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Advances in Mathematics 224 (2010) 1784–1800

ADVANCES IN Mathematics

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Spherical homogeneous spaces of minimal rank

N. Ressayre [1]

Université Montpellier II, CC 51-Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

Received 22 February 2008; accepted 25 January 2010

Available online 24 February 2010

Communicated by David Ben-Zvi

Abstract

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let G/B denote the complete flag variety of G. A G-homogeneous space G/H is said to be *spherical* if H has finitely many orbits in G/B. A class of spherical homogeneous spaces containing the tori, the complete homogeneous spaces and the group G (viewed as a $G \times G$-homogeneous space) has particularly nice properties. Namely, the pair (G, H) is called a *spherical pair of minimal rank* if there exists x in G/B such that the orbit $H.x$ of x by H is open in G/B and the stabilizer H_x of x in H contains a maximal torus of H. In this article, we study and classify the spherical pairs of minimal rank.

Keywords: Spherical varieties; Schubert varieties and generalizations

1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $\mathcal{B}$ denote the complete flag variety of G. Let H be an algebraic subgroup of G which acts on $\mathcal{B}$ with finitely many orbits; the subgroup H and the homogeneous space G/H are said to be *spherical*.

In this article, we study and classify a class of spherical homogeneous spaces containing the tori, the complete homogeneous spaces and the group G viewed as a $G \times G$-homogeneous space. Namely, the pair (G, H) is called a *spherical pair of minimal rank* if there exists x in $\mathcal{B}$ such that the orbit $H.x$ of x by H is open in $\mathcal{B}$ and the stabilizer H_x of x in H contains a maximal

E-mail address: ressayre@math.univ-montp2.fr.

[1] The author was partially supported by the French National Research Agency (ANR-09-JCJC-0102-01).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.01.014

torus of H. In [7], the rank $\mathrm{rk}(G/H)$ of the homogeneous space G/H is defined. Moreover, for any spherical subgroup H of G, we have $\mathrm{rk}(G/H) \geqslant \mathrm{rk}(G) - \mathrm{rk}(H)$ (where $\mathrm{rk}(G)$ and $\mathrm{rk}(H)$ denote the ranks of the groups G and H) with equality if and only if (G, H) is of minimal rank. This explains the name. The spherical pairs (G, H) of minimal rank such that H is a symmetric subgroup of G have been introduced in [5]. During the redaction of this article the compactifications of the spherical homogeneous spaces of minimal rank have been studied in [14] and [1].

Let us state our main result. Propositions 3.1, 3.2 and 4.2 reduce the classification to the special case when G is semisimple adjoint and H is simple. Indeed, any spherical pair of minimal rank is obtained from special ones and toric ones by products, finite covers and parabolic inductions. Next, we prove

Theorem A. *Up to conjugacy by an automorphism of G, the spherical pairs (G, H) of minimal rank with G semisimple adjoint and H simple are:*

(i) $G = H$.
(ii) *H is simple and diagonally embedded in $G = H \times H$.*
(iii) $(\mathrm{PSL}_{2n}, \mathrm{PSp}_{2n})$ *with* $n \geqslant 2$.
(iv) $(\mathrm{PSO}_{2n}, \mathrm{SO}_{2n-1})$ *with* $n \geqslant 4$.
(v) (SO_7, G_2).
(vi) (E_6, F_4).

We denote by $\mathbf{H}(\mathcal{B})$ the set of H-orbit closures in $\mathcal{B}$. If $H = P$ is a parabolic subgroup of G, the elements of $\mathbf{H}(\mathcal{B})$ are the famous Schubert varieties. Most of combinatorial and geometric properties of the Schubert varieties cannot be generalized to the elements of $\mathbf{H}(\mathcal{B})$ if H is only assumed to be spherical. However, if H has minimal rank the elements of $\mathbf{H}(\mathcal{B})$ have nice properties. Let us give details.

The Weyl group W of G acts transitively on the set of Schubert varieties; this action parametrizes these varieties by W/W_P. In general, F. Knop has defined in [7] an action of W in $\mathbf{H}(\mathcal{B})$; but, it seems to be difficult to deduce a parametrization of $\mathbf{H}(\mathcal{B})$ from this action. We show in Proposition 2.1 that G/H is of minimal rank if and only if the action of W is transitive on $\mathbf{H}(\mathcal{B})$. In this case, the isotropy groups are isomorphic to the Weyl group W_H of H; and W/W_H parametrizes $\mathbf{H}(\mathcal{B})$.

The Schubert varieties are normal; but, in general elements of $\mathbf{H}(\mathcal{B})$ are not normal (see [4] or [8] for examples). By a result of Brion, if G/H is of minimal rank, the elements of $\mathbf{H}(\mathcal{B})$ are normal.

In [7], F. Knop also defined an action of a monoid $\tilde{W}$ (constructed from the generators of W) on $\mathbf{H}(\mathcal{B})$. Moreover, the inclusion defines an order on $\mathbf{H}(\mathcal{B})$ which generalizes the Bruhat order for the Schubert varieties. The description of the Bruhat order from the action of $\tilde{W}$ is well known as the cancellation lemma. In general, no such description of this order is known. Corollary 2.1 is a cancellation lemma in the minimal rank case.

The number of Schubert varieties of dimension d equals the number of those of the codimension d. In Proposition 2.3 we show such a symmetry property of $\mathbf{H}(\mathcal{B})$ for any spherical pair (G, H) of minimal rank.

Let us explain another important motivation for this work. Let T be a maximal torus of G and X be a G-equivariant embedding of a spherical homogeneous space G/H of minimal rank. In Proposition 2.4, we show that for all fixed points x of T in X, $G.x$ is complete. This property seems to play a key role in several works about the embeddings of $G \times G/G$ (see for example [13]). We also prove a stability property for the set of spherical homogeneous spaces of minimal rank: any orbit in a toroidal embedding of a spherical homogeneous space of minimal rank is spherical of minimal rank.

In Section 2, we study the properties of $\mathbf{H}(\mathcal{B})$ and of the toroidal embeddings of G/H for the spherical pairs (G, H) of minimal rank. This allows us to give several characterizations of the minimal rank property. In Section 3, we reduce the classification to the case when G and H are semisimple. In Section 4, we classify such pairs by associating to (G, H) an involution on the vertexes of the Dynkin diagram of G.

2. Equivalent definitions and first properties

2.1. Minimal rank and orbits of H in $\mathcal{B}$

2.1.1. Let us fix some general notation. If X is a variety, $\dim(X)$ denotes the dimension of X. If x belongs to X, T_xX denotes the Zariski tangent space of X at x. If Γ is an algebraic group a *Γ-variety* X is a variety endowed with an algebraic action of Γ. Let Γ be an affine algebraic group and X be a Γ-variety. For x a point in X, we denote by Γ_x the isotropy group of x and by $\Gamma.x$ its orbit. We denote by X^Γ the set of fixed points of Γ in X. We denote by Γ^u the unipotent radical of Γ.

2.1.2. Let us recall that G is a connected reductive group, $\mathcal{B}$ its complete flag variety, H a spherical subgroup of G and $\mathbf{H}(\mathcal{B})$ the set of H-orbit closures in $\mathcal{B}$. If V belongs to $\mathbf{H}(\mathcal{B})$, we denote by V° the unique open H-orbit in V.

We recall the definition given in [9] of a graph $\Gamma(G/H)$ whose vertexes are the elements of $\mathbf{H}(\mathcal{B})$. The original construction of $\Gamma(G/H)$ due to M. Brion is equivalent but slightly different (see [4]).

Consider the set Δ of conjugacy classes of minimal non-solvable parabolic subgroups of G. If α belongs to Δ, we denote by $\mathcal{P}_\alpha$ the G-homogeneous space with isotropy α. Then, there exists a unique G-equivariant map $\phi_\alpha : \mathcal{B} \longrightarrow \mathcal{P}_\alpha$ which is a $\mathbb{P}^1$-bundle.

