, as illustrated in two surveys [3,22].

An $L(2,1)$-labeling of a graph G that uses labels in the set $\{0, 1, \ldots, k\}$ is called a k-*labeling*. The minimum k so that G has a k-labeling is called the λ-*number* of G and will be denoted by $\lambda(G)$. Griggs and Yeh [9] conjectured that $\lambda(G) \leq \Delta^2(G)$ where $\Delta(G)$ denotes the maximum degree of G. The best general upper bound yet established is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [8], although Griggs and Yeh's conjecture holds for sufficiently large $\Delta(G)$ [12].

Since the general problem of determining $\lambda(G)$ is NP-hard [7], a significant body of literature on $L(2,1)$-labelings focuses on finding exact λ-numbers, and thereby verifying Griggs and Yeh's conjecture, for particular classes of graphs. One such class that has attracted considerable interest in recent years consists of the generalized Petersen graphs [1,2,5,13,14], defined

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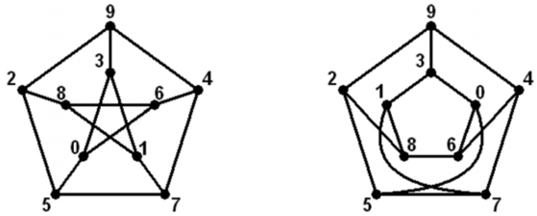

Fig. 1.1. A 9-labeling of the Petersen graph and an alternative drawing.

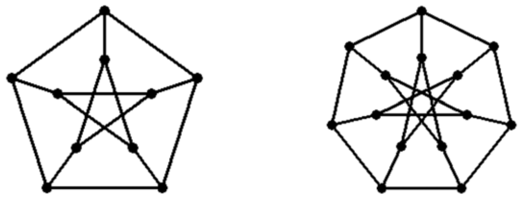

Fig. 1.2. The 5-star (the classical Petersen graph) and the 7-star, respectively.

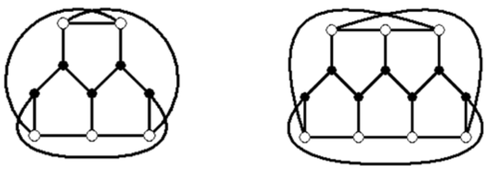

Fig. 1.3. Möbius representations of the 5-star and the 7-star, respectively.

by analogy with the well-known Petersen graph, depicted with a 9-labeling in Fig. 1.1 in its classical form, along with a second rendition suggested by Definition 1.1.

Definition 1.1. A *generalized Petersen graph* (GPG) of order $n \geq 3$ consists of two disjoint cycles C_n, called *outer* and *inner* cycles, so that each vertex on the outer (resp., inner) cycle is adjacent to exactly one vertex on the inner (resp., outer) cycle. Equivalently, if G is a GPG of order n then G has vertices $\{w_0, w_1, \ldots, w_{n-1}\} \bigcup \{v_0, v_1, \ldots, v_{n-1}\}$ with edges $\{w_i, w_{i+1}\}$ and $\{v_i, v_{i+1}\}$ for all $i = 0, 1, \ldots, n-1$, where subscript addition is taken modulo n, and each w_i (resp., v_i), $i = 0, 1, \ldots, n-1$ is adjacent to exactly one v_j (resp., w_j) for some $0 \leq j \leq n-1$. The cycle on vertices $\{w_0, w_1, \ldots, w_{n-1}\}$ (resp., $\{v_0, v_1, \ldots, v_{n-1}\}$) will be called the outer (resp., inner) cycle.

The GPGs may provide a good structure for communications network models based on the criteria of efficiency, network connectivity, and reliability. Every GPG is a minimal 3-connected graph, providing a network topology with a minimal number of connections in which the failures of two transmitters do not interrupt the flow of information.

Georges and Mauro [5] showed that the λ-number for all GPGs is bounded above by 8, excluding the case of the Petersen graph with λ-number 9. They also showed that this upper bound can be lowered to 7 for GPGs of order at most 6 (other than the Petersen graph), and conjectured that the upper bound of 7 also holds for GPGs of orders greater than 6. This conjecture proved to be true for orders 7 and 8 [1]. Recently, a combinatorial approach has been used to verify Georges and Mauro's conjecture for GPGs of orders 9, 10, 11, and 12 [13,14].

Exact λ-numbers for GPGs of order at most 8 are known [1,2]. In addition, some work has focused on determining the λ-numbers of certain infinite subclasses of GPGs. For example, the λ-numbers of *prisms*, a certain symmetrical subclass of GPGs wherein the edges between vertices on the outer and inner cycles are precisely $\{w_i, v_i\}$, $i = 0, 1, \ldots, n-1$, have been completely determined [6,15,17,18]. Then, inspired by the symmetry of the prisms and the original Petersen graph itself, the n-stars were defined and their λ-numbers were also completely determined [1].

Definition 1.2. For each odd n, $n \geq 5$, an *n-star* is a GPG wherein the edges between vertices on the outer and inner cycles are precisely $\{w_{(n-1)i/2}, v_i\}$ for $i = 0, 1, \ldots, n-1$ (or, equivalently, $\{w_j, v_{(n-2)j}\}$ for $j = 0, 1, \ldots, n-1$), where subscripts are taken modulo n and the notation is as introduced in Definition 1.1.

Every n-star has a nice representation on a Möbius strip that was instrumental in the determination of its λ-number. For instance, the 5-star (Petersen graph) and the 7-star of Fig. 1.2 can be drawn on Möbius strips as shown in Fig. 1.3. The vertices in the inner (resp., outer) cycle of the n-stars of Fig. 1.2 appear as black (resp., white) dots in Fig. 1.3. The Möbius representation of an n-star allows us to see its structure in a new light as an interconnection of 5-cycles. Two of these 5-cycles are not obvious from Fig. 1.3 but appear in the 'wrapped around' portion of the Möbius strip.

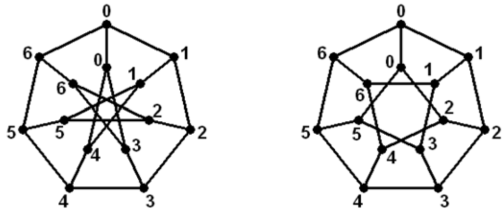

Fig. 2.1. GPGs associated with the permutations [0362514] and [0531642], respectively.

In Section 2, we provide the exact λ-numbers for all GPGs of orders 9, 10, 11, and 12, thereby closing all cases with orders up to 12. More specifically, we show that all but three GPGs of these orders have λ-numbers 5 or 6, improving the upper bound of 7 conjectured by Georges and Mauro [5] and confirmed combinatorially in [13,14] for GPGs of orders 9, 10, 11, and 12. Critical to our success in obtaining these exact λ-numbers is a novel algorithm that identifies GPG-specific isomorphisms. In Section 3, we extend the definition of the n-star introduced in [1] to cover even values of n and provide exact values for their λ-numbers. In Sections 4 and 5, we provide the exact λ-numbers for the n-igloos and n-mosaics, two additional infinite subclasses of the GPGs that have useful representations on the Möbius strip. Our findings are summarized in Section 6.

