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Spectrum is periodic for n-intervals

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Abstract

In this paper we study spectral sets which are unions of finitely many intervals in $\mathbb{R}$. We show that any spectrum associated with such a spectral set Ω is periodic, with the period an integral multiple of the measure of Ω. As a consequence we get a structure theorem for such spectral sets and observe that the generic case is that of the equal interval case.

Keywords: Spectral sets; Spectrum; Tiling; Fuglede's conjecture; Zeros of exponential polynomials; Sets of sampling and interpolation; Landau's density theorem

1. Introduction

In this paper we study the structure of the spectrum associated to a spectral set $\Omega \subset \mathbb{R}$, which is a finite union of intervals. In order to describe our result and its context, we begin with a brief account of some of the relevant history of the problem.

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^d$ with finite positive measure. For $\lambda \in \mathbb{R}^d$, let

$$e_\lambda(x) := |\Omega|^{-1/2} e^{2\pi i\lambda.x} \chi_\Omega(x), \quad x \in \mathbb{R}^d.$$

Ω is said to be a *spectral set* if there exists a subset $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{e_\lambda \colon \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space $L^2(\Omega)$. The set Λ is said to be a *spectrum* for Ω and the pair (Ω, Λ) is called a *spectral pair*.

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The study of spectral properties of sets has its origin in some questions of functional-analysis. It began with the work of B. Fuglede [15], who while investigating a question suggested to him by I. Segal concerning sets Ω which have the 'extension property' (namely, the existence of commuting self-adjoint extensions of the operators $-i\frac{\partial}{\partial x_1},\dots,-i\frac{\partial}{\partial x_n}$ defined on $C_0^\infty(\Omega)$ to a dense subspace of $L^2(\Omega)$) observed that spectral sets have this property. Further it was shown that if Ω is assumed to be connected then having the extension property is equivalent to Ω being a spectral set [15,26,47]. For a detailed and very interesting account of the early history and motivation behind the origin of these problems we refer the interested reader to [11].

In his study of spectral sets Fuglede observed that the spectral pair problem has interesting connections to tiling problems.

A measurable set $T \subset \mathbb{R}^d$, having positive measure is said to be a *prototile* if T tiles $\mathbb{R}^d$ by translations. In other words, we say a set T as above is a prototile if there exists a subset $\mathcal{T} \subset \mathbb{R}^d$ such that $\{T + t\colon t \in \mathcal{T}\}$ forms a partition a.e. of $\mathbb{R}^d$. The set $\mathcal{T}$ is said to be a *tiling set* for T and the pair $(T, \mathcal{T})$ is called a *tiling pair*.

Fuglede proved the following theorem:

Theorem 1.1. *(See Fuglede [15].) Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^d$ and $\mathcal{L}^*$ be the dual lattice. Then $(\Omega, \mathcal{L})$ is a tiling pair if and only if $(\Omega, \mathcal{L}^*)$ is a spectral pair.*

He went on to make the following conjecture, which is also known as the spectral set conjecture:

Conjecture 1.2 *(Fuglede's conjecture). A set $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translations.*

This led to an intense study of spectral and tiling properties of sets. In recent years, this conjecture, in its full generality, has been shown to be false in both directions if the dimension $d \geqslant 3$ [50,34,35,44,14,13]. However, interest in the conjecture is alive and the conjecture has been shown to be true in many cases under additional assumptions.

For example, the case where Ω is assumed to be convex received a lot of attention recently. It is known that if a convex body K tiles $\mathbb{R}^d$ by translations then it is necessarily a symmetric polytope and there is a lattice $\mathcal{L}$ such that $(K, \mathcal{L})$ is a tiling pair [53,45]. Thus the "tiling implies spectral" part of the Fuglede conjecture follows easily from Fuglede's result. In the converse direction, it has been shown that a convex set which is spectral has to be symmetric [31], and such sets do not have a point of curvature [20,32,25] (i.e., they are symmetric polytopes). However it is only in dimension 2 that the "spectral implies tiling" part of the Fuglede conjecture has been proved [21].

In its full generality Fuglede's conjecture remains open in dimensions 1 and 2. In one dimension the conjecture is known to be related to some interesting number theoretic questions and conjectures [6,37,40,51]. It is generally believed that the conjecture is true in dimension 1.

An interesting recent development is the discovery of spectral measures [28,49], these are probability measures μ which have a spectrum Λ (i.e., the set of exponentials E_Λ is an orthonormal basis for $L^2(\mu)$). Research on these problems has led to the spectral theory for fractal measures and has received a lot of attention in recent years [10,7,8,12].

Not surprisingly, spectral sets have found application in various fields, most notably in the study of wavelets. Gabardo and Nashed introduced a generalization of Mallat's classical multiresolution analysis using spectral pairs [16–19,3]. Later in a very influential work Wang [54] studied wavelets with irregular translation and dilation sets and established a surprising connection of this question to that of spectral pair and tiling sets. Starting with the work of Dutkay and Jorgensen [9] the subject of wavelets on spectral measures has gained considerable attention recently [11,4,1,2].

Attempts to answer the question about sets (measures) which admit such Fourier expansions, have revealed a plethora of connections between functional analysis, number theory, representation theory, combinatorics, commutative algebra, dynamical systems, operator theory and Fourier analysis. In both of these instances there are intriguing duality questions related to tiling problems about which we talk in more detail in the next section.

1.1. Spectral-tiling duality

Starting with Fuglede's original work, many results demonstrate that there exists a deep relationship between spectra and tiling sets. For example, when I is the unit cube in $\mathbb{R}^d$, then (I, Γ) is a tiling pair if and only if (I, Γ) is a spectral pair. This was first conjectured by Jorgensen and Pedersen [29] who proved it for $d \leqslant 3$. Subsequently several authors gave proofs of this result using different techniques [38,23,30,43]. It is worth mentioning here that tiling by cubes can be very complicated [39].

In fact there is a dual conjecture due to Jorgensen and Pedersen.

Conjecture 1.3 *(The dual spectral set conjecture [29]). A subset Γ of $\mathbb{R}$ is a spectrum for some spectral set Ω if and only if it is a tiling set for some prototile T.*

Approaching the spectral set conjecture by studying the associated spectra or tiling sets has been very fruitful, specially when these have some additional structure like periodicity.

A set $\Gamma \subset \mathbb{R}^d$ is said to be *periodic* if there exists a full-rank lattice $\mathcal{L}$ of $\mathbb{R}^d$ such that $\Gamma = \mathcal{L} + \{\gamma_1, \dots, \gamma_m\}$, and if, in addition, all coset differences $\gamma_i - \gamma_j$ are commensurate with the lattice $\mathcal{L}$, then Γ is said to be *rational periodic*.

Pedersen [48] gave a classification of spectral sets which have a periodic spectrum expressed in terms of complex Hadamard matrices. On the other hand, Lagarias and Wang [41] gave a characterization of prototiles which tile $\mathbb{R}^d$ by a rational periodic tiling set in terms of factorization of abelian groups. Further, they introduced the concept of universal spectrum [41].

A tiling set $\mathcal{T}$ is said to have a *universal spectrum* $\Lambda_{\mathcal{T}}$, if every set Ω that tiles R^d by $\mathcal{T}$ is a spectral set with spectrum $\Lambda_{\mathcal{T}}$.

