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Qualitative analysis of steady states to the Sel'kov model ☆

Rui Peng

Institute of Nonlinear Complex Systems, College of Science, China Three Gorges University, Yichang City 443002, Hubei Province, PR China

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Abstract

In this work, we are concerned with a reaction–diffusion system well known as the Sel'kov model, which has been used for the study of morphogenesis, population dynamics and autocatalytic oxidation reactions. We derive some further analytic results for the steady states to this model. In particular, we show that no nonconstant positive steady state exists if $0 < p \leqslant 1$ and θ is large, which provides a sharp contrast to the case of $p > 1$ and large θ, where nonconstant positive steady states can occur. Thus, these conclusions indicate that the parameter p plays a crucial role in leading to spatially nonhomogeneous distribution of the two reactants. The a priori estimates are fundamental to our mathematical approaches.

MSC: 35J55; 35B45; 35Q80; 92C15; 92C40

Keywords: Sel'kov model; A priori estimates; Steady states; Nonexistence

1. Introduction

In this paper, we investigate a well-known reaction–diffusion system which was introduced by Sel'kov [17] in 1968 as a model for glycolysis and now has been used in various forms in the study of morphogenesis, population dynamics and autocatalytic oxidation reactions (see [9, 13,18], respectively). In its simplified and nondimensionalized form, the steady-state problem

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E-mail address: pengrui_seu@163.com.

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of this model subject to the homogeneous Neumann boundary condition satisfies the following coupled elliptic system:

$$\begin{cases} -\theta \Delta u = \lambda\big(1 - uv^p\big) & \text{in } \Omega, \\ -\Delta v = \lambda\big(uv^p - v\big) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. The functions u and v represent the concentrations of two reactants or densities of two species, and thus are considered to be nonnegative, θ, λ and p are fixed positive constants. We refer the interested readers to [3,10,17,20] and the references therein for a more detailed discussion of the implication of this system.

It is obviously noted that $(u, v) = (1, 1)$ is the unique constant solution of (1.1). In the recent decades, the system (1.1) has received extensive studies, analytically as well as numerically, and please see [7,11] for one spatial dimension, [1,6] for two spatial dimensions and [3] for n spatial dimensions ($n = 1, 2, 3$). In particular, Eilbeck and Furter [7] performed some numerical bifurcation calculations to show that the one-dimensional problem has nonconstant solutions for suitable ranges of the parameters. Then, in [3], Davidson and Rynne obtained a priori upper bounds of positive classical solutions of (1.1) in the two cases where $0 < p < \infty$ if $n = 1, 2$, or $0 < p < 3$ if $n = 3$. Moreover, letting p and θ be fixed and considering λ as a bifurcation parameter, by the celebrated bifurcation theory due to Crandall and Rabinowitz [2], they studied the existence of nonconstant positive solutions of (1.1) for some certain but quite restricted ranges of p, θ and λ. More recently, Wang [20] established some more refined a priori estimates of upper and lower bonds for the positive solutions of (1.1). Combining these results with the theory of topological degree (see, for example, [14]), the local and global bifurcation proposed by [2,16], the author also discussed the bifurcation and global existence of nonconstant positive solutions with respect to θ and λ, respectively, which greatly improved the conclusions of [3]. One of the most important steps in their arguments is the establishment of a priori estimates for any positive solutions to (1.1). However, for $n \geqslant 3$, in [3,20], the authors had to impose a restricted upper bound on p, namely, $p < n/(n-2)$. In the most recent paper [10], Lieberman showed that this technically mathematical condition is unnecessary by proving some much more delicate a priori estimates for positive solutions of (1.1) for an arbitrary number p in any number of dimensions. The techniques for his proof include some maximum principle arguments and the weak Harnack inequality.

In the present paper, we are also concerned with the Sel'kov model, and attempt to derive some nonexistence results for nonconstant positive solutions of (1.1). In particular, we show that there is no nonconstant positive steady state provided that $0 < p \leqslant 1$ and θ is sufficiently large. This result provides a sharp contrast to the case of $p > 1$ and large θ, where the author [20] confirmed that nonconstant positive steady states may occur. Thus, these conclusions indicate that the parameter p plays a crucial role in leading to spatially nonhomogeneous distribution of the two reactants in (1.1).

The remaining content of this paper is organized as follows. In Section 2, we recall some existing results about (1.1), which are often used in our later discussions. In Section 3, we mainly derive some a priori estimates for the gradients of positive solutions to (1.1). Finally, in Section 4, based on the a priori estimates, we apply two different methods to obtain nonexistence for nonconstant positive solutions of (1.1), and also discuss the asymptotic behavior of nonconstant positive solutions as θ is large enough.

2. Some preliminary results

In this section, we shall recall some existing results for model (1.1), including the a priori estimates of positive solutions and existence for nonconstant positive solutions. These results will be frequently used in the forthcoming sections. To begin with, we first introduce some notations as follows.

