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A quantization of $(2+1)$-gravity related to high-energy Yang–Mills theory ☆

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Abstract

We point out that canonical quantization of the two-body problem in $(2+1)$-gravity is related to the high-energy equation in Yang–Mills theory by a proper ordering of the relevant operators. This feature arises from expanding the Hamiltonian around its conformal limit—or treating running coupling effects in the Yang–Mills case—and yields a peculiar short distance behaviour of the wave functions.

1. Introduction

It is a common feature of several theoretical models to show a (nearly) two-dimensional dynamics either because of fundamental reasons (as in string theory) or due to the importance of certain kinematical configurations. Here we focus on two models of the latter class, namely gauge theories in the high-energy limit and $(2+1)$-gravity with pointlike matter. It is known that both models possess a $(2+1)$-dimensional "configuration" space, with a "time" parameter of quite different meaning in the two cases, and two space dimensions (the transverse ones to the high-energy momenta in the gauge case). It is less known, and we emphasize it here that the Hamiltonian of either model is eventually provided by the same operator and is related to a fundamental scale of the system.

In other words, in this Letter we show that Hamiltonian quantization relates in a nontrivial manner a well-known high-energy tool—the so-called BFKL equation [1,2]—to a quantized (two-body) gravitational system with proper ordering of the relevant operators. Perhaps the reason why two such different systems happen to have eventually the same dynamics is that both admit a conformal limit in which the Hamiltonian is just the dilatation operator. Deviations from the conformal behaviour are then treated by expanding in some mass parameter (with different physical meaning in the two cases).

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It should be noticed that the relationship of gravitational theories to gauge theories and/or conformal theories may have a more general origin [3,4]. For gravity in $2+1$ dimensions [5] this appears from the relevance of Liouville fields in the classical solutions with cosmological constant [6,7] and/or with matter sources [8,9]. It is not known however whether these suggestions may have an impact at quantum level also.

From the point of view of $(2+1)$-gravity, the quantization of the two-body system proposed here provides an alternative to the standard one of Deser, Jackiw and 't Hooft (DJH) [5,10,11], which adds up to those already known [12]. The large-distance properties (in particular phase-shifts and scattering angle) stay the same, but the wave function behaves differently at short distances where it shows an anomalous dimension behaviour as in the gauge theory case.

After introducing Hamiltonian and physics of our two models in Sections 2 and 3, we discuss in detail the ensuing quantum properties in Section 4 and possible developments at many body level in the conclusive Section 5.

2. The $(2+1)$-gravity Hamiltonian

Gravity in three space–time dimensions is characterized by the fact that the Riemann tensor is proportional to the Einstein tensor and thus to the energy–momentum tensor. As a consequence, space–time is flat outside the matter sources. If the latter are pointlike particles, local Minkowskian coordinates can be extended all around them, but are in general multivalued, i.e., carry nontrivial monodromy transformations for parallel transport in a closed loop around each particle site [13,14].

For a spinless particle at rest, the loop integral of the connection is $8\pi Gm$—or just m, in units of the energy $1/8\pi G$, so that the Minkowskian coordinates $X^a = (T, Z, \overline{Z})$ possess a branch-cut characterized by the discontinuity relation

$$Z_{\mathrm{II}} = e^{-im} Z_{\mathrm{I}}, \qquad T_{\mathrm{II}} = T_{\mathrm{I}} \quad (Z = X + iY), \tag{1}$$

between values above and below the cut. This corresponds to a cut-out sector or deficit angle m

$$|\Theta| = |\arg Z| < \pi\alpha, \quad \alpha = 1 - \frac{m}{2\pi} = 1 - 4Gm, \tag{2}$$

typical of a conical space. For moving particles of momenta P_n^a, the relation (1) is boosted to Lorentz transformations—the DJH matching conditions [5]

$$(X_{\mathrm{II}} - X_n)^a = L(P_n)^a_b (X_{\mathrm{I}} - X_n)^b \quad \big(P_n^2 = m_n^2,\ n = 1, \dots, N\big), \tag{3}$$

and the latter *do not commute* for nonvanishing relative velocities.