Lagarias and Wang [41] proved that a large class of tiling sets $\mathcal{T}$ have a universal spectrum and then conjectured that all rational periodic tiling sets have a universal spectrum which is also rational periodic. This is known as the Universal Spectrum conjecture. Given a rational periodic tiling set $\mathcal{T}$ they gave necessary and sufficient conditions for a rational periodic spectrum $\Lambda_{\mathcal{T}}$ to be a universal spectrum for $\mathcal{T}$.

These developments were instrumental in disproving the "tiling implies spectral" part of Fuglede's conjecture. Later Farkas, Matolcsi and Móra [13] proved that the "tiling implies spectral" part of Fuglede's conjecture is equivalent to the Universal Spectrum conjecture in any dimension. Unlike in higher dimensions where the tiling sets can be very irregular (e.g. consider the case of tiling by a cube [39]) the tiling sets in one dimension are quite rigid in that they exhibit a lot of structure. This gives further credence to the belief that Fuglede conjecture may be true in this case.

1.2. Structure of one-dimensional tiling sets and spectrum

Many results are known concerning the structure of tiling sets associated with one-dimensional prototiles. The fundamental work in this setting is due to Lagarias and Wang [40], who gave a complete characterization of the structure of a tiling set $\mathcal{T}$ associated with a compactly supported prototile T whose boundary has measure zero. They proved that in this case $\mathcal{T}$ is always rational periodic and the period is an integral multiple of the measure of T. Equipped with this knowledge they manage to give a characterization of T itself. Further they show that for every tiling pair $(T, \mathcal{T})$ there exists a tiling pair $(T_1, \mathcal{T})$ where T_1 is a cluster i.e., a union of equal intervals, and the problem of finding all possible tiling pairs $(T, \mathcal{T})$ is then related to finding all possible factorizations of finite cyclic groups. Thus, in essence, the entire complexity is contained in the equal interval case itself. Later Kolountzakis and Lagarias extended the periodicity result to all compactly supported prototiles [33].

Given that the tiling sets in one dimension are such highly regular object one would expect the spectra to possess a similar property in this case. Indeed, to date all known spectra associated with one-dimensional spectral sets are rational periodic. But rather surprisingly, comparatively little progress has been made in classifying the structure of spectra associated with one-dimensional spectral sets.

The main result in this direction which the authors could find in the literature is by Jorgensen and Pedersen in [27], where the following assertion is proved: that a spectral set $\Omega \subset \mathbb{R}$ which is a finite union of equal intervals has finitely many distinct spectra, which are all periodic. Further, under an additional hypothesis that such a set Ω is contained in a "small" interval, Laba has proved that the associated spectra for such spectral sets Ω are rational periodic [37].

For the case when Ω is a finite union of intervals (intervals can have unequal length), even less is known. Only the 2-interval case has been completely resolved by Laba [36], where she proved that Fuglede's conjecture holds true. In [5] the 3-interval case was investigated, where it was shown that for such sets "tiling implies spectral" holds; whereas the "spectral implies tiling" part of the conjecture was proved for this case under some additional hypothesis.

The general case of spectral sets Ω which are unions of finitely many intervals (not necessarily equal) was studied in [5]. It was shown there that a spectrum Λ associated with a spectral set Ω, which is a union of n-intervals has a highly "arithmetical structure", namely, if the spectrum Λ contains an arithmetic progression of length $2n$, then the complete arithmetic progression is contained in it.

1.3. Results

Our objective in this paper is to study the structure of a spectrum Λ associated with a spectral set $\Omega \subset \mathbb{R}$, when Ω is a union of n-intervals. We prove that all associated spectra for such spectral sets are periodic.

The essential idea behind our proof is to show that similar to the case of a tiling set a finite section of a spectrum essentially determines the complete spectrum. Theorem 2.2 and Theorem 2.8 are manifestations of this phenomenon and will be central to our proof. The other key ingredient of the proof is a density result of Landau for sets of sampling and interpolation (see Theorem 2.10). In Section 2, we state this theorem, explore the geometry of the zero set of the Fourier transform of a spectral set and prove Theorem 2.2 and Theorem 2.8.

In Section 3 we prove our main theorem

Theorem 1.4. *Let Ω be a union of n intervals, $\Omega = \bigcup_{j=1}^{n} I_j$, such that $|\Omega| = 1$. If (Ω, Λ) is a spectral pair, then Λ is a d-periodic set with $d \in \mathbb{N}$. Thus Λ has the form $\Lambda = \bigcup_{j=1}^{d} \{\lambda_j + d\mathbb{Z}\}$.*

The structure of spectral sets which have a periodic spectrum has been studied in [48] and [41]. As a consequence of Theorem 1.4 we get a structure theorem for such spectral sets and observe that the equal interval case is the generic case.

Theorem 1.5. *Let (Ω, Λ) be a spectral pair such that Ω is a bounded region in $\mathbb{R}$ and Λ is d-periodic. Then there exists a disjoint partition of $[0, 1/d)$ into finite number of sets $E_1, E_2, \dots, E_k$ such that $\Omega = \bigcup_{j=1}^{k} (E_j + A_j)$; $A_j \subseteq \mathbb{Z}/d$. Further, each set $\Omega_j := [0, 1/d) + A_j$ is a spectral set with Λ as a spectrum.*

Our results are based on the study of the geometry of the spectrum, more specifically the study of zero sets of exponential polynomials and Landau's density theorem about sets of sampling and interpolation which we describe in the following section.

2. The geometry of the spectrum

Let (Ω, Λ) be a spectral pair. Since spectral properties of sets are invariant under affine transformations, we will henceforth assume that Ω has measure 1 and that $0 \in \Lambda \subset \Lambda - \Lambda$.

In this paper we will always assume that Ω is bounded. Then $\widehat{\chi_\Omega}$, the Fourier transform of the characteristic function of Ω, is an entire function.

Let $\mathbb{Z}(\widehat{\chi_\Omega})$ be the zero set of $\widehat{\chi_\Omega}$ union $\{0\}$ i.e.,

$$\mathbb{Z}(\widehat{\chi_\Omega}) := \bigl\{\xi \in \mathbb{R}\colon\ \widehat{\chi_\Omega}(\xi) = 0\bigr\} \cup \{0\}.$$

If $\lambda, \lambda' \in \Lambda$, then by orthogonality of e_λ and $e_{\lambda'}$ we have $\lambda - \lambda' \in \mathbb{Z}(\widehat{\chi_\Omega})$. Hence $0 \in \Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}(\widehat{\chi_\Omega})$. Thus the geometry of the zero set of $\widehat{\chi_\Omega}$ plays a crucial role in determining the structure of Λ.

Observe that, as $\widehat{\chi_\Omega}(0) = 1$, there exists a neighborhood around 0, which does not intersect $\mathbb{Z}(\widehat{\chi_\Omega})$ except at 0. Hence, Λ is uniformly discrete. Let Λ_s be the set of spectral gaps for a spectrum Λ i.e.,

$$\Lambda_s := \{\lambda_{n+1} - \lambda_n \mid \lambda_n \in \Lambda\}.$$

Clearly $\Lambda_s \subseteq \Lambda - \Lambda \subseteq \mathbb{Z}(\widehat{\chi_\Omega})$ and Λ_s is bounded below. On the other hand, as a consequence of Landau's density results (see Theorem 2.10 below), we see easily that Λ_s is also bounded above. So, by the analyticity of $\widehat{\chi_\Omega}$ we can conclude that Λ_s is finite. Thus the spectrum can be seen as a bi-infinite word made up of a finite alphabet, in terms of the spectral gaps. When Ω is a union of finite number of intervals, a much more precise estimate is known for spectral gaps [42,22,24].