We denote $\|\cdot\|_q$ and $\|\cdot\|_{m,q}$ to be the usual norms of the Banach spaces $L^q(\Omega)$ and $W^{m,q}(\Omega)$, respectively, and $|\Omega|$ to be the volume of the domain Ω. For our later purpose, from now on, we also let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ be all of the eigenvalues of the Laplacian operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition, and define the algebraic multiplicity of λ_i by $\dim(\lambda_i)$.

In [10], in order to estimate the lower bound of v for any positive solution to (1.1), Lieberman applied the following local result for weak supersolution of linear elliptic equations (also see, for example, [19, Theorem 8.18]).

Lemma 2.1. *(See [10, Lemma 2.1].) Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. Let Λ be a nonnegative constant and suppose that $w\in W^{1,2}(\Omega)$ is a nonnegative weak solution of the inequalities*

$$0\leqslant \Delta w+\Lambda w \quad \text{in } \Omega, \qquad \partial_\nu w\leqslant 0 \quad \text{on } \partial\Omega.$$

Then, for any $q\in[1,n/(n-2))$, there exists a positive constant C_0, depending only on q, Λ and Ω, such that

$$\|w\|_q\leqslant C_0\inf_{\Omega} w.$$

On the other hand, Lieberman adopted the following analogue of [12, Proposition 2.2] to derive the upper bounds of u and v.

Lemma 2.2. *(See [10, Lemma 2.3].) Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, and let $g\in C(\bar{\Omega}\times\mathbf{R})$. If $w\in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta w+g(x,w)\geqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu w\leqslant 0 \quad \text{on } \partial\Omega,$$

and if there is a constant K such that $g(x,z)<0$ for $z>K$, then

$$w\leqslant K \quad \text{a.e. in } \Omega.$$

Combining Lemmas 2.1, 2.2 with a Harnack inequality for weak solutions (see [10, Lemma 2.2]), the author obtained a priori upper and lower bounds for any positive solution (u,v). More precisely, we have

Theorem 2.1. *(See [10, Theorem 3.1].) Let P,Θ and Λ be arbitrary positive constants, and suppose that $0<p\leqslant P$, $0<\theta\leqslant\Theta$ and $0<\lambda\leqslant\Lambda$. Let C_1 be the value of C_0 corresponding to $q=1$ and set $\varepsilon=|\Omega|/C_1$. Then, any positive solution (u,v) to (1.1) satisfies the inequalities*

$$\left(\Theta\varepsilon^{-P}+1\right)^{-P}\leqslant u\leqslant\varepsilon^{-P} \quad \text{and} \quad \varepsilon\leqslant v\leqslant\Theta\varepsilon^{-P}+1.$$

Furthermore, if $P < n/n-2$, the paper [10] improved the above Theorem 2.1 by showing that the estimates are independent of θ. That is,

Theorem 2.2. *(See [10, Theorem 3.2].) Let P and Λ be positive constants with $0 < P < n/(n-2)$, and suppose that $0 < p \leqslant P$ and $0 < \lambda \leqslant \Lambda$. Then, there exists positive constants C_2 and C_3, determined only by P, Λ and Ω, such that any positive solution (u, v) to* (1.1) *satisfies*

$$C_2 < u < C_3 \quad \textit{and} \quad C_2 < v < C_3.$$

As far as the existence of nonconstant positive solutions to (1.1) with respect to the parameter θ is concerned, by use of the argument of topological degree, Wang [20] showed

Theorem 2.3. *(See [20, Theorem 6.1].) Assume that $\lambda(p-1) \in (\lambda_m, \lambda_{m+1})$ for some fixed $\lambda > 0$, $p > 1$ and $m \geqslant 1$. Let $L_m = \sum_{i=1}^{m} \dim(\lambda_i)$. If L_m is odd, then there exists a positive constant Θ such that* (1.1) *admits at least one nonconstant positive solution provided that $\theta \geqslant \Theta$.*

3. A priori estimates for positive solutions to (1.1)

In this section, based on Theorems 2.1 and 2.2, we first establish some a priori estimates for the $W^{1,2}$ norm of u and v. In what follows, we set

$$\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u \,\mathrm{d}x \quad \text{and} \quad \bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v \,\mathrm{d}x,$$

and

$$\phi = u - \bar{u} \quad \text{and} \quad \psi = v - \bar{v}.$$

As in [20], we integrate the two equations in (1.1) and easily find that

$$\int_{\Omega} u v^p \,\mathrm{d}x = \int_{\Omega} v \,\mathrm{d}x = |\Omega|, \quad \bar{v} = 1. \tag{3.1}$$

Using the estimates of Theorem 2.1, we have

$$\left|\theta^{-1}\lambda\left(1 - uv^p\right)\right| \leqslant \theta^{-1}\lambda \max\left\{1, \varepsilon^{-p}\left(\theta\varepsilon^{-p} + 1\right)^p\right\} :\equiv A_1 \tag{3.2}$$

and

$$\left|\lambda\left(uv^p - v\right)\right| \leqslant \lambda \max\left\{\varepsilon^{-p}\left(\theta\varepsilon^{-p} + 1\right)^p, \theta\varepsilon^{-p} + 1\right\} :\equiv A_2. \tag{3.3}$$