We note for clarity that Definition 1.1 differs from the definition for the generalized Petersen graph $GP(n, k)$ studied in [21] defined as the graph with vertices $\{w_0, w_1, \ldots, w_{n-1}\} \bigcup \{v_0, v_1, \ldots, v_{n-1}\}$ and edges $\{w_i, w_{i+1}\}$, $\{w_i, v_i\}$, and $\{v_i, v_{i+k}\}$ for all $i = 0, 1, \ldots, n-1$, where subscript addition is taken modulo n. For example, the 5-star and the 7-star in Fig. 1.2 are $GP(5, 2)$ and $GP(7, 3)$, respectively. However, $GP(6, 2)$ is not a GPG as defined in Definition 1.1 since the vertices $\{v_0, v_1, \ldots, v_5\}$ induce two disjoint cycles with three vertices each.

2. Exact λ-numbers for GPGs of orders 9, 10, 11, 12

In this section, we determine the exact λ-numbers for GPGs of orders 9, 10, 11, and 12 which was possible in large part by presenting a novel algorithm for identifying GPG-specific isomorphisms. This algorithm outperforms the algorithm used in [1] to determine the λ-numbers for GPGs of order 7 and 8, however it uses some of the same foundational elements. As described in [1], with each permutation $A = [a_0\ a_1\ \ldots\ a_{n-1}]$ of the integers $0, 1, \ldots, n-1$ we associate a GPG of order n as follows. Number consecutive vertices on the outer cycle in order, clockwise, with integers $0, 1, \ldots, n-1$, and number each vertex on the inner cycle with the same integer as the vertex on the outer cycle adjacent to it. The permutation A describes the order in which the numbers assigned to the vertices of the inner cycle appear consecutively if you go around the cycle. For example, Fig. 2.1 shows the GPGs associated with the permutations [0362514] and [0531642], respectively. Several different permutations will oftentimes be associated with isomorphic GPGs. For instance, rotating a permutation, taking its reverse, or starting at a different point would clearly generate permutations associated with isomorphic GPGs. However, as observed in [1], these are not the only operations on permutations that lead to isomorphic GPGs. To illustrate, the two GPGs in Fig. 2.1 are isomorphic, as one can be obtained from the other by switching the placement of the inner and outer cycles. This placement switch in the GPG associated with a permutation $A = [a_0\ a_1\ \ldots\ a_{n-1}]$ can be captured by the permutation A^* whose a_ith entry (0th through $(n-1)$th) is equal to i for $i = 0, 1, \ldots, n-1$. For example, given $A = [0362514]$ of Fig. 2.1, $a_0 = 0$, so the a_0th $=$ 0th entry of A^* is equal to 0; $a_1 = 3$, so the a_1st $=$ 3rd entry of A^* is equal to 1; $a_2 = 6$, so the a_2nd $=$ 6th entry of A^* is equal to 2; and so on until we get all of $A^* = [0531642]$. (In fact, if we recall the definition of $GP(n, k)$ provided at the end of Section 1, the two graphs in Fig. 2.1 are $GP(7, 3)$ and $GP(7, 2)$, respectively, which are known to be isomorphic.)

More formally, given a permutation $A = [a_0\ a_1\ \ldots a_{n-1}]$ of the integers $0, 1, \ldots, n-1$ and a nonnegative integer x not exceeding $n-1$, we define the following operations on A:

(i) *Start A at x*th *position*, denoted by A_x, is the permutation $A_x = [a_x\ a_{x+1}\ \ldots\ a_{x+(n-1)}]$, where subscript addition is taken modulo n.
(ii) *Rotation of A by x*, denoted by $A + x$, is the permutation $A + x = [(a_0 + x)\ (a_1 + x)\ \ldots\ (a_{n-1} + x)]$, where addition is taken modulo n.
(iii) *Reverse of A*, denoted by $\bar{A}$, is the permutation $\bar{A} = [a_0\ a_{n-1}\ a_{n-2}\ \ldots\ a_1]$.
(iv) *In-out switch of A*, denoted by A^*, is the permutation with i as its a_ith entry.

We say that two permutations are *similar* if one can be obtained from the other by a series of applications of the operations (i)–(iv). One can verify that this similar relation is indeed an equivalence relation inducing a partition of the permutations into equivalence classes. Moreover, since similar permutations are associated with isomorphic GPGs, each equivalence class under the similar relation must contain permutations associated with isomorphic GPGs. However, two different equivalence classes might not necessarily be associated with non-isomorphic GPGs because it is possible to have a pair of permutations that are not similar but associated with isomorphic GPGs (see [1] for a pair of order 8).

To completely categorize the λ-numbers of GPGs of orders 7 and 8 in [1], a brute-force algorithm examined every permutation and systematically pared down the number of associated GPGs to be examined through the elimination of

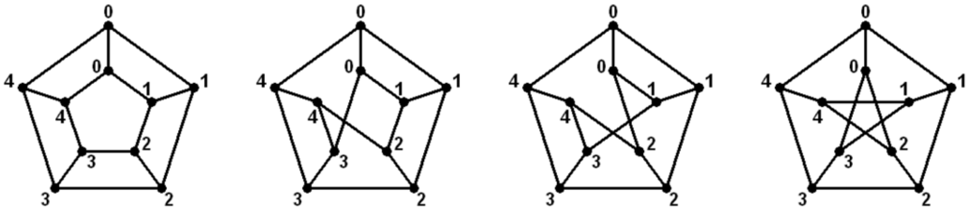

Fig. 2.2. GPGs associated with the permutations [01234], [01243], [01342], and [02413], respectively.

similar permutations. The number of remaining GPGs was manageable (28 for order 7, and 127 for order 8), so it was possible to use a graph labeling program to determine their exact λ-numbers and later check for isomorphisms within each class of GPGs with the same λ-number. Unfortunately, this approach proved infeasible for larger orders as the required computational capacity rapidly increased. In order to determine the λ-numbers for GPGs of orders 9, 10, 11, and 12, we use a novel, more efficient elimination algorithm to pare down the number of GPGs to be considered for each order by first identifying certain GPG-specific isomorphisms, thus reducing the number of graphs of these orders requiring further processing by approximately 95% (see Table 2.2 later in this section). We then determine the λ-numbers of these GPGs using an optimized version of the graph labeling program discussed in [1], and finally use the BLISS algorithm [16] (a successor of the well-known NAUTY algorithm [19]) implemented in *igraph* [4] to further check for general isomorphisms within some classes of GPGs with the same λ-number. As BLISS is well-documented and as the graph labeling program uses straightforward techniques, we omit the discussion of their workings for the sake of brevity. We will describe, however, our own elimination algorithm to detect similar permutations as its efficiency is what allows us to close previously intractably large cases.

Given a permutation $A = [a_0\, a_1\, \ldots\, a_{n-1}]$ of the integers $0, 1, \ldots, n-1$ we define the *difference sequence* of A as the sequence of integers $\text{diff}(A) = (d_0 d_1, \ldots, d_{n-1})$ where $d_i = (a_{i+1} - a_i)$ modulo n for $i = 0, 1, \ldots, n-1$, where the subscript addition is also taken modulo n. We will assume that these sequences are circular, that is, d_0 follows d_{n-1}. In general terms, the elimination algorithm will examine permutations in lexicographic order and use difference sequences to identify the representative of each equivalence class under the similar relation with the smallest difference sequence in lexicographic order. These representatives will be added to a list L of permutations to be later passed to the graph labeling program.

We will now discuss the interplay of the difference sequences and the operations (i)–(iv), defined previously. Note first that only permutations starting with a 0 need to be considered as every permutation is similar to one such permutation by the operation (i). Within these, a permutation $A = [0\, a_1\, \ldots\, a_{n-1}]$ with $a_1 > \lfloor \frac{n}{2} \rfloor$ does not need to be considered because it is similar by operations (i)–(iii) to a permutation where the 1st coordinate is at most $\lfloor \frac{n}{2} \rfloor$ (namely, the reverse of $(A + x)_1$ using the notation introduced for permutations operations where $x = n - a_1$).