From now on we will assume that Ω is a union of a finite number of intervals. Let $\Omega = \bigcup_{i=1}^{n} [a_i, a_i + r_i)$, $\sum_{i=1}^{n} r_i = 1$. Then,

$$\widehat{\chi_\Omega}(\xi) = \frac{\sum_{i=1}^{n} [e^{2\pi i (a_i + r_i)\xi} - e^{2\pi i (a_i)\xi}]}{2\pi i \xi},$$

and $\mathbb{Z}(\widehat{\chi_\Omega})$ is precisely the zero set of the exponential polynomial given by

$$\mathcal{P}_\Omega(\xi) := \sum_{i=1}^{n} \left(e^{2\pi i(a_i+r_i)\xi} - e^{2\pi i(a_i)\xi}\right),$$

which is the numerator in the expression of $\widehat{\chi_\Omega}$. Thus we are naturally led to the study of exponential polynomials and their zeros.

There is a beautiful result by Turan [52,46] which gives size estimates of exponential polynomials along arithmetic progressions. This result has the interesting consequence that if an arithmetic progression $a, a+d, \dots, a+(2n-1)d$ of length $2n$ occurs in $\mathbb{Z}(\widehat{\chi_\Omega})$ then the complete arithmetic progression $a+d\mathbb{Z} \subset \mathbb{Z}(\widehat{\chi_\Omega})$. This suggests that the zero sets of exponential polynomials are highly structured and we are naturally led to ask the question whether Λ inherits this kind of structure?

In the next section we will prove an analog of Turan's lemma for the spectrum.

2.1. Arithmetic progressions in Λ

As we have mentioned before, it was shown in [5] that the existence of an arithmetic progression of length $2n$ in Λ implies that the complete arithmetic progression is in Λ. Here, we improve on that result and using Newton's Identities about symmetric polynomials, give a proof that the occurrence of an arithmetic progression of length $n+1$ in the spectrum ensures that the complete arithmetic progression is in the spectrum. Let

$$P(z) := \prod_{i=1}^{n} (z - \alpha_n) = z^n + S_1 z^{n-1} + S_2 z^{n-2} + \dots + S_n.$$

Let W_k be the sum of kth power of the roots of $P(z)$, namely

$$W_k := \alpha_1^k + \alpha_2^k + \dots + \alpha_n^k; \quad k = 1, \dots, n.$$

Then the coefficients S_i and W_i are related by Newton's Identities:

$$W_k + S_1 W_{k-1} + S_2 W_{k-2} + \dots + S_{k-1} W_1 + k S_1 = 0; \quad k = 1, \dots, n. \tag{1}$$

Thus $W_1, W_2, \dots, W_n$ uniquely determine the polynomial $P(z)$.

Proposition 2.1. *If $\mathbb{Z}(\widehat{\chi_\Omega})$ contains an arithmetic progression of length $n+1$ with its first term 0, say $0, d, \dots, nd \in \mathbb{Z}(\widehat{\chi_\Omega})$ then*

(a) *the whole arithmetic progression $d\mathbb{Z} \subset \mathbb{Z}(\widehat{\chi_\Omega})$,*
(b) *$d \in \mathbb{Z}$, and*
(c) *Ω d-tiles $\mathbb{R}$.*

Proof. Note that if $t \in \mathbb{Z}(\widehat{\chi_\Omega})$, then

$$\sum_{j=1}^{n}\left[e^{2\pi i t(a_j+r_j)} - e^{2\pi i t a_j}\right] = 0.$$

The hypothesis says that $\widehat{\chi_\Omega}(ld) = 0$; $l = 1, \dots, n$, hence

$$\sum_{j=1}^{n}\left[e^{2\pi i l d(a_j+r_j)} - e^{2\pi i l d a_j}\right] = 0, \quad l = 1, \dots, n.$$

We write $\zeta_{2j} = e^{2\pi i d a_j}$; $\zeta_{2j-1} = e^{2\pi i d(a_j+r_j)}$; $j = 1, \dots, n$. Then the above system of equations can be rewritten as

$$\begin{aligned}
\zeta_1 + \zeta_3 + \cdots + \zeta_{2n-1} &= \zeta_2 + \zeta_4 + \cdots + \zeta_{2n} = W_1, \\
\zeta_1^2 + \zeta_3^2 + \cdots + \zeta_{2n-1}^2 &= \zeta_2^2 + \zeta_4^2 + \cdots + \zeta_{2n}^2 = W_2, \\
&\vdots \\
\zeta_1^n + \zeta_3^n + \cdots + \zeta_{2n-1}^n &= \zeta_2^n + \zeta_4^n + \cdots + \zeta_{2n}^n = W_n.
\end{aligned} \tag{2}$$

Let

$$P_1(z) := \prod_{j=1}^{n}(z - \zeta_{2j-1}) \quad \text{and} \quad P_2(z) := \prod_{j=1}^{n}(z - \zeta_{2j}).$$

Then by (1) and (2) we get $P_1(z) = P_2(z)$.

Thus we get a partition of ζ_i's into n distinct pairs (ζ_i, ζ_j) such that $\zeta_i = \zeta_j$; $i \in 1, 3, \dots, 2n-1$ and $j \in 2, 4, \dots, 2n$. We can relabel the ζ_{2j}'s, $j = 1, \dots, n$ so that $\zeta_{2j-1} = \zeta_{2j}$. But then $\zeta_{2j-1}^k = \zeta_{2j}^k$, $\forall k \in \mathbb{Z}$ and we get

$$\widehat{\chi_\Omega}(kd) = \frac{1}{2\pi i k d}\sum_{j=1}^{n}\left(\zeta_{2j-1}^k - \zeta_{2j}^k\right) = 0; \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{3}$$

Thus $d\mathbb{Z} \subset \mathbb{Z}(\widehat{\chi_\Omega})$. Now consider,

$$F(x) = \sum_{k \in \mathbb{Z}} \chi_\Omega(x + k/d), \quad x \in [0, 1/d). \tag{4}$$

Thus F is $\frac{1}{d}$ periodic and integer valued and

$$\widehat{F}(ld) = d\sum_{k \in \mathbb{Z}} \int_0^{\frac{1}{d}} \chi_\Omega(x + k/d)e^{-2\pi i l d x}\,dx = d\widehat{\chi_\Omega}(ld) = d\delta_{l,0}. \tag{5}$$

Thus $F(t) = d$ a.e. so $d \in \mathbb{Z}$ and Ω d-tiles the real line. $\square$

Using Proposition 2.1, we now prove the corresponding result for the spectrum.

