Hyers–Ulam stability of first-order homogeneous linear differential equations with a real-valued coefficient

Masakazu Onitsuka*, Tomohiro Shoji

Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan

Abstract

This paper is concerned with the Hyers–Ulam stability of the first-order linear differential equation \( x' - ax = 0 \), where \( a \) is a non-zero real number. The main purpose is to find an explicit solution \( x(t) \) of \( x' - ax = 0 \) satisfying \( |\phi(t) - x(t)| \leq \varepsilon/|a| \) for all \( t \in \mathbb{R} \) under the assumption that a differentiable function \( \phi(t) \) satisfies \( |\phi'(t) - a\phi(t)| \leq \varepsilon \) for all \( t \in \mathbb{R} \). In addition, the precise behavior of the solutions of \( x' - ax = 0 \) near the function \( \phi(t) \) is clarified on the semi-infinite interval. Finally, some applications to nonhomogeneous linear differential equations are included to illustrate the main result.

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1. Introduction

We consider the first-order homogeneous linear differential equation

\[
x' - ax = 0, \quad t \in I,
\]

where \( I \) is a nonempty open interval of \( \mathbb{R} \); \( a \) is a non-zero real number. We call that Eq. (1) has the “Hyers–Ulam stability” on \( I \) if there exists a constant \( K > 0 \) with the following property: Let \( \varepsilon > 0 \) be a given arbitrary constant. If a differentiable function \( \phi : I \rightarrow \mathbb{R} \) satisfies \( |\phi'(t) - a\phi(t)| \leq \varepsilon \) for all \( t \in I \), then there exists a solution \( x : I \rightarrow \mathbb{R} \) of Eq. (1) such that \( |\phi(t) - x(t)| \leq K\varepsilon \) for all \( t \in I \). We call such \( K \) a “HUS constant” for Eq. (1) on \( I \). It is easy to check that if \( a = 0 \) then Eq. (1) does not have the Hyers–Ulam stability on \( \mathbb{R} \). From this reason, we consider only the case that \( a \neq 0 \).

In 1998, Alsina and Ger [1] studied the Hyers–Ulam stability of the fundamental linear differential equation \( x' - x = 0 \). They proved that the linear differential equation \( x' - x = 0 \) has the Hyers–Ulam stability with a HUS constant 3 on \( I \). After that, many researchers have studied the Hyers–Ulam stability of the various linear differential equations (see [2–16]).

* Corresponding author.
E-mail address: onitsuka@xmath.ous.ac.jp (M. Onitsuka).

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In 2003, Miura, Miyajima and Takahasi [11, Corollary 2.5] gave the following sharp result. The original result can be applied to the Banach space-valued differential equations.

**Theorem A.** Eq. (1) has the Hyers–Ulam stability with a HUS constant \(1/|a|\) on \(\mathbb{R}\). Here, \(1/|a|\) is the minimum of HUS constants for Eq. (1) on \(\mathbb{R}\).

Moreover, using one of the results presented by Jung [7], Miura, Miyajima and Takahasi [11], Takahasi, Miura and Miyajima [14], we see that the solution \(x(t)\) of (1) satisfying \(|\phi(t) - x(t)| \leq \varepsilon/|a|\) for all \(t \in \mathbb{R}\) is the only one (unique). An important question now arises. Can we find an explicit solution corresponding to the above solution \(x(t)\) of (1)? The purpose of this paper is to give the answer to this question. In addition, we will investigate the precise behavior of the solutions of (1) near the function \(\phi(t)\), under the assumption that \(\sup I\) or \(\inf I\) exists. The obtained result is as follows.

