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Gromov's Theorem on Groups of Polynomial Growth and Elementary Logic

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INTRODUCTION

In the fall of 1980 the authors attended Professor Tits' course at Yale University in which he gave an account of Gromov's beautiful proof that every finitely generated group of polynomial growth has a nilpotent subgroup of finite index.

An essential part of Gromov's argument consists of constructing for each group of polynomial growth a locally compact metric space and an action of a subgroup of finite index on that space. The intuitive motivation underlying this construction is fairly clear but it required an elaborate theory of "limits" of metric spaces to be carried out.

It occurred to us to give a simple nonstandard definition of a space which has all the nice properties needed in the rest of Gromov's argument. Besides shortening proofs our construction works for arbitrary finitely generated groups, not only for those of polynomial growth, and it has functorial properties. This enables us to state some of Gromov's lemmas without the restriction of polynomial growth, e.g., (4.2) and (5.5).

We also found a new proof of local compactness of the space, see Section 6, under an a priori weaker hypothesis than polynomial growth, and this led to a slight extension of Gromov's theorem:

If the group Γ with finite generating set X has growth function G_X with $G_X(n) \leqslant c \cdot n^d$ for infinitely many n and positive constants c, d, then Γ has a nilpotent subgroup of finite index. (Gromov's hypothesis is that $G_X(n) \leqslant c \cdot n^d$ for *all* $n > 0$.)

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We have tried to make this paper reasonably self-contained: For the reader's convenience we give all the basic definitions and repeat arguments which occur in the literature.

Sections 1 and 2 contain a proof of the main theorem just quoted, Theorem (1.10), *modulo* a demonstration of the basic properties of the space attached to any finitely generated group. (These properties are only summarized in Section 2.) In Section 3 we define nonstandard extensions and describe its properties, concentrating on those we need later. (This section may seem a bit long, but we are confident that together with the rest of the paper, it will help readers not versed in the subject to acquire an understanding of how nonstandard extensions are actually used in various situations.) In Section 4 we give our (nonstandard) space construction, and in Sections 5 and 6 we derive the properties of the space we had used before in Section 2 in the proof of the main theorem.

In Section 7 we show how another simple application of logic gives an *algorithm*, based on trial and error, to compute bounds related to Gromov's theorem, where previously only the existence of bounds was known.

For other accounts of Gromov's theorem and geometric applications we refer the reader to the original paper [5] and to Tits' Bourbaki seminar lecture [14].

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1. Preliminaries and Previous Results

(1.1) Let Γ be a group generated by a *finite* subset X.
The *length function* $|\ | = |\ |_X : \Gamma \to \mathbb{N}$ is defined as follows:

$$|g| = \text{length of shortest word in } X \cup X^{-1} \text{ representing } g.$$

Properties

(i) $|g| = 0 \Leftrightarrow g = e$ (the empty word represents the identity e).

(ii) $|g| = |g^{-1}|$.

(iii) $|gh| \leqslant |g| + |h|$.

The norm-like properties of $|\ |$ give rise to a metric $d = d_X : \Gamma \times \Gamma \to \mathbb{N}$, defined by $d(g, h) = |g^{-1}h|$. Note that d is invariant under left multiplication: $d(ag, ah) = d(g, h)$.

(1.2) We define the *growth function*

$$G = G_X : \mathbb{N} \to \mathbb{N} \text{ of } (\Gamma, X)$$

by:

> $G(n) = \#B_e(n) =$ number of group elements representable as words in $X \cup X^{-1}$ of length $\leqslant n$.

Here and in the following, $B_p(r)$ denotes the *closed* ball of raldius r and center p in a given metric space.

(1.3) EXAMPLES (from [5]).

(a) $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$, $X = \{(1, 0), (0, 1)\}$. Then $G(n) = 2n^2 + 2n + 1$.

(b) $\Gamma =$ free group on $X = \{a, b\}$, $a \neq b$. Then $G(n) = 2 \cdot 3^n - 1$.

(1.4) The two examples illustrate two different ways in which a group can grow. This is formalized in the notions (1), (2) below due to Milnor, who introduced them in connection with problems in differential geometry.

(1.5) DEFINITIONS. (1) Γ is of *growth degree* $\leqslant d$ $(d \in \mathbb{N})$ if there is $c > 0$ such that $G(n) \leqslant c \cdot n^d$ for $n = 1, 2, 3, \ldots$.

Γ is of *polynomial growth* if Γ is of growth degree $\leqslant d$, for some d.

(2) Γ is of *exponential growth* if there is $c > 1$ such that $G(n) \geqslant c^n$ for $n = 1, 2, \ldots$.

For our strengthening of Gromov's theorem we also define:

(3) Γ is of *near growth degree* $\leqslant d$ $(d \in \mathbb{N})$ if there is $c > 0$ such that $G(n) \leqslant c \cdot n^d$ for infinitely many n. Γ is of *near polynomial growth* if there is d such that Γ is of near growth degree $\leqslant d$.

(1.6) *Remarks.* (i) Let us first check that these notions are independent of the finite generating set X. Indeed, let X' also be a finite set of generators for Γ. Put $b = \max\{|x'|_X : x' \in X'\}$. Then clearly $G_{X'}(n) \leqslant G_X(bn)$, and reversing the roles of X and X' gives a similar inequality. This shows that being of (near) growth degree $\leqslant d$ does not depend on the choice of X. To prove this for the notion of exponential growth, suppose that $G_{X'}(n) \geqslant (c')^n$ for some $c' > 1$ and all $n \geqslant 1$. Then $G_X(n) \geqslant G_{X'}([n/b]) \geqslant (c')^{[n/b]} \geqslant c^n$ for some $c > 1$ and all $n \geqslant 1$.

(ii) Let H be a finitely generated (f.g. for short) subgroup of Γ. Taking X so that $X \cap H$ generates H we get $G_{X \cap H} \leqslant G_X$. Hence, if Γ is of (near) growth degree $\leqslant d$, so is H; similarly, if H is of exponential growth, so is Γ.

(iii) Let H be a subgroup of finite index in Γ. Take a finite set Y of generators for H and adjoin to it a set of coset representatives of Γ/H to obtain a generating set X for Γ. Then, see [14], there is a positive integer b

such that $G_X(n) \leqslant \#(X) \cdot G_Y(bn)$ for $n = 1, 2, 3, \ldots$. Hence, H has (near) growth degree $\leqslant d$ (resp. exponential growth) if and only if Γ has.

(1.7) The easy fact that f.g. abelian groups are of polynomial growth was generalized by Wolf as follows:

A f.g. nilpotent group has polynomial growth.

(The precise growth degree was given by Bass; see [14].)

(1.8) Milnor and Wolf also proved [11]:

If Γ is solvable, then Γ is either of exponential growth or has a nilpotnent subgroup of finite index.

(1.9) These theorems characterize the groups of polynomial growth among the f.g. solvable groups. Gromov managed to remove the hypothesis of solvability, cf. [5].

If Γ is of polynomial growth, it has a nilpotent subgroup of finite index.

We will slightly weaken the hypothesis of Gromov's theorem and prove:

(1.10) *If Γ is of near polynomial growth, it has a nilpotent subgroup of finite index.*

(1.11) A rough sketch of Gromov's remarkable proof is as follows: Consider the sequence of discrete metric spaces $(\Gamma, (1/n)\, d)$. (As n increases one moves, so to speak, away from the space (Γ, d) so that its points seem to get closer together.) In case Γ is of polynomial growth Gromov shows that some subsequence $(\Gamma, (1/n_i)\, d)$ "converges" to a metric space Y with the following properties:

(i) Y is homogeneous (for any two points there is an isometry carrying one to the other).

(ii) Y is connected and locally connected.

(iii) Y is complete.

(iv) Y is locally compact and finite dimensional.

From the solution of Hilbert's fifth problem, it then follows that the isometry group of Y is a Lie group. Now one can let a subgroup of finite index of Γ act on Y in such a way that, using that Isom(Y) is a Lie group and theorems of Jordan and Tits on linear groups, one obtains a homomorphism of this subgroup onto $\mathbb{Z}$ (assuming Γ is infinite). It then follows that the kernel is of polynomial growth of lower degree. An inductive assumption allows us to conclude that Γ has a solvable subgroup of finite index so that an application of the theorem of Milnor–Wolf finishes the proof.

(1.12) Our proof of (1.10) follows the same lines. The difference is mainly in the construction of the space Y, which we obtain in Section 4 by a very simple and general nonstandard argument.

The next section, Section 2, just assembles the relevant properties of the space Y and shows how (1.10) follows.

2. Proof of Gromov's Theorem Assuming Properties of the Space Y

Γ continues to denote a finitely generated group (with finite generating set X). The following algebraic lemma is essentially due to Milnor.