Let $V \in \mathbf{H}(\mathcal{B})$ and $\alpha \in \Delta$. We assume that the restriction of ϕ_α to V° is finite and we denote its degree by $d(V, \alpha)$. Then, $\phi_\alpha^{-1}(\phi_\alpha(V))$ is an element denoted V' of $\mathbf{H}(\mathcal{B})$; in this case, we say that α *raises V to V'*. One of the three following cases occurs.

- Type U: H has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V^\circ))$ (V° and V'°) and $d(V, \alpha) = 1$.
- Type T: H has three orbits in $\phi_\alpha^{-1}(\phi_\alpha(V^\circ))$ and $d(V, \alpha) = 1$.
- Type N: H has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V^\circ))$ (V° and V'°) and $d(V, \alpha) = 2$.

Definition. Let $\Gamma(G/H)$ be the oriented graph with vertexes the elements of $\mathbf{H}(\mathcal{B})$ and edges labeled by Δ, where V is joined to V' by an edge labeled by α if α raises V to V'. This edge is simple (resp. double) if $d(V, \alpha) = 1$ (resp. 2). Following the above cases, we say that an edge has *type U, T or N*.

2.1.3. Let us fix a Borel subgroup B of G. Let Y be a B-variety. The *character group $\mathcal{X}(Y)$* of Y is the set of characters of B that arise as weights of eigenvectors of B in the function field $\mathbb{K}(Y)$. Then $\mathcal{X}(Y)$ is a free abelian group of finite rank $\mathrm{rk}(Y)$, *the rank of Y* (see [7]). It is well known that a B-orbit $\mathcal{O}$ is isomorphic as a variety to $\mathbb{K}^l \times (\mathbb{K}^*)^r$ where $r = \mathrm{rk}(\mathcal{O})$ and $l = \dim(\mathcal{O}) - \mathrm{rk}(\mathcal{O})$.

If V belongs to $\mathbf{H}(\mathcal{B})$, we set:

$$V_H = \{gH \in G/H\colon\ g^{-1}B \in V\}.$$

Then, V_H is a B-orbit closure in G/H. Moreover, the map $V \longmapsto V_H$ is a bijection from $\mathbf{H}(\mathcal{B})$ onto the set of the B-orbit closures in G/H. The rank of V_H is also denoted by $\mathrm{rk}(V)$ and called *the rank of V*.

2.1.4. Let T be a maximal torus of B. Let W denote the Weyl group of T. Every α in Δ has a unique representative P_α which contains B. Moreover, there exists a unique s_α in W such that $Bs_\alpha B$ is dense in P_α; and this s_α is a simple reflexion of W. The map $\Delta \longrightarrow W$, $\alpha \longmapsto s_\alpha$ is a bijection from Δ onto the set of simple reflexions of W.

F. Knop defined in [7] an action of W on the set $\mathbf{H}(\mathcal{B})$ by describing the action of the s_α's, for any $\alpha \in \Delta$:

- Type U: s_α exchanges the two vertexes of an edge of type U labeled by α.
- Type T: If α raises V_1 and V_2 to V, then $s_\alpha V_1 = V_2$ and $s_\alpha V = V$.
- Type N: s_α fixes the two vertexes of a double edge labeled by α.
- s_α fixes all other vertexes of $\Gamma(G/H)$.

2.1.5. We can now characterize the spherical pairs of minimal rank in terms of $\mathbf{H}(\mathcal{B})$:

Proposition 2.1. *With the above notation, the following are equivalent:*

(i) *There exists $x \in \mathcal{B}$ such that $H.x$ is open in $\mathcal{B}$ and H_x contains a maximal torus of H.*
(ii) $\mathrm{rk}(G) - \mathrm{rk}(H) = \mathrm{rk}(G/H)$.
(iii) *All the elements of $\mathbf{H}(\mathcal{B})$ have the same rank.*
(iv) *All the edges in $\Gamma(G/H)$ have type U.*
(v) *W acts transitively on $\Gamma(G/H)$.*

If (G, H) satisfies these properties, we say that (G, H) is of minimal rank.

Proof. The equivalence between the two first assertions follows from [10, Corollary 3.1].

Let us recall some properties of the graph $\Gamma(G/H)$ from [4]. If α raises V to V' by an edge of type U (resp. T or N) then $\mathrm{rk}(V') = \mathrm{rk}(V)$ (resp. $\mathrm{rk}(V') = \mathrm{rk}(V) + 1$). Moreover, any V in $\mathbf{H}(\mathcal{B})$ is joined to $\mathcal{B}$ by an increasing path in $\Gamma(G/H)$ (property of connectedness). Finally, the rank of a closed H-orbit in $\mathcal{B}$ equals $\mathrm{rk}(G) - \mathrm{rk}(H)$.

Now, one easily checks the equivalence between assertions (ii), (iii), (iv) and (v). □

2.1.6. Let (G, H) be a spherical pair of minimal rank. Then, the elements of $\mathbf{H}(\mathcal{B})$ can be parametrized. Indeed, let W_0 be the stabilizer of $\mathcal{B}$ for the action of W. In [10], it is shown that W_0 is isomorphic to the Weyl group W_H of H. Moreover, Proposition 2.1 shows that Knop's action gives a bijection between W/W_0 and $\mathbf{H}(\mathcal{B})$. In particular, we have: $|\mathbf{H}(\mathcal{B})| = \frac{|W|}{|W_H|}$, where $|E|$ denotes the cardinality of the finite set E.

Each orbit closure V of H in $\mathcal{B}$ is multiplicity-free in the sense of [4, p. 284]. In particular, by [4, Theorem 5] V is normal.

2.1.7. In this paragraph, we are interested in reading the generalized Bruhat order off the graph $\Gamma(G/H)$. Let us start by showing the following nice property of this graph:

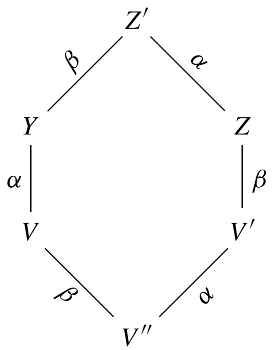
Proposition 2.2. *There exists a unique closed orbit of H in $\mathcal{B}$ and it is the only minimal element of $\Gamma(G/H)$.*

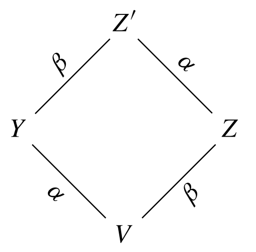
Proof. Let V_0 be a closed orbit of H in $\mathcal{B}$. Let $\mathbf{H}_0$ be the set of H-orbit closures in $\mathcal{B}$ linked with V_0 by an oriented path in $\Gamma(G/H)$. It is sufficient to prove that $\mathbf{H}_0 = \mathbf{H}(\mathcal{B})$.

We assume that $\mathbf{H}(\mathcal{B}) - \mathbf{H}_0$ is not empty. Since $\mathcal{B}$ belongs to $\mathbf{H}_0$ and all the orbits are joined to $\mathcal{B}$ by an oriented path, there exist $Z \in \mathbf{H}(\mathcal{B}) - \mathbf{H}_0$ and $\alpha \in \Delta$ such that α raises Z to an element Z' of $\mathbf{H}_0$ (it is sufficient to take Z of maximal dimension in $\mathbf{H}(\mathcal{B}) - \mathbf{H}_0$). Let us fix such a pair (Z, α) such that Z is of minimal dimension.

Since $Z' \neq V_0$, there exist $\beta \in \Delta$ and $Y \in \mathbf{H}_0$ such that β raises Y to Z'. Since the edges of $\Gamma(G/H)$ are of type U and $Y \neq Z$, we have $\beta \neq \alpha$.