In the static case (1), it is straightforward to construct single-valued coordinates $x^\mu = (t, z, \bar{z})$ around each particle, by a coordinate transformation of the type

$$T = t, \qquad Z = z\left(\frac{z}{\lambda}\right)^{-\mu} \sim z^\alpha \quad \left(\mu = \frac{m}{2\pi} = 4Gm\right), \tag{4}$$

so that $Z \to e^{-im} Z$ when $z \to e^{2i\pi} z$. The corresponding metric is nontrivial, with line element

$$ds^2 = dt^2 - \alpha^2 \left|\frac{z}{\lambda}\right|^{-2\mu} |dz|^2 \quad \big(|\operatorname{Arg} z| \leqslant \pi\big), \tag{5}$$

and yields the conformal-gauge description. Finally, the scale change $\rho = \lambda(r/\lambda)^\alpha$ (we denote by r and θ the modulus and argument of z) brings Eq. (5) into the canonical DJH form

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 \rho^2\, d\theta^2, \tag{6}$$

in which the geometry of a cone with aperture $2\pi\alpha = 2\pi - 2\pi\mu$ is transparent.

We have introduced in the above equations a scale parameter λ which is arbitrary at this stage, but becomes dependent on the dynamical variables of the system if the metric (5) is interpreted as the asymptotic metric[1] for $|z| \gg \lambda(z_n, p_n)$ of a system of many particles of coordinates $\{z_n, p_n\}$ and invariant mass $M = 2\pi\mu$.

The simplest quantization procedure is that of a test particle in the conical space of Eqs. (5) and (6), described by the eigenfunctions of the Laplace operator on the cone [10,11]

$$\Delta|_{\text{cone}} = \left|\frac{z}{\lambda}\right|^{2\mu} \Delta_z = \left|\frac{z}{\lambda}\right|^{2\mu} |p|^2, \tag{7}$$

where $p = -i\partial_z$. Generalizing such approach to the dynamical many-body system [17] is nontrivial, due to the noncommutativity of the monodromies (3). At the classical level, a single-valued metric of conformal type was obtained in the instantaneous gauge [8] by Bellini, Valtancoli and one of us (M.C.) [17], and by Welling [18]. The same solution was exhibited by Menotti and Seminara [9,19] in a canonical formalism, which allows the derivation of the classical two-body Hamiltonian[2]

$$\mathcal{H} = \mu \log|z|^2 + \log|p|^2 = \mu \log \lambda^2(z, p) \quad \left(\mu = \frac{M}{2\pi}\right). \tag{8}$$

Here $z = z_2 - z_1$ is the relative coordinate of the two particles, and p is the conjugate momentum, while $\lambda(z, p)$ turns out to be [19] the scale parameter introduced in Eq. (5), interpreted as the asymptotic metric of the 2-body system.

The expression (8) is formally the logarithm of (7) and thus may lead to a quantization much similar to that of Deser, Jackiw and 't Hooft. However, the decomposition (8) as a sum of two contributions suggests a quantization with different ordering, related to the BFKL equation with running coupling, as we shall see in the following.

3. The high-energy evolution equation

The basic question of high-energy QCD is to find the total cross section $\mathcal{G}(k, k_0; Y)$ for the scattering of two gluons at scales k, k_0 and relative rapidity $Y = \log(s/(kk_0))$, s being their center-of-mass squared energy. The pioneering work of Balitsky, Fadin, Kuraev and Lipatov [1,2] showed that the perturbative high-energy behaviour could be resummed in the leading logarithmic approximation by an evolution equation involving a two dimensional Hamiltonian $\mathcal{H}_0$ [20,21]