Hence, multiplying the first equation of (1.1) by ϕ and integrating over Ω, we see

$$\int_{\Omega} |\nabla\phi|^2 \,\mathrm{d}x = \int_{\Omega} \theta^{-1}\lambda\left(1 - uv^p\right)\phi \,\mathrm{d}x \leqslant A_1 \int_{\Omega} |\phi| \,\mathrm{d}x \leqslant A_1 |\Omega|^{1/2} \left(\int_{\Omega} |\phi|^2 \,\mathrm{d}x \right)^{1/2}. \tag{3.4}$$

Applying the well-known Poincaré inequality

$$\int_\Omega \phi^2 \, \mathrm{d}x \leqslant \lambda_1^{-1} \int_\Omega |\nabla \phi|^2 \, \mathrm{d}x,$$

we yield from (3.4) that

$$\int_\Omega |\nabla \phi|^2 \, \mathrm{d}x \leqslant A_1 \lambda_1^{-1/2} |\Omega|^{1/2} \biggl(\int_\Omega |\nabla \phi|^2 \, \mathrm{d}x \biggr)^{1/2},$$

which, together with the Poincaré inequality again, implies the following:

$$\int_\Omega |\nabla \phi|^2 \, \mathrm{d}x \leqslant A_1^2 \lambda_1^{-1} |\Omega| \quad \text{and} \quad \int_\Omega \phi^2 \, \mathrm{d}x \leqslant A_1^2 \lambda_1^{-2} |\Omega|. \tag{3.5}$$

In a similar manner, we can use the equation for v in (1.1) to claim that

$$\int_\Omega |\nabla \psi|^2 \, \mathrm{d}x \leqslant A_2^2 \lambda_1^{-1} |\Omega| \quad \text{and} \quad \int_\Omega \psi^2 \, \mathrm{d}x \leqslant A_2^2 \lambda_1^{-2} |\Omega|. \tag{3.6}$$

Therefore, it follows from (3.2), (3.3), (3.5) and (3.6) that

Theorem 3.1. *Suppose that* (u, v) *is any positive solution to* (1.1). *Then, the following estimates hold*:

$$\|\phi\|_{1,2} \leqslant \bigl(1 + \lambda_1^{1/2}\bigr) \lambda_1^{-1} \theta^{-1} \lambda \max\bigl\{1, \varepsilon^{-p}\bigl(\theta \varepsilon^{-p} + 1\bigr)^p\bigr\} |\Omega|^{1/2}$$

and

$$\|\psi\|_{1,2} \leqslant \bigl(1 + \lambda_1^{1/2}\bigr) \lambda_1^{-1} \theta^{-1} \lambda \max\bigl\{\varepsilon^{-p}\bigl(\theta \varepsilon^{-p} + 1\bigr)^p, \theta \varepsilon^{-p} + 1\bigr\} |\Omega|^{1/2}.$$

In the following, we establish the relationship of the gradients of u and v. It will be seen that our proof does not depend on the previous estimates.

Theorem 3.2. *Assume that* (u, v) *is any positive solution to* (1.1). *Then, the gradients of u and v satisfy*

$$\theta^{-2} \int_\Omega |\nabla v|^2 \, \mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \, \mathrm{d}x \leqslant \bigl(2 + 2\lambda_1^{-1}\lambda + \lambda_1^{-2}\lambda^2\bigr) \theta^{-2} \int_\Omega |\nabla v|^2 \, \mathrm{d}x.$$

Proof. To prove our statement, we need to define $w = \theta u + v$. Thus, (1.1) and (3.1) show

$$-\Delta w = \lambda(1 - v) = \lambda(\bar{v} - v) = -\lambda \psi \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

Since $\int_\Omega \phi \, \mathrm{d}x = \int_\Omega \psi \, \mathrm{d}x = 0$, it follows from (3.7) that

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x = -\lambda \int_\Omega w\psi\,\mathrm{d}x = -\lambda\theta \int_\Omega (u-\bar u)\psi\,\mathrm{d}x - \lambda \int_\Omega (v-\bar v)\psi\,\mathrm{d}x$$
$$= -\lambda\theta \int_\Omega \phi\psi\,\mathrm{d}x - \lambda \int_\Omega \psi^2\,\mathrm{d}x. \tag{3.8}$$

On the other hand, we have

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x = \int_\Omega \big|\nabla(\theta u + v)\big|\,\mathrm{d}x = \theta^2 \int_\Omega |\nabla\phi|^2\,\mathrm{d}x + 2\theta \int_\Omega \nabla\phi\cdot\nabla\psi\,\mathrm{d}x + \int_\Omega |\nabla\psi|^2\,\mathrm{d}x.$$