Theorem 2.2. *Let (Ω, Λ) be a spectral pair. If for some $a, d \in \mathbb{R}$, an arithmetic progression of length $n+1$, say $a, a+d, \ldots, a+nd \in \Lambda$, then the complete arithmetic progression $a+d\mathbb{Z} \subseteq \Lambda$. Further $d \in \mathbb{Z}$ and Ω d-tiles $\mathbb{R}$.*

Proof. Since $a, a+d, \ldots, a+nd \in \Lambda$, shifting Λ by a we get that $\Lambda_1 = \Lambda - a$ is a spectrum for Ω and

$$0, d, \ldots, nd \in \Lambda_1 \subset \Lambda_1 - \Lambda_1 \subset \mathbb{Z}(\widehat{\chi_\Omega}).$$

Thus surely $d\mathbb{Z} \subset \mathbb{Z}(\widehat{\chi_\Omega})$ by Proposition 2.1.

Now, let $\lambda \in \Lambda_1$. Then by orthogonality,

$$-\lambda, d-\lambda, 2d-\lambda, \ldots, nd-\lambda \in \mathbb{Z}(\widehat{\chi_\Omega}).$$

Put

$$\xi_{2j} = e^{-2\pi i\lambda a_j}, \qquad \xi_{2j-1} = e^{-2\pi i\lambda(a_j+r_j)}; \quad j = 1, \ldots, n,$$
$$\zeta_{2j} = e^{2\pi i d a_j}, \qquad \zeta_{2j-1} = e^{2\pi i d(a_j+r_j)}; \quad j = 1, \ldots, n.$$

Since $\widehat{\chi_\Omega}(kd-\lambda) = 0$, for $k = 0, \ldots, n$ we have

$$\xi_1\zeta_1^k - \xi_2\zeta_2^k + \cdots + \xi_{2n-1}\zeta_{2n-1}^k - \xi_{2n}\zeta_{2n}^k = 0 \quad \text{for } k = 0, \ldots, n. \tag{6}$$

But the ζ_i's can be partitioned into n disjoint pairs (ζ_i, ζ_j) such that $\zeta_i = \zeta_j$ where $i \in 1, 3, \ldots, 2n-1$ and $j \in 2, 4, \ldots, 2n$. Without loss of generality, we relabel the ζ_{2j}'s and simultaneously, the corresponding ξ_{2j}'s so that $\zeta_{2j-1} = \zeta_{2j}$, $j = 1, \ldots, n$. Thus from (6) we get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_3 & \cdots & \zeta_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{n-1} & \zeta_3^{n-1} & \cdots & \zeta_{2n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_3 - \xi_4 \\ \vdots \\ \xi_{2n-1} - \xi_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

Now, if $[\xi_1 - \xi_2, \xi_3 - \xi_4, \ldots, \xi_{2n-1} - \xi_{2n}]^t$ is the trivial solution, i.e., $\xi_{2j-1} - \xi_{2j} = 0$, $\forall j = 1, \ldots, n$ then $\forall k \in \mathbb{Z}$, we have

$$\begin{aligned} \widehat{\chi_\Omega}(kd-\lambda) &= \frac{1}{2\pi i(kd-\lambda)}\left[\xi_1\zeta_1^k - \xi_2\zeta_2^k + \cdots + \xi_{2n-1}\zeta_{2n-1}^k - \xi_{2n}\zeta_{2n}^k\right] \\ &= \frac{1}{2\pi i(kd-\lambda)}\left[\zeta_1^k(\xi_1 - \xi_2) + \cdots + \zeta_{2n-1}^k(\xi_{2n-1} - \xi_{2n})\right] = 0. \end{aligned}$$

Thus $d\mathbb{Z} - \lambda \in \mathbb{Z}(\widehat{\chi_\Omega})$. If, however, $[\xi_1 - \xi_2, \xi_3 - \xi_4, \ldots, \xi_{2n-1} - \xi_{2n}]^t$ is not the trivial solution, then $\zeta_{2l-1} = \zeta_{2k-1}$ for some $l, k \in 1, \ldots, n$; $l \neq k$.

Removing all the redundant variables and writing the remaining variables as η_{2j+1}^l, $j, l = 0, 1, \ldots, k-1$, we get a non-singular Vandermonde matrix satisfying

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \eta_1 & \eta_3 & \cdots & \eta_{2k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1^{k-1} & \eta_3^{k-1} & \cdots & \eta_{2k-1}^{k-1} \end{pmatrix} \begin{pmatrix} \sum_1 \\ \sum_3 \\ \vdots \\ \sum_{2k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{8}$$

where

$$\sum\nolimits_k = \sum_{j:\ \zeta_{2j-1}=\eta_k} \xi_{2j-1} - \xi_{2j}.$$

Then each of the $\sum_i = 0$; $i = 1, \dots, k$. But, then once again $\forall p \in \mathbb{Z}$,

$$\widehat{\chi_\Omega}(pd - \lambda) = \frac{1}{2\pi i(pd - \lambda)}\left[\eta_1^p \sum\nolimits_1 + \eta_3^p \sum\nolimits_3 + \cdots + \eta_{2k-1}^p \sum\nolimits_{2k-1}\right] = 0.$$

Thus $d\mathbb{Z} - \lambda \subseteq \mathbb{Z}(\widehat{\chi_\Omega})$. We already have $d\mathbb{Z} \subseteq \mathbb{Z}(\widehat{\chi_\Omega})$ and now we have seen if $\lambda \in \Lambda_1$ then $d\mathbb{Z} - \lambda \in \mathbb{Z}(\widehat{\chi_\Omega})$. Thus $d\mathbb{Z} \subseteq \Lambda_1$, hence $a + d\mathbb{Z} \subset \Lambda$. That $d \in \mathbb{Z}$ and Ω d-tiles $\mathbb{R}$ follow from Proposition 2.1. □

Remark 2.3. Theorem 2.2 is the best possible result in this direction, as existence of an arithmetic progression of shorter length in a spectrum does not ensure the complete arithmetic progression is in the spectrum. For example, consider $\Omega = [0, 1/3] \cup [1, 4/3] \cup [2, 7/3]$ then $\Lambda = \{0, 1/3, 2/3\} + 3\mathbb{Z}$ is a spectrum for Ω which contains the 3 term arithmetic progression $0, 1/3, 2/3$ but clearly the complete arithmetic progression $\mathbb{Z}/3 \nsubseteq \Lambda$.

2.2. *Embedding Λ in a vector space*

In this section we will investigate the spectrum in a geometric manner. The setting is again that of a set Ω, which is a union of finitely many intervals, namely, $\Omega = \bigcup_1^n [a_j, a_j + r_j]$. We assume that Ω is spectral with a spectrum Λ. We will embed Λ in a vector space and incorporate the orthogonality of the corresponding set $E_\Lambda = \{e_\lambda\colon \lambda \in \Lambda\}$, via a conjugate linear form.

Consider the $2n$-dimensional vector space $\mathbb{C}^n \times \mathbb{C}^n$. We write its elements as $\underline{v} = (v_1, v_2)$ with $v_1, v_2 \in \mathbb{C}^n$. For $\underline{v}, \underline{w} \in \mathbb{C}^n \times \mathbb{C}^n$ define

$$\underline{v} \odot \underline{w} := \langle v_1, w_1\rangle - \langle v_2, w_2\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{C}^n$. Note that this conjugate linear form is degenerate, i.e., there exists $\underline{v} \in \mathbb{C}^n \times \mathbb{C}^n$, $\underline{v} \neq 0$ such that $\underline{v} \odot \underline{v} = 0$. We call such a vector a *null-vector*. For example, every element of $\mathbb{T}^n \times \mathbb{T}^n$ is a null-vector.