For each examined permutation A, the algorithm computes the difference sequences for A, for its reverse, for its in–out switch and for the reverse of this in–out switch, that is, $\text{diff}(A)$, $\text{diff}(\bar{A})$, $\text{diff}(A^*)$, and $\text{diff}(\bar{A}^*)$, respectively. Suppose there exists a permutation B similar to A among the permutations being considered such that $\text{diff}(B)$ is smaller than $\text{diff}(A)$ in lexicographic order. Thus, $\text{diff}(B)$ must be equal, up to a shift of its starting point, to one of $\text{diff}(A)$, $\text{diff}(\bar{A})$, $\text{diff}(A^*)$, and $\text{diff}(\bar{A}^*)$. This is because applying operation (i) on a permutation causes its difference sequence to just shift its starting point, while applying operation (ii) on a permutation does not have any effect on its difference sequence. So the existence of such $\text{diff}(B)$ is equivalent to the existence of a prefix of length k of $\text{diff}(A)$, for some $k = 1, 2, \ldots, n-1$, larger in lexicographic order than a subsequence of the same length within $\text{diff}(C)$, for some $C = (c_0 c_1 \ldots c_{n-1})$ in $\{A, \bar{A}, A^*, \bar{A}^*\}$. If this subsequence starts at the jth entry of $\text{diff}(C)$, then $B = (C + x)_j$ where $x = n - c_j$ is similar to A with $\text{diff}(B)$ smaller than $\text{diff}(A)$ in lexicographic order. Hence, the elimination algorithm compares each prefix of length k of $\text{diff}(A)$ for $k = 1, 2, \ldots, n-1$ to each subsequence of the same length within $\text{diff}(A)$, $\text{diff}(\bar{A})$, $\text{diff}(A^*)$, and $\text{diff}(\bar{A}^*)$, starting at the jth entry for each $j = 1, 2, \ldots, n-1, 0$. If a subsequence smaller in lexicographic order than the given prefix is found then A should not be added to the list L; if no such subsequence is found, A should be added to the list L. We can conclude that the elimination algorithm keeps the representative of each equivalence class under the similar relation with the smallest difference sequence in lexicographic order.

For example, Table 2.1 provides the details generated by the algorithm when $n = 5$. The 1st, 2nd, 4th, and 11th permutations in column 1 (shaded) are added to L since their respective difference sequences in column 2 have prefixes that are always smaller in lexicographic order than any other subsequences of the four difference sequences in columns 2, 4, 6, 8. These permutations are associated, respectively, with the four GPGs of order 5 in Fig. 2.2 (these are actually all of the non-isomorphic GPGs of order 5; the final step using the BLISS algorithm would not eliminate further GPGs). The 3rd permutation $A = [01324]$ in column 1 is not added to L because $\text{diff}(A) = (12421)$ in column 2 has prefix $\underline{12}$, and $\text{diff}(C) = \text{diff}(A)$ contains a subsequence $\underline{11}$, starting at the jth $= 4$th entry of $\text{diff}(C)$, which is smaller in lexicographic order, indicating that if $x = n - c_j = 5 - 4 = 1$, the permutation $B = (C + x)_j = (A + 1)_4 = [01243]$ in column 10 is in the same equivalence class as A with $\text{diff}(B) = (11242)$ in column 11 smaller than $\text{diff}(A)$ in lexicographic order. On the other hand, the 6th permutation $A = [01432]$ in column 1 is not added to L because $\text{diff}(A) = (13443)$ in column 2 has prefix $\underline{13}$, $C = \bar{A} = [02341]$ in column 3, and $\text{diff}(C) = \text{diff}(\bar{A}) = (21124)$ in column 4, contains a subsequence $\underline{11}$, starting at the jth $= 1$st entry of $\text{diff}(C)$, which is smaller in lexicographic order, indicating that if $x = n - c_j = 5 - 2 = 3$, the permutation

Table 2.1
Elimination algorithm simulation for $n = 5$.

A	diff(A)	$\bar{A}$	diff($\bar{A}$)	A^*	diff(A^*)	$\bar{A}^*$	diff($\bar{A}^*$)	In L ?	B	diff(B)
[0 1 2 3 4]	(1 1 1 1 1)	[0 4 3 2 1]	(4 4 4 4 4)	[0 1 2 3 4]	(1 1 1 1 1)	[0 4 3 2 1]	(4 4 4 4 4)	**Y**		
[0 1 2 4 3]	(1 1 2 4 2)	[0 3 4 2 1]	(3 1 3 4 4)	[0 1 2 4 3]	(1 1 2 4 2)	[0 3 4 2 1]	(3 1 3 4 4)	**Y**		
[0 1 3 2 4]	(**1** **2** 4 2 **1**)	[0 4 2 3 1]	(4 3 1 3 4)	[0 1 3 2 4]	(1 2 4 2 1)	[0 4 2 3 1]	(4 3 1 3 4)	N	[0 1 2 4 3]	(1 1 2 4 2)
[0 1 3 4 2]	(1 2 1 3 3)	[0 2 4 3 1]	(2 2 4 3 4)	[0 1 4 2 3]	(1 3 3 1 2)	[0 3 2 4 1]	(3 4 2 2 4)	**Y**		
[0 1 4 2 3]	(**1** **3** 3 **1** **2**)	[0 3 2 4 1]	(3 4 2 2 4)	[0 1 3 4 2]	(1 2 1 3 3)	[0 2 4 3 1]	(2 2 4 3 4)	N	[0 1 3 4 2]	(1 2 1 3 3)
[0 1 4 3 2]	(**1** **3** 4 4 3)	[0 2 3 4 1]	(2 **1** **1** 2 4)	[0 1 4 3 2]	(1 3 4 4 3)	[0 2 3 4 1]	(2 1 1 2 4)	N	[0 1 2 4 3]	(1 1 2 4 2)
[0 2 1 3 4]	(**2** 4 2 1 **1**)	[0 4 3 1 2]	(4 4 3 1 3)	[0 2 1 3 4]	(2 4 2 1 1)	[0 4 3 1 2]	(4 4 3 1 3)	N	[0 1 2 4 3]	(1 1 2 4 2)
[0 2 1 4 3]	(**2** 4 3 4 2)	[0 3 4 1 2]	(3 **1** 2 1 3)	[0 2 1 4 3]	(2 4 3 4 2)	[0 3 4 1 2]	(3 1 2 1 3)	N	[0 1 3 4 2]	(1 2 1 3 3)
[0 2 3 1 4]	(**2** **1** 3 3 1)	[0 4 1 3 2]	(4 2 2 4 3)	[0 3 1 2 4]	(3 3 1 2 1)	[0 4 2 1 3]	(4 3 4 2 2)	N	[0 1 4 2 3]	(1 3 3 1 2)
[0 2 3 4 1]	(**2** **1** 1 2 4)	[0 1 4 3 2]	(1 3 4 4 3)	[0 4 1 2 3]	(4 2 1 1 2)	[0 3 2 1 4]	(3 4 4 3 1)	N	[0 1 2 4 3]	(1 1 2 4 2)
[0 2 4 1 3]	(**2** 2 2 2 2)	[0 3 1 4 2]	(3 3 3 3 3)	[0 3 1 4 2]	(3 3 3 3 3)	[0 2 4 1 3]	(2 2 2 2 2)	**Y**		
[0 2 4 3 1]	(**2** 2 4 3 4)	[0 1 3 4 2]	(**1** 2 1 3 3)	[0 4 1 3 2]	(4 2 2 4 3)	[0 2 3 1 4]	(2 1 3 3 1)	N	[0 1 4 2 3]	(1 3 3 1 2)