$$-\frac{\partial}{\partial Y}\mathcal{G}(Y) = \bar{\alpha}_s \mathcal{H}_0 \mathcal{G}(Y), \qquad \mathcal{G}(k, k_0; 0) = \delta^2(k - k_0), \tag{9}$$

where

$$\mathcal{H}_0 = \log|k|^2 + \log|z|^2 - 2\psi(1), \tag{10}$$

$k = k_1 + ik_2$, $z \equiv i\partial_k = i(\partial_1 - i\partial_2)/2$ is the variable conjugated to the 2-dimensional momentum k (its Hermitian conjugate operator is $\bar{z} \equiv i\partial_{\bar{k}}$) and $\bar{\alpha}_s = \alpha_s N_c/\pi$ is the QCD squared coupling constant.

The BFKL evolution equation (9) is solved in terms of the eigenfunctions of the Hamiltonian (10), which is actually scale-invariant, so that it has characteristic function $-\chi_n(\nu)$, where

$$\chi_n(\nu) = 2\psi(1) - 2\,\mathcal{R}\mathrm{e}\,\psi\left(\frac{1+|n|}{2} + i\nu\right) = 2\psi(1) - 2\,\mathcal{R}\mathrm{e}\,\psi\left(\frac{1-|n|}{2} + i\nu\right), \tag{11}$$

[1] Actually, the asymptotic metric takes up a more general ADM form [15] if the angular momentum of the system is nonvanishing [5]. In Ref. [16], the case of a more complicated topology of space was also investigated.

[2] In a general time gauge the Hamiltonian may differ from (8) by a constant factor, which for a "local" observer is $\sim(\mu - \mu_1 - \mu_2)$, as in the equations of motion of Ref. [17].

on the power-behaved eigenfunctions $\Psi_{n,\nu}(k,\bar{k}) = k^{\gamma-1}\bar{k}^{\tilde{\gamma}-1}$ ($\gamma = \tilde{\gamma} - n = (1-n)/2 + i\nu$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$). Thus, the evolution equation (9) and Eq. (11) show that the 2-gluon correlator is exponentially behaved in Y (or power-behaved in s), with exponent $\omega_{\mathrm{P}} = \bar{\alpha}_s \chi_0(0)$, yielding the so-called hard pomeron behaviour of the cross section.

Let us outline the proof of these classical results. The fact that the eigenfunctions $\Psi_{n,\nu}$ are powers of k and $\bar{k}$ is due to scale invariance of the Hamiltonian (10). Single-valuedness of the cross section imposes that $\tilde{\gamma} - \gamma$ be an integer. The eigenvalues (11) are then computed by using the relation of $\mathcal{H}_0$ to the sum of $\log|k|^2$ and $\log|\partial_k|^2$ which defines our ordering. We start from the definition of the $\log|z|^2 = \log|\partial_k|^2$ operator acting on a given function $\Psi(k,\bar{k})$:

$$\log|z|^2 \Psi\big(k,\bar{k}\big) = \int \frac{d^2\rho}{(2\pi)^2}\, e^{i(k\bar{\rho}+\bar{k}\rho)/2} \log\left(\frac{|\rho|^2}{4}\right) \widetilde{\Psi}\big(\rho,\bar{\rho}\big), \tag{12}$$

where $\widetilde{\Psi}(\rho,\bar{\rho})$ is the Fourier transform of $\Psi(k,\bar{k})$. It is useful to take the following representation for the logarithm:

$$\log\left(\frac{|\rho|^2}{4}\right) = -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(\frac{|\rho|^2}{4}\right)^{-\epsilon}. \tag{13}$$