A subset $S \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is called a set of *mutually null-vectors* if $\forall \underline{v}, \underline{w} \in S$ we have $\underline{v} \odot \underline{w} = 0$. It is clear from the definition that elements of a set of mutually null-vectors are themselves null-vectors.

Lemma 2.4. *Let $S = \{\underline{v}^1, \underline{v}^2, \dots, \underline{v}^m\}$ be a set of mutually null-vectors in $\mathbb{C}^n \times \mathbb{C}^n$. Let V be the linear subspace spanned by S. Then, V is a set of mutually null-vectors and $dim(V) \leqslant n$.*

Proof. Let $\underline{v}, \underline{w} \in V$. Since the subspace V is spanned by S, we have $\underline{v} = \sum_{i=1}^{m} a_i \underline{v}^i$ and $\underline{w} = \sum_{j=1}^{m} b_j \underline{v}^j$. Now, as the set S is a set of mutually null-vectors $(\underline{v}^i \odot \underline{v}^j) = 0$; $\forall i, j = 1, \dots, m$ and so, we have $\underline{v} \odot \underline{w} = \sum_{i,j=1}^{m} a_i \overline{b_j} (\underline{v}^i \odot \underline{v}^j) = 0$. Hence, V is a set of mutually null-vectors.

Let $\underline{w}^j := (e_j, 0)$, $j = 1, \dots, n$ where e_j's are the standard basis vectors of $\mathbb{C}^n$. Consider the subspace W of $\mathbb{C}^n \times \mathbb{C}^n$ spanned by the vectors $\underline{w}^j$, $j = 1, \dots, n$. Since, these vectors are linearly independent in $\mathbb{C}^n \times \mathbb{C}^n$, $dim(W) = n$. Further, note that for $\underline{w} \in W$, $\underline{w} \neq 0$ we have $\underline{w} \odot \underline{w} > 0$. Thus $W \cap V = \{0\}$ and hence $dim(V) \leqslant n$. □

Suppose $\Omega = \bigcup_{j=1}^{n} [a_j, a_j + r_j)$ is a union of n disjoint intervals with $a_1 = 0 < a_1 + r_1 < a_2 < a_2 + r_2 < \cdots < a_n < a_n + r_n$ and $\sum_1^n r_j = 1$.

We define a map φ_Ω from $\mathbb{R}$ to $\mathbb{T}^n \times \mathbb{T}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n$ by

$$x \to \varphi_\Omega(x) = \big(\varphi_1(x), \varphi_2(x)\big),$$

where

$$\varphi_1(x) = \big(e^{2\pi i (a_1 + r_1)x}, e^{2\pi i (a_2 + r_2)x}, \dots, e^{2\pi i (a_n + r_n)x}\big)$$

and

$$\varphi_2(x) = \big(1, e^{2\pi i a_2 x}, \dots, e^{2\pi i a_n x}\big).$$

The following lemma, which is immediate from the definitions, makes clear the connection between a spectral pair (Ω, Λ) and the image of Λ under the map φ_Ω.

Lemma 2.5. *Let Ω be a union of n intervals, as above, and suppose $\Gamma \subseteq \mathbb{R}$. Then the set of exponentials $E_\Gamma = \{e_\gamma \colon \gamma \in \Gamma\}$ is an orthogonal set in $L^2(\Omega)$ if and only if $\varphi_\Omega(\Gamma) := \{\varphi_\Omega(\gamma) \colon \gamma \in \Gamma\}$ is a set of mutually null-vectors.*

Thus, if (Ω, Λ) is a spectral pair, $\varphi_\Omega(\Lambda)$ is a set of mutually null-vectors. What about the converse? We will now try to find some criterion to decide whether a given pair (Ω, Λ) is a spectral pair.

First, observe that from Lemma 2.4, we already know that if (Ω, Λ) is a spectral pair then the vector space $V_\Omega(\Lambda) := span\{\varphi_\Omega(\lambda) \colon \lambda \in \Lambda\}$ has dimension at most n. We will now show that Λ has a "local finiteness property", in the sense that there exists a finite subset $\mathcal{B}$ of Λ, $\#\mathcal{B} \leqslant n$, such that Λ gets uniquely determined by $\mathcal{B}$.

Lemma 2.6. *Let (Ω, Λ) be a spectral pair and $\mathcal{B} = \{y_1, \dots, y_m\} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B}) := \{\varphi_\Omega(y_1), \dots, \varphi_\Omega(y_m)\}$ forms a basis of $V_\Omega(\Lambda)$. Then $x \in \Lambda$ iff $\varphi_\Omega(x) \odot \varphi_\Omega(y_i) = 0$, $\forall i = 1, \dots, m$.*

Proof. Let $x \in \Lambda$. Since $\mathcal{B} \subseteq \Lambda$, by orthogonality we have $\langle e_x, e_{y_i} \rangle = 0$, $\forall y_i \in \mathcal{B}$ and the result follows from Lemma 2.5.

For the converse, let $dim(V_\Omega(\Lambda)) = m$ and $\mathcal{B} = \{y_1, \dots, y_m\} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B})$ is a basis for $V_\Omega(\Lambda)$. Suppose there exists some $x \notin \Lambda$ such that $\varphi_\Omega(x) \odot \varphi_\Omega(y_j) = 0, \forall y_j \in \mathcal{B}$. Since $\varphi_\Omega(\mathcal{B})$ is a basis for $V_\Omega(\Lambda)$, we have for any $\lambda \in \Lambda$, $\varphi_\Omega(\lambda) = \sum_{j=1}^{m} a_j \varphi_\Omega(y_j)$. Now by linearity we get $\varphi_\Omega(x) \odot \varphi_\Omega(\lambda) = \sum_{j=1}^{m} \overline{a_j}(\varphi_\Omega(x) \odot \varphi_\Omega(y_j)) = 0$. Hence by Lemma 2.5 we get $\langle e_x, e_\lambda \rangle = 0, \forall \lambda \in \Lambda$. But $E_\Lambda = \{e_\lambda \colon \lambda \in \Lambda\}$ is total in $L^2(\Omega)$, and $e_x \not\equiv 0$, a contradiction. Thus x must be in Λ. □

The following lemma, gives a rather nice criterion for a spectrum Λ to be periodic.

Lemma 2.7. *Let $dim(V_\Omega(\Lambda)) = m \leqslant n$ and $\mathcal{B} = \{y_1, \dots, y_m\} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B})$ is a basis for $V_\Omega(\Lambda)$. If for some $d \in \mathbb{R}$, we have $\mathcal{B} + d = \{y_1 + d, \dots, y_m + d\} \subseteq \Lambda$ then Λ is d-periodic, i.e., $\Lambda = \{\lambda_1, \dots, \lambda_d\} + d\mathbb{Z}$.*

Proof. By Lemma 2.6 $x \in \Lambda$ iff $\varphi_\Omega(x) \odot \varphi_\Omega(y_j) = 0$, $j = 1, \dots, m$. Let $\lambda \in \Lambda$, since $\mathcal{B} + d \subseteq \Lambda$ we get $\varphi_\Omega(\lambda) \odot \varphi_\Omega(y_j + d) = 0$, $j = 1, \dots, m \Leftrightarrow \varphi_\Omega(\lambda - d) \odot \varphi_\Omega(y_j) = 0$, $j = 1, \dots, m \Leftrightarrow$ $\lambda - d \in \Lambda$ and hence Λ is d-periodic. By Theorem 2.2 we get $d \in \mathbb{N}$ and since Λ has density 1 by Theorem 2.10, we conclude that $\Lambda = \{\lambda_1, \dots, \lambda_d\} + d\mathbb{Z}$. □

Recall, that if Γ is periodic, has density 1 and $\varphi_\Omega(\Gamma)$ is a set of mutually null-vectors, then by [48,41] (Ω, Γ) is a spectral pair.