(2.1) Lemma. *Let $1 \to K \to \Gamma \to^h \mathbb{Z} \to 0$ be exact and Γ not of exponential growth. Then K is finitely generated.*

Moreover:

(1) *If Γ has near growth degree $\leqslant d+1$, then K has near growth degree $\leqslant d$;*

(2) *if K has a solvable subgroup of finite index, then Γ has one, too.*

Proof. Take $\gamma \in \Gamma$ with $h(\gamma) = 1$, and take $e_1, \ldots, e_k \in K$ such that $\Gamma = \langle \gamma, e_1, \ldots, e_k \rangle$. Define $\gamma_{m,i} = \gamma^m e_i \gamma^{-m}$ for $m \in \mathbb{Z}$, $i = 1, \ldots, k$. Then one easily checks that K is generated by the $\gamma_{m,i}$. Fix an i in $\{1, \ldots, k\}$. For $m > 0$ consider the elements of Γ of the form $\gamma_{0,i}^{\varepsilon_0} \cdots \gamma_{m,i}^{\varepsilon_m}$, $\varepsilon_i = 0$ or 1. There are 2^{m+1} words on $\{\gamma, e_1, \ldots, e_k\}$ here, each of length $\leqslant 2m$. The assumption of nonexponential growth implies that for some $m > 0$ two of those words represent the same element, say $\gamma_{0,i}^{\varepsilon_0} \cdots \gamma_{m,i}^{\varepsilon_m} = \gamma_{0,i}^{\delta_0} \cdots \gamma_{m,i}^{\delta_m}$ and $\varepsilon_m \neq \delta_m$. Then $\gamma_{m,i} \in \langle \gamma_{0,i}, \ldots, \gamma_{m-1,i} \rangle$. Conjugating this relation by γ we see that $\gamma_{m+1,i} \in \langle \gamma_{1,i}, \ldots, \gamma_{m,i} \rangle \subset \langle \gamma_{0,i}, \ldots, \gamma_{m-1,i} \rangle$ and by induction we obtain

$$\gamma_{p,i} \in \langle \gamma_{0,i}, \ldots, \gamma_{m-1,i} \rangle \qquad \text{for all } p \geqslant 0.$$

A similar argument for negative m gives us that K is generated by a finite set

$$\{\gamma_{m,i} : 1 \leqslant i \leqslant k, |m| \leqslant M\}, \qquad M \in \mathbb{N}.$$

To prove (1), let $c > 0$ and $S \subset \mathbb{N}$ infinite such that $G_X(n) \leqslant c \cdot n^{d+1}$ for all $n \in S$. Without loss of generality, see (1.6)(i), we may assume that $X = Y \cup \{\gamma\}$, where Y generates K. Let $n \in S$ and let g_i, $i = 1, \ldots, G_Y([n/2])$ be the distinct elements in K of Y-length $\leqslant [n/2]$. Then the $n \cdot G_Y([n/2])$ elements $g_i \gamma^j$, $i = 1, \ldots, G_Y([n/2])$, $-[n/2] \leqslant j \leqslant [n/2]$ are distinct and of X-

length $\leqslant n$. So $n \cdot G_Y([n/2]) \leqslant g_X(n) \leqslant c \cdot n^{d+1}$, i.e., $G_Y([n/2]) \leqslant c \cdot n^d \leqslant c' \cdot [n/2]^d$ for a suitable constant $c' > 0$ independent of $n \in S$. This shows that K has near growth degree $\leqslant d$.

For (2), suppose that K has a solvable subgroup of finite index. Taking the intersection of all subgroups of that index we even obtain a *characteristic solvable* subgroup K' of finite index in K. Let $\Gamma' = \langle K', \gamma \rangle$. As K' is normal in Γ' we see that the kernel of $h \mid \Gamma'$ is K', in other words $K \cap \Gamma' = K'$. Also $K \cdot \Gamma' = \Gamma$, so $[\Gamma : \Gamma'] = [K : K'] < \infty$. Moreover Γ' is solvable because $1 \to K' \to \Gamma' \to \mathbb{Z} \to 0$ is exact, and K' is solvable.

(2.2) This lemma suggests that one should try to construct a morphism of Γ (or of a subgroup of finite index in Γ) onto $\mathbb{Z}$. Gromov succedes in this through the intermediary of the isometry group of a certain metric space Y attached to Γ (if Γ is of polynomial growth).

(2.3) Definition. Given a metric space Y with metric d and distinguished point e we make its isometry group $\mathrm{Isom}(Y)$ into a topological group by taking the $U_{k,\varepsilon}$ $k \in \mathbb{N}^{>0}$, $\varepsilon > 0$, as a basis of neighborhoods of the identity 1_Y, where $U_{k,\varepsilon} = \{\sigma \in \mathrm{Isom}(Y) : d(\sigma y, y) \leqslant \varepsilon$ for all y with $d(y, e) \leqslant k\}$.

Note. If Y is locally compact and homogeneous then the topology on $\mathrm{Isom}(Y)$ coincides with the so called compact-open topology, cf. [1].

(2.4) In Sections 4, 5, 6 we will prove the following basic result.

> *To each finitely generated group Γ one can associate a metric space $Y = Y(\Gamma)$ and a homomorphism $l : \Gamma \to \mathrm{Isom}(Y)$ with the following properties*:

(I) *Y is homogeneous* (for any two points there is an isometry carrying one to the other).

(II) *Y is connected and locally connected.*

(III) *Y is complete.*

(IV) *In case $l(\Gamma)$ is finite and Γ has no abelian subgroup of finite index, the group $\Gamma' =$ kernel(l) has for each neighborhood U of 1_Y a homomorphic image in $\mathrm{Isom}(Y)$ intersecting $U \backslash \{1_Y\}$.*

(V) *If Γ is of near polynomial growth, then Y is locally compact and finite dimensional.*

(VI) *If Γ is of exponential growth, then Y is not locally compact.*

In this section, we will simply assume (I)–(VI) and derive our version (1.10) of Gromov's theorem from it. First an intermediate result.

(2.5) THEOREM. *Suppose $Y = Y(\Gamma)$ is locally compact and finite dimensional, and Γ is infinite. Then Γ has a subgroup of finite index which has $\mathbb{Z}$ as a homomorphic image.*

Proof. If Γ has an abelian subgroup of finite index, the conclusion is immediate and from now on we assume that we are not in this case. The hypothesis of the theorem, together with (I), (II), (III) above, allows us to use the deep results of Gleason–Mongomery–Zippin on Hilbert's fifth problem. In fact, we use [12, 6.3] and [1, p. 606] to conclude:[1]

$\mathrm{Isom}(Y)$ *is a Lie group with finitely many connected components.* $(*)$

Let L be the connected component of the identity. So L is a connected Lie group of finite index in $\mathrm{Isom}(Y)$. We claim:

Γ contains a subgroup Δ of finite index which has arbitrarily large homomorphic images in L. $(**)$

("arbitrarily large": for each $n \in \mathbb{N}$ there is one of cardinality $\geqslant n$). The claim holds trivially if $l(\Gamma) \subset \mathrm{Isom}(Y)$ is infinite. (Take $\Delta = l^{-1}(L) \cap \Gamma$.) So from now on we suppose that $l(\Gamma)$ is finite. In particular, $\Gamma' = \mathrm{kernel}(l)$ is of finite index in Γ. As Γ has no abelian subgroup of finite index, we can use property (IV), which implies that Γ' has homomorphic images in $\mathrm{Isom}(Y)$ containing elements $\neq 1_Y$ arbitrarily close to 1_Y. Now, as a Lie group, $\mathrm{Isom}(Y)$ has the property that for each $n > 0$ a suitable neighborhood of 1_Y contains no elements $\neq 1_Y$ of order $\leqslant n$. (The "no small subgroups" property.) It follows that Γ' has arbitrarily large homomorphic images in $\mathrm{Isom}(Y)$. Now there are only finitely many subgroups of Γ' of any given index, so at least one of the subgroups of Γ' of index $\leqslant [\mathrm{Isom}(Y): L]$, say Δ, has arbitrarily large homomorphic images in L. Claim $(**)$ is proved.

Let C be the center of L. So L/C embeds into $GL_n(\mathbb{C})$, where $n = \dim(L)$ (by a fundamental property of connected Lie groups).

Consider the morphisms $\Delta \to L/C$ obtained by composing the morphisms $\Delta \to L$ with the natural map $L \to L/C$. If all of these have images of order bounded by q, say, then their kernels are subgroups of Δ of index $\leqslant q$ which have arbitrarily large images in C, and so the intersection of those kernels is a subgroup Δ' of finite index in Δ with arbitrarily large abelian homomorphic images. Hence the commutator subgroup of Δ' has infinite index in Δ', and it follows that Δ' which is of finite index in Γ and therefore finitely generated, has $\mathbb{Z}$ as a homomorphic image, so the conclusion of the theorem holds.

[1] It is useful to have some familiarity with the subject treated in [12] to see that the theorems we refer to apply.

So from now on we assume that the morphisms $\Delta \to L/C \subset GL_n(\mathbb{C})$ referred to above have arbitrarily large images. We distinguish two cases:

(a) the morphisms $\Delta \to GL_n(\mathbb{C})$ have arbitrarily large finite images.

(b) there is a morphism $\Delta \to GL_n(\mathbb{C})$ with an infinite image $\bar{\Delta}$.

In case (a), Theorem (2.5) follows by very similar arguments as above using the following theorem of Jordan [3, 36.13]:

There is an integer $q = q(n)$ such that each finite subgroup of $GL_n(\mathbb{C})$ has an abelian subgroup of index $\leqslant q$.

Case (b) is handled by a deep result of J. Tits, cf. [13]:

A finitely generated subgroup of $GL_n(\mathbb{C})$ has either a free subgroup of rank 2 or has a solvable subgroup of finite index.

If $\bar{\Delta}$ has a free subgroup of rank 2, then $\bar{\Delta}$, hence Δ and Γ are of exponential growth, which is excluded by the hypothesis of the theorem and property (VI) of (2.4). So $\bar{\Delta}$ has a solvable subgroup of finite index, and replacing, if necessary, Δ by a suitable subgroup of finite index, we may as well assume that $\bar{\Delta}$ is solvable, and that its commutator subgroup has infinite index. Then $\bar{\Delta}$, hence Δ, has $\mathbb{Z}$ as a homomorphic image. The proof of the theorem is finished.