Using [4, Lemma 3], one easily checks that one of the two following graphs is a subgraph of $\Gamma(G/H)$:





In the first case, Z, V' and V'' do not belong to $\mathbf{H}_0$. By minimality of the dimension of Z and by considering (V'', β) we deduce that V does not belong to $\mathbf{H}_0$. Now, the pair (V, α) contradicts the minimality of the dimension of Z. A similar argument works in the second case. $\square$

Let V be in $\mathbf{H}(\mathcal{B})$ and V_0 denote the unique closed H-orbit $\mathcal{B}$. By Proposition 2.2, there exists an increasing path in $\Gamma(G/H)$ from V_0 to V. Let $(\alpha_1, \ldots, \alpha_s)$ be the sequence of the labels of the edges of such a path. Notice that $s = \dim(V) - \dim(V_0)$. The inclusion relation in $\mathbf{H}(\mathcal{B})$ can be read off the graph $\Gamma(G/H)$ by the following cancellation corollary:

Corollary 2.1. *We use the above notation and fix V' in $\mathbf{H}(\mathcal{B})$. Set $k = \dim(V') - \dim(V_0)$. Then, $V' \subset V$ if and only if there exist $i_1 < \cdots < i_k$ such that the increasing path starting from V_0 and with labels $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ ends at V'.*

Proof. Using Proposition 2.2, the proof of [12] works here. $\square$

2.1.8. Let d_G (resp. d_H) denote the dimension of the complete flag variety of G (resp. H). Then, we have the following "symmetry" on the set $\mathbf{H}(\mathcal{B})$:

Proposition 2.3. *Here we assume that H is connected. For all $0 \leqslant \delta \leqslant d_G - d_H$, the number of elements in $\mathbf{H}(\mathcal{B})$ of dimension $d_G - \delta$ equals the number of those of dimension $d_H + \delta$.*

Proof. Consider $P_G(t)$ and $P_H(t)$ the Poincaré polynomials of the complete flag varieties of G and H. By Poincaré duality, they are symmetric polynomials of degrees d_G and d_H; that is, $t^{d_G} P_G(1/t) = P_G(t)$ and $t^{d_H} P_H(1/t) = P_H(t)$.

Consider the following polynomial:

$$Q(t) = \sum_{V \in \mathbf{H}(\mathcal{B})} t^{\dim(V) - d_H}.$$

We claim that $Q(t).P_H(t) = P_G(t)$. The claim implies that $Q(t)$ is symmetric and so the proposition.

Let $\mathcal{B}_H$ denote the complete flag variety of H. For any $x \in \mathcal{B}$, H_x is a solvable subgroup of H containing a maximal torus of H. It follows that H_x is contained in a Borel subgroup of H: consider $\varphi_x : H.x \longrightarrow \mathcal{B}_H$ the map induced by this inclusion. The fiber $\varphi_x^{-1}(\varphi_x(x))$ is homogeneous under the Borel subgroup B_x of H corresponding to $\varphi_x(x)$. Since x is fixed by a maximal torus of B_x, it follows that $\varphi_x^{-1}(\varphi_x(x))$ is isomorphic to an affine space. Moreover, this space has dimension $\dim(H.x) - d_H$.

We choose one point in each orbit of H in $\mathcal{B}$ and consider the associated morphisms φ_x. There exists a finitely generated extension K of $\mathbb{Q}$ such that G, H, the inclusion of H in G, the chosen points in $\mathcal{B}$, the morphisms φ_x, the isomorphisms between the fibers of the φ_x's and corresponding affine spaces are all defined. By taking an extension if necessary, we may (and shall) also assume that the Schubert cells (for fixed Borel subgroups of G and H) of $\mathcal{B}$ and $\mathcal{B}_H$ are defined and isomorphic to affine spaces over K.

Now, we consider a finite quotient $\mathbb{F}_q$ of K and the points $\mathcal{B}(\mathbb{F}_{q^n})$ of $\mathcal{B}$ over $\mathbb{F}_{q^n}$ for all positive integers n. By using the decompositions of $\mathcal{B}$ and $\mathcal{B}_H$ in Schubert cells, one obtains:

$$\left|\mathcal{B}(\mathbb{F}_{q^n})\right| = P_G\left(q^n\right) \quad \text{and} \quad \left|\mathcal{B}_H(\mathbb{F}_{q^n})\right| = P_H\left(q^n\right).$$

Now, we count the points in $\mathcal{B}(\mathbb{F}_{q^n})$ by using the decomposition in H-orbits:

$$\left|\mathcal{B}(\mathbb{F}_{q^n})\right| = \sum_{V \in \mathbf{H}(\mathcal{B})} \left|V^\circ(\mathbb{F}_{q^n})\right| = \sum_{V \in \mathbf{H}(\mathcal{B})} \left|\mathcal{B}_H(\mathbb{F}_{q^n})\right|.\left(q^n\right)^{\dim(V) - d_H} = P_H\left(q^n\right).Q\left(q^n\right).$$

The claim follows. $\square$

2.2. Minimal rank and toroidal embeddings

2.2.1. In this subsection, (G, H) is a spherical pair not necessarily of minimal rank. An *embedding of G/H* is a pair (X, x) where X is a normal and irreducible G-variety and x is a point of X such that $G.x$ is open in X and $G_x = H$. Such an embedding is said to be *toroidal* if any irreducible B-stable divisor of X which contains a G-orbit is G-stable.

Lemma 2.1. *Let G/H be a spherical homogeneous space (not necessarily of minimal rank). Let (X, x) be a toroidal embedding of G/H and y be a point in X.*

Then, we have the inequality:

$$\mathrm{rk}(G/H) + \mathrm{rk}(H) \geqslant \mathrm{rk}(G.y) + \mathrm{rk}(G_y).$$

In particular, if G/H is of minimal rank, $G.y$ is.

Proof. Firstly, we prove that it is sufficient to show the lemma when $\dim(G.y) = \dim(X) - 1$. By [2, Lemma 2.1.1], since X is toroidal, there exist G-orbits $\mathcal{O}_0, \dots, \mathcal{O}_s$ such that $G.y = \mathcal{O}_0 \subset \overline{\mathcal{O}_1} \subset \cdots \subset \overline{\mathcal{O}_s} = X$ (where $\overline{\mathcal{O}_i}$ denotes the closure of $\mathcal{O}_i$ in X) and $\dim(\mathcal{O}_{i+1}) = \dim(\mathcal{O}_i) + 1$ for all $i = 0, \dots, s-1$. For each i, we fix a point y_i in $\mathcal{O}_i$. Since $\overline{\mathcal{O}_i}$ is normal, we can apply the case when $\dim(G.y) = \dim(X) - 1$ to each $\mathcal{O}_i \subset \overline{\mathcal{O}_{i+1}}$ showing that

$$\mathrm{rk}(G.y_{i+1}) + \mathrm{rk}(G_{y_{i+1}}) \geqslant \mathrm{rk}(G.y_i) + \mathrm{rk}(G_{y_i}).$$

The inequality of the lemma follows.

We now assume that $\dim(G.y) = \dim(X) - 1$. Set $\mathcal{O} = G.y$. Consider the linear action of the group G_y on the quotient $T_yX/T_y\mathcal{O}$ of T_yX by $T_y\mathcal{O}$. Since X is normal, it is smooth at y and $T_yX/T_y\mathcal{O}$ is a line. So, the action of G_y defines a character $\chi : G_y \longrightarrow \mathbb{K}^*$.

Let T_y be a maximal torus of G_y. Let S denote the identity component of the kernel of the restriction of χ to T_y. We claim that S has fixed points in $G.x$. Set $\Omega = G.x \cup \mathcal{O}$; it is open in X and hence it is a smooth variety. By a result of Białynicki–Birula, we have $T_y(\Omega^S) = (T_y\Omega)^S$. In particular, Ω^S is not contained in $\mathcal{O}$. This proves the claim.

By the claim, a subgroup conjugated to S fixes x and

$$\mathrm{rk}(H) \geqslant \dim(S) = \dim(T_y) - 1 = \mathrm{rk}(G_y) - 1.$$

Moreover, since X is toroidal $\mathrm{rk}(G/H) = \mathrm{rk}(\mathcal{O}) + 1$. The lemma follows. □

2.2.2. The fixed points of a maximal torus of G in the toroidal embeddings of spherical homogeneous spaces of minimal rank are easy to localize. Indeed, we have:

Proposition 2.4. *Let (G, H) be a spherical pair and T be a maximal torus of G. The following are equivalent:*

(i) *(G, H) is of minimal rank.*
(ii) *There exists a complete toroidal embedding (X, x) of G/H such that for any $x \in X^T$ $G.x$ is complete.*
(iii) *For any complete toroidal embedding (X, x) of G/H and for any $x \in X^T$, $G.x$ is complete.*

Proof. We assume that (G, H) is of minimal rank and fix a complete toroidal embedding (X, x) of G/H. Let $y \in X^T$. Lemma 2.1 shows that $\mathrm{rk}(G.y) = 0$; that is (see for example [3, Corollaire 1]), $G.y$ is complete. This proves that assertion (i) implies assertion (iii).