To get rid of the term $\int_\Omega \nabla\phi\cdot\nabla\psi\,\mathrm{d}x$, multiplying the equation of w by ψ and integrating over Ω, we find that

$$-\lambda \int_\Omega \psi^2\,\mathrm{d}x = \int_\Omega \nabla w\cdot\nabla\psi\,\mathrm{d}x = \theta \int_\Omega \nabla\phi\cdot\nabla\psi\,\mathrm{d}x + \int_\Omega |\nabla\psi|^2\,\mathrm{d}x.$$

As a result, we obtain

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x = \theta^2 \int_\Omega |\nabla\phi|^2\,\mathrm{d}x - 2\lambda \int_\Omega \psi^2\,\mathrm{d}x - \int_\Omega |\nabla\psi|^2\,\mathrm{d}x. \tag{3.9}$$

Consequently, we yield

$$\theta^{-2} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla\phi|^2\,\mathrm{d}x,$$

which is equivalent to the left-hand side inequality of Theorem 3.2.

Next, we verify the right-hand side inequality of Theorem 3.2. From (3.8) and (3.9), it follows that

$$\int_\Omega |\nabla\phi|^2\,\mathrm{d}x = -\lambda\theta^{-1} \int_\Omega \phi\psi\,\mathrm{d}x + \theta^{-2} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x + \lambda\theta^{-2} \int_\Omega \psi^2\,\mathrm{d}x$$
$$\leqslant \lambda\theta^{-1} \int_\Omega |\phi||\psi|\,\mathrm{d}x + \big(1+\lambda_1^{-1}\lambda\big)\theta^{-2} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x$$
$$\leqslant \frac{1}{2}\lambda_1 \int_\Omega \phi^2\,\mathrm{d}x + \frac{1}{2}\lambda_1^{-1}\lambda^2\theta^{-2} \int_\Omega \psi^2\,\mathrm{d}x + \big(1+\lambda_1^{-1}\lambda\big)\theta^{-2} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x$$
$$\leqslant \frac{1}{2} \int_\Omega |\nabla\phi|^2\,\mathrm{d}x + \left(\frac{1}{2}\lambda_1^{-2}\lambda^2 + \lambda_1^{-1}\lambda + 1\right)\theta^{-2} \int_\Omega |\nabla\psi|^2\,\mathrm{d}x.$$

This implies the desired result. We now finish the proof. □

Remark 3.1. We like to mention that, if (u, v) is a nonconstant positive solution to (1.1), applying Theorem 3.2 and the Poincaré inequality, we also have

$$\lambda_1(1+\lambda_1)^{-1}\theta^{-2} \leqslant \frac{\int_\Omega (|\nabla\phi|^2+\phi^2)\,\mathrm{d}x}{\int_\Omega (|\nabla\psi|^2+\psi^2)\,\mathrm{d}x} \leqslant \lambda_1^{-1}\big(2+2\lambda_1^{-1}\lambda+\lambda_1^{-2}\lambda^2\big)\theta^{-2}.$$

4. Nonexistence of nonconstant positive solutions to (1.1)

This section is devoted to the nonexistence of nonconstant positive solutions to (1.1) in the two cases: (a) arbitrary p and λ is small; (b) $0 < p \leqslant 1$ and θ is large. To this aim, our proof depends on two different mathematical approaches, that is, the energy integration and the application of the implicit function theorem. Moreover, we also study the asymptotic behavior of nonconstant positive solutions to (1.1) as θ converges to infinity if λ and p enter into some certain ranges. As the by-product of this result, we also yield a sufficient condition for existence of nonconstant positive solutions to a nonlocal elliptic equation.

4.1. Nonexistence of nonconstant positive solutions for any p and small λ

In this subsection, we derive some results of nonexistence of nonconstant positive solutions for arbitrary p and small λ through integration of equations and energy estimates. Obviously, theses results improve Theorems 4.1 and 4.2 in [20].

Theorem 4.1. *Suppose that p, θ and λ satisfy one of the following conditions, then* (1.1) *has no nonconstant positive solution*:

(i) $1 < p$ *and* $\lambda p\varepsilon^{-p}(\theta\varepsilon^{-p}+1)^p[1+\theta^2(\theta\varepsilon^{-p}+1)^{-2}] < 2\lambda_1\theta$.
(ii) $0 < p \leqslant 1$ *and* $\lambda p\varepsilon^{-1}(1+\theta^2) < 2\lambda_1\theta$.

Proof. Let (u, v) be a positive solution of (1.1), and denote $\bar{u}, \bar{v}, \phi$ and ψ as in Section 3. Then, multiplying the first equation of (1.1) by ϕ, integrating over Ω and using Theorem 2.2, we have that

$$\begin{aligned}\theta\int_\Omega |\nabla\phi|^2\,\mathrm{d}x &= \lambda\int_\Omega \big(1-uv^p\big)\phi\,\mathrm{d}x = \lambda\int_\Omega \big(\bar{u}\bar{v}^p-uv^p\big)\phi\,\mathrm{d}x \\ &= -\lambda\int_\Omega \big\{v^p\phi^2+p\bar{u}\xi^{p-1}\phi\psi\big\}\,\mathrm{d}x, \end{aligned} \tag{4.1}$$

where $\xi(x)$ lies between v and $\bar{v}$.