(2.6) *Proof of* (1.10). Given that Γ has near growth degree $\leqslant d$ for some $d \in \mathbb{N}$, we have to show that Γ has a nilpotent subgroup of finite index. The proof is by induction on d.

If $d = 0$, then Γ is finite, and we are done.

Suppose Γ is of near growth degree $\leqslant d + 1$, and Γ is infinite. Now we use property (V) of (2.4), and apply Theorem (2.5) and (1.6)(iii) to reduce to the case that there is a surjective morphism $h: \Gamma \to \mathbb{Z}$. Let $K = \text{kernel}(h)$. By (2.1)(1) and the induction hypothesis K has a nilpotent, hence solvable, subgroup of finite index. By (2.1)(2) Γ has a solvable subgroup of finite index. An application of the Milnor–Wolf theorem, cf. (1.8), to this subgroup complete the proof. ∎

3. Some Introductory Remarks on Nonstandard Extensions[2]

(3.1) As already remarked in the introduction we are going to use the theory of nonstandard extensions to construct a space Y having the

[2] The reader already familiar with nonstandard methods can skip this section, although we shall occasionally refer to results described here, and use notation introduced here, in the sequel.

properties listed in (2.4). While we cannot give a full account of the foundations of this theory here, we hope that the following remarks will give an adequate idea of the principal novelty in nonstandard analysis compared to such similar sounding subjects as nonarchimedean analysis.

(3.2) The general idea then is to uniformly extend all structures under consideration (in our application these will be just $\mathbb{N}$, $\mathbb{R}$ and the group Γ) in such a way that (a) enough properties of the original structures which are relevant to the problem at hand, are preserved in the larger structures, but (b) certain iterated limit constructions performable on the original structures can be succinctly replaced by use of a single element of the larger structure which "codes" an infinite amount of information.[3] Because of its algebraic flavour we have chosen to describe how the ultrapower method achieves these aims.

(3.3) Let I be a countably infinite (index) set. Fix a nonprincipal ultrafilter D on I. That is, D is a collection of subsets of I having the following properties:

(i) D contains no finite sets.

(ii) $A \in D$ and $B \in D \Rightarrow A \cap B \in D$.

(iii) $A \in D$ and $A \subset B \subset I \Rightarrow B \in D$.

(iv) For all $A \subset I$, either $A \in D$ or $I \backslash A \in D$.

(Note that the collection of cofinite subsets of I satisfies (i)–(iii). Further, given (i)–(iii), (iv) is easily seen to be equvalent to: (iv)′ D is maximal with properties (i)–(iii). Hence nonprincipal ultrafilters exist by Zorn's lemma.)

These properties readily imply that (for $n \in \mathbb{N}$) $A_1 \cup \cdots \cup A_n = I \Rightarrow A_i \in D$ for some i, and so D can be thought of as a $\{0, 1\}$-valued, finitely additive measure defined on all subsets of I. We thus say that a property $\cdots i \cdots$ of elements of I holds p.p.i. ("for almost all i") or just "almost everywhere" if $\{i \in I: \cdots i \cdots\} \in D$.

(3.4) Now suppose S is any set (or, more precisely, any structure, i.e., with functions and relations, that may be defined on S presently). Let S^I denote the set of all functions from I to S, and identify two functions $f, g \in S^I$ if they agree almost eveywhere, i.e., if $f(i) = g(i)$ p.p.i. By (3.3) (ii), (iii) this identification is an equivalence relation, and we define $S^* = S^I/D = \{f/D: f \in S^I\}$ = the set of equivalence classes. For $s \in S$, define $\hat{s} \in S^I$ by $\hat{s}(i) = s$ $(i \in I)$. Then the map $\nu: S \to S^*: s \mapsto \hat{s}/D$ is 1–1 (by 3.3(i)). We identify S with its image under ν from now on, so that $S \subset S^*$. The set S^* is called the nonstandard extension of S (by D) and elements of $S^* \backslash S$ are

[3] We are indebted to the referee for this concise remark.

called nonstandard elements. We also sometimes refer to the elements of S as standard elements in this context. We leave the reader to verify:

(3.5) $$S = S^* \qquad \text{iff } S \text{ is finite.}$$

It is not literally true that

(3.6) $$T \subset S \Rightarrow T^* \subset S^*$$

since if $h \in T^I$ (so $h \in S^I$) then h/D evaluated in T^* is in general a proper subset of h/D evaluated in S^*. However, identifying these two equivalence classes is completely harmless (since any fuction in the first class is equal, almost everywhere, to any function in the second class) and we shall do it, so that (3.6) holds. This also implies (together with the identification of S and $\nu(S)$) that

(3.7) $$T \subset S \Rightarrow T^* \cap S = T.$$

(3.8) We can generalize (3.7) as follows. Given sets $S_1,..., S_n$ and $V \subset S_1 \times \cdots \times S_n$ define $V^* = \{\langle f_1/D,..., f_n/D\rangle \in S_1^* \times \cdots \times S_n^* : \langle f_i(i),..., f_n(i)\rangle \in V, \text{ p.p.i.}\}$.

Then $V^* \subset S_1^* \times \cdots \times S_n^*$ and it is easy to check that

(3.9) $$V^* \cap (S_1 \times \cdots \times S_n) = V.$$

Further, if V happens to be a function $S_1 \times \cdots \times S_{n-1} \to S_n$ (so we write $V(x_1,..., x_{n-1}) = x_n$ for $\langle x_1,..., x_n\rangle \in V$), then we also have

(3.10)
$$V^* \text{ is a function } S_1^* \times \cdots \times S_{n-1}^* \to S_n^* \quad \text{and} \quad V^* \restriction S_1 \times \cdots \times S_{n-1} = V.$$

In fact, $V^*(f_1/D,..., f_{n-1}/D) = V(f_1,..., f_{n-1})/D$, where $V(f_1,..., f_n) \in S_n^I$ is of course defined by $V(f_1,..., f_{n-1})(i) = V(f_1(i),..., f_{n-1}(i))$.

Note that (3.9) and (3.10) tell us that S is a sub-structure of S^* (more precisely, ν is an embedding) with respect to all functions and relations defined on S. This partially justifies remark(a) of (3.2) but we need something much stronger. (For example, we shall need to know that if $\circ$ is a group operation on S, then $\langle S^*, \circ^*\rangle$ is also a group.) To this end we define a subset W of $S_1^* \times \cdots \times S_n^*$ to be *internal* if membership to W can be computed co-ordinatewise almost everywhere, i.e., if there exists for each $i \in I$, a subset W_i of $S_1 \times \cdots \times S_n$ such that for all $f_1 \in S_1^I,..., f_n \in S_n^I$:

(3.11) $$\langle f_1/D,..., f_n/D\rangle \in W \Leftrightarrow \langle f_i(i),..., f_n(i)\rangle \in W_i \text{ p.p.i.}$$

We refer to $(W_i)_{i \in I}$ as a family of components for W. We leave the reader to check that (3.11) is a well-defined equivalence, and that if $(W_i')_{i \in I}$ is another family of components for W, then $W_i = W_i'$ p.p.i.

(3.12) It is also immediate that if we are given any family $(W_i)_{i \in I}$ of subsets of $S_1 \times \cdots \times S_n$, then (3.11) uniquely defines a (necessarily internal) subset of $S_1^* \times \cdots \times S_n^*$ with components W_i.

Note that if $V \subset S_1 \times \cdots \times S_n$, then V^* is an internal subset of $S_1^* \times \cdots \times S_n^*$ (take all the components to be V), but in general not all internal sets are of this form.

The definition of *internal function* can be obtained from (3.11) (as (3.10) was from (3.9)) and turns out to be equivalent to:

(3.13) $F: S_1^* \times \cdots \times S_{n-1}^* \to S_n^*$ is internal iff there exists for each $i \in I$ a function $F_i: S_1 \times \cdots \times S_{n-1} \to S_n$ such that for all $f_1 \in S_1^I, \ldots, f_{n-1} \in S_{n-1}^I$ $F(f_1/D, \ldots, f_n/D) = (i \to F_i(f_1(i), \ldots, f_{n-1}(i)))/D$.

We shall need the following lemmas later; they are examples of remark (b) of (3.2).

(3.14) LEMMA. (i) *Suppose W is an internal subset of S^*, with components W_i, and $n \in \mathbb{N}$ and $\#W_i \leqslant n$* p.p.i. *Then $\#W \leqslant n$.*

(ii) *No infinite subset of S is an internal subset of S^*.*

Proof. We leave the proof of (i) to the reader. For (ii) suppose $A \subset S$, A infinite and internal. Let $(A_i)_{i \in I}$ be a family of components for A. Suppose $a_1, a_2, \ldots, a_n, \ldots$ are distinct elements of A, and say $I = \{i_1, i_2, \ldots, i_n, \ldots\}$ (recall that I is countable). Define $f \in S^I$ by $f(i_n) = a_j$, where j is maximal such that $a_j \in A_{i_n}$, if j exists, a_{n+j} otherwise, where j is minimal such that $a_{n+j} \in A_{i_n}$; since $f(i) \in A_i$ for all $i \in I$, we have $f/D \in A$. Therefore $f/D = a_m$ for some $m \in \mathbb{N}$, i.e., $f/D = \hat{a}_m/D$, i.e., $f(i) = a_m$ p.p.i. However, this clearly implies $a_{m+1} \in A_i$ p.p.i., so $\hat{a}_{m+1}/D \notin A$, i.e., $a_{m+1} \notin A$—contradiction. ∎

(3.15) LEMMA. *Suppose $g_n \in S^*$ for $n \in \mathbb{N}$. Then there is an internal function $F: \mathbb{N}^* \to S^*$ such that $F(n) = g_n$ for all $n \in \mathbb{N}$.* (We make no claim here for the values of $F(n)$ when n is nonstandard, except of course that they lie in S^*.)