Conversely, let (X, x) satisfy assertion (ii). It remains to prove that (G, H) is of minimal rank.

Let λ be a one-parameter subgroup of T such that T is the centralizer of the image of λ (that is, λ is regular) and $X^\lambda = X^T$ (where X^λ denote the set of fixed points of the image of λ). Since

λ is regular, the set of $g \in G$ such that $\lim_{t\to 0}\lambda(t)g\lambda(t^{-1})$ exists in G is a Borel subgroup of G denoted by $B(\lambda)$. By Proposition 2.1, it is sufficient to prove that for any $y \in G.x$ we have $\mathrm{rk}(B(\lambda).y) = \mathrm{rk}(G/H)$. This holds by Lemma 2.2 below since the rank of a complete G-orbit equals zero. □

Lemma 2.2. *Let (X, x) be a complete toroidal embedding of the spherical homogeneous space G/H. Let y be a point in the open G-orbit in X. Let λ be a regular one-parameter subgroup of T such that $X^\lambda = X^T$. Set $z = \lim_{t\to 0}\lambda(t)y$.*

Then, we have

$$\mathrm{rk}(G/H) - \mathrm{rk}\big(B(\lambda).y\big) = \mathrm{rk}(G.z).$$

Proof. Let us introduce some material and notation from [2]. There exists a parabolic subgroup P of G containing T such that P_z is reductive. Let Q denote the parabolic subgroup of G containing T and opposite to P. We have $G_z^u \subset Q^u$ and there exists a closed subvariety $A \subset Q^u$, P_z-stable, such that the product in G induces an isomorphism $A \times Q_z^u \longrightarrow Q^u$. By [2, Lemma 1.1], there exists a locally closed affine normal and irreducible subvariety S of X such that $S \cap G.z = \{z\}$, S is P_z-stable and the morphism $G \times S \longrightarrow X$ induced by the action is smooth at $(1, z)$. In particular, we have

$$\dim(S) = \dim(G/H) - \dim(G.z) = \mathrm{rk}(G/H) - \mathrm{rk}(G.z). \tag{1}$$

Let $P \times_{P_z} (A \times S)$ denote the quotient of $P \times A \times S$ by the action of P_z defined by $p.(q, a, s) = (qp^{-1}, pap^{-1}, ps)$, where $p \in P_z$, $(q, a, s) \in P \times A \times S$. The group P acts naturally on this variety. By [2, Theorem 5], the morphism

$$\begin{aligned} \Theta : P \times_{P_z} (A \times S) &\longrightarrow X \\ \big(p : (a, s)\big) &\longmapsto (pa).s \end{aligned}$$

is an open immersion.

Consider the Białynicki–Birula cell

$$X(\lambda, z) = \Big\{p \in X\colon \lim_{t\to 0}\lambda(t)p = z\Big\}.$$

Notice that $y \in X(\lambda, z)$. By [2, Propositions 2.1 and 2.3], $X(\lambda, z) \cap G.x = B(\lambda).y$ and $G.x \cap S = T.y'$ for some $y' \in B(\lambda).y$. Then, the proof of [2, Proposition 2.3] shows that Θ induces by restriction an isomorphism:

$$\big(P \cap B(\lambda)\big) \times_{B(\lambda)_z} \big(\big(A \cap B(\lambda)\big) \times T.y'\big) \longrightarrow B(\lambda).y.$$

Since T is contained in $B(\lambda)_z$, this isomorphism implies that

$$\mathrm{rk}\big(B(\lambda).y\big) = \dim\big(T.y'\big) = \dim S. \tag{2}$$

The lemma follows from equalities (1) and (2). □

3. Reduction to the case when G and H are semisimple

The goal of this section is to reduce the classification of the spherical pairs (G, H) to those with G semisimple adjoint and H semisimple.

Proposition 3.1. *Recall that G is a connected reductive group and H is a subgroup of G. Then, (G, H) is a spherical pair of minimal rank if and only if there exist a parabolic subgroup P of G with a Levi decomposition $P = P^u L$ and a reductive subgroup K of L such that:*

(i) $H = P^u K$,
(ii) (L, K) *is a spherical pair of minimal rank.*

Proof. We first assume that $H = P^u K$ with P, L and K as in the statement. Let $\mathcal{B}_L$ denote the variety of Borel subgroups of L. The variety $\mathcal{B}$ contains a P-stable open subset isomorphic to $P^u \times \mathcal{B}_L$. Let x in $\mathcal{B}_L$ be such that K_x contains a maximal torus of K and such that $K.x$ is dense in $\mathcal{B}_L$. One easily checks that $H.x$ is open in $\mathcal{B}$ and H_x contains a maximal torus of H. So, (G, H) is a spherical pair of minimal rank.

Conversely, let us assume that (G, H) is a spherical pair of minimal rank. We can write $H = H^u K$, where K is a reductive subgroup of H. But, by [6, 30.3], there exists a parabolic subgroup $P = P^u L$ of G such that $H^u \subseteq P^u$ and $K \subseteq L$. We claim that P and L satisfy the proposition.

Let us firstly prove that $P^u = H^u$.

Let $T(H)$ be a maximal torus of K (and hence of H). The variety $\mathcal{B}$ contains an open subset stable by P (and hence by H) isomorphic to $P^u \times \mathcal{B}_L$. By assumption, there exists a point x in $\mathcal{B}$ fixed by $T(H)$ such that $H.x$ is open in $\mathcal{B}$. But, $x = (u, y)$ belongs to $P^u \times \mathcal{B}_L$. Since the H-orbit of x is open in $P^u \times \mathcal{B}_L$, so is its intersection with $P^u \times \{y\}$. Hence, the set of $hkuk^{-1} \in P^u$ such that $h \in H^u$ and $k \in K_y$ is open and dense in P^u. In particular, the K_y-orbit $K_y.(uH^u)$ is open and dense in P^u/H^u.

Since $x \in \mathcal{B}^{T(H)}$, $uH^u \in (P^u/H^u)^{T(H)}$. But, K_y is a solvable group with $T(H)$ as maximal torus. So, $K_y^\circ.uH^u$ is one orbit of the unipotent radical K_y^u of K_y°. In particular, it is closed in the affine variety P^u/H^u. But it is also open. We deduce that K_y acts transitively on P^u/H^u.

But K_y^u is contained in K and normalizes H^u. So, H^u is a fixed point of K_y in P^u/H^u. We deduce that P^u/H^u has only one point; that is, that $P^u = H^u$.

On the other hand, $K.y$ is open in $\mathcal{B}_L$ and y is fixed by the maximal torus $T(H)$ of K. We deduce that (L, K) is a spherical pair of minimal rank. □

Since the parabolic subgroups of a given reductive group are very well known, Proposition 3.1 reduces the problem of classification of the spherical pairs (G, H) of minimal rank to the case when H is reductive.

Proposition 3.2. *Let G be a connected reductive group. Set $G_{\mathrm{ad}} = G/Z(G)$ and consider the projection $p : G \longrightarrow G_{\mathrm{ad}}$. Let H be a reductive subgroup of G. Then:*

(i) *The pair (G, H) is spherical of minimal rank if and only if the pair $(G_{\mathrm{ad}}, p(H))$ is.*
(ii) *The pair (G, H) is spherical of minimal rank if and only if the pair (G, H°) is.*
(iii) *If G/H is of minimal rank, the identity component $p(H)^\circ$ of $p(H)$ is semisimple.*

Proof. Assertions (i) and (ii) are obvious from assertion (i) of Proposition 2.1.