We specialize to the power-functions $\Psi_{n,\nu}(k,\bar{k}) = k^{\gamma-1}\bar{k}^{\tilde{\gamma}-1}$. Their Fourier transform $\widetilde{\Psi}_{n,\nu}(\rho,\bar{\rho})$ can be computed using the formula

$$\int d^2k\, k^{\gamma-1}\bar{k}^{\tilde{\gamma}-1} e^{-ik\cdot\rho} = \pi i^{\gamma-\tilde{\gamma}} \left(\frac{2}{\rho}\right)^{\tilde{\gamma}} \left(\frac{2}{\bar{\rho}}\right)^{\gamma} \frac{\Gamma(\tilde{\gamma})}{\Gamma(1-\gamma)}, \tag{14}$$

which holds for $\tilde{\gamma} - \gamma$ positive (the case $\tilde{\gamma} - \gamma$ negative is obtained by exchanging γ and $\tilde{\gamma}$). The integral over ρ in Eq. (12) is performed using again the same formula. Finally one takes the derivative with respect to ϵ to obtain

$$\big(\log|z|^2 + \log|k|^2\big)\Psi_{n,\nu}\big(k,\bar{k}\big) = \big(\psi(\gamma) + \psi(1-\tilde{\gamma})\big)\Psi_{n,\nu}\big(k,\bar{k}\big). \tag{15}$$

As $1 - \tilde{\gamma} = \bar{\gamma}$, this equation proves that the functions $\Psi_{n,\nu}(k,\bar{k}) = k^{\gamma-1}\bar{k}^{\tilde{\gamma}-1}$ are eigenfunctions of $\mathcal{H}_0$ with eigenvalues given by Eq. (11). Note that Eq. (15) can be formally written as an identity between operators: $\log|\partial_k|^2 + \log|k|^2 = \psi(1 + k\partial_k) + \psi(-\bar{k}\partial_{\bar{k}})$. Consequently, the BFKL Hamiltonian can be expressed as a function of the dilatation operators $k\partial_k$ and $\bar{k}\partial_{\bar{k}}$, besides admitting the customary integral operator formulation [1,2].

Finally, solutions to the BFKL evolution (9) are linear combinations of the eigenfunctions to which one applies the evolution operator

$$\mathcal{G}(k,k_0;Y) = \sum_n \int \frac{d\nu}{2\pi}\, \overline{\Psi}_{n,\nu}\big(k,\bar{k}\big) e^{-\bar{\alpha}_s \mathcal{H}_0 Y} \Psi_{n,\nu}\big(k_0,\bar{k}_0\big) = \frac{1}{|kk_0|} \sum_n \int \frac{d\nu}{2\pi} \left|\frac{k}{k_0}\right|^{-2i\nu} \left(\frac{k}{\bar{k}}\frac{\bar{k}_0}{k_0}\right)^{n/2} e^{\bar{\alpha}_s \chi_n(\nu) Y}. \tag{16}$$

The dominant large Y behaviour is given by the saddle point of the azimuthally symmetric $n = 0$ component, which lies at $\nu = 0$ so that $\mathcal{G}(Y) \sim \exp(\bar{\alpha}_s \chi_0(0)\cdot Y)$, as stated before.

The model just outlined shows conformal invariance at nonvanishing momentum transfer, and can be generalized to many-gluon correlators [21] and connected to an integrable 2-dimensional model [22].

When subleading logs corrections [23–25] are taken into account, the picture changes qualitatively, because at this level the QCD coupling acquires renormalization group (RG) evolution in the form

$$\bar{\alpha}_s\big(k^2\big) = \frac{1}{b \log k^2/\Lambda^2}, \tag{17}$$

where Λ^2 is the RG invariant QCD scale. Therefore, by introducing the variable ω conjugated to Y, Eq. (9) takes the form

$$\big[(1 + b\omega)\log|k|^2 + \log|z|^2\big]\mathcal{G} = b\omega \log \Lambda^2 \mathcal{G}, \tag{18}$$

apart from a delta function source $b\log(k_0^2/\Lambda^2)\delta^2(k-k_0)$. Notice that Eq. (18) is no longer scale-invariant, and involves a new kind of Hamiltonian, of the type

$$\mathcal{H}=\log|k|^2+\mu\log|z|^2 \quad \left(\mu=\frac{1}{1+b\omega}\right), \tag{19}$$

whose eigenvalues are related to $b\omega$ and to the QCD scale Λ^2 itself. This Hamiltonian is exactly the same as the one given in Eq. (8), except that (19) is already quantized: indeed k and z are operators satisfying to the commutation relation $[z,k]=i$.