Proof. Say $g_n = f_n/D$, where $f_n \in S^I$, for $n \in \mathbb{N}$. For each $i \in I$ define the function $F_i: \mathbb{N} \to S$ by $F_i(n) = f_n(i)$ ($n \in \mathbb{N}$). Let F be the function $\mathbb{N}^* \to S^*$ (necessarily internal—see (3.2)) with components $\{F_i : i \in I\}$. Then for $n \in \mathbb{N}$,

$$F(\hat{n}/D) = \frac{i \mapsto F_i(\hat{n}(i))}{D} \qquad \text{(by 3.13)},$$

$$= \frac{i \mapsto F_i(n)}{D} \qquad \text{(by definition of } \hat{n}\text{)},$$

$$= \frac{i \mapsto f_n(i)}{D} \qquad \text{(by definition of } F_i\text{)},$$

$$= f_n/D = g_n, \qquad \text{which gives us the required result, by the identification of } \hat{n}/D \text{ with } n. \quad \blacksquare$$

(3.15) There comes a time in any exposition of nonstandard analysis when one cannot avoid some simple logical distinctions. In our approach, the force of remark (a) of (3.2) is contained in Łoš's theorem, which states that any property of a structure, that can be expressed in the language of elements and sets, is preserved (when suitably reconstrued) to the nonstandard extension of that structure.[4] To (roughly) explain this let us fix sets $S_1, \ldots, S_n$ in whose structure we are interested. Suppose $W_1, \ldots, W_m$ are internal sets (i.e., each W_j is an internal subset of some finite Cartesian product of the S_k^*'s) and $f_1/D, \ldots, f_k/D$ are each an element of some S_k^*. Suppose Φ is some property of $W_{1i}, \ldots, W_{mi}, f_1(i), \ldots, f_l(i)$ (the ith components), which can be expressed over the S_k's in the language of sets and elements. That is Φ can be expressed using the set quantifiers "$\exists X \subset \Pi$," "$\forall X \subset \Pi$" (where Π is some finite Cartesian product of the S_k's), the quantifiers "$\exists x \in \Pi$," "$\forall x \in \Pi$," equality "$=$" membership "$\in$," and the usual Boolean operations "$\wedge$" (and), "$\vee$" (or), "$\neg$" (not), "$\rightarrow$" (implies). If the set quantifiers are not needed to express Φ, then Φ is called *elementary*.

(3.16) Examples. (i) Suppose $S_1 = \mathbb{R}$, and $W_1 = \leqslant^*$, ($\leqslant$ denotes the usual ordering of $\mathbb{R}$—regarded as a subset of $\mathbb{R} \times \mathbb{R}$, but we write "$x \leqslant y$" for "$\langle x, y \rangle \in \leqslant$"— and $\leqslant^*$ denotes its nonstandard extension as given by (3.8)) so we take each component, W_{1i}, of $\leqslant^*$ to be just $\leqslant$. Let Φ_1 be the property "$\leqslant$ is a total ordering." Then Φ_1 is elementary, since it may be expressed as: $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}((x \leqslant y \wedge y \leqslant x) \rightarrow x = y) \wedge \forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \forall z \in \mathbb{R}\ ((x \leqslant y \wedge y \leqslant z) \rightarrow x \leqslant z) \wedge \forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x \leqslant y \vee y \leqslant x)$.

(ii) Consider now the property, Φ_2, of $\leqslant$, which says "$\leqslant$ is complete," i.e., "every nonempty subset of $\mathbb{R}$ with an upper bound, has a supremum." This can be written as: $\forall X \subset \mathbb{R}[(\exists x \in \mathbb{R}(x \in X) \wedge \exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ (x \in X \rightarrow x \leqslant y)) \rightarrow \exists z \in \mathbb{R}\ (\forall x \in \mathbb{R}\ (x \in X \rightarrow x \leqslant z) \wedge \forall t \in \mathbb{R}\ ((t \leqslant z \wedge \neg t = z) \rightarrow \exists u \in \mathbb{R}\ (u \in X \wedge t \leqslant u)))]$, which shows Φ_2 to be expressible in the language of elements and sets.

(iii) We leave the reader to write out in the language of elements and sets, the property of $\leqslant$ and $\mathbb{N}$ (Φ_3, say) which expresses "every nonempty subset of $\mathbb{N}$ has a least element."

(3.17) Let Φ^*, the "nonstandard interpretation of Φ", be that property

[4] Łoš's theorem is also true for higher order languages, but we shall not need this fact in our proof of Gromov's theorem, so we do not discuss it here.

of $W_1,\ldots, W_m, f_1/D,\ldots,f_l/D$ which results from Φ by changing "$\exists x \in S_k$", "$\forall x \in S_k$" to "$\exists x \in S_k^*$," "$\forall x \in S_k^*$," respectively, and "$\exists X \subset \Pi$," "$\forall X \subset \Pi$" to "there is an *internal* subset of Π^*..." and "for all internal subsets of Π^*...," respectively; here, if Π is, say $S_1 \times S_2$, then Π^* is $S_1^* \times S_2^*$. (For example, let us see what Φ_1^*, Φ_2^*, and Φ_3^* (of (3.16)) say. Φ_1^* asserts that the relation $\leqslant^*$ totally orders $\mathbb{R}^*$; Φ_2^* asserts that every nonempty internal subset of $\mathbb{R}^*$ which is ($\leqslant^*$ –) bounded above has a ($\leqslant^*$ –) supremum; Φ_3^* asserts that every nonempty internal subset of $\mathbb{N}^*$ has a ($\leqslant^*$ –) least element.) Then Łoš's theorem states:

(3.18) *$W,\ldots, W_m$, $f_1/D,\ldots,f_l/D$ have property Φ^* iff $W_{1i},\ldots, W_{mi}$, $f_1(i),\ldots,f_l(i)$ have property Φ* p.p.i. *In particular, Φ holds of* (the standard sets and elements) $V_1,\ldots, V_m$, $s_1,\ldots, s_l$ *iff Φ^* holds of* $V_1^*,\ldots, V_m^*$ $s_1,\ldots, s_l$.

A full discussion of Łoš's theorem (in the elementary case) may be found in [2]; see also [9, Chap. 1]. However, the proof of (3.18) for some particular Φ's conveys the flavour of the general result.

Since Φ_1 (of (3.16)) holds of $\leqslant$, we must show Φ_1^* holds of $\leqslant^*$, i.e., we must show $\leqslant^*$ totally orders $\mathbb{R}^*$. So suppose $x, y \in R^*$, $x \leqslant^* y$ and $y \leqslant^* x$. Say $x = f/D$, $y = g/D$. Then $f(i) \leqslant g(i)$ p.p.i. and $g(i) \leqslant f(i)$ p.p.i. (by definition of $\leqslant^*$). Hence by (3.3)(ii), (iii), $f(i) = g(i)$ p.p.i., so $x = y$. We leave the proof of the other two conjuncts in Φ_1^* (the third requires (3.3)(iv)) to the reader.

Let us now show that Φ_2^* (of 3.16)) holds of $\leqslant^*$. Let $X \subset \mathbb{R}^*$ be internal and assume $f/D \in \mathbb{R}^*$ is an ($\leqslant^*$ –) upper bound for X. Let $(X_i)_{i\in I}$ be a family of components for X. We claim that X_i is bounded above (in $\mathbb{R}$) p.p.i. For otherwise X_i would be unbounded above p.p.i. (by (3.3)(iv)) and so we could choose, for each $i \in \{i \in I\colon X_i$ unbounded above$\}$ an element $g(i) \in X_i$ such that $f(i) < g(i)$. Setting $g(i) = 0$ (say) if X_i is bounded above, gives $f(i) < g(i)$ p.p.i., and $g(i) \in X_i$ p.p.i., and hence $f/D <^* g/D$ and $g/D \in X$, which contradicts the assumption that f/D is an $\leqslant^*$ – upper bound for X.

Now define

$$\eta \in R^I \quad \text{by} \quad \eta(i) = \begin{cases} \sup X_i & \text{if} \quad X_i \text{ is } (\leqslant -) \text{ bounded above,} \\ 0 \quad (\text{say}) & \text{otherwise.} \end{cases}$$

We leave the reader to check that η/D is the $\leqslant^*$ – supremum of X.

As a further example, we recommend the exercise of proving (3.18) for the property Φ_3 of (3.16).

We hope these examples go some way towards convicing the reader why the definition of internal set, and the restriction of set quantifiers to these sets, guarantees the truth of (3.18). Of course, in using (3.18) we shall not always write out the property Φ under consideration in strict logical notation, since we hope it will be fairly clear what Φ^* is saying.

Indeed, if Φ is in fact elementary, then Φ^* expresses the same property of the nonstandard extension as Φ does of the original structure, although more care must be taken if Φ is not known to be elementary. For example, it is *not* the case that every subset of $\mathbb{R}^*$, which is bounded above, has a supremum as we shall see below.