Table 2.2
Computation times and results for the elimination algorithm.

n	GPGs not in L	GPGs in L	% eliminated	CPU-time
9	1.8962e+004	1,198	94.7746	2.59 s
10	1.92216e+005	9,384	94.703	22.99 s
11	1.73067e+006	83,729	95.6643	3 min 34.91 s
12	2.09328e+007	839,955	95.6299	48 min 0.27 s

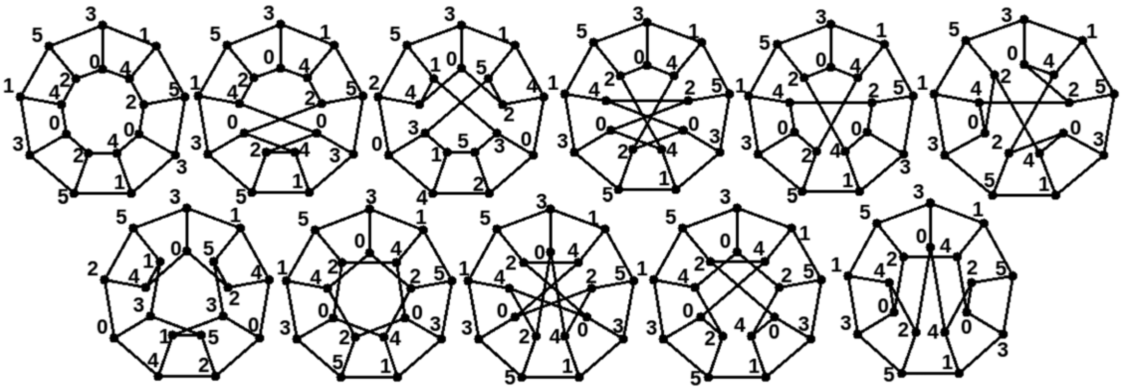

Fig. 2.3. The 11 non-isomorphic GPGs of order 9 with λ-number 5 (labels represent 5-labelings).

$B = (C + x)_j = (\bar{A} + 3)_1 = [01243]$ in column 10 is in the same equivalence class as A with diff(B) $= (11242)$ in column 11 smaller than diff(A) in lexicographic order. Table 2.1 does not provide instances where $C = A^*$ or $\bar{A}^*$.

After applying the elimination algorithm, we used the graph labeling program to identify which of the remaining GPGs had λ-number 5, 6, and 7 for each order 9 through 12. For orders 9, 10, and 12, the vast majority have λ-number 6, relatively few have λ-number 5, and none have λ-number 7. Thus we were able to use the BLISS algorithm to eliminate isomorphisms within the class of GPGs with λ-number 5. For GPGs of order 11, exactly three isomorphism classes have λ-number 7 and the rest all have λ-number 6. These findings are summarized in Theorems 2.1–2.4. Computation data for the elimination algorithm for orders 9, 10, 11, and 12 are summarized in Table 2.2, showing that we initially reduced the number of potentially non-isomorphic GPGs of these orders by approximately 95%, thus requiring only a reasonable number of graphs to be labeled. An even smaller number of graphs needed to be checked subsequently for further isomorphisms.

Theorem 2.1. *Let G be a GPG of order 9. Then $\lambda(G) = 5$ if G is isomorphic to any of the 11 graphs in Fig. 2.3, otherwise $\lambda(G) = 6$.*

Theorem 2.2. *Let G be a GPG of order 10. Then $\lambda(G) = 5$ if G is isomorphic to any of the 298 GPGs associated with the permutations in Table 2.3, otherwise $\lambda(G) = 6$.*

Theorem 2.3. *Let G be a GPG of order 11. Then $\lambda(G) = 7$ if G is isomorphic to one of the three graphs in Fig. 2.4, otherwise $\lambda(G) = 6$.*

Table 2.3
The 298 permutations associated with non-isomorphic GPGs of order 10 with λ-number 5.

[0123458976]	[0123468957]	[0123478596]	[0123569748]	[0123579648]	[0123587496]	[0123596478]	[0123597468]
[0123679485]	[0123679854]	[0123758946]	[0123759864]	[0123768954]	[0123769584]	[0123895674]	[0124358976]
[0124367985]	[0124589376]	[0124637985]	[0124658973]	[0124659387]	[0124659783]	[0124695387]	[0124786593]
[0124795683]	[0124859367]	[0124867935]	[0124893576]	[0124893675]	[0124958367]	[0124973586]	[0124985376]
[0124985673]	[0125467398]	[0125468397]	[0125493768]	[0125497368]	[0125498367]	[0125643798]	[0125648793]
[0125649783]	[0125689743]	[0125693748]	[0125694387]	[0125694738]	[0125698743]	[0125798643]	[0125849637]
[0125867394]	[0125893764]	[0125963847]	[0125967843]	[0126458793]	[0126458973]	[0126479538]	[0126579348]
[0126597438]	[0126597834]	[0126598473]	[0126743895]	[0126758943]	[0126895374]	[0126897435]	[0126897534]
[0126958743]	[0127485693]	[0127495683]	[0127496583]	[0127569834]	[0127596834]	[0127649853]	[0127685943]
[0127845396]	[0127849635]	[0127854936]	[0127869354]	[0127869534]	[0127895643]	[0127985643]	[0128659743]
[0128795643]	[0128975643]	[0132579486]	[0132594867]	[0132657984]	[0132684957]	[0132685947]	[0132687954]
[0132697854]	[0132796485]	[0132957684]	[0134268597]	[0134295867]	[0134752986]	[0134756982]	[0134759682]
[0134782956]	[0134786952]	[0134865297]	[0134895267]	[0135294867]	[0135297864]	[0135479268]	[0135492768]
[0135496728]	[0135624798]	[0135687924]	[0135742986]	[0135864297]	[0135867294]	[0135879624]	[0135892764]
[0135894267]	[0135896427]	[0135897264]	[0135924768]	[0135924786]	[0135942687]	[0135968742]	[0135978624]
[0136428975]	[0136458972]	[0136472985]	[0136598472]	[0136729584]	[0136754928]	[0136849527]	[0136895724]
[0136957284]	[0137268594]	[0137295864]	[0137298564]	[0137459826]	[0137462985]	[0137465982]	[0137489562]
[0137495682]	[0137594682]	[0137596842]	[0137598264]	[0137642985]	[0137648925]	[0137865294]	[0137985642]
[0138264597]	[0138267594]	[0138297564]	[0138472596]	[0138564297]	[0138569724]	[0138594267]	[0138597264]
[0138624795]	[0138645927]	[0138649725]	[0138695742]	[0138742956]	[0138794526]	[0138794625]	[0138924756]
[0138927465]	[0138956472]	[0139276458]	[0139452768]	[0139465287]	[0139472586]	[0139485276]	[0139564287]
[0139584762]	[0139852764]	[0139854672]	[0139864257]	[0143579862]	[0143597268]	[0143869572]	[0143895276]
[0143895762]	[0143975628]	[0145372698]	[0145372968]	[0145379826]	[0145382796]	[0145389726]	[0145392786]
[0145829376]	[0145879326]	[0145893276]	[0145896372]	[0145897263]	[0145932687]	[0145937682]	[0145938672]
[0145972368]	[0145972638]	[0145978362]	[0145982637]	[0146273598]	[0146275398]	[0146287935]	[0146298357]
[0146385972]	[0146395872]	[0146529783]	[0146582937]	[0146598327]	[0146739285]	[0146829375]	[0146832957]
[0146839257]	[0146875392]	[0146892753]	[0146895372]	[0146928357]	[0146932875]	[0146935782]	[0147265398]
[0147596382]	[0147983265]	[0147983652]	[0148275693]	[0148362975]	[0148365972]	[0148375962]	[0148635972]
[0148672935]	[0148932675]	[0148962375]	[0148965372]	[0148972365]	[0148972635]	[0148975362]	[0149627358]
[0149832657]	[0149862537]	[0149863527]	[0153429786]	[0153627498]	[0153694728]	[0153792486]	[0153794286]
[0153972486]	[0154962387]	[0157364982]	[0157382946]	[0157634982]	[0157648293]	[0157936842]	[0158467293]
[0158472693]	[0158942673]	[0158946372]	[0158973642]	[0159386742]	[0159672438]	[0159832674]	[0163752498]
[0163847592]	[0163857942]	[0163958472]	[0164538297]	[0164835297]	[0164857392]	[0164937582]	[0165837294]
[0168475392]	[0168534297]	[0168537294]	[0168597243]	[0168597342]	[0168742593]	[0168972534]	[0168975243]
[0173586294]	[0173594862]	[0174859362]	[0175396482]	[0175938642]	[0175948362]	[0179465283]	[0179534268]
[0179538624]	[0241683957]	[0247319586]	[0247951386]	[0247951683]	[0247961835]	[0248315796]	[0248573196]
[0248579316]	[0248613957]	[0249163857]	[0253971684]	[0263719584]	[0268317594]	[0268493175]	[0268497153]
[0269417385]	[0274916385]						