Let us solve the eigenvalue equation $\mathcal{H}\phi_E=E\phi_E$ for a coordinate-dependent wave-function $\phi_E(z,\bar z)$. Although the Hamiltonian is no more scale invariant, we shall take the functions $\Psi_{n,\nu}(z,\bar z)=z^{-\gamma}\bar z^{-\tilde\gamma}/(\pi\sqrt{2})$ as a basis for its eigenfunctions, where $\gamma(n,\nu)$ and $\tilde\gamma(n,\nu)$ were introduced in the previous section, thus spanning an L_2 space. The normalization is chosen so that the $\Psi_{n,\nu}$ are orthonormal $\int d^2z\,\Psi_{n,\nu}(z,\bar z)\,\overline{\Psi}_{n',\nu'}(z,\bar z)=\delta_{n,n'}\delta(\nu-\nu')$. Since the Ψ's are eigenfunctions of $\mathcal{H}(\mu=1)$ with the eigenvalues in Eq. (15), we can set $\mathcal{H}=\mathcal{H}(\mu=1)+(\mu-1)\log|z|^2$ and we notice the representation $\log|z|^2\to-\partial/\partial(i\nu)$. It is then easy to see that the linear combination

$$\phi_E(z,\bar z)=\sum_n\int\frac{d\nu}{2\pi}\,\Psi_{n,\nu}(z,\bar z)\,f_{E,n,\nu}, \tag{20}$$

is an eigenfunction of $\mathcal{H}$ if the coefficients of the expansion are:

$$f_{E,n,\nu}=\sqrt{\frac{2\pi}{1-\mu}}\exp\left(-\frac{i\nu E}{1-\mu}-\frac{X_n(\nu)}{1-\mu}\right), \tag{21}$$

where

$$X_n(\nu)=\int_0^{i\nu}d(i\nu')\big(\chi_n(\nu')-2\psi(1)\big)=\log\frac{\Gamma\left(\frac{1+|n|}{2}-i\nu\right)}{\Gamma\left(\frac{1+|n|}{2}+i\nu\right)}. \tag{22}$$

The normalization has been fixed by requiring that the set of functions ϕ_E be orthonormal:

$$\int d^2z\,\phi_E(z,\bar z)\bar\phi_{E'}(z,\bar z)=\delta(E-E').$$

The method just outlined is the so-called "γ-representation" widely used in high-energy physics. The expansion (20) is the natural equivalent of a Fourier expansion in the case in which the Hamiltonian is a function of the dilatation operator $z\partial_z$. The set of functions $\Psi_{n,\nu}(z,\bar z)$ for the expansion plays the role of the Fourier basis $e^{ik\cdot z}$: the latter are eigenfunctions of the translation operator ∂_z while the former are eigenfunctions of the dilatation operator $z\partial_z$. Finally, the energy E is fixed in this case by Eq. (18) to be $E=(1-\mu)\log\Lambda^2$ and is thus related to the QCD scale.

4. Quantum scattering solutions

We have just shown that the cross section for gluon scattering at high-energy obeys a Schrödinger equation. Its Hamiltonian is classically the same as the one describing the diffusion of 2 massive particles in $2+1$ dimensions (see Section 2).

We shall now take advantage of this equivalence to investigate the new quantization scheme for gravity which comes from the ordering induced by Eq. (15). We shall study the properties of the wave function obtained in the previous section.