(3.19) An immediate corollary of (3.18) is that any subset of $S_1^* \times \cdots \times S_n^*$ which can be defined from internal sets using (our restricted) quantifiers and boolean operations is also internal and hence, if it is infinite, must contain a nonstandard element (by (3.14)(ii)). This latter phenomenon is called overspill. It is crucial in many applications of nonstandard analysis because the nonstandard elements that arise in this way often turn out to do the coding mentioned in remark (3.2)(b).

(3.20) We now look more closely at the structure of $\mathbb{R}^*$ and Γ^* (the nonstandard extension of the group Γ) in the light of (3.16). For convenience of giving examples, we take our index set I to be the set of natural numbers.

(3.18) implies that the nonstandard extension to $\mathbb{R}^*$: $+^*, -^*, \cdot^*, \leqslant^*$, etc., of the usual operations and relations on $\mathbb{R}$, makes $\mathbb{R}^*$ into an ordered field (this is an elementary property). Suppose $\eta \in \mathbb{R}^*$. If $-r \leqslant^* \eta \leqslant^* r$ for some $r \in \mathbb{R}$, η is called *finite*; otherwise, it is called *infinite*. For example, $i \mapsto 3 + (1/i + 1)/D$ is a finite element of $\mathbb{R}^*$ (which is not in $\mathbb{R}$); $i \mapsto i/D$ is infinite (by (3.3)(i)). If $-r \leqslant^* \eta \leqslant^* r$ for *all* positive $r \in \mathbb{R}$, η is called infinitesimal. Thus $i \mapsto 1/i + 1/D$ is an example of a nonzero infinitesimal. Define $\mathbb{R}^{\text{fin}} = \{\eta \in \mathbb{R}^*: \eta \text{ finite}\}$, $\mathbb{R}^0 = \{\eta \in \mathbb{R}^*: \eta \text{ infinitesimal}\}$. Warning: these sets are *not* internal subsets of $\mathbb{R}^*$. (Proof: They are both bounded above (in $\mathbb{R}^*$) but neither has a supremum.) Clearly $\mathbb{R} \subseteq \mathbb{R}^{\text{fin}}$. We leave the reader to check that $\mathbb{R}^{\text{fin}}$ is a subring of $\mathbb{R}^*$, that $\mathbb{R}^0$ is a maximal ideal in $\mathbb{R}^{\text{fin}}$ and that the map $\rho: \mathbb{R} \to \mathbb{R}^{\text{fin}}/\mathbb{R}^0: r \mapsto r + \mathbb{R}^0$ is a (field) isomorphism. Let $h: \mathbb{R}^{\text{fin}} \to \mathbb{R}^{\text{fin}}/\mathbb{R}^0$ be the natural homomorphism. The homomorphism $\rho^{-1} \circ h: \mathbb{R}^{\text{fin}} \to \mathbb{R}$ is called the standard-part map and is usually denoted by st. Thus for each $\eta \in \mathbb{R}^{\text{fin}}$, st$(\eta)$ is the unique real number "infinitesimally close to η", i.e., it satisfies st$(\eta) - \eta \in \mathbb{R}^0$.

Let us also note here that $\mathbb{N}^* \cap \mathbb{R}^{\text{fin}} = \mathbb{N}$, so that all nonstandard elements of $\mathbb{N}^*$ are infinite.

(3.20) We now investigate Γ^*. Let us suppose Γ is infinite so that $\Gamma^* \neq \Gamma$ (by (3.5)). By (3.18) Γ^* is certainly a group under the nonstandard extension, $\circ^*$, of the group operation, $\circ$, on Γ (the reader may like to verify this directly from (3.10)) and Γ is a subgroup of Γ^* by (3.10). Now suppose X is a *finite* generating set for Γ. By (3.5) $X = X^*$; but now we appear to have a conflict with (3.18). While the statement "X generates Γ" is true, the statement "X^* (i.e., X) generates Γ^*" cannot be literally true (since $\Gamma \subsetneq \Gamma^*$). The clue is that "$X$ generates Γ" is not an elementary statement about X and

Γ (see the remarks immediately proceeding section (3.19)). It turns out that this statement can be adequately expressed in the language of elements and sets. In this way, we discover the appropriate nonstandard interpretation. The following formulation of "X generates Γ" will do:

> "for each $g \in \Gamma$, there is $n \in \mathbb{N}$ and functions $x: \{m \in \mathbb{N}: m < m\} \to X \cup X^{-1}$, $y: \{m \in \mathbb{N}: m \leqslant n\} \to \Gamma$ such that $y(0) = e$ (the identity of Γ) and for all $m < n, y(m+1) = y(m) \circ x(m)$, and $y(n) = g$." (In ordinary notation, $y(m) = x(0)\,x(1)\,x(2) \cdots x(m-1)$, $g = y(n) = x(0)\,x(1) \cdots x(n-1)$.)

The nonstandard interpretation of "X generates Γ" (which must be true by (3.18)) should now be clear. It states that if $g \in \Gamma^*$, then there is $n \in \mathbb{N}^*$ and *internal* functions $x: \{m \in \mathbb{N}^*: m \leqslant^* n\} \to X \cup X^{-1}$ and $y: \{m \in \mathbb{N}^*: m \leqslant^* n\} \to \Gamma^*$ such that for all $m <^* n$ $(m \in \mathbb{N}^*)$, $y(m+1) = y(m) \circ^* x(m)$ and $y(n) = g$. (Of course, if $g \in \Gamma^* \backslash \Gamma$, then the n here will be in $\mathbb{N}^* \backslash \mathbb{N}$, i.e., it will be infinite.)

In other words, every element, g, of Γ^* may be written as an "internal word" (the function x) in the generators, although the length of this word may be an "infinite natural number." (The function y, of course, will enumerate the "initial segments" of this word.) Note also that the set of all $n \in \mathbb{N}^*$ for which such a word exists (for fixed $g \in \Gamma^*$) is internal (by the remarks at the beginning of (3.19)) so it will have $a \leqslant^* -$ least element (see Φ_3^* of (3.17)), and this element will of course be $|g|^*$, i.e., the value of non-standard extension of the length function, $|\ |^*: \Gamma^* \to \mathbb{N}^*$, applied to g (again, this follows by a suitable aplication of (3.18)).

(3.21) The preceding discussion illustrates a principal difference between nonstandard extensions and more familiar extensions obtained by forming, say, completions, or by identifying "infinitesimal" objects like tangent vectors with derivations on a local ring of functions. What the latter do not do is to pick out *infinite integers* which are constantly used in nonstandard analysis for "counting," "coding," etc. We recognize the importance of the "familiar" extensions: they generally give *canonical* objects. But then it is to be noted that these objects can always be interpreted quite intuitively as (equivalence classes of) suitably chosen objects in a nonstandard extension. In particular, the points of the space we are going to associate in the next section to Γ will be equivalence classes of elements in Γ^*. We shall use nonstandard counting in Section 6 to obtain the local compactness of that space, if Γ has near polynomial growth.

4. Nonstandard Construction of the Space Y

(4.1) We now return to the situation of Section 1, so that Γ is a group with finite generating subset X, and length function $|\ |: \Gamma \to \mathbb{N}$. Our aim in this section is to construct a space Y having the properties listed in (2.4). To this end we consider (uniform) non-standard extensions, Γ^*, $\mathbb{R}^*$, $\mathbb{N}^*$ etc., of Γ, $\mathbb{R}$, $\mathbb{N}$, as described in Section 3, although we shall now use the same symbols to denote the nonstandard extensions of familiar functions and relations defined on these sets. (For example, we shall use just $|\cdot|$ for $|\cdot|^*$, $\leqslant$ for $\leqslant^*$.)

Fix a positive infinite hyperreal number R, i.e., $R \in \mathbb{R}^*$ and $R > n$ for all $n \in \mathbb{N}$, and define $\Gamma^{(R)}$ as the subgroup $\{g \in \Gamma^* \mid |g|/R \leqslant c$ for some $c > 0$, $c \in \mathbb{R}\}$ of Γ^*, and let $\mu = \mu^{(R)}$ be the subgroup $\{g \in \Gamma^* \mid |g|/R \leqslant c$ for all $c \in \mathbb{R}$, $c > 0\}$ of $\Gamma^{(R)}$. The quotient $|\cdot|/R$ defines a map $\Gamma^{(R)} \to \mathbb{R}^{\text{fin}} = \{x \in R^*: -c \leqslant x \leqslant c$, some $c \in \mathbb{R}\}$, and clearly $|g|/R - |h|/R$ is infinitesimal, whenever $g\mu = h\mu$ in the set of *left* cosets $\Gamma^{(R)}/\mu$. So we can factor out μ and apply the standard map st: $\mathbb{R}^{\text{fin}} \to \mathbb{R}$ to obtain a commuting diagram:

$$\begin{array}{ccc} \Gamma^{(R)} & \xrightarrow{|\cdot|/R} & \mathbb{R}^{\text{fin}} \\ \downarrow & & \downarrow \text{st} \\ \Gamma^{(R)}/\mu & \xrightarrow{\|\cdot\|} & \mathbb{R} \end{array} \qquad \|g\mu\| = \text{st}(|g|/R).$$

(Note that the construction here is rather similar to the discussion of $\mathbb{R}$ in (3.20).)