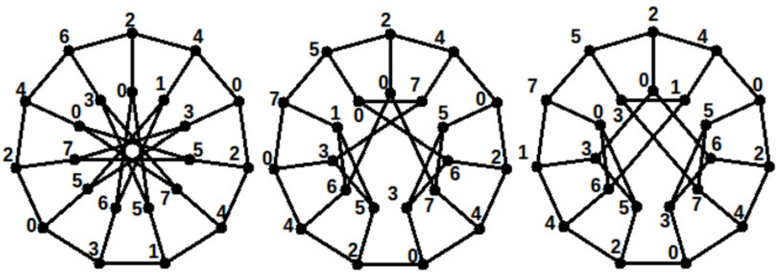

Fig. 2.4. The three non-isomorphic GPGs of order 11 with λ-number 7 (labels represent 7-labelings).

Theorem 2.4. *Let G be a GPG of order* 12*. Then* $\lambda(G) = 5$ *if G is isomorphic to any of the* 190 *GPGs associated with the permutations in Table* 2.4*, otherwise* $\lambda(G) = 6$*.*

We note that computing the results in this section took a substantially longer time for each successive order due to the factorial increase in complexity. For this reason, we observe that while the λ-numbers of the next few orders of GPGs might perhaps be categorized using similar methods, even a computer-assisted proof would be currently infeasible for higher orders.

3. A subclass of GPGs: the even *n*-stars

As described in Section 1, the *n*-stars were originally defined only for odd values of n, $n \geq 5$ [1]. By slightly changing the structure of the GPG, we can extend this definition to include even values of n as well for $n \geq 4$. This new subclass, the even *n*-stars, maintains some of the resemblance to the classical Petersen graph with its distinctive "star" arrangement of the vertices on the inner cycle.

Table 2.4
The 190 permutations associated with non-isomorphic GPGs of order 12 with λ-number 5, where $A = 10$ and $B = 11$.

[01237564B9A8]	[0123756AB948]	[01237594B6A8]	[0123759AB648]	[01237864B9A5]	[0123786AB945]	[01237894B6A5]	[01294867B3A5]
[0129486AB375]	[01294B6783A5]	[01294B6A8375]	[01297564B3A8]	[01297864B3A5]	[01297B6483A5]	[0129A564B378]	[01237B6489A5]
[01237B6A5948]	[01237B6A8945]	[01237B9486A5]	[01237B9A5648]	[01237B9A8645]	[0129A864B375]	[0129AB645378]	[0129AB648375]
[0129AB675348]	[0129AB678345]	[0123A567B948]	[0123A597B648]	[0123AB675948]	[0123AB678945]	[0123AB975648]	[0123AB978645]
[0123456789AB]	[01234567B9A8]	[0123456AB978]	[01234597B6A8]	[0123459AB678]	[01234867B9A5]	[0123486AB975]	[01234897B6A5]
[0123489AB675]	[01453786B9A2]	[0145379B68A2]	[01234B6789A5]	[01234B6A8975]	[01234B9786A5]	[01234B9A8675]	[01537264B9A8]
[01537294B6A8]	[01537B6429A8]	[01537B6A2948]	[01537B6A8942]	[01537B9426A8]	[01537B9486A2]	[01537B9A8642]	[0153A267B948]
[0153A297B648]	[0153A864B972]	[0153A867B942]	[0153A894B672]	[0153A897B642]	[0153AB648972]	[0153AB672948]	[0153AB948672]
[0153AB972648]	[0153AB978642]	[01564237B9A8]	[01564297B3A8]	[01564837B9A2]	[0156483AB972]	[01564897B3A2]	[01564B3729A8]
[01564B3A8972]	[01564B9723A8]	[01564B9A8372]	[0156A237B948]	[0156A297B348]	[0156A834B972]	[0156A837B942]	[0156A894B372]
[0156A897B342]	[0156AB348972]	[0156AB372948]	[0156AB948372]	[0156AB978342]	[01579BA32468]	[01594267B3A8]	[01594837B6A2]
[01594867B3A2]	[01594B3726A8]	[01594B3786A2]	[01594B3A8672]	[01594B6723A8]	[01594B6A8372]	[01597264B3A8]	[01597834B6A2]
[01597864B3A2]	[01597B3426A8]	[01597B3486A2]	[01597B3A2648]	[01597B3A8642]	[01597B6423A8]	[01597B6483A2]	[01597B6A8342]
[0159A267B348]	[0159A834B672]	[0159A837B642]	[0159A864B372]	[0159A867B342]	[01684A2B3957]	[01834B6729A5]	[01834B6A2975]
[01834B9726A5]	[01837594B6A2]	[01837B6A5942]	[01837B9426A5]	[0183A597B642]	[01864B3A2975]	[01864B3A5972]	[01864B9753A2]
[01867594B3A2]	[0186A534B972]	[0186A537B942]	[0186A594B372]	[0186A597B342]	[01894B3726A5]	[01894B6723A5]	[01894B6753A2]
[01897534B6A2]	[01897564B3A2]	[01897B6423A5]	[01897B6453A2]	[01B35798A642]	[01B3759486A2]	[01B3A5978642]	[01B3A8675942]
[01B3A8975642]	[01B6453A8972]	[01B6483A5972]	[01B6A5948372]	[01B6A8375942]	[01B6A8975342]	[01B9486753A2]	[01B9486A5372]
[01B9A8675342]	[02468195A3B7]	[02468A3519B7]	[02468A3B1957]	[01264537B9A8]	[0126453AB978]	[02495168A3B7]	[02495831B6A7]
[02495A3816B7]	[02495A6813B7]	[02496A7B8135]	[0249B168A357]	[01264597B3A8]	[01264837B9A5]	[0126483AB975]	[01264897B3A5]
[02735A6819B4]	[0273B195A684]	[02795A3816B4]	[02795A6813B4]	[02795B3186A4]	[02A3B7951684]	[05A3816B4927]	[01267534B9A8]
[0126753AB948]	[01267594B3A8]	[01267834B9A5]	[0126783AB945]	[01267B3489A5]	[01267B3A5948]	[01267B3A8945]	[01267B9453A8]
[01267B9483A5]	[01267B9A5348]	[01267B9A8345]	[0126A537B948]	[0126A594B378]	[0126A597B348]	[0126A894B375]	[0126A897B345]
[0126AB375948]	[0126AB378945]	[0126AB945378]	[0126AB948375]	[0126AB975348]	[0126AB978345]		