We will consider separately each component of given angular momentum n of the wave function. Introducing the modulus r and argument θ of the transverse coordinate vector z, we write

$$\phi_E(r,\theta) = \sum_n \varphi_n(r,\theta|E). \tag{23}$$

We recast the partial waves in the following form:

$$\varphi_n(r,\theta) = \frac{e^{in\theta}}{\sqrt{\pi(1-\mu)}}\frac{1}{r}\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\,e^{\mathcal{E}_n(i\nu|r)}, \tag{24}$$

where

$$\mathcal{E}_n(i\nu|r) = -2i\nu\log\bigl(\kappa(E)r\bigr) - \frac{1}{1-\mu}\log\frac{\Gamma\bigl(\frac{1+|n|}{2}-i\nu\bigr)}{\Gamma\bigl(\frac{1+|n|}{2}+i\nu\bigr)}, \tag{25}$$

and where we have singled out the scale of distances by defining

$$\kappa^2(E) = e^{E/(1-\mu)}. \tag{26}$$

For sake of simplicity, the E-dependence will be implicit in most of the following equations which involve κ.

We note that in the particular case $\mu = 0$, the wave function (24) reduces to a Bessel function:

$$\varphi_n(r,\theta)|_{\mu=0} = \frac{\kappa e^{in\theta}}{\sqrt{\pi}} J_{|n|}(2\kappa r). \tag{27}$$

For the general case, we will investigate the behaviour of the wave function (24) in different limits of the parameter κr. Technically, we will use the steepest descent method. The equation $\partial\mathcal{E}_n(i\nu|r)/\partial(i\nu) = 0$ defines the saddle points $i\nu_s$ as roots of the equation

$$-2\log(\kappa r) + \frac{1}{1-\mu}\left\{\psi\left(\frac{1+|n|}{2}+i\nu_s\right) + \psi\left(\frac{1+|n|}{2}-i\nu_s\right)\right\} = 0. \tag{28}$$

As we shall only be interested in the leading terms in $|\log(\kappa r)|$, we will solve this equation by taking the relevant approximations for the ψ function. The wave function reads in the saddle point approximation

$$\varphi_n(r,\theta) \simeq \frac{e^{in\theta}}{\sqrt{\pi(1-\mu)}}\frac{1}{r}\cdot\sum_{i\nu_s}\frac{e^{\mathcal{E}_n(i\nu_s|r)}}{\sqrt{2\pi(\partial^2\mathcal{E}_n(i\nu|r)/\partial(i\nu)^2)|_{i\nu=i\nu_s}}} \times\left\{1 + 3\frac{\partial^4\mathcal{E}_n(i\nu|r)/\partial(i\nu)^4|_{i\nu=i\nu_s}}{(\partial^2\mathcal{E}_n(i\nu|r)/\partial(i\nu)^2)^2|_{i\nu=i\nu_s}} + \cdots\right\}. \tag{29}$$

The sum goes over all the roots of Eq. (28). We have singled out the dominant contribution as well as the first correction to it.

First, the study of the large distance behaviour of the wave function ($\kappa r \gg 1$) enables to identify the phase shift due to the scattering [10]. Since $\log(\kappa r)$ is large and positive, the saddle point defined by Eq. (28) sits at large $i\nu_s$. One can use the approximation $\psi(z)\sim\log z$ which then gives $(\kappa r)^{2(1-\mu)} = ((1+|n|)/2)^2 + \nu_s^2$ up to terms of relative order $1/(\kappa r)^{(1-\mu)}$. Taking into account the two roots of this equation, we use formula (29) and we obtain

$$\varphi_n(r,\theta) \underset{\kappa r\gg 1}{\simeq} \frac{\kappa e^{in\theta}}{\pi}(\kappa r)^{-(1+\mu)/2}\cos\left(\frac{2(\kappa r)^{1-\mu}}{1-\mu} - \frac{|n|\pi}{2(1-\mu)} - \frac{\pi}{4}\right), \tag{30}$$

for the leading contribution. The first relative correction is $3/4\cdot(1-\mu)(\kappa r)^{-(1-\mu)}$ from Eq. (29). It is subleading in $|\log(\kappa r)|$, which justifies the method.