**Theorem 1.** Let \(\varepsilon > 0\) be a given arbitrary constant. Suppose that a differentiable function \(\phi : I \to \mathbb{R}\) satisfies \(|\phi'(t) - a\phi(t)| \leq \varepsilon\) for all \(t \in I\). Then one of the following holds:

(i) if \(a > 0\) and \(\sup I\) exists, then \(\lim_{t \to 0^-} \phi(t)\) exists where \(\tau = \sup I\), and any solution \(x(t)\) of (1) with 
\[|\lim_{t \to 0^-} \phi(t) - x(\tau)| < \varepsilon/a\]
satisfies that \(|\phi(t) - x(t)| < \varepsilon/a\) for all \(t \in I\);

(ii) if \(a > 0\) and \(\sup I\) does not exist, then \(\lim_{t \to -\infty} \phi(t)e^{-at}\) exists, and there exists exactly one solution 
\(x(t) = (\lim_{t \to -\infty} \phi(t)e^{-at})e^{at}\) of (1) such that \(|\phi(t) - x(t)| \leq \varepsilon/a\) for all \(t \in I\);

(iii) if \(a < 0\) and \(\inf I\) exists, then \(\lim_{t \to \sigma^-} \phi(t)\) exists where \(\sigma = \inf I\), and any solution \(x(t)\) of (1) with 
\[|\lim_{t \to \sigma^-} \phi(t) - x(\sigma)| < \varepsilon/|a|\]
satisfies that \(|\phi(t) - x(t)| < \varepsilon/|a|\) for all \(t \in I\);

(iv) if \(a < 0\) and \(\inf I\) does not exist, then there exists exactly one solution 
\(x(t) = (\lim_{t \to -\infty} \phi(t)e^{-at})e^{at}\) of (1) such that \(|\phi(t) - x(t)| \leq \varepsilon/|a|\) for all \(t \in I\).

From Theorem 1, we can establish the following result.

**Corollary 2.** Eq. (1) has the Hyers–Ulam stability with a HUS constant \(1/|a|\) on \(I\).

**Remark 1.** In the special case that \(a = 1\), a HUS constant for Eq. (1) on \(I\) is one from Corollary 2. That is, we can conclude that our theorem is an improvement of the result of Alsina and Ger [1].

In the case that \(I = \mathbb{R}\), we can state the following result from the assertions (ii) and (iv) in Theorem 1.

**Corollary 3.** Eq. (1) has the Hyers–Ulam stability with a HUS constant \(1/|a|\) on \(\mathbb{R}\). Furthermore, the solution \(x(t)\) of (1) satisfying \(|\phi(t) - x(t)| \leq \varepsilon/|a|\) for all \(t \in \mathbb{R}\) is the only one, which written as 
\[x(t) = (\lim_{t \to -\infty} \phi(t)e^{-at})e^{at}\]
if \(a > 0\) (resp., \(x(t) = (\lim_{t \to -\infty} \phi(t)e^{-at})e^{at}\) if \(a < 0\)).

**Remark 2.** Let \(\varepsilon > 0\) be a given arbitrary constant. We consider the nonhomogeneous differential equation
\[x' - ax = -\varepsilon\]
on \(\mathbb{R}\), where \(a\) is a non-zero real number. We can easily see that the function \(\phi(t) = \varepsilon/a + ce^{at}\) for \(t \in \mathbb{R}\) is the general solution of this nonhomogeneous differential equation, where \(c\) is an arbitrary constant. Since \(ce^{at}\) is a solution of (1), \(|\phi(t) - x(t)| = \varepsilon/|a|\) holds for all \(t \in \mathbb{R}\). From this fact and the assertion in Corollary 3, we can conclude that \(1/|a|\) is the minimum of HUS constants for Eq. (1) on \(\mathbb{R}\). Moreover, this example shows that it is not possible to weaken the condition \(|\lim_{t \to -\tau^-} \phi(t) - x(\tau)| < \varepsilon/a\) in (i) of Theorem 1 to \(|\lim_{t \to -\tau^-} \phi(t) - x(\tau)| \leq \varepsilon/a\), in order to satisfy \(|\phi(t) - x(t)| < \varepsilon/a\) for \(t \in I\).
When restricted to the case that $I$ is finite interval, using the assertions (i) and (iii) in Theorem 1, we can verify the following fact.

**Corollary 4.** Let $J$ be a finite nonempty open interval of $\mathbb{R}$ and $\varepsilon > 0$ be a given arbitrary constant. If a differentiable function $\phi : J \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \leq \varepsilon$ for all $t \in J$, then there exists a solution $x : J \to \mathbb{R}$ of (1) such that $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in J$.