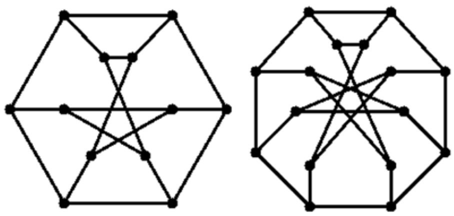

Fig. 3.1. The 6-star and 8-star, respectively.

Definition 3.1. For each even n, $n \geq 4$, an *even n-star* is a GPG wherein the edges between vertices on the outer and inner cycles are precisely $\{w_i, v_{2i-\lfloor 2i/n \rfloor}\}$ for $i = 0, 1, \ldots, n-1$, where subscripts are taken modulo n and the notation is as introduced in Definition 1.1.

It follows that the even n-stars are well defined since $\{2i - \lfloor 2i/n \rfloor \bmod n, i = 0, 1, \ldots, n-1\} = \{0, 1, 2, \ldots, n-1\}$ for all even n. We present the 6-star and 8-star in Fig. 3.1 in a way that illustrates the connection with the 5-star and the 7-star in Fig. 1.2, respectively.

Note that if in each of the even n-stars of Fig. 3.1 we first contract (as defined in [11]) the two edges that are simultaneously on the inner (resp., outer) cycle and on the only 4-cycle, and later remove any redundant edges resulting from these contractions, we obtain the $(n-1)$-stars of Fig. 1.2. The corresponding reversed operation of converting or *splitting* an edge connecting a vertex on the inner and outer cycles of a $(n-1)$-star into a 4-cycle of an even n-star as described will be useful in Section 3. More formally, assume without loss of generality that such an edge in the $(n-1)$-star for n even is $\{w_0, v_0\}$. To construct an even n-star we *split* the edge $\{w_0, v_0\}$, that is, we first delete edges $\{w_0, w_1\}$ and $\{v_0, v_{n-2}\}$, then add two new vertices w and v and the five new edges $\{w, v\}$, $\{w, w_0\}$, $\{w, w_1\}$, $\{v, v_0\}$, and $\{v, v_{n-2}\}$.

Every even n-star has a nice representation on a Möbius strip. For instance, the 6-star and the 8-star of Fig. 3.1 can be drawn on Möbius strips as shown in Fig. 3.2, with the vertices in the inner cycle of the n-stars appearing as black dots and the vertices on the outer cycle as white dots. The Möbius representation of an even n-star allows us to see its structure as an interconnection of 5-cycles, one 4-cycle, and one 6-cycle. The latter two cycles are not obvious from Fig. 3.2, but appear in the 'wrapped around' portion of the Möbius strip. Fig. 3.3 shows rotations of the Möbius strips in Fig. 3.2 where these cycles in the 6-star and 8-star are more apparent.

This Möbius representation is essential in the proof of Theorem 3.1, the main result of this section.

Theorem 3.1. *Let G be an even n-star. Then* $\lambda(G) = 7$ *if* $n = 4$ *and* $\lambda(G) = 6$ *otherwise.*

Proof. Let G be an even n-star. Prior results imply that the 4-star has $\lambda(G) = 7$ [5], the 6-star has $\lambda(G) = 6$ [2], and the 8-star also has $\lambda(G) = 6$ [1]. Now suppose that n is even and $n \geq 10$. Consider the graphs A_m, $m = 6, 8, 10$, and B in Fig. 3.4 with the given 6-labelings. If $n = 10$, connect the right-most diamond-shaped vertex ♦, triangular-shaped vertex ▲, and square-shaped vertex ■ of A_{10} to the left-most ♦, ▲, ■ vertices of A_{10}, respectively, to obtain one possible 6-labeling of a

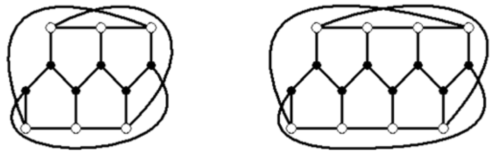

Fig. 3.2. Möbius representations of the 6-star and 8-star, respectively.

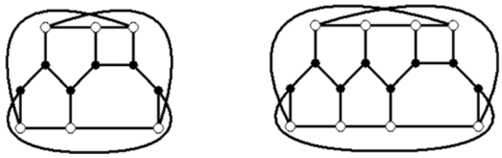

Fig. 3.3. Alternative Möbius representations of the 6-star and 8-star, respectively.

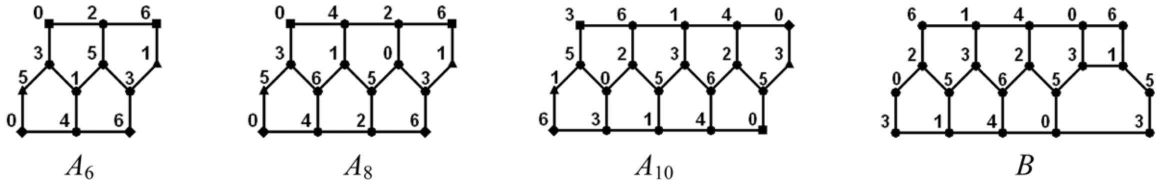

Fig. 3.4. 6-labelings of A_m, $m = 6, 8, 10$, and B.

Möbius representation of G. If $n > 10$, then n can be written as the sum $6p + 8q$ with nonnegative integers p and q. Construct a Möbius representation of G as follows. Arrange p consecutive copies of A_6 followed by q consecutive copies of A_8 in a row and connect each copy to the copy immediately to its right by connecting the right-most ♦, ▲, ■ vertices of the former to the respective left-most ♦, ▲, ■ vertices of the latter. Finalize the construction by connecting the right-most ♦, ▲, ■ vertices of the whole sequence of p A_6's and q A_8's to the left-most ■, ▲, ♦ vertices of the sequence, respectively (note that, unlike the previous connections, vertices ♦ and ■ will be connected). By inspection, one can verify that this construction provides a 6-labeling of G. In addition, since B is a subgraph of all even n-stars of order $n \geq 10$ and $\lambda(B) = 6$ by using the same graph labeling program of Section 2, we can conclude that $\lambda(G) = 6$ for all remaining orders. □

Note: We have discovered a small oversight in the proof of Lemma 11 on pages 1322–1323 of [1], which is the result for odd n corresponding to our Theorem 3.1 for even n. That proof can be easily corrected by using a modified labeling for the graph B given there when generating 6-labelings for the case of n-stars when n is not a multiple of 3: the two vertices in the bottom right corner of B labeled 0 and 2 should be re-labeled with 6 and 0, respectively.

