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Error Bounds for Least Squares Approximation by Polynomials

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Let $f \in C^{n+1}[-1, 1]$ and let $H[f](x)$ be the nth degree weighted least squares polynomial approximation to f with respect to the orthonormal polynomials $\{q_k\}$ associated with a distribution $d\alpha$ on $[-1, 1]$. It is shown that if $\|q_{n+1}\|/\|q_n\| \geqslant \max(q_{n+1}(1)/q_n(1), \; -q_{n+1}(-1)/q_n(-1))$, then $\|f - H[f]\| \leqslant \|f^{(n+1)}\| \cdot \|q_{n+1}\| / \|q_{n+1}^{(n+1)}\|$, where $\|\cdot\|$ denotes the supremum norm. Furthermore, it is shown that in the case of Jacobi polynomials with distribution $(1-t)^\alpha (1+t)^\beta \, dt$, $\alpha, \beta > -1$, the condition on $\|q_{n+1}\|/\|q_n\|$ is satisfied when either $\max(\alpha, \beta) \geqslant -1/2$ or $-1 < \alpha = \beta < -1/2$. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $\{q_k\}$ be the orthonormal polynomials associated with the distribution $d\alpha$ on the interval $[-1, 1]$. Let $f \in C^{n+1}[-1, 1]$. The weighted least squares approximation to f is given by

$$H[f](x) = \sum_{k=0}^{n} q_k(x) \int_{-1}^{1} f(t) \, q_k(t) \, d\alpha(t). \tag{1.1}$$

Brass [1] has shown that if the distribution $d\alpha$ has the symmetry property that for all continuous functions g

$$\int_{-1}^{1} g(t) \, d\alpha(t) = \int_{-1}^{1} g(-t) \, d\alpha(t)$$

and if $\|q_k\| = q_k(1)$, $k = 0, 1, ..., n+1$, then a bound for the error, $f(x) - H[f](x)$, in this approximation is given by

$$\|f - H[f]\| \leqslant \frac{\|q_{n+1}\|}{\|q_{n+1}^{(n+1)}\|} \, \|f^{(n+1)}\|, \tag{1.2}$$

where $\|\cdot\|$ denotes the supremum norm on $[-1, 1]$.

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With regard to the distribution $(1-t)^{\alpha}\ (1+t)^{\beta}\, dt$, $\alpha, \beta > -1$, and the associated normalised Jacobi polynomials, the conditions required by Brass are satisfied only if $\alpha = \beta \geqslant -1/2$.

Previously Paget [2] has given bounds of the form (1.2) in the Jacobi polynomial case for α, β such that either $\max(\alpha, \beta) \geqslant -1/2$ or $-1 < \alpha = \beta < -1/2$.

It is the purpose of this present paper to show that the method of Brass [1] may be extended to include all those cases considered in Paget [2].

2. The Theorem of Brass Extended

With the s-norm for functionals Q on $C^s[-1, 1]$ defined by

$$\|Q\|_s = \sup_{\|f^{(s)}\| \leqslant 1} \|Q[f]\| \tag{2.1}$$

and for $x \in [-1, 1]$ the functional R_x defined on $C^{n+1}[-1, 1]$ by

$$R_x[f] = f(x) - H[f](x), \tag{2.2}$$

Brass [1] has shown that

$$\|R_x\|_{n+1} \leqslant \frac{\delta_n}{(n+1)\,\delta_{n+1}} \max(\|\bar{q}_{n+1} C_n + \bar{q}_n C_{n+1}\|_n, \|\bar{q}_{n+1} C_n - \bar{q}_n C_{n+1}\|_n), \tag{2.3}$$

where $\delta_k > 0$ is the coefficient of x^k in $q_k(x)$, $\bar{q}_k = \|q_k\|$, and the functional C_k is defined on $C^n[-1, 1]$ by

$$C_k[g] = \int_{-1}^{1} g(t)\, q_k(t)\, d\alpha(t). \tag{2.4}$$

Now

$$(\bar{q}_{n+1} C_n \pm \bar{q}_n C_{n+1})[g] = \int_{-1}^{1} g(t)(\bar{q}_{n+1} q_n(t) \pm \bar{q}_n q_{n+1}(t))\, d\alpha(t), \tag{2.5}$$

so that we need to consider polynomials of the form $q_{n+1}(t) - c q_n(t)$ where c is a constant.

Theorem 2.1 (Szegö [3, p. 46]). *Let c be an arbitrary real constant, then the polynomial*

$$q_{n+1}(t) - c q_n(t)$$

has $n+1$ distinct real zeros. If $c>0$ $(c<0)$ these zeros lie in $(-1, 1)$ with the exception of the greatest (least) zero which lies in $[-1, 1]$ if and only if

$$c \leqslant q_{n+1}(1)/q_n(1) \qquad (c \geqslant q_{n+1}(-1)/q_n(-1)).$$

Using this result the theorem of Brass may be extended in the following way.