Let (Ω, Γ) be such that $\varphi_\Omega(\Gamma)$ is a set of mutually null-vectors. The natural queston is: Can we extend Γ to a spectrum of Ω? The following theorem gives a criterion for periodic orthogonal extension of a set Γ and will be central to our proof of periodicity of a spectrum in the next section.

Theorem 2.8. *Let $\Gamma \subset \mathbb{R}$ be such that the set of exponentials E_Γ is orthogonal in $L^2(\Omega)$. Let $dim(V_\Omega(\Gamma)) = r$ and $\mathcal{B}_0 = \{\mu_1, \dots, \mu_r\}$ be such that $\varphi_\Omega(\mathcal{B}_0)$ forms a basis of $V_\Omega(\Gamma)$. Further suppose a translate of $\mathcal{B}_0$ is contained in Γ, i.e., $\mathcal{B}_1 = \mathcal{B}_0 + d \subseteq \Gamma$. Then Γ can be extended periodically to obtain a d-periodic subset $\Gamma_d \subseteq \mathbb{R}$ such that the set of exponentials E_{Γ_d} are orthogonal in $L^2(\Omega)$.*

Proof. Let $\Gamma_d := \Gamma + d\mathbb{Z}$. As in Lemma 2.7, we will prove that $\varphi_\Omega(\Gamma_d)$ is a mutually null set. We will first show by induction that

$$\varphi_\Omega(\mu_k) \odot \varphi_\Omega(\mu_j + ld) = 0 \quad \text{for all } l \in \mathbb{Z}, \text{ and } j, k = 1, \dots, r.$$

Observe that both $\varphi_\Omega(\mathcal{B}_0)$ and $\varphi_\Omega(\mathcal{B}_1)$ span the same vector space $V_\Omega(\Gamma)$. Let us assume that the orthogonality relations hold for all $s = 1, \dots, l - 1$ i.e.,

$$\varphi_\Omega(\mu_k) \odot \varphi_\Omega(\mu_j + sd) = 0 \quad \text{for all } j, k = 1, \dots, r.$$

We have to show

$$\varphi_\Omega(\mu_k) \odot \varphi_\Omega(\mu_j + ld) = 0 \quad \text{for all } j, k = 1, \dots, r.$$

But by the induction hypothesis, we have

$$\varphi_\Omega(\mu_k + d) \odot \varphi_\Omega(\mu_j + ld) = \varphi_\Omega(\mu_k) \odot \varphi_\Omega\big(\mu_j + (l-1)d\big) = 0, \quad \forall j, k = 1, \dots, r.$$

But, we know that $\varphi_\Omega(\mathcal{B}_0) \subseteq span\{\varphi_\Omega(\mathcal{B}_1)\}$. Hence,

$$\varphi_\Omega(\mu_k) \odot \varphi_\Omega(\mu_j + ld) = 0, \quad \forall j, k = 1, \ldots, r.$$

Now if $\gamma, \gamma' \in \Gamma_d$, then $\gamma = \gamma_p + ld$, $\gamma' = \gamma'_p + l'd$ for some $\gamma_p, \gamma'_p \in \Gamma$ and $l, l' \in \mathbb{Z}$. Since $\varphi_\Omega(\gamma_p), \varphi_\Omega(\gamma'_p) \in V_\Omega(\Gamma) = Span\{\varphi_\Omega(\mathcal{B}_0)\}$, we have

$$\varphi_\Omega(\gamma_p) = \sum_{k=1}^{r} \alpha_k \varphi_\Omega(\mu_k) \quad \text{and} \quad \varphi_\Omega(\gamma'_p) = \sum_{j=1}^{r} \alpha'_j \varphi_\Omega(\mu_j).$$

Now,

$$\begin{aligned}
\varphi_\Omega(\gamma) \odot \varphi_\Omega(\gamma') &= \varphi_\Omega(\gamma_p + ld) \odot \varphi_\Omega(\gamma'_p + l'd) = \varphi_\Omega\big(\gamma_p + (l - l')d\big) \odot \varphi_\Omega(\gamma'_p) \\
&= \varphi_\Omega\big(\gamma_p + (l - l')d\big) \odot \left(\sum_{1}^{r} \alpha'_j \varphi_\Omega(\mu_j)\right) \\
&= \sum_{j=1}^{r} \overline{\alpha'_j} \varphi_\Omega(\gamma_p) \odot \varphi_\Omega\big(\mu_j + (l' - l)d\big) \\
&= \sum_{j=1}^{r} \sum_{k=1}^{r} \overline{\alpha'_j} \alpha_k \varphi_\Omega(\mu_k) \odot \varphi_\Omega\big(\mu_j + (l' - l)d\big) = 0. \qquad \square
\end{aligned}$$

Remark 2.9. Under the assumption of Lemma 2.7 the Λ_d obtained in Theorem 2.8 is Λ itself.

2.3. Density of the spectrum

Let $\Gamma \subset \mathbb{R}$ be a uniformly discrete set. Then we define $n^+(R)$, $n^-(R)$ respectively, as the largest and smallest number of elements of Γ contained in any interval of length R, i.e.,

$$\begin{aligned}
n^+(R) &= \max_{x \in \mathbb{R}} \#\big\{\Gamma \cap [x - R, x + R]\big\}, \\
n^-(R) &= \min_{x \in \mathbb{R}} \#\big\{\Gamma \cap [x - R, x + R]\big\}.
\end{aligned}$$

A uniformly discrete set Γ is called a *set of sampling* for $L^2(\Omega)$, if there exists a constant K such that $\|f\|_2^2 \leqslant K \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2$, $\forall f \in L^2(\Omega)$, and Γ is called a *set of interpolation* for $L^2(\Omega)$, if for every square summable sequence $\{a_\gamma\}_{\gamma \in \Gamma}$, there exists an $f \in L^2(\Omega)$ with $\hat{f}(\gamma) = a_\gamma$, $\gamma \in \Gamma$.

Clearly if (Ω, Λ) is a spectral pair, then Λ is both a set of sampling and a set of interpolation for $L^2(\Omega)$. The following result of Landau, regarding sets of sampling and interpolation gives an estimate on the numbers $n^+(R)$ and $n^-(R)$ for a spectrum Λ, when Ω is a union of a finite number of intervals.

Theorem 2.10. *(See Landau [42].) Let Ω be a union of a finite number of intervals with total measure* 1*, and Λ a uniformly discrete set. Then*:

(1) *If Λ is a set of sampling for $L^2(\Omega)$,*

$$n^-(R) \geqslant R - A\log^+ R - B.$$

(2) *If Λ is a set of interpolation for $L^2(\Omega)$,*

$$n^+(R) \leqslant R - A\log^+ R - B$$

where A and B are constants independent of R.