To prove the last assertion, it is sufficient to prove that the connected center S of H is contained in the center of G. There exists $x \in \mathcal{B}$ fixed by S such that $H.x$ is open in $\mathcal{B}$. Since, $H \subset G^S$, $G^S.x$ is open in $\mathcal{B}$. But, $G^S.x$ is an irreducible component of $\mathcal{B}^S$ by [11]. Therefore, $\mathcal{B}^S = \mathcal{B}$ and S is central in G. $\square$

Proposition 3.2 reduces the problem of classification of the spherical pairs (G, H) of minimal rank to the case when G is semisimple adjoint and H is semisimple. From now on, we only consider such pairs.

4. Classification of Lie algebras

Let (G, H) be a spherical pair of minimal rank with G semisimple adjoint and H semisimple. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of G (resp. H).

4.1. Root systems of $\mathfrak{g}$ *and* $\mathfrak{h}$

Let $T(H)$ be a maximal torus of H. Let $T \supset T(H)$ be a maximal torus of G. Let $\mathcal{X}(T) = \mathrm{Hom}(T, \mathbb{K}^*)$ (resp. $\mathcal{X}(T(H)) = \mathrm{Hom}(T(H), \mathbb{K}^*)$) denote the character group of T (resp. $T(H)$). Let $\phi_{\mathfrak{g}} \subset \mathcal{X}(T)$ (resp. $\phi_{\mathfrak{h}} \subset \mathcal{X}(T(H))$) be the set of roots of $\mathfrak{g}$ (resp. $\mathfrak{h}$). Let $\rho : \mathcal{X}(T) \longrightarrow \mathcal{X}(T(H))$ be the restriction map.

In this subsection, we will prove some very constraining relations between $\phi_{\mathfrak{g}}$, $\phi_{\mathfrak{h}}$ and ρ.

4.1.1. A reduction

The following stability of the set of spherical pairs of minimal rank will be used to localize the study over some fixed roots of $\mathfrak{h}$:

Lemma 4.1. *Let S be a subtorus of H.*
Then, (G^S, H^S) is a spherical pair of minimal rank.

Proof. Let $T(H)$ be a maximal torus of H which contains S. Let x be a fixed point of $T(H)$ in $\mathcal{B}$ such that $V := H.x$ is open in $\mathcal{B}$. Since $V \cap G^S.x$ is open in $G^S.x$, it is irreducible. So, it is an irreducible component of V^S. Now, [11, Theorem A] implies that $V \cap G^S.x = H^S.x$. In particular, $H^S.x$ is open in $G^S.x \simeq \mathcal{B}_{G^S}$ and x is fixed by the maximal torus $T(H)$ of H^S. The lemma follows. $\square$

Lemma 4.2. *With the above notation, we have $\rho(\phi_{\mathfrak{g}}) = \phi_{\mathfrak{h}}$.*

Proof. Let $\alpha \in \phi_{\mathfrak{g}}$. Set $S = \mathrm{Ker}(\rho(\alpha))^\circ \subset T(H)$. By Lemma 4.1, H^S is a spherical subgroup of G^S and $\mathrm{rk}(G^S/H^S) = \mathrm{rk}(G^S) - \mathrm{rk}(H^S)$. Since the semisimple rank of G^S is one, this implies that $H^{S\circ}$ is not a torus. So, $\rho(\alpha)$ is a root of $\mathfrak{h}$.

Moreover, since $\mathfrak{h} \subset \mathfrak{g}$, $\phi_{\mathfrak{h}} \subset \rho(\phi_{\mathfrak{g}})$. $\square$

By Lemma 4.2, we can define the map $\rho_\phi : \phi_{\mathfrak{g}} \longrightarrow \phi_{\mathfrak{h}}, \alpha \longmapsto \rho(\alpha)$.

Lemma 4.3. *The spherical pairs (G, H) of minimal rank with G semisimple adjoint, H connected and $\mathfrak{h} = \mathfrak{sl}_2$ are:*

(i) $(\mathrm{PSL}_2, \mathrm{PSL}_2)$.
(ii) PSL_2 *diagonally embedded in* $\mathrm{PSL}_2 \times \mathrm{PSL}_2$.

Proof. By assertion (i) of Proposition 2.1, the dimension of $\mathcal{B}$ is at most 2. We deduce that $G = \mathrm{PSL}_2$ or $\mathrm{PSL}_2 \times \mathrm{PSL}_2$. The lemma follows easily. $\square$

Lemma 4.4. *Let* $\alpha \in \phi_{\mathfrak{h}}$.

Then, $\rho_\phi^{-1}(\alpha)$ *contains either one root of* $\mathfrak{g}$ *or two orthogonal roots of* $\mathfrak{g}$. *Moreover, if* $\rho_\phi^{-1}(\alpha) = \{\alpha^\circ\}$ *then* $\mathfrak{h}_\alpha = \mathfrak{g}_{\alpha^\circ}$*; and if* $\rho_\phi^{-1}(\alpha) = \{\alpha^-, \alpha^+\}$ *with* $\alpha^- \neq \alpha^+ \in \phi_{\mathfrak{g}}$ *then* $\mathfrak{h}_\alpha \neq \mathfrak{g}_{\alpha^\pm}$.

Proof. Set $S = \ker(\alpha)^\circ$. Since $\mathfrak{h}^S$ has semisimple rank one and by Lemma 4.1, we can apply Lemma 4.3 to $(G^S/S, H^S/S)$. The lemma follows immediately. $\square$

Lemma 4.4 divides the set of roots of $\mathfrak{h}$ in two parts:

$$\phi_{\mathfrak{h}}^1 := \bigl\{\alpha \in \phi_{\mathfrak{h}}\colon \bigl|\rho_\phi^{-1}(\alpha)\bigr| = 1\bigr\} \quad \text{and} \quad \phi_{\mathfrak{h}}^2 := \bigl\{\alpha \in \phi_{\mathfrak{h}}\colon \bigl|\rho_\phi^{-1}(\alpha)\bigr| = 2\bigr\}.$$

We denote by W_H the Weyl group $N_H(T(H))/T(H)$ of H.

Lemma 4.5. *The sets* $\phi_{\mathfrak{h}}^1$ *and* $\phi_{\mathfrak{h}}^2$ *are stable by the action of* W_H.

Proof. By Lemma 4.1, $(G^{T(H)}, T(H))$ is a spherical pair of minimal rank. So, $G^{T(H)}$ is a torus and $G^{T(H)} = T$. In particular, $N_H(T(H))$ is contained in $N_G(T)$; this inclusion induces an injection of $W_H = N_H(T(H))/T(H)$ into $W = N_G(T)/T$. By this injection, we obtain an action of W_H on $\mathcal{X}(T)$ such that ρ is W_H-equivariant. The lemma follows. $\square$

4.1.2. Simple roots

In Section 4.1.1, we just proved that ρ induces a map from $\phi_{\mathfrak{g}}$ onto $\phi_{\mathfrak{h}}$. In this section, we will prove that ρ induces a map from the Dynkin diagram of $\mathfrak{g}$ onto that of $\mathfrak{h}$.

Let us fix a choice $\phi_{\mathfrak{h}}^+$ of positive roots for $\mathfrak{h}$. Set $\phi_{\mathfrak{g}}^+ = \rho_\phi^{-1}(\phi_{\mathfrak{h}}^+)$. Note that, since the pullback by ρ of a half space in a half space, $\phi_{\mathfrak{g}}^+$ is a choice of positive roots for $\phi_{\mathfrak{g}}$. Let $\Delta_{\mathfrak{g}}$ (resp. $\Delta_{\mathfrak{h}}$) be the set of simple roots of $\phi_{\mathfrak{g}}$ (resp. $\phi_{\mathfrak{h}}$).

Lemma 4.6. *Let* α *be a root of* $\mathfrak{g}$. *Then,* $\alpha \in \Delta_{\mathfrak{g}}$ *if and only if* $\rho_\phi(\alpha) \in \Delta_{\mathfrak{h}}$.

Proof. Since $\alpha \in \phi_{\mathfrak{g}}^+$ if and only if $\rho(\alpha) \in \phi_{\mathfrak{h}}^+$, we may assume that $\alpha \in \phi_{\mathfrak{g}}^+$.