THEOREM 2.2. *If q_n, q_{n+1} are such that*

$$\bar{q}_{n+1}/\bar{q}_n \geqslant \max(q_{n+1}(1)/q_n(1), \qquad -q_{n+1}(-1)/q_n(-1)), \tag{2.6}$$

then for $f \in C^{n+1}[-1, 1]$

$$\|f - H[f]\| \leqslant \|f^{(n+1)}\| \; \|q_{n+1}\| / \|q_{n+1}^{(n+1)}\|. \tag{2.7}$$

Proof. Consider the polynomial

$$p_{n+1}^-(t) = \bar{q}_{n+1} q_n(t) - \bar{q}_n q_{n+1}(t). \tag{2.8}$$

By Theorem 2.1 p_{n+1}^- has $n+1$ distinct zeros, n of which lie in $(-1, 1)$. If $\bar{q}_{n+1}/\bar{q}_n = q_{n+1}(1)/q_n(1)$ then 1 is a zero of p_{n+1}^-. If $\bar{q}_{n+1}/\bar{q}_n > q_{n+1}(1)/q_n(1)$ then the greatest zero of p_{n+1}^- is greater than 1. Thus we may write

$$p_{n+1}^-(t) = -\bar{q}_n \delta_{n+1} \prod_{k=1}^{n+1} (t - \eta_k), \tag{2.9}$$

where

$$-1 < \eta_1 < \eta_2 < \cdots < \eta_n < 1 \leqslant \eta_{n+1}. \tag{2.10}$$

A similar argument using Theorem 2.1 with $c<0$ yields

$$p_{n+1}^+(t) = \bar{q}_{n+1} q_n(t) + \bar{q}_n q_{n+1}(t) = \bar{q}_n \delta_{n+1} \prod_{k=0}^{n} (t - \zeta_k), \tag{2.11}$$

where

$$\zeta_0 \leqslant -1 < \zeta_1 < \zeta_2 < \cdots < \zeta_n < 1. \tag{2.12}$$

Again, following Brass, let $L_{n-1}[g]$ denote the $(n-1)$th degree interpolation polynomial coinciding with g at $\eta_1, \eta_2, ..., \eta_n$. Then

$$\begin{aligned}(\bar{q}_{n+1} C_n - \bar{q}_n C_{n+1})[g] &= \int_{-1}^{1} g(t)\, p_{n+1}^-(t)\, d\alpha(t) \\ &= \int_{-1}^{1} (g(t) - L_{n-1}[g](t))\, p_{n+1}^-(t)\, d\alpha(t) \\ &= \int_{-1}^{1} \frac{g^{(n)}(\xi(t))}{n!} \prod_{k=1}^{n} (t - \eta_k) \cdot p_{n+1}^-(t)\, d\alpha(t),\end{aligned}$$

where $\xi(t) \in (-1, 1)$. From (2.9) and (2.10) we see that $\prod_{k=1}^{n} (t-\eta_k) \cdot p_{n+1}^{-}(t)$ does not change sign on $[-1, 1]$. On applying the mean value theorem and using the orthogonality of $\{q_k\}$ we deduce that

$$(\bar{q}_{n+1} C_n - \bar{q}_n C_{n+1})[g] = \frac{g^{(n)}(\xi)}{n!} \frac{\bar{q}_{n+1}}{\delta_n},$$

for some $\xi \in (-1, 1)$. Thus

$$\|\bar{q}_{n+1} C_n - \bar{q}_n C_{n+1}\|_n = \frac{\bar{q}_{n+1}}{n!\, \delta_n}. \tag{2.13}$$

Similarly, by constructing the interpolation polynomial to coincide with g at $\zeta_1, \zeta_2, ..., \zeta_n$ (see (2.11), (2.12)) it may be shown that

$$\|\bar{q}_{n+1} C_n + \bar{q}_n C_{n+1}\|_n = \frac{\bar{q}_{n+1}}{n!\, \delta_n}. \tag{2.14}$$

Then from (2.3), (2.13), and (2.14) we have that

$$\|R_x\|_{n+1} \leqslant \frac{\bar{q}_{n+1}}{(n+1)!\, \delta_{n+1}} = \|q_{n+1}\| / \|q_{n+1}^{(n+1)}\|.$$

The result (2.7) then follows directly from Definitions (2.1) and (2.2). ∎

We note that in this theorem the distribution symmetry condition of Brass' theorem is not required and also that the maximum value of $|q_k(x)|$ in $[-1, 1]$ may be attained at an interior point provided that (2.6) is satisfied.