If $p > 1$, together with Theorems 2.1, 3.2, it follows from (4.1), the Cauchy inequality and Poincaré inequality that

$$\begin{aligned}\theta\int_\Omega |\nabla\phi|^2\,\mathrm{d}x &\leqslant \lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^{p-1}\int_\Omega |\phi||\psi|\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^p\int_\Omega \phi^2\,\mathrm{d}x+\frac{1}{2}\lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^{p-2}\int_\Omega \psi^2\,\mathrm{d}x\end{aligned}$$

$$\leqslant \frac{1}{2}\lambda_1^{-1}\lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^p\int\limits_\Omega |\nabla\phi|^2\,\mathrm{d}x$$

$$+\frac{1}{2}\lambda_1^{-1}\lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^{p-2}\int\limits_\Omega |\nabla\psi|^2\,\mathrm{d}x$$

$$\leqslant \frac{1}{2}\lambda_1^{-1}\lambda p\varepsilon^{-p}\big(\theta\varepsilon^{-p}+1\big)^p\big[1+\theta^2\big(\theta\varepsilon^{-p}+1\big)^{-2}\big]\int\limits_\Omega |\nabla\phi|^2\,\mathrm{d}x.$$

Under our assumption, the inequality becomes impossible except that $u=\bar{u}$ is a positive constant. Furthermore, Theorem 3.2 implies that $v=\bar{v}$ is also a positive constant. Thus, this confirms our conclusion (i).

For $0<p\leqslant 1$, by Eq. (4.1), Theorems 2.1 and 3.2 again, the analysis as above shows

$$\theta\int\limits_\Omega |\nabla\phi|^2\,\mathrm{d}x \leqslant \lambda p\varepsilon^{-1}\int\limits_\Omega |\phi||\psi|\,\mathrm{d}x \leqslant \frac{1}{2}\lambda p\varepsilon^{-1}\int\limits_\Omega \big(\phi^2+\psi^2\big)\,\mathrm{d}x$$

$$\leqslant \frac{1}{2}\lambda_1^{-1}\lambda p\varepsilon^{-1}\big(1+\theta^2\big)\int\limits_\Omega |\nabla\phi|^2\,\mathrm{d}x.$$

Consequently, (1.1) has no nonconstant positive solution when (ii) holds. The proof is complete. $\square$

It is clear that (1.1) has no nonconstant positive solution if λ is small. We also remark here that, by the equation for v, Theorems 2.1 and 3.2, we can also obtain some other conditions for nonexistence of nonconstant positive solution to (1.1). The details are left to the interested reader.

4.2. Nonexistence of nonconstant positive solutions for $0<p\leqslant 1$ and large θ

In this subsection, we shall analyze the nonexistence of nonconstant positive solutions to (1.1) via the implicit function theorem provided that $0<p\leqslant 1$ and θ is sufficiently large. As a preliminary, we need some basic lemmas. Firstly, we can claim a counterpart of Lemma 2.2.

Lemma 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, and let $g\in C(\bar{\Omega}\times\mathbf{R})$. If $w\in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta w+g(x,w)\leqslant 0 \quad \text{in } \Omega, \qquad \partial_\nu w\geqslant 0 \quad \text{on } \partial\Omega,$$

and if there is a constant K such that $g(x,z)>0$ for $z<K$, then

$$w\geqslant K \quad \text{a.e. in } \Omega.$$

Proof. The proof is similar to that of Lemma 2.3 in [10]. In fact, multiplying the inequality by the test function $(K-w)^+$, we have

$$\int\limits_{\{w<K\}} g(x,w)(K-w)\,\mathrm{d}x \leqslant -\int\limits_{\{w<K\}} |\nabla w|^2\,\mathrm{d}x.$$

Due to our hypothesis, it is necessary that $\{w < K\}$ has zero measure, and this indicates $w \geqslant K$ a.e. in Ω. $\square$

Applying Lemmas 2.2 and 4.1, one easily obtains a useful result, whose stronger form was stated in [15] (see Lemma 3.2 there).

Lemma 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, and let $g \in C(\bar{\Omega} \times \mathbf{R}^+)$. If $w \in W^{1,2}(\Omega)$ is a weak solution of the equation*

$$\Delta w + g(x, w) = 0 \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

and if there is a positive constant K such that $g(x, z) > 0$ for $0 < z < K$ and $g(x, z) < 0$ for $z > K$, then

$$w = K.$$

To study the nonexistence of nonconstant solutions for large θ by the implicit function theorem, we also have to investigate the asymptotic behavior of positive solutions to (1.1) as $\theta \to \infty$.