It follows from Theorem 2.10 that Λ has asymptotic density 1, that is

$$\rho(\Lambda) := \lim_{R\to\infty} \frac{\#(\Lambda \cap [-R+x, R+x])}{2R} = 1, \quad \text{uniformly in } x \in \mathbb{R}.$$

3. Proof of periodicity of the spectrum

Once again in this section $\Omega \subset \mathbb{R}$ is a union of finitely many intervals, $\Omega = \bigcup_1^n [a_j, a_j + r_j]$, $\sum_{j=1}^n r_j = 1$. We assume that Ω is spectral with a spectrum Λ. We will continue to use the notations introduced in Section 2.

We begin with some definitions.

Let $\Lambda = \{\lambda_j\}_{j\in\mathbb{Z}}$ where $\lambda_j < \lambda_{j+1}$ and $\lambda_0 = 0$. Recall that the consecutive distance set of Λ, namely

$$\Lambda_s = \{\lambda_{j+1} - \lambda_j\colon\ j \in \mathbb{Z}\}$$

is finite. So we can view Λ as an infinite word with a finite alphabet $\Lambda_s = \{d_1, d_2, \ldots, d_l\}$. For a finite word $W = [d_{j_1}, d_{j_2}, \ldots, d_{j_n}]$, $d_{j_i} \in \Lambda_s$ we write $length(W) = \sum_{i=1}^n d_{j_i}$.

Suppose $dim(V_\Omega(\Lambda)) = m \leqslant n$ and let $\{\mu_1, \mu_2, \ldots, \mu_m\}$ be such that $\{\varphi_\Omega(\mu_j),\ j = 1, 2, \ldots, m\}$ is a basis for $V_\Omega(\Lambda)$.

Choose L_0 such that $\{\mu_1, \mu_2, \ldots, \mu_m\} \subseteq [0, L_0]$ and then for any $L \geqslant L_0$, partition $\mathbb{R}$ as

$$\mathbb{R} = \bigcup_{k\in\mathbb{Z}} \big[kL, (k+1)L\big).$$

Let

$$\Lambda_k^L = \Lambda \cap \big[kL, (k+1)L\big).$$

Now, for each $k \in \mathbb{Z}$, Λ_k^L, corresponds to a finite word of length at most L, and there are only finitely many, say N_L, words of length at most L. Let

$$V_k^L = Span\big\{\varphi_\Omega(\lambda)\colon\ \lambda \in \Lambda_k^L\big\}.$$

Let us first consider the **special case** that for some large enough L we have

$$dim\big(V_k^L\big) = m \quad \text{for every } k \in \mathbb{Z}. \tag{9}$$

In this case, each Λ_k^L has a set of m elements $\mathcal{B}_k := \{\mu_1^k, \mu_2^k, \ldots, \mu_m^k\}$ such that $\varphi_\Omega(\mathcal{B}_k) := \{\varphi_\Omega(\mu_1^k), \ldots, \varphi_\Omega(\mu_m^k)\}$ forms a basis of $V_\Omega(\Lambda)$. Also by the remarks above, at least two of the words $\Lambda_{k_1}^L$ and $\Lambda_{k_2}^L$ must be the same. Hence for some $d \in \mathbb{R}$, $\Lambda_{k_2}^L = \Lambda_{k_1}^L + d$. In particular, there exists k_0, such that $\Lambda_{k_0}^L$ contains a set of elements $\{\mu_1^{k_0}, \ldots, \mu_m^{k_0}\}$ which form a basis of $V_\Omega(\Lambda)$ and also $\{\mu_1^{k_0}, \ldots, \mu_m^{k_0}\} + d \subseteq \Lambda$. Thus the hypothesis of Lemma 2.7 holds, and so Λ is d-periodic.

Observe that in the above argument, we do not require as much as (9). It would be enough if $\{k\colon \mathit{dim}(V_k^L) = m\}$ is an infinite set, or for that matter, has at least $N_L + 1$ elements. But once we conclude that Λ is periodic, it will follow that for some, possibly larger L' (if d is the period $L' = 3d$ will do) that $\mathit{dim}(V_k^{L'}) = m$, $\forall k \in \mathbb{Z}$.

For the general case, let $1 \leqslant s \leqslant m$, and $L > 0$ and write

$$E_s^L = \bigl\{k\colon \mathit{dim}\bigl(V_k^L\bigr) \geqslant s\bigr\}.$$

We have just seen that if for some $L > 0$, $E_m^L = \mathbb{Z}$, then Λ is periodic. Suppose this is not the case. Then we need the following lemma:

Lemma 3.1. *Let $m' \leqslant m$ be the largest integer such that there exists an $L' > 0$ so that $E_{m'}^{L'} = \mathbb{Z}$. Then m' itself will occur infinitely often in the set $\{\mathit{dim}(V_k^{L'})\}_{k\in\mathbb{Z}}$.*

Proof. First note that for $s = 1$, we can choose $L' > \max\{d_j\}$, and then $E_1^{L'} = \mathbb{Z}$ so clearly $m' \geqslant 1$. If $\mathit{dim}(V_k^{L'}) = m'$ only for finitely many k's then we can take $\tilde{L}$ large enough so that $\mathit{dim}(V_k^{\tilde{L}}) = m'$ for precisely one interval of the partition $\{[k\tilde{L}, (k+1)\tilde{L}]\}$. Let $L'' = 2\tilde{L}$, then observe that $E_{m'+1}^{L''} = \mathbb{Z}$, and this contradicts maximality of m'. (Without loss of generality we may choose $L' \in \mathbb{N}$.) $\square$

We will now prove Theorem 1.4.

Proof of Theorem 1.4. Step 1. We will first prove that the spectrum Λ can be modified to a set Λ_d which is d-periodic and is such that (Ω, Λ_d) is a spectral pair. For this we use Landau's density result to extract a "patch" from Λ which has some periodic structure and has a large enough density. Then we use Theorem 2.8 to show that a suitable periodization of this patch is a spectrum.

With L' as above, let

$$\epsilon_{L'} = \frac{1}{2L'(N_{L'} + 1)}.$$

Then choose $L^* > \frac{1}{2\epsilon_{L'}} = L'(N_{L'} + 1)$ such that $n^-(L^*)/L^* > 1 - \epsilon_{L'}$.

In the case under consideration, we know that $E_{m'}^{L^*} = \mathbb{Z}$ and also that the cardinality of the set $\{p\colon \mathit{dim}(V_p^{L^*}) = m'\}$ is infinite. We choose and fix one such p such that $\mathit{dim}(V_p^{L^*}) = m'$. By the choice of L^*, the interval $[pL^*, (p+1)L^*)$ contains at least $(N_{L'} + 1)$ disjoint intervals of length L'. Now for $j = 1, \ldots, N_L + 1$ each of the $\Lambda_j^{L'} \subset [pL^*, (p+1)L^*)$ has a word W_j of length at most L' associated with it. Further, observe that by the choice of L', each of these

$\Lambda_j^{L'}$ contains at least m' elements whose image under φ_Ω is a linearly independent set, and that, by the choice of p, there can be at most m' such elements. Notice this implies $V_p^{L^*} = V_j^{L'}$, $j = 1, \ldots, N_L + 1$.