4. A second subclass of GPGs: the *n*-igloos

Just as the n-stars can be seen on the Möbius strip as interconnections of 5-cycles and the even n-stars as interconnections of 5-cycles, one 4-cycle and one 6-cycle, certain other repeating patterns also generate GPGs with interesting structures. One such example occurs when the pattern is an interconnection of alternating 4-cycles and 6-cycles, defined below as an n-igloo.

Definition 4.1. For each $n \geq 6$, where $n - 2$ is a multiple of 4, an *n-igloo* is a GPG constructed by taking the $\frac{n}{2}$-star and splitting (as defined in the paragraph following Definition 3.1) every edge matching a vertex on the inner to a vertex on the outer cycle.

In Fig. 4.1, we provide the 10-igloo and its representation on a Möbius strip where the interconnection of alternating 4-cycles and 6-cycles is apparent. The "igloo" moniker derives from the distinctive shape of the juxtaposed 4-cycle and 6-cycle that tessellates to form the representation of the GPG on a Möbius strip; the profile of one such "igloo" is shown shaded in Fig. 4.1.

Theorem 4.1. *Let G be an n-igloo. Then $\lambda(G) = 6$.*

Proof. Let G be an n-igloo with $n \geq 6$ and $n - 2$ is a multiple of 4. The 6-igloo has $\lambda(G) = 6$ by prior results [2], as does the 10-igloo by Theorem 2.2, as it is associated with the permutation [0145892367] which is not in Table 2.2. Consider the graphs A_m, $m = 8, 14, 18, 22$ and B in Fig. 4.2 with the given 6-labelings. For each $n = 14, 18$, and 22, connect the right-most diamond-shaped vertex ♦, triangular-shaped vertex ▲, and square-shaped vertex ■ of each A_n to the left-most ♦, ▲, ■ vertices of A_n, respectively, to obtain one possible 6-labeling of a Möbius representation of the n-igloo G. In particular, it can be verified that the 14-igloo has $\lambda(G) = 6$ using the same graph labeling program mentioned in Section 2. If $n > 22$, then

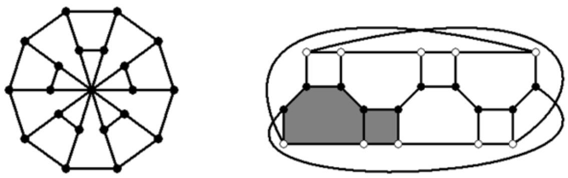

Fig. 4.1. The 10-igloo and its Möbius representation, respectively.

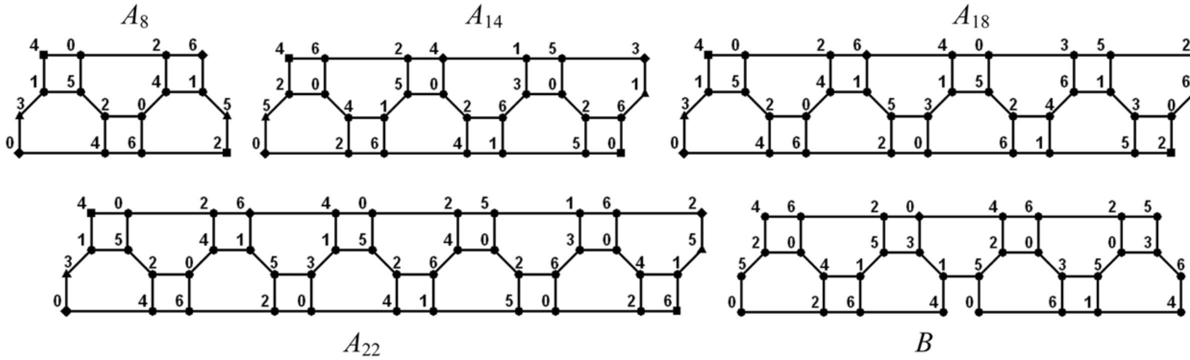

Fig. 4.2. 6-labelings of A_m, $m = 8, 14, 18, 22$ and B.

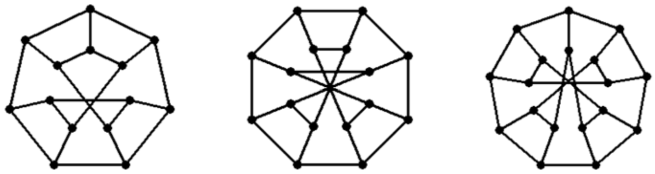

Fig. 4.3. The n-igloos for $n = 7, 8$, and 9, respectively.

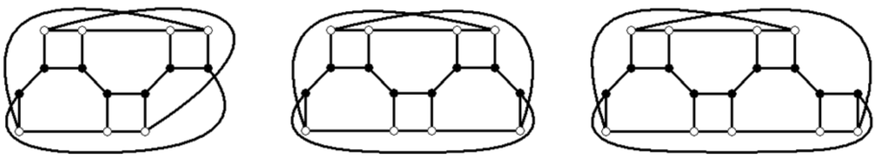

Fig. 4.4. The Möbius representation of the n-igloos for $n = 7, 8$, and 9, respectively.

n can be written as the sum $8p + 18$ or $8p + 22$ for some nonnegative integer p. Construct a Möbius representation of G as follows. Arrange p consecutive copies of A_8 followed by a copy of A_{18} or a copy of A_{22} in a row and connect each copy to the copy immediately to its right by connecting the right-most ♦, ▲, ■ vertices of the former to the left-most ■, ▲, ♦ vertices of the latter, respectively (note that vertices ♦ and ■ will be connected). Finalize the construction by connecting the right-most ♦, ▲, ■ vertices of the whole sequence of p A_8's and an A_{18} or A_{22} to the respective left-most ♦, ▲, ■ vertices of the sequence (note that vertices of the same kind will now be connected). By inspection, one can verify that this construction provides a 6-labeling of G (note that the labeling of A_8 coincides with the labeling of the left-most vertices in A_{18} and A_{22}). In addition, since B is a subgraph of all n-igloos of order $n \geq 18$ and $\lambda(B) = 6$ again using the same graph labeling program, we can conclude that $\lambda(G) = 6$ for all remaining orders. □

The definition of n-igloos can be extended to any integer $n \geq 4$. Although these generalized definitions are accompanied by a loss of symmetry on the structure of their Möbius strip representations, we were still able to verify that the λ-number of any such n-igloo is 6, except for the 4-igloo which has λ-number 7. For the sake of brevity, we will not formally introduce these definitions and results since they are analogous to the case studied in this section (results are available upon request). Fig. 4.3 shows examples of such generalized 7-, 8-, and 9-igloos and Fig. 4.4 shows their respective representations on a Möbius strips.

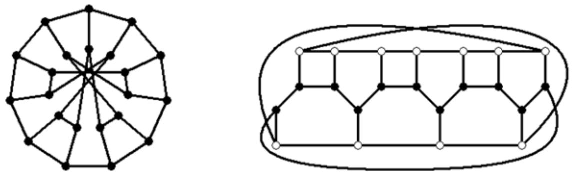

Fig. 5.1. The 11-mosaic and its Möbius representation, respectively.