Lemma 4.3. *Let p, λ be fixed with $0 < p \leqslant 1$ and assume that (u_θ, v_θ) is the positive solution of (1.1), then $(u_\theta, v_\theta) \to (1, 1)$ in $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $\theta \to \infty$.*

Proof. As $0 < p \leqslant 1$, using the equations in (1.1), from Theorem 2.2, the standard L^p and Schauder's estimates of elliptic equations and embedding theory guarantee that there exists a sequence θ_i with $\theta_i \to \infty$ as $i \to \infty$ and the corresponding positive solution $(u_{\theta_i}, v_{\theta_i})$ of (1.1) for $\theta = \theta_i$ such that $(u_{\theta_i}, v_{\theta_i}) \to (\tilde{u}, \tilde{v})$ in $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $i \to \infty$, where $\tilde{u}$ is a positive constant and $\tilde{v}$ is a positive solution of

$$-\Delta\tilde{v} = \lambda\big(\tilde{u}\tilde{v}^p - \tilde{v}\big) \quad \text{in } \Omega, \qquad \partial_\nu\tilde{v} = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Thanks to Lemma 4.2, Eq. (4.2) has a unique positive solution $\tilde{v} = (\tilde{u})^{1/(1-p)}$ if $0 < p < 1$. Thus, the first identity of (3.1) implies $(\tilde{u}, \tilde{v}) = (1, 1)$. For $p = 1$, the positivity of $\tilde{v}$ and the characterization for the first eigenvalue give $\tilde{u} = 1$ and $\tilde{v}$ is a constant, which also leads to $\tilde{v} = 1$ by (3.1) again. This verifies our result. $\square$

Theorem 4.2. *Let $p, \underline{\lambda}, \bar{\lambda}$ be arbitrary fixed positive numbers with $0 < p \leqslant 1$ and $\underline{\lambda} < \bar{\lambda}$. Then there exists $\Theta > 0$, which depends only on $p, \underline{\lambda}, \bar{\lambda}$ and Ω, such that (1.1) has no nonconstant solution provided that $\theta \geqslant \Theta$ and $\underline{\lambda} \leqslant \lambda \leqslant \bar{\lambda}$.*

Proof. To our end, we make the following decomposition:

$$u = \xi + w \quad \text{with} \int_\Omega w \, \mathrm{d}x = 0 \text{ and } \xi \in \mathbf{R}^+,$$

and denote

$$L_0^2(\Omega) = \left\{ g \in L^2(\Omega) \;\middle|\; \int_\Omega g \, \mathrm{d}x = 0 \right\} \quad \text{and} \quad W_\nu^{2,2} = \big\{ g \in W^{2,2}(\Omega) \;\big|\; \partial_\nu g = 0 \text{ on } \partial\Omega \big\}.$$

Hence, finding the positive solution of (1.1) is equivalent to solving the problem:

$$\begin{cases} \Delta w + \lambda\rho\mathbf{P}\big(1-(\xi+w)v^p\big)=0 & \text{in } \Omega, \\ \int_\Omega \big\{1-(\xi+w)v^p\big\}\,\mathrm{d}x = 0, & \\ \Delta v + \lambda\big((\xi+w)v^p - v\big)=0 & \text{in } \Omega, \\ \partial_\nu w = \partial_\nu v = 0 & \text{on } \partial\Omega, \\ \xi > 0,\ v(x) > 0 & \text{in } \Omega, \end{cases} \tag{4.3}$$

where

$$\rho = \theta^{-1} \quad \text{and} \quad \mathbf{P}z = z - \frac{1}{|\Omega|}\int_\Omega z\,\mathrm{d}x,$$

i.e., $\mathbf{P}$ is the projective operator from $L^2(\Omega)$ to $L_0^2(\Omega)$.

Obviously, $(0, 1, 1)$ is a solution of (4.3). From the above analysis, to verify our assertion, by the finite covering argument, it is enough to prove that for any fixed $\tilde\lambda \in [\underline{\lambda}, \bar\lambda]$, there exists a small positive constant δ_0 such that if $\rho \in (0, \delta_0)$, $\lambda \in (\tilde\lambda - \delta_0, \tilde\lambda + \delta_0)$, then $(0, 1, 1)$ is the unique solution of (4.3).

For this purpose, we need to define

$$\begin{aligned} &F(\rho, w, \xi, v) \\ &\quad = (f_1, f_2, f_3) : \mathbf{R}^+ \times \big(L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)\big) \times \mathbf{R}^+ \times W_\nu^{2,2}(\Omega) \to L_0^2(\Omega) \times \mathbf{R} \times L^2(\Omega), \end{aligned}$$

with

$$\begin{aligned} f_1(\rho, w, \xi, v) &= \Delta w + \tilde\lambda\rho\mathbf{P}\big\{1-(\xi+w)v^p\big\}, \\ f_2(\rho, w, \xi, v) &= \int_\Omega \big\{1-(\xi+w)v^p\big\}\,\mathrm{d}x, \\ f_3(\rho, w, \xi, v) &= \Delta v + \tilde\lambda\big\{(\xi+w)v^p - v\big\}. \end{aligned}$$