From the definitions it follows that $\|g\mu\| = 0 \Leftrightarrow g \in \mu$, so by putting $\bar{d}(g\mu, h\mu) = \|g^{-1}h\mu\|$, we obtain a metric space $(\Gamma^{(R)}/\mu, \bar{d})$ which we denote by $Y^{(R)}$, or simply by Y if no confusion results. The reader can easily verify that example (1.3)(a) leads to the space $Y = (\mathbb{R}^2$, *N.Y.*-distance).

(4.2) Proposition. *The metric space Y has the following properties.*

(a) *Y is homogeneous*;

(b) *for each two points $p, q \in Y$ with $\bar{d}(p,q) = r$ there is an isometry of $[0, r]$ into Y sending 0 to p and r to q; so Y is connected and locally connected*;

(c) *Y is complete.*

Proof. (a) $\Gamma^{(R)}$ acts on Y on the left by isometries:

$$\bar{d}(ag\mu, ah\mu) = |h^{-1}a^{-1}ag\mu| = |h^{-1}g\mu| = \bar{d}(g\mu, h\mu),$$

and clearly this action is transitive.

(b) For simplicity we assume that $p = e\mu$, $q = g\mu$ and $\bar{d}(p, q) = \mathrm{st}(|g|/R) = 1$ (the general case is very similar). Replacing if necessary g by another member of $g\mu$, we may assume that $|g| = [R]$, $[\ ]$ denoting the integral part operator defined on $\mathbb{R}^*$ (with values in $\mathbb{Z}^*$). So g has a shortest representation as a (nonstandard) word $g_1 \cdots g_{[R]}$, where all $g_i \in X \cup X^{-1}$. (See the discussion in (3.20).) We define f: $[0, 1] \to Y$ by $f(r) = g_1 \cdots g_{[rR]}$. A straightforward computation shows that f is the required isometry.

(c) Let $(g_n\mu)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. For simplicity we assume $\|g_n\mu\| \leqslant 1$ for all n. As in (b), there is no loss of generality in further assuming that $|g_n| \leqslant R$. Extend $(g_n)_{n\in\mathbb{N}}$ to an internal sequence $(g_n)_{n\in\mathbb{N}^*}$, this being possible by Lemma (3.15). For each $k \in \mathbb{N}^{>0}$, take $M(k) \in \mathbb{N}$ such that $|g_m^{-1}g_n| < R/k$ for all (standard) integers $m, n > M(k)$. By "overspill" this remains true for all $m, n \in \mathbb{N}^*$ greater than $M(k)$ but less than some infinite $N(k) \in \mathbb{N}^*$. (See (3.19). We are applying the remark there to the internal set $\{t \in \mathbb{N}^*: \forall m \in \mathbb{N}^*\ \forall n \in \mathbb{N}^*(M(k) < m, k < t \to |g_m^{-1} g_n| < R/k)\}$.) By a similar argument using (3.15) and (3.19) there is $\omega \in \mathbb{N}^*$ greater than all $M(k)$ and less than all $N(k)$, $k \in \mathbb{N}^{>0}$. Clearly we have $\lim g_n\mu = g_\omega\mu$ in Y. ∎

(4.3) *Remarks.* (1) We have to keep in mind that the metric space Y depends not only on the hyperreal number R but also on the generating set X, so let us (temporarily) write Y_X and its metric as $\bar{d}_X$. Take another finite generating set X' (but keep the same R). Then, with

$$C = \max(\{|g|_X : g \in X'\} \cup \{|g|_{X'} : g \in X\})$$

we have (for $\Gamma \neq \{e\}$):

$$C^{-1}|g|_X \leqslant |g|_{X'} \leqslant C|g|_X \qquad \text{for all } g \in \Gamma, \qquad \text{hence for all } g \in \Gamma^*.$$

So $\Gamma^{(R)}$ does not change, nor does $\mu^{(R)}$, and the metric spaces Y_X and $Y_{X'}$ have the same underlying set $\Gamma^{(R)}/\mu$, and their metrics are related by: $C^{-1}\bar{d}_X \leqslant \bar{d}_{X'} \leqslant C\bar{d}_X$.

(2) The functoriality mentioned in the introduction amounts to the following: Let $\varphi: (\Gamma_1, X_1) \to (\Gamma_2, X_2)$ be a morphism of groups with distinguished finite generating sets X_1, X_2, i.e., $\varphi(X_1) \subset X_2$. Clearly $|g| \geqslant |\varphi(g)|$ for all $g \in \Gamma_1$ (the norms are taken w.r.t. the generating sets X_1 and X_2, respectively). So φ induces naturally a group morphism $\Gamma_1 \to \Gamma_2^{(R)}$ sending $\mu_1^{(R)}$ into $\mu_2^{(R)}$. Hence φ induces a map $\bar{\varphi}$: $Y_1 \to Y_2$, where Y_1, Y_2 are the spaces attached to (Γ_1, X_1, R) and (Γ_2, X_2, R), respectively. Clearly we have $\|g\mu\| \geqslant \|\bar{\varphi}(g\mu)\| = \|(\varphi g)\mu\|$, $g \in \Gamma_1^{(R)}$. So the assignments $(\Gamma, X) \to Y$, $\varphi \to \bar{\varphi}$ define a functor, still depending on R, from the category of groups with distinguished *finite* generating sets to the category of metric spaces with

distance decreasing maps. Note also that if $\varphi(X_1) = X_2$, then $\bar{\varphi}: Y_1 \to Y_2$ is surjective.

(3) It would be nice to know more about the spaces Y. It seems plausible that Y is homeomorphic with some $\mathbb{R}^n$ if Γ is nilpotent. If Γ is free on X, then the space Y is "tree-like": for any two points p, q there is essentially only one way to go from p to q; more precisely, Y contains no subspace homeomorphic with a circle.

Question. Do there exist finitely generated groups Γ for which Y is not simply connected?

5. Actions of Subgroups of Γ on Y

(5.1) The purpose of this section is to establish property (IV) of Section 2. As we saw in the proof of (4.2)(a) the group $\Gamma^{(R)}$ acts isometrically by left multiplication on $Y = \Gamma^{(R)}/\mu$, and this action naturally induces a morphism $\Gamma^{(R)} \to \text{Isom}(Y)$, which we shall write as $\gamma \mapsto l_\gamma$. Unfortunately, many $\gamma \in \Gamma$ may act trivially on Y: the kernel of l is the largest normal subgroup of $\Gamma^{(R)}$ contained in μ. For instance, if Γ is abelian, this kernel is μ itself, so contains Γ.

However, under the hypotheses of property (IV) in Section 2 the kernel Γ' of $l \mid \Gamma$ (of finite index in Γ) has a *conjugate* $\beta^{-1}\Gamma'\beta$, $\beta \in \Gamma^*$, which is contained in $\Gamma^{(R)}$ and acts "usefully" on Y. As $\Gamma' \cong \beta^{-1}\Gamma'\beta$, the action of $\beta^{-1}\Gamma'\beta$ can be transported to an action of Γ' on Y.

(5.2) In the rest of this section we assume that $l(\Gamma) \subset \text{Isom}(Y)$ is finite (one of the assumptions in (IV), Section 2), and we put $\Gamma' = \text{kernel}(l \mid \Gamma')$. So Γ' is of finite index in Γ, and we fix a finite generating set S of Γ' such that $S^{-1} \subset S$.

(5.3) Lemma. *If Γ has no abelian subgroup of finite index, then the set of lengths $|\gamma^{-1}s\gamma|$ $(\gamma \in \Gamma', s \in S)$ is unbounded.*

Proof. Otherwise each $s \in S$ has only finitely many Γ'-conjugates, in other words, the Γ'-centralizer of s has finite index in Γ'. Therefore the center of Γ', being the intersection of the centralizers of the $s \in S$, has finite index in Γ', hence in Γ. ∎

(5.4) Before we state the next proposition which is a precise version of property (IV) of Section 2, we remind the reader that the group Isom(Y) is topologized by taking the $U_{k,\varepsilon}$, $k \in \mathbb{N}^{>0}$, $0 < \varepsilon \in \mathbb{R}$, as basic (closed) neighborhoods of 1_Y, cf. (2.3).

(5.5) PROPOSITION. *Suppose Γ has no abelian subgroups of finite index. Then for each neighborhood U of 1_Y there are $\beta \in (\Gamma')^*$ and $s \in S$ such that $\beta^{-1}\Gamma'\beta \subset \Gamma^{(R)}$ and $l_{\beta^{-1}s\beta} \in U \setminus \{1_Y\}$.*

Proof. For $\gamma \in \Gamma^*$ and $0 < r \in \mathbb{R}^*$ we put $\delta(\gamma, r) = \max\{d(\gamma a, a): |a| \leqslant r\}$ (maximum "displacement" effected by γ among the points of $B_e(r)$). We claim:

$$\delta(g^{-1}\gamma g, r) \leqslant \delta(\gamma, r) + 2\,|g| \qquad (g \in \Gamma^*). \tag{1}$$

Let $|a| \leqslant r$. Then $d(g^{-1}\gamma ga, a) = d(\gamma ga, ga) \leqslant \delta(\gamma, |g| + r) \leqslant \delta(\gamma, r) + 2\,|g|$. The last inequality follows by writing an element of $B_e(|g| + r)$ as bc with $|b| \leqslant r$, $|c| \leqslant |g|$; then

$$\begin{aligned} d(\gamma bc, bc) &\leqslant d(\gamma bc, \gamma b) + d(\gamma b, b) + d(b, bc) \\ &= d(\gamma b, b) + 2d(bc, b) \leqslant \delta(\gamma, r) + 2\,|g|. \end{aligned}$$

Inequality (1) is established.