2. Preliminaries

In this section, we give some preparations.

**Lemma 1.** Let $\varepsilon > 0$ be a given arbitrary constant and let $\phi : I \to \mathbb{R}$ be a differentiable function. Then the inequality $|\phi'(t) - a\phi(t)| \leq \varepsilon$ holds for all $t \in I$ if and only if the inequality

$$0 \leq \left( \phi(t) - \frac{\varepsilon}{a} \right) e^{-at}' \leq 2\varepsilon e^{-at}$$

holds for all $t \in I$.

**Proof.** The statement of Lemma 1 is clearly true since the equality

$$\left( \phi(t) - \frac{\varepsilon}{a} \right) e^{-at}' = (\phi'(t) - a\phi(t) + \varepsilon)e^{-at}$$

holds for all $t \in I$. \qed

**Proposition 2.** Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \leq \varepsilon$ for all $t \in I$. Then there exist a nondecreasing differentiable function $u : I \to \mathbb{R}$ and a nonincreasing differentiable function $v : I \to \mathbb{R}$ such that

$$\phi(t) = u(t)e^{at} + \frac{\varepsilon}{a} = v(t)e^{at} - \frac{\varepsilon}{a}$$

and one of the following hold:

(i) if $a > 0$ and the supremum of $I$ exists, then $\lim_{t \to \tau-0} u(t)$ and $\lim_{t \to \tau-0} v(t)$ exist, and

$$u(t) \leq \lim_{t \to \tau-0} u(t) < \lim_{t \to \tau-0} v(t) \leq v(t)$$

holds for all $t \in I$, where $\tau = \sup I$;

(ii) if $a > 0$ and the supremum of $I$ does not exist, then $\lim_{t \to -\infty} u(t)$ and $\lim_{t \to -\infty} v(t)$ exist, and

$$u(t) \leq \lim_{t \to -\infty} u(t) = \lim_{t \to -\infty} v(t) \leq v(t)$$

holds for all $t \in I$;

(iii) if $a < 0$ and the infimum of $I$ exists, then $\lim_{t \to \sigma+0} u(t)$ and $\lim_{t \to \sigma+0} v(t)$ exist, and

$$v(t) \leq \lim_{t \to \sigma+0} v(t) < \lim_{t \to \sigma+0} u(t) \leq u(t)$$

holds for all $t \in I$, where $\sigma = \inf I$.
(iv) if \( a < 0 \) and the infimum of \( I \) does not exist, then \( \lim_{t \to -\infty} u(t) \) and \( \lim_{t \to -\infty} v(t) \) exist, and

\[
v(t) \leq \lim_{t \to -\infty} v(t) = \lim_{t \to -\infty} u(t) \leq u(t)
\]  

holds for all \( t \in I \).

**Proof.** Let

\[
u(t) = \left( \phi(t) - \frac{\varepsilon}{a} \right) e^{-at} \quad \text{and} \quad v(t) = \left( \phi(t) + \frac{\varepsilon}{a} \right) e^{-at}
\]

for \( t \in I \), then clearly (2) holds. Therefore, we have

\[
u(t) = v(t) - \frac{2\varepsilon}{a} e^{-at}
\]

and

\[
u(t) = \begin{cases} v(t) & \text{if } a > 0, \\ > v(t) & \text{if } a < 0 \end{cases}
\]

for \( t \in I \). Using (7) and the assertion in Lemma 1, we obtain the inequalities

\[
0 \leq u'(t) \leq 2\varepsilon e^{-at} \quad \text{and} \quad -2\varepsilon e^{-at} \leq v'(t) \leq 0
\]

for \( t \in I \). Therefore, we can conclude that \( u(t) \) is a nondecreasing function and \( v(t) \) is a nonincreasing function. It follows from these facts and (8) that assertions (i) and (iii) are true.