It is clear that (4.3) is equivalent to solving $F(\rho, w, \xi, v) = 0$. Moreover, (4.3) has a unique solution $(w, \xi, v) = (0, 1, 1)$ when $\rho = 0$. By a simple computation, we have

$$D_{(w,\xi,v)}F(0,0,1,1) : \big(L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)\big) \times \mathbf{R} \times W_\nu^{2,2}(\Omega) \to L_0^2(\Omega) \times \mathbf{R} \times L^2(\Omega),$$

where

$$D_{(w,\xi,v)}F(0,0,1,1)(y, \zeta, z) = \begin{pmatrix} \Delta y \\ \int_\Omega \{-y - \zeta - pz\}\,\mathrm{d}x \\ \Delta z + \tilde\lambda(p-1)z + \tilde\lambda y + \tilde\lambda\zeta \end{pmatrix}.$$

We observe that $\Delta : L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega) \to L_0^2(\Omega)$ is invertible, thus $D_{(w,\xi,v)}F(0,0,1,1)$ is invertible if and only if

$$L(\zeta, z) = \begin{pmatrix} \int_\Omega \{-\zeta - pz\}\,\mathrm{d}x \\ \Delta z + \tilde{\lambda}(p-1)z + \tilde{\lambda}\zeta \end{pmatrix}$$

is invertible. Note that $\zeta \in \mathbf{R}$. Thus, it is not hard to verify that $L : \mathbf{R} \times W_\nu^{2,2}(\Omega) \to \mathbf{R} \times L^2(\Omega)$ is invertible if $0 < p \leqslant 1$. Moreover, direct computations show that $D_{(w,\xi,v)}F(0,0,1,1)$ is also a surjection under this assumption.

By the implicit function theorem, there exist positive constants ρ_0 and δ_0 such that, for each $\rho \in [0, \rho_0]$, $(0,1,1)$ is the unique solution of $F(\rho, w, \xi, v) = 0$ in $B_{\delta_0}(0,1,1)$, where $B_{\delta_0}(0,1,1)$ is the ball in $(L_0^2(\Omega) \cap W_\nu^{2,2}(\Omega)) \times \mathbf{R} \times W_\nu^{2,2}(\Omega)$ centered at $(0,1,1)$ with radius δ_0. Let $(w_\rho, \xi_\rho, v_\rho)$ be any solution of (4.3) for small $\rho > 0$, Lemma 4.3 shows that $(w_\rho, \xi_\rho, v_\rho) \to (0,1,1)$ as $\rho \to 0^+$. As a result, $(1,1)$ is the unique solution of (1.1) as θ is sufficiently large. This finishes our proof. $\square$

Now, we make some remarks in the case of $1 < p < n/(n-2)$ and large θ. Let (u_θ, v_θ) be a positive solution to (1.1). In this case, Theorem 2.2 says that the upper and lower bounds of (u_θ, v_θ) are independent of θ. Then, the standard theory implies that $(u_\theta, v_\theta) \to (u, v)$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $\theta \to \infty$, where u is a positive constant and v is a positive function on $\bar{\Omega}$. In addition, using the first identity of (3.1) and the second equation of (1.1), $(\tilde{u}, \tilde{v})$ solves

$$\tilde{u} = |\Omega| \biggl(\int_\Omega \tilde{v}^p \,\mathrm{d}x \biggr)^{-1} \tag{4.4}$$

and

$$-\Delta \tilde{v} = \lambda \biggl(|\Omega| \biggl(\int_\Omega \tilde{v}^p \,\mathrm{d}x \biggr)^{-1} \tilde{v}^p - \tilde{v} \biggr) \quad \text{in } \Omega, \qquad \partial_\nu \tilde{v} = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

The same analysis as in the proof of Lemma 4.3 shows that $v = 1$ is the unique positive solution of (4.5) if $0 < p \leqslant 1$. Suppose that (4.5) has only the constant positive solution $v = 1$, thus Lemma 4.3 holds. Further suppose that

$$\lambda(p-1) \neq \lambda_i, \quad i = 1, 2, 3, \ldots,$$

is satisfied. By checking the proof of Theorem 4.2, it is not difficult to find that for such fixed p and λ, the implicit function theorem still works and so (1.1) has no nonconstant positive solution for large θ. On the other hand, under the conditions of Theorem 2.3, nonconstant positive solutions exist for (1.1) provided that θ is large enough. Based on this consideration, we can conclude the following

Theorem 4.3. *Assume that* $1 < p < n/(n-2)$ *and* $\lambda(p-1) \in (\lambda_m, \lambda_{m+1})$ *for some fixed* $m \geqslant 1$. *Let* $L_m = \sum_{i=1}^m \dim(\lambda_i)$ *and suppose that* L_m *is odd. Then, the nonlocal elliptic problem* (4.5) *admits at least one nonconstant positive solution. In addition, for any nonconstant positive solution* (u_θ, v_θ) *to* (1.1), *there exists a subsequence* $(u_{\theta_i}, v_{\theta_i})$ *of* (u_θ, v_θ) *such that* $(u_{\theta_i}, v_{\theta_i})$

converges to $(\tilde{u}, \tilde{v})$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $\theta_i \to \infty$, where $\tilde{v}$ is a nonconstant positive solution of (4.5) *and $\tilde{u}$ is uniquely determined by* (4.4).