Fix a neighborhood $U = U_{k,\varepsilon}$, $k \in \mathbb{N}^{>0}$, $\varepsilon > 0$. A nonstandard translation of Lemma (5.3) gives us $s \in S$ and $g \in (\Gamma')^*$ such that:

$$|g^{-1}sg| > \varepsilon R. \tag{2}$$

Write $g = s_1 \cdots s_t$, $s_i \in S$, $t \in \mathbb{N}^*$. For $0 \leqslant i \leqslant t$ we put:

$$g_i = s_1 \cdots s_i \qquad \text{and} \qquad M_i = \max\{\delta(g_i^{-1}sg_i, kR): s \in S\}.$$

Further we let $C = \max\{|s|: s \in S\}$, so $C \in \mathbb{N}$. Then we have:

$$M_o < \varepsilon R \qquad \text{(because } \Gamma' \text{ acts trivially on } Y\text{)}, \tag{3}$$

$$M_t > \varepsilon R \qquad \text{(by (2))}, \tag{4}$$

$$|M_{i+1} - M_i| \leqslant 2C \qquad \text{for} \quad 0 \leqslant i \leqslant t-1 \text{ (by 1))}. \tag{5}$$

From (3), (4) and (5) we derive the existence of an $i \in \{0,..., t\}$ with:

$$|M_i - \varepsilon R| \leqslant 2C. \tag{6}$$

For this i we define $\beta = g_i$, so $\beta \in (\Gamma')^*$. Note that if $\gamma \in \Gamma'$, then $\beta^{-1}\gamma\beta$ is a finite product of elements of the form $\beta^{-1}s\beta$, $s \in S$, each of which is in $\Gamma^{(R)}$ by (6), so:

$$\beta^{-1}\Gamma'\beta \subset \Gamma^{(R)}. \tag{7}$$

From (6) we also obtain the existence of $s \in S$ such that $\delta(\beta^{-1}s\beta, kR)$ differs from εR by at most $2C$. For $\sigma = l_{\beta^{-1}s\beta}$ this means that $\sigma \neq 1_Y$. Moreover, for

$|a| \leqslant kR$: $\bar{d}(\sigma(a\mu), a\mu) = \mathrm{st}(d(\sigma a, a)/R) \leqslant \varepsilon$, by (6), so $\sigma \in U$. The proposition is proved. ∎

6. Further Properties of Y Resulting from Growth Restrictions on Γ

(6.1) The properties of the space $Y = Y^{(R)}$ discussed in Sections 4 and 5 hold for any finitely generated group Γ, and any positive infinite $R \in \mathbb{R}^*$.

We now show that if Γ has near polynomial growth, then R can be chosen so that $Y^{(R)}$ is locally compact and of finite dimension. The proof of the following lemma contains the crucial argument.

(6.2) Lemma. *Let R_0 be positive infinite and suppose $G(R_0) \leqslant c \cdot R_0^d$ where $0 < c \in \mathbb{R}$, $d \in \mathbb{N}$. Then there is a positive infinite $S \leqslant R_0$, such that for every $i \in \mathbb{N}$, $i \geqslant 4$, the following property $P_i(S)$ holds*:

> $P_i(S)$: if $g_1,\ldots, g_t \in B_e(S/4)$, $t \in \mathbb{N}^*$, *and* $B_{g_1}(S/i),\ldots, B_{g_t}(S/i)$ *are pairwise disjoint, then* $t \leqslant i^{d+1}$.

Proof. Suppose the lemma is false. Thus, for all $S \in \mathbb{R}^*$ with $\log R_0 \leqslant S \leqslant R_0$ there is some $i \in \mathbb{N}$, $i \geqslant 4$, such that $P_i(S)$ fails. In fact, clearly the function mapping S to the least i such that $P_i(S)$ fails ($\log R_0 \leqslant S \leqslant R_0$), is internal, so its range must be internal; since this range is a subset of $\mathbb{N}$, it is bounded by some $K \in \mathbb{N}$ (by (3.14)(ii)). Hence we may define internally, by induction, natural numbers $i_1,\ldots, i_u$, $u \in \mathbb{N}^*$ to be chosen below, and elements $g(l,j) \in \Gamma^*$, for $1 \leqslant l \leqslant u$, $1 \leqslant j \leqslant t_l$, where $t_l = [i_l^{d+1}] + 1$, such that for $l = 1,\ldots, u$:

$$4 \leqslant i_l \leqslant K, \tag{1}$$

$$g(l,j) \in B_e\left(\frac{R_0}{4i_1 \cdots i_{l-1}}\right) \qquad \text{for} \quad 1 \leqslant j \leqslant t_l, \tag{2}$$

$$B_{g(l,j)}\left(\frac{R_0}{i_1 \cdots i_{l-1} i_l}\right) \cap B_{g(l,j')}\left(\frac{R_0}{i_1 \cdots i_{l-1} i_l}\right) = \varnothing, \qquad \text{for} \quad 1 \leqslant j < j' \leqslant t_l. \tag{3}$$

(As l goes from 1 to u, the radii $R_0/i_l \cdots i_{l-1}$ represent decreasing values of S.) Clearly the obvious inductive definition of the i_l's and $g(l,j)$'s may proceed as long as the condition $P_i(R_0/i_1 \cdots i_{l-1})$ fails, and this will be guaranteed if we choose u to satisfy

$$\frac{R_0}{i_1 \cdots i_{u-1}} \geqslant \log R_0 > \frac{R_0}{i_1 \cdots u_{u-1} i_u}. \tag{4}$$

Now let $T = \{(s_1, ..., s_u) : s_l \in \mathbb{N},\ 1 \leqslant s_l \leqslant t_l, \text{ for } l = 1, ..., u\}$, and for $s = (s_1, ..., s_u) \in T$ define $g_s = g(1, s_1)\, g(2, s_2) \cdots g(u, s_u)$. Note that

$$\begin{aligned} |g_s| &\leqslant \sum_{l=1}^{u} |g(l, s_l)| \leqslant \sum_{l=1}^{u} \frac{R_0}{4 i_1 \cdots i_{l-1}} && \text{(by (2))}, \\ &\leqslant \sum_{l=1}^{u} R_0/4^l && \text{(by (1))}, \\ &< R_0. \end{aligned}$$

Hence:

$$\{g_s : s \in T\} \subset B_e(R_0). \tag{5}$$

Suppose $s, s' \in T$ and $s \neq s'$. We shall show that:

$$g_s \neq g_{s'}. \tag{6}$$

For if $g_s = g_{s'}$, then $g(v, s_v) \cdots g(u, s_u) = g(v, s'_v) \cdots g(u, s'_u)$ for some $v < u$, with $s_v \neq s'_v$. Hence:

$$\begin{aligned} g(v, s'_v)^{-1} g(v, s_v) = g(v+1, s'_{v+1}) \cdots g(u, s'_u)\, g(u, s_u)^{-1} \\ \cdots g(v+1, s_{v+1})^{-1}. \end{aligned} \tag{7}$$

But

$$\begin{aligned} \frac{R_0}{i_1 \cdots i_v} &\leqslant |g(v, s'_v)^{-1} g(v, s_v)| && \text{(by (3))}, \\ &\leqslant 2 \sum_{l=v+1}^{u} R_0/4 i_1 \cdots i_{l-1} && \text{(by (2)), using (7))}, \\ &\leqslant \frac{R_0}{2 i_1 \cdots i_v} \left(1 + \sum_{l=v+2}^{u} \frac{1}{4^{l-v-1}}\right) && \text{(by (1))}, \\ &< \frac{R_0}{i_1 \cdots i_v}. \end{aligned}$$

This contradiction establishes (6), which clearly implies

$$\begin{aligned} \#\{g_s : s \in T\} = \#T &\geqslant \prod_{l=1}^{u} i_l^{d+1} && \text{(by definition of } T), \\ &> (R_0/\log R_0)^{d+1} && \text{(by (4))}, \\ &> c \cdot R_0^d && \text{(since } R_0 \text{ is infinite)}. \end{aligned}$$

Together with (5) this contradicts $G(R_0) \leqslant c \cdot R_0^d$, and this establishes the lemma. ∎

(6.3) PROPOSITION. *Let Γ be of near growth degree $\leqslant d$, $d \in \mathbb{N}$. (See (1.5)(3).) Then there exists positive infinite $R \in \mathbb{R}^*$ such that the metric space $Y^{(R)}$ is locally compact of dimension $\leqslant d+1$.*

Proof. By "overspill" there is a positive infinite $R_0 \in \mathbb{R}^*$ such that $G(R_0) \leqslant c \cdot R_0^d$. Let $S \in \mathbb{R}^*$ be positive infinite such that the property $P_i(S)$ of (6.2) holds for all $i \in \mathbb{N}$, $i \geqslant 4$. Set $R = S/4$. We shall show that $Y^{(R)}$ satisfies the conditions. Let $k \in \mathbb{N}^{>0}$. From (6.2) we conclude: if $g_1, ..., g_t \in B_e(R)$, $t \in \mathbb{N}^*$, and the balls $B_{g_1}(R/k), ..., B_{g_t}(R/k)$ are pairwise disjoint, then $t \leqslant (4k)^{d+1}$, in particular t is finite. Taking t maximal, the balls $B_{g_i}(2 \cdot R/k)$ necessarily cover $B_e(R)$: if g were not in their union then $B_g(R/k)$ would be disjoint from all $B_{g_i}(R/k)$. Applying the coset map $\Gamma^{(R)} \to Y^{(R)}$ we see that the closed ball of radius 1 and center $e\mu$ in $Y^{(R)}$ is covered by at most $(4k)^{d+1}$ balls of radius $2/k$, for any $k \in \mathbb{N}^{>0}$. As $Y^{(R)}$ is complete and homogeneous this immediately implies that all closed balls of radius 1 are compact and of Hausdorff dimension $\leqslant d+1$. (See [8, Chap. 7] for the notion of Hausdorff dimension and its connections with topological dimension.) Hence $Y^{(R)}$ is locally compact and of dimension $\leqslant d+1$. ∎

(6.4) *Remarks.* (1) In the arguments of (6.2) and (6.3) the term $d+1$ can actually be replaced by $d+\varepsilon$, where ε is an arbitrary positive real. Hence the space $Y^{(R)}$ in (6.3) can in fact be taken to be of dimension $\leqslant d$. Note that we have now established property (V) of (2.4) for the space $Y = Y^{(R)}$, where R is as in (6.3).