This result should be compared to the wave function for scattering in flat space: the partial wave of angular momentum n is given by $J_{|n|}(2\kappa r) \sim \sqrt{1/\pi\kappa r}\cos(2\kappa r - |n|\pi/2 - \pi/4)$. In this case, the wave front is rotated by an angle π in the scattering process. In our case, by analogy, we see on Eq. (30) that the wave front is rotated by $\pi/(1-\mu)$. Thus the scattering angle is $\pi\mu/(1-\mu)$, and corresponds to the deficit angle of the effective conical space in which the particles are moving. Both phase shifts in Eq. (30) and scattering angle agree with the results of DJH.

The small distance behaviour of the wave function is also of interest because in the context of high-energy scattering, this regime corresponds to a configuration in which the interacting gluons have large virtualities. As $\log(\kappa r)$ is large and negative, the saddle point defined by Eq. (28) sits near the pole of the ψ function at $(1+|n|)/2 + i\nu = 0$. The ψ functions are approximated by $\psi(z) \sim -1/z$. The equation for the saddle point then gives $(1+|n|)/2 + i\nu_s = -1/(2(1-\mu)\log(\kappa r))$. Applying once again formula (29), one obtains the leading term

$$\varphi_n(r,\theta) \underset{\kappa r \ll 1}{\simeq} \frac{\kappa e^{in\theta}}{2\sqrt{2}\pi(1-\mu)} \left(\frac{2e(1-\mu)}{\Gamma(1+|n|)}\right)^{1/(1-\mu)} (\kappa r)^{|n|} \left|\log(\kappa r)\right|^{\mu/(1-\mu)}. \tag{31}$$

A bit of care is in order in this case. Indeed, the first relative correction to this approximation (second term in the parenthesis in Eq. (29)) is $18(1-\mu)$. Hence the saddle point method only provides the coefficient of the leading behaviour of the wave function up to terms of relative order $1-\mu$.

We note that in the particular case in which $1/(1-\mu)$ is an integer, the wave function is a Meijer function. All subleading orders can be computed by expanding it in a power series of κr. This can be done by picking up the successive poles of the integrand in the upper ν-plane. For $\mu = 1/2$, we obtain the following result:

$$\begin{aligned}\varphi_n(r,\theta) = 2\kappa e^{in\theta}\sqrt{\frac{2}{\pi}}\,(\kappa r)^{|n|}\Bigg(&\frac{|\log(\kappa r)|}{\Gamma^2(1+|n|)}\,{}_0F_3\big(1, 1+|n|, 1+n, (\kappa r)^2\big) \\ &+ \sum_{k=0}^{\infty}(\kappa r)^{2k}\frac{\psi(1+k)+\psi(1+|n|+k)}{\Gamma^2(1+k)\Gamma^2(1+|n|+k)}\Bigg).\end{aligned} \tag{32}$$

For $\mu \neq 0$, the behaviour of the wave function for small κr computed in Eqs. (31), (32) is to be contrasted with the one found in Refs. [10,19], within the DJH quantization scheme

$$\varphi_n(r,\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}} J_{\frac{|n|}{1-\mu}}\left(\frac{(\kappa r)^{1-\mu}}{1-\mu}\right) \underset{\kappa r \ll 1}{\simeq} \frac{e^{in\theta}}{\sqrt{2\pi}} \frac{(\kappa r)^{|n|}}{2^{\frac{|n|}{1-\mu}}\Gamma\big(1+\frac{|n|}{1-\mu}\big)}, \tag{33}$$

which does not show the logarithmic corrections of Eqs. (31), (32). The latter are due to the Yang–Mills anomalous dimension which in turn are embodied in the small γ behaviour of the eigenvalue function (15). The occurrence of such logarithms in the weak coupling regime $M \sim \hbar\kappa \ll 1/G$ can possibly be ascribed to quantum loops, but no comparison with consistent calculations is actually possible, because string-gravity loops have been studied only close to $D = 4$ [26], due to the occurrence of an infrared anomaly at $D = 3$. Eq. (31) provides also the resummation of such logarithms, formally valid for $\mu = 4GM < 1$. However, before the conformal limit $\mu = 1$ is reached, effects related to the Planck scale $\hbar G$ could become relevant also. Furthermore, inelastic effects of matter radiation—not included in the quantum mechanical Hamiltonian—are expected to affect the low partial waves $n = 0, 1, \ldots,$ too [27].