Hence by the pigeon hole principle, there exists k_1 and k_2 such that the words $\Lambda_{k_1}^{L'}$ and $\Lambda_{k_2}^{L'}$ are the same, and therefore $\Lambda_{k_2}^{L'} = \Lambda_{k_1}^{L'} + d$ for some $d \in \mathbb{R}$, where $d \leqslant (N_{L'} + 1)L' = \frac{1}{2\epsilon_{L'}}$.

To complete the proof, we will need the following lemma:

Lemma 3.2. *Let Λ_d be the d-periodization of $\Lambda_p^{L^*}$, i.e. $\Lambda_d = \{\Lambda_p^{L^*} + d\mathbb{Z}\}$. Then Λ_d is orthogonal.*

Proof. Let $\mathcal{B}_0 = \{\mu_1, \ldots, \mu_{m'}\} \subseteq \Lambda_{k_1}^{L'} \subseteq \Lambda_p^{L^*}$ be such that $\varphi_\Omega(\mathcal{B}_0) := \{\varphi_\Omega(\mu_1), \ldots, \varphi_\Omega(\mu_{m'})\}$ is a basis of $V_p^{L^*}$ and also of $V_{k_1}^{L'}$. Now since $\Lambda_{k_1}^{L'} + d = \Lambda_{k_2}^{L'}$, $\mathcal{B}_1 = \mathcal{B}_0 + d \subseteq \Lambda_{k_2}^{L'}$, this subset again gives a basis for $V_p^{L^*}$. By Theorem 2.8 we see that the set of exponentials E_{Λ_d} are mutually orthogonal in $L^2(\Omega)$. $\square$

Now since Λ_d is orthogonal it is a set of interpolation and by Landau's density theorem we get $\rho(\Lambda_d) \leqslant 1$. But by our choice of L^* we get $\rho(\Lambda_d) > n^-(L^*)/L^* > 1 - \epsilon_{L'}$. On the other hand, since Λ_d is d-periodic, if $\rho(\Lambda_d) < 1$, we have $\rho(\Lambda_d) \leqslant 1 - \frac{1}{d} < 1 - 2\epsilon_{L'}$ as $\frac{1}{d} \geqslant 2\epsilon_{L'}$. This is a contradiction.

It follows that Λ_d is a periodic set whose density is 1 and E_{Λ_d} is orthogonal in $L^2(\Omega)$. Thus we get Λ_d is a spectrum for Ω [48,41]. Since Λ_d has density of 1 and is d-periodic it can be written in the form $\Lambda_d = \bigcup_{j=1}^{d}(\mu_j + d\mathbb{Z})$.

Step 2. We now prove that Λ itself is periodic. Once again we will be using Landau's density theorem and Theorem 2.8 along with Theorem 2.2 which will be crucial.

Choose L^* as above, so that $\{p\colon dim(V_p^{L^*}) = m'\}$ is infinite.

Then let L^{**} be such that

$$n^-(L^{**})/L^{**} > 1 - \frac{1}{2(n+1)L^*} \quad \text{and} \quad L^{**} \gg (n+1)L^*.$$

(Recall that n is the number of intervals in Ω.) Here by $\gg$ we mean that many blocks of intervals, each of length $(n+1)L^*$ are contained in any interval of the L^{**}-grid.

Then we can find a p such that $dim(V_p^{L^{**}}) = m'$ (since there are infinitely many such). Now extend $\Lambda_p^{L^{**}}$ d-periodically to a spectrum Λ_d^* of Ω, where $d < L^*$. Write $\Lambda_d^* = \bigcup_{j=1}^{d}(\mu_j + d\mathbb{Z})$, with $\mu_1, \mu_2, \ldots, \mu_d \in [pL^{**}, (p+1)L^{**})$.

We end the proof by showing that in fact $\Lambda_d^* = \Lambda$. For this it will be enough to show that for each μ_j, there are $(n+1)$ consecutive terms from the arithmetic progression $\mu_j + d\mathbb{Z}$ in Λ. Suppose this is not the case, then for each $a \in \mathbb{Z}$ such that $[\mu_j + ad, \mu_j + (a+n)d] \subset [pL^{**}, (p+1)L^{**}]$ we have at least one element from the $n+1$ length AP $\mu_j + ad, \mu_j + (a+1)d, \ldots, \mu_j + (a+n)d$ is missing from $\Lambda_p^{L^{**}}$. But that will affect the density, so that $n^-(L^{**})/L^{**} \leqslant 1 - \frac{1}{(n+1)d} \leqslant 1 - \frac{1}{(n+1)L^*}$, which is a contradiction. Now By Theorem 2.2 we get that Λ is indeed periodic. $\square$

The structure of spectral sets Ω which have a periodic spectrum is well known (see [48, 41]). Here for the sake of completeness we give a structure theorem for Ω using a result of Kolountzakis.

Theorem. *(See Kolountzakis [30].) Let Ω be a bounded open set, Λ a discrete set in $\mathbb{R}^d$, and $\delta_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$. Then $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling if and only if Λ has uniformly bounded density and*

$$(\Omega - \Omega) \cap supp(\widehat{\delta_\Lambda}) = \{0\}.$$

We will now prove Theorem 1.5.

Proof of Theorem 1.5. Recall that (Ω, Λ) is a spectral pair if and only if $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling. Further if Λ is d-periodic, then Ω d-tiles $\mathbb{R}$, i.e. $\sum_n \chi_\Omega(x + n/d) = d$.

In particular,

$$d\chi_{[0,\frac{1}{d})}(x) = \chi_{[0,\frac{1}{d})}\sum_{k\in\mathbb{Z}} \chi_\Omega(x + k/d).$$

So for each $x \in [0, \frac{k}{d})$, the set $A_x = \{k \in \mathbb{Z}\colon\ x + k/d \in \Omega\}$ has cardinality d. Define an equivalence relation $\approx$ on $[0, 1/d)$ by $x \approx y$ if and only if $A_x = A_y$.

Since Ω is bounded, the above equivalence relation gives a partition of $[0, 1/d)$ into finitely many equivalence classes $E_1, E_2, \ldots, E_k$. For each E_j we write A_j for the common set defined above.

Then $\Omega = \bigcup_{j=1}^k (E_j + A_j)$ and $[0, 1/d) = \bigcup_{j=1}^k E_j$ and we may assume $|E_j| > 0$, $\forall j = 1, 2, \ldots, k$. Now let $\Omega_j := [0, 1/d) + A_j$. Our claim is (Ω_j, Λ) is a spectral pair. We will need the above mentioned theorem due to Kolountzakis [30].

Now as $\sharp(A_j) = d$ we have $|\Omega_j| = 1$. If $\Lambda = \Gamma + d\mathbb{Z}$ with $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$, then $supp(\widehat{\delta_\Lambda}) = \{k/d\colon\ \widehat{\delta_\Lambda}(k/d) \neq 0\} \subseteq \mathbb{Z}/d$ and $supp(\Omega_j - \Omega_j) \subseteq (-1/d, 1/d) + A_j - A_j$. But $A_j - A_j$ is $1/d$-separated, so $supp(\Omega_j - \Omega_j) \cap supp(\widehat{\delta_\Lambda}) = \{0\}$ for otherwise as $E_j + A_j \subseteq \Omega$ and $|E_j| > 0$ we get $supp(\Omega - \Omega) \cap supp(\widehat{\delta_\Lambda}) \neq \{0\}$ and thus (Ω, Λ) cannot be a spectral set. □

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