(2) It is an easy exercise to show that if $Y^{(R)}$ is locally compact, R positive infinite, then every closed bounded subset of $Y^{(R)}$ is compact. (Assuming that closed balls of radius $r > 0$ in Y are compact, use (4.2) to show that closed balls of radius $(3/2) \cdot r$ are covered by finitely many balls of radius r.)

(3) It seems plausible (and would be nice to prove) that, conversely, local compactness of $Y^{(R)}$ for some positive infinite R, implies that Γ is of polynomial growth. Let us show here only the following fact which we stated in (2.4) as property (VI) of the space Y, putting $Y = Y^{(R)}$.

(6.5) *If Γ is of exponential growth, then for no positive infinite R is the space $Y^{(R)}$ locally compact.*

By an observation of Milnor [10, p. 2] $\lim G(n)^{1/n}$ exists, and it is >1 iff Γ is of exponential growth. So if Γ is of exponential growth there is a real number $r > 1$ such that for any positive infinite R we have:

$\mathrm{st}(G(R)^{1/R}) = \mathrm{st}(G(2R)^{1/2R}) = r$, so $\mathrm{st}((G(2R)/G(R))^{1/R}) = r$, which implies that $G(2R)/G(R)$ is infinite: hence $B_e(2R)$ cannot be covered by finitely many balls $B_g(R)$, so that closed balls of radius 2 in $Y^{(R)}$ are not compact, and therefore $Y^{(R)}$ is not locally compact. ∎

(6.6) *Concluding Remark.* We have finally proved all the properties stated in (2.4) for the space $Y = Y^{(R)}$, where R is chosen as in (6.3). Hence Gromov's theorem (1.10) has now been established.

7. Effective Bounds

Gromov proved also a finite version of his theorem by means of a topological compactness argument:

(7.1) Theorem. *Let positive integers d and k be given. Then there are positive M, i, N such that for any group Γ with finite generating set X we have:*

if $G_X(n) \leqslant k \cdot n^d$ for $n = 1, \ldots, M$, *then*

$G_X(n) \leqslant k \cdot n^d$ *for all n, and Γ has a subgroup of index $\leqslant i$ and nilpotency class $\leqslant N$.*

(7.2) We will give a new proof of this theorem by means of a model theoretic finiteness argument which moreover shows how M, i, N can be effectively computed from d and k (in the sense of recursion theory; we do not claim efficiency of the algorithm).

(7.3) *Proof of* (7.1). Let T be the first order theory of groups with k distinguished elements $x_1, \ldots, x_k$, not necessarily distinct. For each positive integer M, let σ_M be a sentence in the language of T saying (for each model of T) that $G_X(n) \leqslant k \cdot n^d$ for $n = 1, \ldots, M$ in the subgroup Γ generated by $X = \{x_1, \ldots, x_k\}$ in the model. It is clear how to construct such a sentence for given d, k and M.

Similarly, let $\tau_{i,N}$ be the sentence in the language of T saying (for each model of T) that subgroup generated by $x_1, \ldots, x_k$ has a subgroup of index at most i and nilpotency class at most N.

To see that such a sentence $\tau_{i,N}$ exists and can be effectively constructed from k, d, i, N we use the following two facts:

(*) A group generated by r elements $y_1, \ldots, y_r$ is nilpotent of class N iff all commutators $[y_{i_1}, \ldots, y_{i_N}]$ $(1 \leqslant i_j \leqslant r)$ equal the identity. See [7, 10.2.3].

(**) If Γ is generated by $\{x_1, \ldots, x_k\}$ then there are at most $A(i, k)$ subgroups of index at most i (and exactly $A(i, k)$ such subgroups if Γ is free on the k-element set $\{x_1, ..., x_k\}$); here $A: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is effectively computable, and for all i, k one can effectively specify $A(i, k)$ finite sets of generators for these subgroups, each generator given as a word on $\{X_1, ..., X_k, X_1^{-1}, ..., X_k^{-1}\}$, where the letter X_i is to be interpreted as x_i. (Obviously, we only have to do this for the case that Γ is free on the k elements $x_1, ..., x_k$, and for that case one may consult [6]; in general we allow the possibility that several of the $A(i, k)$ finite sets define the same subgroup of Γ).

Now Gromov's theorem (in its "infinite" form) says:

$$T \models \bigwedge_{M \geqslant 1} \sigma_M \leftrightarrow \bigvee_{i \geqslant 1, N \geqslant 1} \tau_{i,N}.$$

By the compactness theorem of model theory (see [4, pp. 148–149] for the version used here) it follows that there are $M, i, N \geqslant 1$ such that

$$T \models \sigma_M \leftrightarrow \tau_{i,N}.$$

By Gödel's completeness theorem we will eventually find such M, i, N by systematically checking proofs from T until we find one of $T \models \sigma_M \leftrightarrow \tau_{i,N}$. ∎

(7.4) *Remark.* Note that (7.1) is a finite version of Gromov's original theorem. The method of (7.3) also gives a finite version of our improvement (1.10) of Gromov's theorem:

There is a (recursive) functional which to any triple (α, d, k), with $\alpha: \mathbb{N} \to \mathbb{N}$ and $d, k \in \mathbb{N}^{>0}$, assigns positive integers M, i, N such that for any group Γ with finite generating set X we have:

> *if $G_X(\alpha(n)) \leqslant k \cdot (\alpha(n))^d$ for $n = 1, ..., M$, and $\alpha(1) < \cdots < \alpha(M)$, then there is $d' \in \mathbb{N}$ such that $G_X(n) \leqslant k \cdot n^{d'}$ for all $n > 0$, and Γ has a subgroup of index $\leqslant i$ and nilpotency class $\leqslant N$.*

(7.5) One of the most interesting unsolved problems concerning the growth of finitely generated groups is to determine whether or not every such group has either polynomial or exponential growth.[5] In the last result of this paper we use (7.1) to construct an effectively computable function growing faster than any polynomial and such that every growth function majorized by it is of polynomial growth.

(7.6) THEOREM. *There is an effectively computable, nondecreasing*

[5] We have been told that R. Grigorčuk, in Moscow, has recently constructed finitely generated groups whose growth is neither polynomial nor exponential.

function $g: \mathbb{N} \to \mathbb{N}$, *such that* $g(n) \to \infty$ *as* $n \to \infty$, *with the following property*:

> *if* Γ *is any group with finite generating set* X, *and the growth function* G_X *of* Γ *satisfies* $G_X(n) \leqslant k \cdot n^{g(n)}$ *for* $n = 1, 2,\ldots$, *where* k *is a constant, then* Γ *has polynomial growth.*

Proof. Define $g(n) = (\frac{1}{2}) \cdot \#\{d \in N: d \geqslant 1$ and $d + \max_{1 \leqslant s,u \leqslant d} M(s,u) \leqslant n\}$, where the function $M = M(k,d)$ is given by (7.1).

Clearly g is nondecreasing, $g(n) \to \infty$ as $n \to \infty$ and g is effectively computable (because M is).

Now let Γ be any group with finite generating set X, and suppose $k \in N$, $k \geqslant 1$ is such that $G_X(n) \leqslant k \cdot n^{g(n)}$ for $n = 1, 2,\ldots$.

Let $d = \max\{2 \cdot \#X + 1, 2k\}$.

Then $n^d \geqslant k \cdot n^{d/2}$ for $n = 2, 3,\ldots$. However, from the definition of g we see that $g(n) \leqslant d/2$ for $n \leqslant \max_{1 \leqslant s,u \leqslant d} M(s,u)$, thus, in particular, we obtain

$$n^d \geqslant k \cdot n^{g(n)} \qquad \text{for} \quad n = 2, 3,\ldots, M(2 \cdot \#X + 1, d),$$

so that

$$n^d \geqslant G_X(n) \qquad \text{for} \quad n = 2, 3,\ldots, M(2 \cdot \#X + 1, d).$$

Since $G_X(1) \leqslant 2 \cdot \#X + 1$ we deduce that

$$G_X(n) \leqslant (2 \cdot \#X + 1) \cdot n^d \qquad \text{for} \quad n = 1, 2,\ldots, M(2 \cdot \#X + 1, d)$$

and so, by (7.1), Γ has polynomial growth, as required. ∎

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