Proof. The first part of Theorem 4.3 has been verified as above. Now, we prove the remaining part by an indirect argument.

Suppose that there exists a sequence $\{\theta_i\}$ with $\theta_i \to \infty$ as $i \to \infty$ and the corresponding nonconstant positive solution (u_i, v_i) to (1.1) for $\theta = \theta_i$, such that our statement is not true. Then, combined with Theorem 2.2, the standard L^p and Schauder's estimates and the embedding theorems show that there is a subsequence of (u_i, v_i), still labelled by itself, such that $(u_i, v_i) \to (1,1)$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $i \to \infty$.

Set

$$h_i = u_i - 1, \qquad k_i = v_i - 1,$$

and

$$\tilde{h}_i = \frac{h_i}{\|h_i\|_\infty + \|k_i\|_\infty}, \qquad \tilde{k}_i = \frac{k_i}{\|h_i\|_\infty + \|k_i\|_\infty}.$$

Thus, it is easy to check that

$$-\Delta \tilde{h}_i = -\theta_i^{-1} \lambda \left(v_i^p \tilde{h}_i + p \xi_i^{p-1} \tilde{k}_i \right) \quad \text{in } \Omega, \qquad \partial_\nu \tilde{h}_i = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

and

$$-\Delta \tilde{k}_i = \lambda \left(v_i^p \tilde{h}_i + p \xi_i^{p-1} \tilde{k}_i - \tilde{k}_i \right) \quad \text{in } \Omega, \qquad \partial_\nu \tilde{k}_i = 0 \quad \text{on } \partial\Omega, \tag{4.7}$$

where ξ_i lies between 1 and v_i for each $i \geqslant 1$.

Since the right-hand sides of (4.6) and (4.7) are L^∞ bounded by Theorem 2.2 and $\|\tilde{h}_i\|_\infty, \|\tilde{k}_i\|_\infty \leqslant 1$, it follows from the L^p estimates and the Sobolev embedding theorems that $\tilde{h}_i \to \tilde{h}$ and $\tilde{k}_i \to \tilde{k}$ in $C^1(\bar{\Omega})$. Noting that $(u_i, v_i) \to (1,1)$ on $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ as $i \to \infty$ and passing to the limit in (4.6) and (4.7), we see

$$-\Delta \tilde{h} = 0, \qquad -\Delta \tilde{k} = \lambda (p-1) \tilde{k} + \lambda \tilde{h} \quad \text{in } \Omega, \qquad \partial_\nu \tilde{h} = \partial_\nu \tilde{k} = 0 \quad \text{on } \partial\Omega.$$

Hence, $\tilde{h}$ is a constant and so our assumption indicates $\tilde{k} = \tilde{h}/(1-p)$. Furthermore, $\|\tilde{h}_i\|_\infty + \|\tilde{k}_i\|_\infty = 1$ for each $i \geqslant 1$ gives

$$\left(1 + \left|1/(1-p)\right|\right)|\tilde{h}| = 1.$$

However, by the second identity of (3.1), it is necessary that $\int_\Omega \tilde{k}\, \mathrm{d}x = 0$, leading to $\tilde{h} = 0$. A contradiction occurs, and this completes our proof. □

Remark 4.1. In [20, Remark 2.1], since the constant positive solution $(u, v) = (1,1)$ is uniformly asymptotically stable for the corresponding reaction–diffusion system of (1.1) in the sense of [8] if $0 < p \leqslant 1$, the authors pointed out that it is difficult to expect the bifurcation of (1.1) near $(u, v) = (1,1)$ in this case. Our Theorem 4.2 shows that no bifurcation will happen for (1.1) provided that $0 < p \leqslant 1$ and θ is sufficiently large because the existence of bifurcation will imply the local existence of nonconstant positive solutions to (1.1).

Remark 4.2. As pointed out in the proof of Theorem 3.1 in [10], for any $p > 0$, the upper and lower bounds of positive solutions (u_θ, v_θ) to (1.1) are independent of the lower bound of θ. Therefore, we can claim that for any fixed p and λ, $(u_\theta, v_\theta) \to (1, 1)$ uniformly on $\bar{\Omega}$ as $\theta \to 0$. Our proof depends on the following result: Assume that $\mu > 0$ is a constant and $a(x)$ is a continuous positive function on $\bar{\Omega}$. Then, the problem

$$-\Delta w = \mu\big(1 - a^{-1}(x)w\big) \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

has a unique positive solution w_μ, and $w_\mu \to a(x)$ uniformly on $\bar{\Omega}$ as $\mu \to \infty$. For the similar details of proof of this result, please refer to [5]. We also like to mention that Du and Ma [4] established a similar result when the homogeneous Dirichlet boundary condition is concerned (see Lemma 2.2 there). However, we have to mention that we are unable to derive some results for the existence or nonexistence of nonconstant positive solutions to (1.1) when θ is small enough.

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