Let us assume that $\alpha \notin \Delta_{\mathfrak{g}}$. Then, there exist β and γ in $\phi_{\mathfrak{g}}^+$ such that $\alpha = \beta + \gamma$. By applying ρ, we see that $\rho(\alpha)$ does not belong to $\Delta_{\mathfrak{h}}$.

Let us assume that $\alpha \in \Delta_{\mathfrak{g}}$. By absurd, we assume that there exist β and γ in $\phi_{\mathfrak{h}}^+$ such that $\rho(\alpha) = \beta + \gamma$. Three cases occur:

Case 1: β and γ belong to $\phi_{\mathfrak{h}}^1$.

Let β° and γ° be in $\phi_{\mathfrak{g}}$ such that $\rho(\beta^\circ) = \beta$ and $\rho(\gamma^\circ) = \gamma$. By Lemma 4.3, $\mathfrak{g}_{\beta^\circ} = \mathfrak{h}_\beta$ and $\mathfrak{g}_{\gamma^\circ} = \mathfrak{h}_\gamma$. So, we have

$$[\mathfrak{g}_{\beta^\circ}, \mathfrak{g}_{\gamma^\circ}] = [\mathfrak{h}_\beta, \mathfrak{h}_\gamma] = \mathfrak{h}_{\rho(\alpha)}.$$

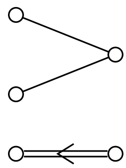


Fig. 1. Dynkin diagram of $(\mathrm{PSL}_4, \mathrm{PSp}_4)$.

In particular this bracket is non-zero and $\beta^\circ + \gamma^\circ$ is a root of $\mathfrak{g}$. Moreover, $\mathfrak{g}_{\beta^\circ+\gamma^\circ} = \mathfrak{h}_{\rho(\alpha)}$. So, Lemma 4.3 shows that $\rho(\alpha) \in \phi^1_{\mathfrak{h}}$. But α and $\beta^\circ + \gamma^\circ$ belong to $\rho_\phi^{-1}(\rho(\alpha))$. So, $\alpha = \beta^\circ + \gamma^\circ$; and this root is not simple.

Case 2: $\beta \in \phi^1_{\mathfrak{h}}$ and $\gamma \in \phi^2_{\mathfrak{h}}$.

Let β° be as above. We can write $\rho_\phi^{-1}(\gamma) = \{\gamma^+, \gamma^-\}$. We have

$$\mathfrak{g}_{\beta^\circ+\gamma^+} + \mathfrak{g}_{\beta^\circ+\gamma^-} \supset [\mathfrak{g}_{\beta^\circ}, \mathfrak{g}_{\gamma^+} + \mathfrak{g}_{\gamma^-}] \supset [\mathfrak{g}_{\beta^\circ}, \mathfrak{h}_\gamma] = [\mathfrak{h}_\beta, \mathfrak{h}_\gamma] = \mathfrak{h}_{\beta+\gamma} = \mathfrak{h}_{\rho(\alpha)}.$$

Moreover, since $\mathfrak{h}_\gamma$ is different from $\mathfrak{g}_{\gamma^+}$ and $\mathfrak{g}_{\gamma^-}$, $\mathfrak{h}_{\rho(\alpha)}$ is different from $\mathfrak{g}_{\beta^\circ+\gamma^+}$ and $\mathfrak{g}_{\beta^\circ+\gamma^-}$. In particular, $\beta^\circ + \gamma^+$ and $\beta^\circ + \gamma^-$ are roots of $\mathfrak{g}$; and $\rho_\phi^{-1}(\rho(\alpha)) = \{\beta^\circ + \gamma^+, \beta^\circ + \gamma^-\}$. So, $\alpha = \beta^\circ + \gamma^+$ or $\beta^\circ + \gamma^-$; and this root is not simple.

Case 3: β and γ belong to $\phi^2_{\mathfrak{h}}$.

With obvious notation, we have

$$\mathfrak{g}_{\beta^++\gamma^+} + \mathfrak{g}_{\beta^++\gamma^-} + \mathfrak{g}_{\beta^-+\gamma^+} + \mathfrak{g}_{\beta^-+\gamma^-} \supset [\mathfrak{h}_\beta, \mathfrak{h}_\gamma] = \mathfrak{h}_{\beta+\gamma}.$$

If $\mathfrak{h}_{\beta+\gamma}$ equals one of the four spaces $\mathfrak{g}_{\beta^\pm+\gamma^\pm}$, Lemma 4.4 shows that α equals $\beta^\pm + \gamma^\pm$ and is not a simple root. Otherwise, two of the four spaces $\mathfrak{g}_{\beta^\pm+\gamma^\pm}$ are not zero and α equals one of the two corresponding roots; in particular α is not simple. $\square$

Consider the map

$$\begin{aligned} \rho_\Delta : \Delta_{\mathfrak{g}} &\longrightarrow \Delta_{\mathfrak{h}} \\ \alpha &\longmapsto \rho_\phi(\alpha). \end{aligned}$$

Set $\Delta^2_{\mathfrak{h}} = \Delta_{\mathfrak{h}} \cap \phi^2_{\mathfrak{h}}$ and $\Delta^1_{\mathfrak{h}} = \Delta_{\mathfrak{h}} \cap \phi^1_{\mathfrak{h}}$.

On the Dynkin diagram of $\mathfrak{h}$, we color in black the simple roots in $\Delta^2_{\mathfrak{h}}$. The so obtained diagram is called the *colored Dynkin diagram of* $\mathfrak{h}$ and is denoted by $\mathcal{D}_{\mathfrak{h}}$. From now on, when we draw the Dynkin diagram $\mathcal{D}_{\mathfrak{g}}$ of $\mathfrak{g}$, two simple roots α and β are placed on the same vertical line if and only if $\rho_\Delta(\alpha) = \rho_\Delta(\beta)$; in such a way, ρ_Δ identifies with the vertical projection. Note that by Lemma 4.4, α and β are orthogonal. For $(\mathrm{PSL}_4, \mathrm{PSp}_4)$, we obtain Fig. 1.

By exchanging the simple roots in a fiber of ρ_Δ, we define an involution $\sigma_{\mathfrak{h}}$ on the set of vertexes of the Dynkin diagram of $\mathfrak{g}$. Note that $\sigma_{\mathfrak{h}}$ is not necessarily an automorphism of $\mathcal{D}_{\mathfrak{g}}$ (see the pair (SO_7, G_2)).

4.2. A result of unicity

Proposition 4.1. *For a fixed pair $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$ there exists at most one (up to conjugacy by an element of G) spherical pair (G, H) where G is adjoint and H connected.*

Proof. Obviously, G is determined by $\mathcal{D}_{\mathfrak{g}}$. Let us fix a Borel subgroup B of G and a maximal torus T of B. Let $\Delta_{\mathfrak{g}}$ denote the set of simple roots of G. For any $\alpha \in \Delta_{\mathfrak{g}}$, we fix an $\mathfrak{sl}_2$-triple $(X_\alpha, Y_\alpha, H_\alpha)$. Consider

$$\begin{aligned} \Theta : T &\longrightarrow \left(\mathbb{K}^*\right)^{\Delta_{\mathfrak{h}}^2} \\ t &\longmapsto \left(\beta(t)\alpha\left(t^{-1}\right)\right)_{\substack{\alpha \neq \beta \in \Delta_{\mathfrak{g}} \\ \rho_\Delta(\alpha) = \rho_\Delta(\beta)}}. \end{aligned}$$

The identity component S of the kernel of Θ is a subtorus of T of dimension $|\Delta_{\mathfrak{g}}| - |\Delta_{\mathfrak{h}}^2| = |\Delta_{\mathfrak{h}}|$. Moreover, Θ is surjective.

Let H be a semisimple subgroup of G such that (G, H) is a spherical pair of minimal rank with $(\mathcal{D}_{\mathfrak{g}}, \mathcal{D}_{\mathfrak{h}}, \rho_\Delta)$ as associated triple. Let $T(H)$ be a maximal torus of H. Up to conjugacy, we may assume that $T(H)$ is contained in T. But, $T(H)$ is contained in S; by a dimension argument we conclude that $T(H) = S$.