Anyway, it is amusing to note that the relevant regimes of the two theories are somehow exchanged. The short distance behaviour, which in $(2+1)$-gravity is dependent on the quantization procedure and is expected to be of strong-coupling nature is related to the perturbative anomalous dimension regime of the gauge model. On the other hand, the large distance behaviour, fixed in $(2+1)$-gravity by the semiclassical limit, is related to the large $|\operatorname{Im}\gamma|$ behaviour of the characteristic function, which in the gauge model is expected to be affected by higher order corrections. This interchange of weak- and strong-coupling regimes is analogous to what happens in duality transformations.

Finally, we check the consistency of the wave function that we have constructed by taking its classical limit. We get the time evolution of the system by applying the operator $e^{i\mathcal{H}t}$ to a wave packet. We choose to construct the latter by combining all the wave functions corresponding to different values of the energy E with an equal weight: $\Phi_n(t,r,\theta) = e^{i\mathcal{H}t}\int_{-\infty}^{+\infty} dE\, \phi_n(r,\theta|E)$. This enables to express the variable ν as a function of t: $\nu = (1-\mu)t$ and therefore to get rid of the integration over ν. We obtain:

$$\Phi_n(t,r,\theta) = e^{in\theta}\sqrt{\frac{1-\mu}{\pi}}\exp\left\{-\big(1+2it(1-\mu)\big)\log r - \frac{1}{1-\mu}\log\frac{\Gamma\big(\frac{1+|n|}{2}+i(1-\mu)t\big)}{\Gamma\big(\frac{1+|n|}{2}-i(1-\mu)t\big)}\right\}. \tag{34}$$

The classical limit of the wave function corresponds to large angular momenta, i.e., large $|n|$. In this limit, the phase is stationary for:

$$(1-\mu)\theta = \tan^{-1}\frac{2(1-\mu)t}{|n|}, \qquad \left(\frac{z}{z_0}\right)^{1-\mu} = 1 + i\frac{2(1-\mu)t}{|n|}, \tag{35}$$

where we have reintroduced the complex coordinate vector z ($r^2 = z\bar{z}$), and z_0 is its value for $t=0$. Eq. (35) give the classical trajectory for the effective particle which agrees with previous results [9,17]. One sees that in the conformal limit of small $1-\mu$, the angle θ grows linearly with time while the radius r is fixed: we have a circular motion of body 2 around body 1.

5. Outlook

The preceding analysis shows that in the two body case, the Hamiltonian of both $(2+1)$-gravity and the high-energy model are related to a basic scale of the problem which, in the conformal limit, takes the form $\sim \log|p|^2|z|^2$. Thus canonical quantization, with proper ordering of operators, leads to analogous features, even when scaling violations $\mathrm{O}(1-\mu)$ are turned on.

A natural question is whether this analogy is kept in the many-body case. In the conformal limit of the high-energy model, some exact three-body solutions are known (the “odderon” [21,28]). Much less is known in the case of running coupling (or scaling violations). On the other hand, the general structure of the Hamiltonian is known [19] in $(2+1)$-gravity too, but becomes tractable in the small speed limit only [17]. Furthermore, the conformal limit $\mu \to 1$ is ambiguous, in the sense that it depends on the mass parameters μ_i also, and it is conceivable that the high-energy model may correspond to some special mass configuration. Therefore, a deeper analysis is needed in order to understand whether the amusing correspondence found here survives in the general case.

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