3. Application to Jacobi Polynomials

We show that Condition (2.6) of Theorem 2.2 is satisfied by the normalised Jacobi polynomials provided that either $\max(\alpha, \beta) \geqslant -1/2$ or $-1 < \alpha = \beta < -1/2$. This result shows that Theorem 2.2 is a significant extension of the theorem of Brass [1] which, for the Jacobi polynomials, only covers the case $\alpha = \beta \geqslant -1/2$.

The distribution being considered is $(1-t)^{\alpha} (1+t)^{\beta}\, dt$ with $\alpha, \beta > -1$, and the associated orthonormal polynmials are

$$q_k(t) = h_k^{-1/2} P_k^{(\alpha,\beta)}(t), \tag{3.1}$$

where

$$h_k = \int_{-1}^{1} (P_k^{(\alpha,\beta)}(t))^2 (1-t)^{\alpha} (1+t)^{\beta}\, dt. \tag{3.2}$$

For all α, $\beta > -1$ we have that

$$\frac{q_{n+1}(1)}{q_n(1)} = \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{P_{n+1}^{(\alpha,\beta)}(1)}{P_n^{(\alpha,\beta)}(1)} = \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{n+1+\alpha}{n+1}$$

and

$$-\frac{q_{n+1}(-1)}{q_n(-1)} = -\left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{P_{n+1}^{(\alpha,\beta)}(-1)}{P_n^{(\alpha,\beta)}(-1)} = \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{n+1+\beta}{n+1}.$$

Therefore

$$\max\left(\frac{q_{n+1}(1)}{q_n(1)}, -\frac{q_{n+1}(-1)}{q_n(-1)}\right) = \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{n+1+\max(\alpha,\beta)}{n+1}. \tag{3.3}$$

Case 1. $\max(\alpha, \beta) \geqslant -1/2$. From Szegö [3, p. 168] we have that

$$\frac{\bar{q}_{n+1}}{\bar{q}_n} = \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{n+1+\max(\alpha,\beta)}{n+1}. \tag{3.4}$$

Thus from (3.3) and (3.4) we see Condition (2.6) is satisfied.

Case 2. $-1 < \alpha = \beta < -1/2$. This case is more complicated because we have no precise expression for $\bar{q}_k$ when k is odd.

For k even we have the expression (see [3, p. 171])

$$\bar{q}_k = h_k^{-1/2} \, |P_k^{(\alpha,\alpha)}(0)| = \frac{h_k^{-1/2}\Gamma(k+\alpha+1)}{2^k(k/2)!\,\Gamma((k/2)+\alpha+1)}, \qquad k \text{ even}. \tag{3.5}$$

For k odd we note that $P_k^{(\alpha,\alpha)}$ is an odd function and $P_k^{(\alpha,\alpha)}(0) = 0$. We use a particular case of Sonin's theorem ([3, p. 166]) and an adaptation of it.

Let g be defined by

$$g(x) = (P_k^{(\alpha,\alpha)}(x))^2 + \frac{1-x^2}{k(k+2\alpha+1)} \left(\frac{d}{dx} P_k^{(\alpha,\alpha)}(x)\right)^2. \tag{3.6}$$

Using the differential equation for $P_k^{(\alpha,\alpha)}$ we have

$$g'(x) = \frac{2(2\alpha+1)}{k(k+2\alpha+1)} x \left(\frac{d}{dx} P_k^{(\alpha,\alpha)}(x)\right)^2. \tag{3.7}$$

Since $2\alpha+1<0$ we see that g is non-decreasing in $(-1, 0)$ and non-increasing in $(0, 1)$. It follows that $|P_k^{(\alpha,\alpha)}(x)|$ achieves its maximum value at $\pm x_k^*$, the two stationary points of $P_k^{(\alpha,\alpha)}$ closest to zero. Since $g(x_k^*) < g(0)$,

$$|P_k^{(\alpha,\alpha)}\,(x_k^*)| < (k(k+2\alpha+1))^{-1/2} \, |P_k^{(\alpha,\alpha)\prime}(0)|. \tag{3.8}$$

Let h be defined by

$$h(x) = (1 - x^2)^{2\alpha + 1}\, g(x). \tag{3.9}$$

Then using (3.6) and (3.7) we have that

$$h'(x) = -2(2\alpha + 1)\, x(1 - x^2)^{2\alpha}\, (P_k^{(\alpha,\alpha)}(x))^2. \tag{3.10}$$

We see that h is non-increasing in $(-1, 0)$ and non-decreasing in $(0, 1)$. Thus $h(x_k^*) > h(0)$ and

$$|P_k^{(\alpha,\alpha)}(x_k^*)| > (k(k + 2\alpha + 1))^{-1/2}\, (1 - x_k^{*2})^{-(\alpha + 1/2)}\, |P_k^{(\alpha,\alpha)\prime}(0)|. \tag{3.11}$$