For all $\alpha \in \Delta_{\mathfrak{g}}^1$, we have $\mathfrak{g}_\alpha = \mathfrak{h}_{\rho(\alpha)}$. Moreover, we claim that up to conjugacy, we may assume that for all $\alpha \neq \beta \in \Delta_{\mathfrak{g}}$ such that $\rho_\Delta(\alpha) = \rho_\Delta(\beta)$ we have $\mathfrak{h}_{\rho(\alpha)} = \mathbb{K}.(X_\alpha + X_\beta)$.

We write $\Delta_{\mathfrak{h}}^2 = \{\alpha_1, \ldots, \alpha_k\}$ and $\Delta_{\mathfrak{g}}^2 = \{\alpha_1^-, \ldots, \alpha_k^-\} \cup \{\alpha_1^+, \ldots, \alpha_k^+\}$ such that for all $i = 1, \ldots, k$, $\rho(\alpha_i^\pm) = \alpha_i$. By Lemma 4.4, there exist $x_1, \ldots, x_k \in \mathbb{K}^*$ such that for all $i = 1, \ldots, k$, $\mathfrak{h}_{\alpha_i} = \mathbb{K}.(X_{\alpha_i^-} + x_i X_{\alpha_i^+})$. Since Θ is surjective, there exists $t \in T$ such that $\Theta(t) = (x_1, \ldots, x_k)$. By conjugating H by t, we obtain the claim.

Let $i \in \{1, \ldots, k\}$. There exists $y \in \mathbb{K}^*$ such that $\mathfrak{h}_{-\alpha_i} = \mathbb{K}.(Y_{\alpha_i^-} + yY_{\alpha_i^+})$. Since α_i^- and α_i^+ are orthogonal, $\xi := [X_{\alpha_i^-} + X_{\alpha_i^+}, Y_{\alpha_i^-} + yY_{\alpha_i^+}] = H_{\alpha_i^-} + yH_{\alpha_i^+}$. But, ξ belongs to the Lie algebra of $T(H) = S$, so $(\alpha_i^- - \alpha_i^+)(\xi) = 0$. We conclude that $y = 1$.

Finally, since $\mathfrak{h}$ is generated as Lie algebra by the $\mathfrak{h}_{\pm\alpha}$ for $\alpha \in \Delta_{\mathfrak{h}}$; $\mathfrak{h}$ is generated by

$$\{X_\alpha\}_{\alpha \in \Delta_{\mathfrak{g}}^1} \cup \{Y_\alpha\}_{\alpha \in \Delta_{\mathfrak{g}}^1} \cup \{X_{\alpha_i^-} + X_{\alpha_i^+}\}_{\alpha \in \Delta_{\mathfrak{h}}^2} \cup \{Y_{\alpha_i^-} + Y_{\alpha_i^+}\}_{\alpha \in \Delta_{\mathfrak{h}}^2}.$$

In particular, $\mathfrak{h}$ only depends on the pair $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$. □

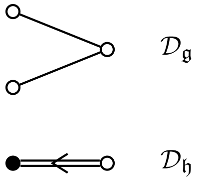
4.2.1. The case when $\mathrm{rk}(H) = 2$

Lemma 4.3 considers the case when $\mathrm{rk}(H) = 1$. We now consider the case when $\mathrm{rk}(H) = 2$:

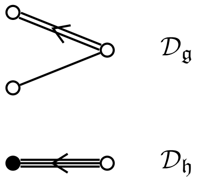
Lemma 4.7. *We assume that* $\mathrm{rk}(H) = 2$. *Then, the possibilities for $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$ up to conjugacy by an automorphism of $\mathcal{D}_{\mathfrak{g}}$ are:*

(i) $(\mathcal{D}_{\mathfrak{h}}, \text{Identity})$ *obtained with $G = H$.*
(ii) $(\mathcal{D}_{\mathfrak{h}} \cup \mathcal{D}_{\mathfrak{h}}, \text{Exchange})$ *obtained with H embedded diagonally in $H \times H$.*
(iii) $\mathcal{D}_{\mathfrak{g}} = A_1 \cup A_1 \cup A_1$ *and $\sigma_{\mathfrak{h}}$ exchanges the two first copies. This case corresponds to* $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $(\xi, \eta) \mapsto (\xi, \xi, \eta)$.

(iv) $\mathcal{D}_{\mathfrak{g}} = A_3$ *and* $\sigma_{\mathfrak{h}}$ *fixes the central vertex and exchanges the two others; obtained with* $H = \mathrm{PSp}_4 \subset G = \mathrm{PSL}_4$*:*

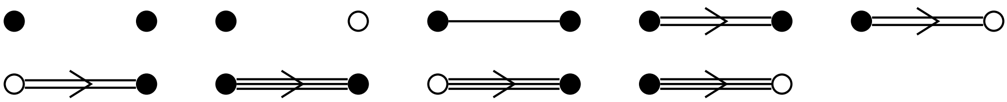


(v) $\mathcal{D}_{\mathfrak{g}} = B_3$ *and* $\sigma_{\mathfrak{h}}$ *fixes the central vertex and exchanges the two others; obtained with* $G_2 \subset \mathrm{SO}_7$*:*



Proof. If $\phi_{\mathfrak{h}}^2$ is empty, the dimensions of $\mathfrak{g}$ and $\mathfrak{h}$ are equal and hence $\mathfrak{g} = \mathfrak{h}$. From now on, we assume that $\phi_{\mathfrak{h}}^2$ is not empty. Since $\phi_{\mathfrak{h}}^2$ is stable by the action of the Weyl group W_H of H, $\Delta_{\mathfrak{h}}^2$ is nonempty.

By invariance by W_H the possibilities for colored Dynkin diagrams of $\mathfrak{h}$ are:



In each case, using the action of W_H, one can determine $\phi_{\mathfrak{h}}^1$ and $\phi_{\mathfrak{h}}^2$ and thus, compute the cardinality $|\phi_{\mathfrak{g}}|$ of $\phi_{\mathfrak{g}}$ which equals $|\phi_{\mathfrak{h}}^1| + 2|\phi_{\mathfrak{h}}^2|$. For example, assume that the colored Dynkin diagram of $\mathfrak{h}$ is the first one in the above list. Then, $\phi_{\mathfrak{h}}^2 = \phi_{\mathfrak{h}}$. Then, the cardinality of $\phi_{\mathfrak{g}}$ is 8, and $\mathfrak{g}$ has rank 4. We deduce that $\mathfrak{g} = \mathfrak{sl}_2^4$; and that, $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$ corresponds to the diagonal embedding of $(\mathfrak{sl}_2)^2$ in $\mathfrak{sl}_2^4$.

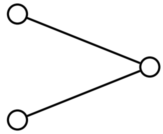
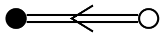
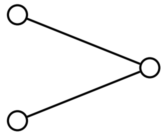
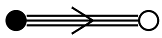
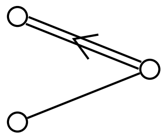
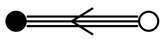
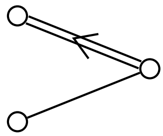
In a similar way, one can check that the second case corresponds to $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2, (\xi, \eta) \mapsto (\xi, \xi, \eta)$.

Consider a remaining case. Since there is no nontrivial morphism from $\mathfrak{h}$ to $\mathfrak{sl}_2$, the group H acts trivially on $\mathbb{P}^1$. Since H has an open orbit in $\mathcal{B}$, this implies that $\mathbb{P}^1$ is not a factor of $\mathcal{B}$. So, $\mathcal{D}_{\mathfrak{g}}$ cannot have isolated vertex.