We next prove assertion (ii). Let \( s \in I \) be a fixed number. From (8) with \( a > 0 \), we have

\[
u(t) < v(s)
\]

for \( t \in I \). Hence, \( u(t) \) is bounded above and nondecreasing. From this reason, we conclude that \( \lim_{t \to -\infty} u(t) \) exists. Moreover, we can easily see that

\[
\lim_{t \to -\infty} u(t) = \lim_{t \to -\infty} v(t)
\]

holds from (7). Since \( u(t) \) is a nondecreasing function, \( v(t) \) is a nonincreasing function and the above equality holds, (4) is satisfied for \( t \in I \).

Using the same argument in the proof of assertion (ii), we can easily see that assertion (iv) is true. The proof of Proposition 2 is now complete. \( \Box \)

3. Proof of the main theorem

In this section, we give the proof of Theorem 1.

**Proof of Theorem 1.** First we prove case (i). It follows from assertion (i) in Proposition 2 that there exist two differentiable functions \( u : I \to \mathbb{R} \) and \( v : I \to \mathbb{R} \) such that (2) and (3) hold for \( t \in I \). Since \( \lim_{t \to -\tau} u(t) \) exists and (2) holds for \( t \in I \), \( \lim_{t \to -\tau} \phi(t) \) also exists, where \( \tau = \sup I \). Let \( \lim_{t \to -\tau} u(t) < c_1 < \lim_{t \to -\tau} \phi(t) \) be arbitrary. We consider the function \( x(t) = c_1 e^{at} \) for \( t \in I \). Then, from (2), (3) and \( x(\tau) = c_1 e^{a\tau} \), we see that \( x(t) \) is a solution of Eq. (1) satisfying \( |\lim_{t \to -\tau} \phi(t) - x(\tau)| < \varepsilon / a \).

Using (2) and (3) again, we have

\[
\phi(t) - x(t) \leq \left( \lim_{t \to -\tau} u(t) - c_1 \right) e^{at} + \frac{\varepsilon}{a} \leq \frac{\varepsilon}{a}
\]
and 
\[\phi(t) - x(t) \geq \left( \lim_{t \to t_0} v(t) - c_1 \right) e^{at} - \frac{\varepsilon}{a} > \frac{-\varepsilon}{a}\]
for \(t \in I\). Thus, we obtain the inequality \(|\phi(t) - x(t)| < \varepsilon/a\) for \(t \in I\).

Next we prove case (ii). By means of assertion (ii) in Proposition 2, there exist two functions \(u : I \to \mathbb{R}\) and \(v : I \to \mathbb{R}\) such that (2) and (4) hold for \(t \in I\). Since \(\lim_{t \to \infty} u(t)\) exists and (2) holds for \(t \in I\), the function \(\phi(t) e^{-at}\) also has the same limiting value. Let
\[c_2 = \lim_{t \to \infty} u(t) = \lim_{t \to \infty} \phi(t) e^{-at},\]
and consider the function \(x(t) = c_2 e^{at}\) for \(t \in I\). Then \(x(t)\) is a solution of Eq. (1). From (2) and (4), we have
\[\phi(t) - x(t) = (u(t) - c_2) e^{at} + \frac{\varepsilon}{a} \leq \frac{-\varepsilon}{a}\]
and
\[\phi(t) - x(t) = (v(t) - c_2) e^{at} - \frac{\varepsilon}{a} \geq \frac{-\varepsilon}{a}\]
for \(t \in I\). Hence, we get the inequality \(|\phi(t) - x(t)| < \varepsilon/a\) for \(t \in I\). Note here that if we choose a constant \(c\) so that \(c \neq c_2\), then the function \(x(t) = ce^{at}\) is a solution of Eq. (1), however, it does not satisfy (9) or (10) for \(t\) sufficiently large. Thus, \(x(t) = c_2 e^{at}\) is exactly one solution of (1) satisfying \(|\phi(t) - x(t)| < \varepsilon/a\) for \(t \in I\).