From (3.8) and (3.11) we have, for k odd,

$$\begin{aligned} |P_k^{(\alpha,\alpha)}(x_k^*)| &= D_k \frac{|P_k^{(\alpha,\alpha)\prime}(0)|}{(k(k + 2\alpha + 1))^{1/2}} \\ &= D_k \frac{(k + 2\alpha + 1)^{1/2}\, \Gamma(k + \alpha + 1)}{k^{1/2} 2^k ((k-1)/2)!\, \Gamma((k/2) + \alpha + (3/2))}, \end{aligned} \tag{3.12}$$

where

$$(1 - x_k^{*2})^{-(\alpha + 1/2)} < D_k < 1. \tag{3.13}$$

From (3.5) and (3.12) it follows that

$$\frac{\bar{q}_{n+1}}{\bar{q}_n} = \begin{cases} \left(\dfrac{h_n}{h_{n+1}}\right)^{1/2} \dfrac{n + \alpha + 1}{n + 1} \left(\dfrac{n + 1}{n + 2\alpha + 2}\right)^{1/2} D_{n+1} & \text{if } n \text{ is even,} \\ \left(\dfrac{h_n}{h_{n+1}}\right)^{1/2} \dfrac{n + \alpha + 1}{n + 1} \left(\dfrac{n}{n + 2\alpha + 1}\right)^{1/2} \dfrac{1}{D_n} & \text{if } n \text{ is odd,} \end{cases} \tag{3.14}$$

with bounds for D_n, D_{n+1} given by (3.13).

Recalling that $2\alpha + 1 < 0$, we see from (3.3), (3.13), and (3.14) that for n odd

$$\frac{\bar{q}_{n+1}}{\bar{q}_n} > \left(\frac{h_n}{h_{n+1}}\right)^{1/2} \frac{n + \alpha + 1}{n + 1} = \max\left(\frac{q_{n+1}(1)}{q_n(1)}, -\frac{q_{n+1}(-1)}{q_n(-1)}\right),$$

so that the Condition (2.6) is satisfied.

For n even we need to look closer at D_{n+1}. Since

$$\frac{d}{dx} P_{n+1}^{(\alpha,\alpha)}(x) = \frac{1}{2}(n + 2\alpha + 2)\, P_n^{(\alpha+1,\alpha+1)}(x)$$

the n stationary points of $P_{n+1}^{(\alpha,\alpha)}$ are precisely the n zeros of $P_n^{(\alpha+1,\alpha+1)}$. Thus we may take x_{n+1}^* to be the smallest positive zero of $P_n^{(\alpha+1,\alpha+1)}$. From [3, p. 139] we have

$$x_{n+1}^* < \cos\frac{(n+\alpha+(1/2))\,\pi}{2(n+\alpha+(3/2))} = \sin\frac{\pi}{2n+2\alpha+3} < \frac{\pi}{2n+2\alpha+3}.$$

Thus

$$1 - x_{n+1}^{*2} > 1 - \frac{\pi^2}{(2n+2\alpha+3)^2}.$$

Now, for $0<a,\ b<1$ it may be shown that

$$(1-a)^b > 1 - \frac{ab}{1-a}.$$

Therefore, since $0 < -(2\alpha+1) < 1$,

$$\begin{aligned} D_{n+1}^2 &> (1 - x_{n+1}^{*2})^{-(2\alpha+1)} \\ &> \left(1 - \frac{\pi^2}{(2n+2\alpha+3)^2}\right)^{-(2\alpha+1)} \\ &> 1 + \frac{(2\alpha+1)\,\pi^2}{(2n+2\alpha+3)^2 - \pi^2} \\ &> \frac{n+2\alpha+2}{n+1}, \qquad \text{provided } n \geqslant 4. \end{aligned}$$

The proviso for this last inequality is algebraically obtained as

$$n > \max_{-1<\alpha<-1/2} ((1/8)(\pi^2 - 12 - 8\alpha) + (\pi/8)(\pi^2 + 8 - 16\alpha)^{1/2}) > 3.019.$$

For $n=2$ the maximum value of $P_3^{(\alpha,\alpha)}$ can be evaluated ($P_3^{(\alpha,\alpha)}((2\alpha+5)^{-1/2}) = (1/6)(\alpha+2)(\alpha+3)(2\alpha+5)^{-1/2}$) so that from (3.12)

$$D_3^2 = \frac{8(\alpha+2)}{3(2\alpha+5)} > \frac{2\alpha+4}{3},$$

the inequality being valid for $-1<\alpha<-1/2$.
Thus for all even integers n we have

$$D_{n+1} > \left(\frac{n+2\alpha+2}{n+1}\right)^{1/2},$$

so that from (3.14) and (3.3) it follows that Condition (2.6) is satisfied. This completes case 2.

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