Consider now the third colored Dynkin diagram. The Dynkin diagram of $\mathfrak{g}$ has 4 vertexes, two over each vertex of $\mathcal{D}_{\mathfrak{h}}$. Since $|\phi_{\mathfrak{g}}| = 2|\phi_{\mathfrak{h}}| = 12$, $\mathcal{D}_{\mathfrak{g}}$ cannot have a triple edge (G_2 has 12 roots!). Assume now that $\mathcal{D}_{\mathfrak{g}}$ has a double edge. Since $\mathfrak{sp}_4$ has 8 roots, $\mathcal{D}_{\mathfrak{g}}$ has no more edge; which contradicts the fact the $\mathcal{D}_{\mathfrak{g}}$ have no isolated vertex. Finally, $\mathcal{D}_{\mathfrak{g}}$ has only simple edges. Moreover, Lemma 4.3 shows that the two simple roots of $\mathfrak{g}$ which map on a given simple root of $\phi_{\mathfrak{h}}^2$ are orthogonal. This implies that $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$ is that of $\mathfrak{sl}_3 \subset \mathfrak{sl}_3 \times \mathfrak{sl}_3$.

In the remaining cases, $\mathfrak{h}$ is either $\mathfrak{sp}_4$ or G_2. We claim that if $\mathfrak{g}$ is not simple then $\mathfrak{h}$ is diagonally embedded in $\mathfrak{g} = \mathfrak{h} \times \mathfrak{h}$. Assume that $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, with nontrivial $\mathfrak{g}_1$ and $\mathfrak{g}_2$. We

Table 1

Case	$\mathfrak{h}$	Colored Dynkin diagram of $\mathfrak{h}$	$(\lvert\phi^1_{\mathfrak{h}}\rvert, \lvert\phi^2_{\mathfrak{h}}\rvert)$	$\lvert\phi_{\mathfrak{g}}\rvert$	Dynkin diagram of $\mathfrak{g}$	$\mathfrak{g}$
1	$\mathfrak{sp}_4$		$(4, 4)$	12		$\mathfrak{sl}_4$
2	$\mathfrak{sp}_4$		$(4, 4)$	12		$\mathfrak{sl}_4$
3	G_2		$(6, 6)$	18		$\mathfrak{so}_7$
4	G_2		$(6, 6)$	18		$\mathfrak{so}_7$

already seen that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ cannot have rank one; we deduce that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have rank two. By projection, one obtains two nontrivial (otherwise, H cannot act on $\mathcal{B}$ with an open orbit) morphisms from $\mathfrak{h}$ to $\mathfrak{g}_1$ and $\mathfrak{g}_2$. It follows that $\mathfrak{g}_1 \simeq \mathfrak{g}_2 \simeq \mathfrak{h}$. Finally, one easily checks that $\mathfrak{h}$ is diagonally embedded in $\mathfrak{h}^2$.

From now on, we assume that $\mathfrak{g}$ is simple; that is, that $\mathcal{D}_{\mathfrak{g}}$ is connected. The same arguments (cardinality of $\phi_{\mathfrak{g}}$ and no edges between two vertexes mapped on one of $\mathcal{D}_{\mathfrak{h}}$) as above allow to show easily that the only possibilities for $(\mathcal{D}_{\mathfrak{g}}, \mathcal{D}_{\mathfrak{h}}, \rho_\Delta)$ with $\mathfrak{h}$ equals to $\mathfrak{sp}_4$ or of type G_2 are those enumerated in Table 1.

Consider Case 1. Let α (resp. β) denote the short (resp. long) simple root of $\mathfrak{h}$. Set $\alpha^\circ = \rho_\Delta^{-1}(\alpha)$. By Lemma 4.5, the short root $\alpha+\beta$ belongs to $\phi^1_{\mathfrak{h}}$; we set $(\alpha+\beta)^\circ = \rho_\phi^{-1}(\alpha+\beta)$. So,

$$\mathfrak{h}_{2\alpha+\beta} = [\mathfrak{h}_\alpha, \mathfrak{h}_{\alpha+\beta}] = [\mathfrak{g}_{\alpha^\circ}, \mathfrak{g}_{(\alpha+\beta)^\circ}] = \mathfrak{g}_{\alpha^\circ+(\alpha+\beta)^\circ}.$$

Now, Lemma 4.4 shows that $2\alpha+\beta \in \phi^1_{\mathfrak{h}}$. With Lemma 4.5, this contradicts $\beta \in \phi^2_{\mathfrak{h}}$.

By elimination, the inclusion of PSp_4 in PSL_4 corresponds to Case 2.

Consider Case 3. Let α (resp. β) denote the short (resp. long) simple root of $\mathfrak{h}$. By Lemma 4.5, $\beta+2\alpha$ belongs to $\phi^1_{\mathfrak{h}}$. By the argument used in Case 1 before, one easily checks that $\beta+3\alpha = (\beta+2\alpha)+\alpha$ belongs to $\phi^1_{\mathfrak{h}}$. This contradicts Lemma 4.5, since $\beta+3\alpha$ is a long root.

Case 4 corresponds to the inclusion of G_2 in SO_7. $\square$

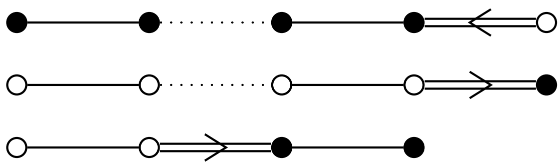
We may now assume that H is simple. Indeed, we have:

Proposition 4.2. *Let (G, H) be a spherical pair of minimal rank with G semisimple adjoint and H connected. If H is not simple then there exist two spherical pairs (G_1, H_1) and (G_2, H_2) of minimal rank such that $G = G_1 \times G_2$ and $H = H_1 \times H_2$.*

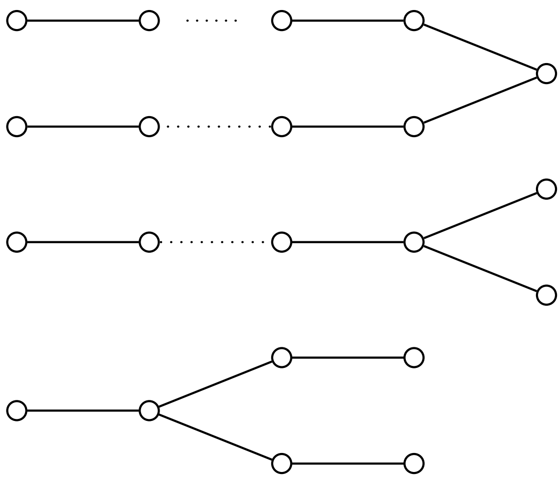
Proof. By assumption, $\mathcal{D}_{\mathfrak{h}}$ is the disjoint union of two Dynkin diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$. By Lemmas 4.7 and 4.1, for all $\alpha, \beta \in \Delta_{\mathfrak{g}}$ such that $\rho(\alpha)$ and $\rho(\beta)$ are orthogonal, α and β are orthogonal. We deduce that $\mathcal{D}_{\mathfrak{g}}$ is the disjoint union of $\rho_{\Delta}^{-1}(\mathcal{D}_1)$ and $\rho_{\Delta}^{-1}(\mathcal{D}_2)$. The proposition follows. □

By Proposition 4.2, to classify all the spherical pairs (G, H) of minimal rank with G semisimple adjoint and H semisimple, we may assume that H is simple. Theorem 1 stated in the introduction lists all such spherical pairs. We can now prove this classification.

Proof of Theorem A. By Proposition 4.1, it is sufficient to classify the possible triples $(\mathcal{D}_{\mathfrak{g}}, \sigma_{\mathfrak{h}})$, up to conjugacy by an automorphism of $\mathcal{D}_{\mathfrak{g}}$. By Lemma 4.7, we may assume that $\mathrm{rk}(H) \geqslant 3$. Moreover, we may assume that $\Delta_{\mathfrak{h}}^2$ is nonempty and different from $\Delta_{\mathfrak{h}}$. Let $\alpha \in \Delta_{\mathfrak{h}}^2$ and $\beta \in \Delta_{\mathfrak{h}}^1$. By Lemma 4.7, either α and β are orthogonal or α is the short root joined to the long root β by a double edge. One easily deduces that the colored Dynkin diagram of $\mathfrak{h}$ is one of the following:



One easily deduces from Lemmas 4.1 and 4.7 than in the three preceding cases the Dynkin diagram $\mathcal{D}_{\mathfrak{g}}$ is respectively:



The theorem follows. □

Acknowledgment

The author would like to thank the referees of this paper for their helpful suggestions.

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