To prove case (iii), we choose a constant \(c_3\) so that
\[\lim_{t \to \infty} v(t) < c_3 < \lim_{t \to \infty} u(t), \quad \sigma = \inf I,\]
where \(u(t)\) and \(v(t)\) satisfy (2) and (5) for \(t \in I\) from assertion (iii) in Proposition 2. From (2) and (5), we can consider the function \(x(t) = c_3 e^{at}\) which becomes a solution of Eq. (1) satisfying \(\lim_{t \to \sigma + 0} \phi(t) - x(\sigma) < \varepsilon/|a|\). By (2) and (5) again, we obtain
\[-\frac{\varepsilon}{|a|} < \left( \lim_{t \to \sigma + 0} u(t) - c_3 \right) e^{at} + \frac{\varepsilon}{a} \leq \phi(t) - x(t) \leq \left( \lim_{t \to \sigma + 0} v(t) - c_3 \right) e^{at} - \frac{\varepsilon}{a} < \frac{\varepsilon}{|a|}\]
for \(t \in I\). That is, we have \(|\phi(t) - x(t)| < \varepsilon/|a|\) for \(t \in I\).

Finally we prove case (iv). By means of assertion (iv) in Proposition 2, there exist two functions \(u(t)\) and \(v(t)\) satisfying (2) and (6) for \(t \in I\). Let \(c_4 = \lim_{t \to -\infty} u(t)\), and consider the function \(x(t) = c_4 e^{at}\) for \(t \in I\). Then, from (2) and (6), we have the inequality \(|\phi(t) - x(t)| < \varepsilon/|a|\) for \(t \in I\). Using the same argument as in the proof of case (ii), we can conclude that \(x(t) = c_4 e^{at}\) is exactly one solution of (1) satisfying \(|\phi(t) - x(t)| < \varepsilon/|a|\) for \(t \in I\). This completes the proof of Theorem 1. 

4. Applications to nonhomogeneous linear differential equations

In this section, we give some applications to illustrate the main result.

Example 1. We consider the nonhomogeneous differential equation
\[x' + x = 2 \cos t.\] (11)
It is easy to check that the function \(\phi(t) = (\phi(0) - 1)e^{-t} + \cos t + \sin t\) is a solution of (11). Using (iii) in Theorem 1 with \(\varepsilon = 2\), any solution \(x(t)\) of (1) with \(a = -1\) and initial condition \(|\phi(0) - x(0)| < 2\) satisfies that \(|\phi(t) - x(t)| < 2\) for all \(t \geq 0\). Fig. 1 shows that all solutions of (1) with \(a = -1\) and \(2 < x(0) < 6\) satisfy
Consider the nonhomogeneous differential equation

\[ x' - ax = f(t), \quad (12) \]

where the real-valued function \( f(t) \) is continuous for \( t \geq 0 \). We can clarify the asymptotic behavior of any solution of (12) by using Theorem 1.

**Corollary 5.** Suppose that there exists an \( \varepsilon > 0 \) such that \( |f(t)| \leq \varepsilon \) for \( t \geq 0 \). Then any solution \( \phi(t) \) of (12) satisfies one of the following:

(i) if \( a > 0 \) then \( \lim_{t \to \infty} \phi(t)e^{-at} \) exists.

(ii) if \( a < 0 \) then \( \limsup_{t \to \infty} |\phi(t)| \leq \varepsilon/|a| \).

**Proof.** Assertion (i) is an immediate consequence from (ii) in Theorem 1. Next, we prove assertion (ii). By way of contradiction, we suppose that there exists a solution \( \phi(t) \) of (12) satisfying \( \limsup_{t \to \infty} |\phi(t)| > \varepsilon/|a| \). Using the assumption and (iii) in Theorem 1, we can find a solution \( x(t) \) of (1) with \( |\phi(0) - x(0)| < \varepsilon/|a| \) satisfies that \( |\phi(t) - x(t)| < \varepsilon/|a| \) for \( t \geq 0 \). Since \( a < 0 \) and \( x(t) \) written as \( x(0)e^{at} \), we see that

\[ \limsup_{t \to \infty} |\phi(t)| = \limsup_{t \to \infty} |\phi(t) - x(t)| \leq \frac{\varepsilon}{|a|}. \]

This is a contradiction. The proof of Corollary 5 is now complete. \( \square \)

**References**