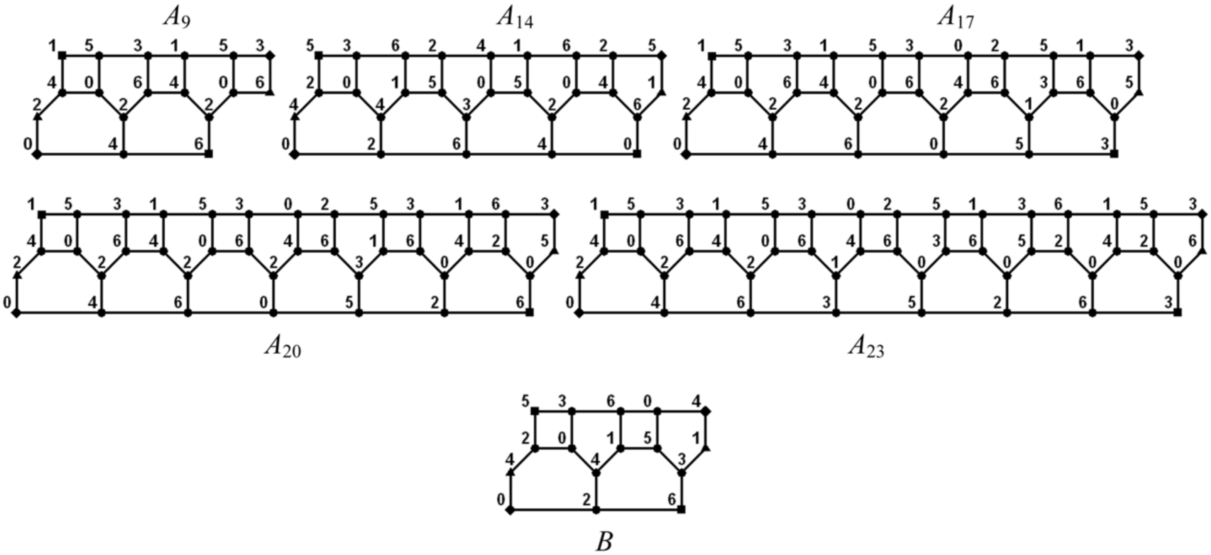

Fig. 5.2. 6-labelings of A_m, $m = 9, 14, 17, 20, 23$ and B.

5. A third subclass of GPGs: the *n*-mosaics

In this section, we introduce the n-mosaics, GPGs whose Möbius strip representations contain tessellations of 4-cycles, 5-cycles, and 6-cycles.

Definition 5.1. For each $n \geq 5$, where $n-2$ is a multiple of 3, an *n-mosaic* is a GPG constructed by taking the m-star, where $m = 2\frac{n-2}{3} + 1$, and splitting every edge $\{w_i, v_i\}$ for $i = \{0, 1, 2, \ldots, \frac{n-2}{3}\}$ and the notation is as introduced in Definition 1.2.

In Fig. 5.1, we provide the 11-mosaic and its representation on a Möbius strip where the tessellation of 4-, 5-, and 6-cycles mentioned previously is apparent.

Theorem 5.1. *Let G be an n-mosaic. Then $\lambda(G) = 6$.*

Proof. Let G be an n-mosaic with $n \geq 5$ and $n-2$ is a multiple of 3. The 5- and 8-mosaics have $\lambda(G) = 6$ by prior results [1], as does the 11-mosaic by Theorem 2.3 as it is not isomorphic to any of the three graphs in Fig. 2.4. Consider the graphs A_m, $m = 9, 14, 17, 20, 23$ and B in Fig. 5.2 with given 6-labelings. For each $n = 14, 17, 20$ and 23, connect the right-most diamond-shaped vertex ♦, triangular-shaped vertex ▲, and square-shaped vertex ■ of each A_n to the left-most ♦, ▲, ■ vertices of A_n, respectively, to obtain one possible 6-labeling of a Möbius representation of the n-mosaic G. If $n > 23$, then n can be written as the sum $9p + 17$ or $9p + 20$ or $9p + 23$ for some nonnegative integer p. Construct a Möbius representation of G as follows. Arrange p consecutive copies of A_9 followed by a copy of one of A_{17}, A_{20}, or A_{23} in a row and connect each copy to the copy immediately to its right by connecting the right-most ♦, ▲, ■ vertices of the former to the left-most ■, ▲, ♦ vertices of the latter, respectively (note that vertices ♦ and ■ will be connected). Finalize the construction by connecting the right-most ♦, ▲, ■ vertices of the whole sequence of p A_9's and an A_{17}, A_{20}, or A_{23} to the respective left-most ♦, ▲, ■ vertices of the sequence (note that vertices of the same kind will now be connected). By inspection, one can verify that this construction provides a 6-labeling of G (note that the labeling of A_9 coincides with the labeling of the left-most vertices in A_{17}, A_{20}, and A_{23}). In addition, since B is a subgraph of all n-mosaics of order $n \geq 8$ and $\lambda(B) = 6$ again using the same graph labeling program mentioned in Section 2, we can conclude that $\lambda(G) = 6$ for all remaining orders. □

Similarly as with the case of n-igloos, the definition of n-mosaics can be generalized to any integer $n \geq 5$. Although these generalized definitions are accompanied by a loss of symmetry on the structure of their Möbius strip representations, we were still able to verify that the λ-number is 6 for such generalized n-mosaics. For the sake of brevity, we will not formally

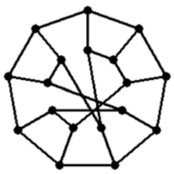

Fig. 5.3. The n-mosaics for $n = 9$ and 10, respectively.

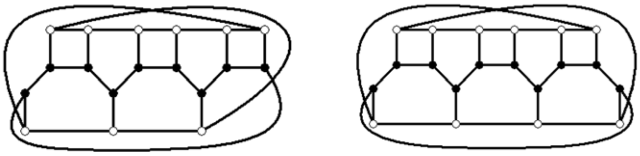

Fig. 5.4. The Möbius representation of the n-mosaics for $n = 9$ and 10, respectively.

introduce these definitions and results since they are analogous to the case studied in this section (results available upon request). Fig. 5.3 shows examples of such generalized 9- and 10-mosaics and Fig. 5.4 shows their respective representations on Möbius strips.

6. Conclusions

Motivated by the channel assignment problem, we studied the λ-number of several subclasses of GPGs. We first determined the exact λ-numbers for all GPGs of orders 9, 10, 11, and 12, thereby closing all remaining open cases up to order $n = 12$. By proving that all but three GPGs of these orders have λ-numbers 5 or 6, we improved the recently obtained upper bound of 7 for GPGs of orders 9, 10, 11, and 12 [13,14]. The method utilized to find these λ-numbers involved generating GPGs of each order associated with non-similar permutations using our novel GPG-specific elimination algorithm. An optimized version of the graph labeling algorithm of [1] was then used to categorize these GPGs according to their λ-numbers. Excess isomorphic GPGs with λ-number 5 were then removed using the BLISS graph isomorphism algorithm [16].

Next, we extended the definition of n-stars [1], a subclass of GPGs with symmetry inspired by the prisms and the Petersen graph, to account for even values of n, and we determined the exact λ-numbers for all even n-stars.

Finally, we defined two more subclasses of GPGs, the n-igloos and the n-mosaics, which also have useful representations on the Möbius strip and symmetry comparable to that of the n-stars. We determined the exact λ-numbers for all n-igloos and n-mosaics. This suggests future work studying additional families of graphs that have nice symmetry properties on Möbius strips or on other topological manifolds.

Future work may also involve answering the following open question: Is there a GPG of order greater than 11 with λ-number at least 7? We are aware of seven GPGs of order at most 11 with λ-number at least 7: the 4-star, the three GPGs of order 11 given in Fig. 2.4, and the two GPGs of order 8 (see [1]) have λ-number 7, while the Petersen graph has λ-number 9. If there is no such GPG for any order greater than 11, then the λ-number for GPGs of orders greater than 6 would be at most 6, lower than the upper bound of 7 conjectured by Georges and Mauro [5].

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