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Differential torsion theory

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Abstract

Differential torsion theories are introduced and it is shown that for a hereditary torsion theory τ every derivation on an R-module M has a unique extension to its module of quotients if and only if τ is a differential torsion theory. Dually, we show that when τ is cohereditary, every derivation on M can be lifed uniquely to its module of coquotients.

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The purpose of this paper is to introduce the concept of a differential torsion theory on $\mathbf{Mod}_R$ and to use this notion to study derivations on modules and their extension to modules of quotients. After obtaining the main result concerning such extensions we turn our attention to the problem lifting derivations on modules to modules of coquotients.

Throughout R will denote an associative ring with identity, all modules will be unitary right R-modules and $\mathbf{Mod}_R$ will denote the category of unitary right R-modules. A function $\delta : R \to R$ is a *derivation on* R if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. If δ is a derivation on R and M is an R-module, then a function $d : M \to M$ is a *δ-derivation* if $d(x+y) = d(x) + d(y)$ and $d(xa) = d(x)a + x\delta(a)$ for all $x, y \in M$ and all $a \in R$. We now assume that δ is a fixed but arbitrary derivation on R and that every derivation under consideration is a δ-derivation. Also, if N is a submodule of an R-module M, then for any $x \in M$, $(N : x)$ will denote the right ideal of R given by $\{a \in R \mid xa \in N\}$.

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A torsion theory τ on $\mathbf{Mod}_R$ is a pair (T, F) of classes of R-modules such that the following conditions hold.

1. $\mathsf{T} \cap \mathsf{F} = 0$.
2. If $M \to N \to 0$ is an exact sequence in $\mathbf{Mod}_R$ and $M \in \mathsf{T}$, then $N \in \mathsf{T}$.
3. If $0 \to M \to N$ is an exact sequence in $\mathbf{Mod}_R$ and $N \in \mathsf{F}$, then $M \in \mathsf{F}$.
4. For each R-module M, there is a short exact sequence $0 \to T \to M \to F \to 0$ in $\mathbf{Mod}_R$ with $T \in \mathsf{T}$ and $F \in \mathsf{F}$.

It follows that the class T is closed under factor modules, direct sums and extensions and that F is closed under submodules, direct products and extensions. A class C of R-modules is said to be *closed under extensions* if whenever $0 \to N_1 \to N \to N_2 \to 0$ is a short exact sequence in $\mathbf{Mod}_R$ and N_1 and N_2 are in C, then N is in C. Modules in T will be called *τ-torsion* and those in F are called *τ-torsion free*. Each R-module has a largest and necessarily unique τ-torsion submodule given by $t_\tau(M) = \Sigma_{N \in \mathsf{S}} N$, where S is the set of τ-torsion submodules of M. A torsion theory will be called *hereditary* if T is closed under submodules and it will be called *cohereditary* if F is closed under factor modules. Standard results and terminology on torsion theory can be found in [4,9], while general information on rings and modules can be found in [2].

1. Differential filters

A nonempty collection $\mathscr{F}$ of right ideals of R is said to be a (Gabriel) *filter* [7] if the following two conditions hold.

1. If $K \in \mathscr{F}$, then $(K : a) \in \mathscr{F}$ for each $a \in R$.
2. If I is a right ideal of R and $K \in \mathscr{F}$ is such that $(I : a) \in \mathscr{F}$ for each $a \in K$, then $I \in \mathscr{F}$.

It can be shown that each filter of right ideals of R also satisfies the following three conditions.

3. If $J \in \mathscr{F}$ and K is a right ideal of R such that $J \subseteq K$, then $K \in \mathscr{F}$.
4. If $J, K \in \mathscr{F}$, then $J \cap K \in \mathscr{F}$.
5. If $J, K \in \mathscr{F}$, then $JK \in \mathscr{F}$.

If $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, then $\mathscr{F}_\tau = \{K \mid K$ is a right ideal of R and $R/K \in \mathsf{T}\}$ is a filter. An element x of an R-module M is said to be a *τ-torsion element* of M if there is a $K \in \mathscr{F}_\tau$ such that $xK = 0$. The set of all τ-torsion elements of M is the τ-torsion submodule $t_\tau(M)$ of M mentioned earlier. Moreover, an R-module M is τ-torsion if $t_\tau(M) = M$ and τ-torsion free if $t_\tau(M) = 0$. Conversely, if $\mathscr{F}$ is a filter of right ideals of R and $t(M) = \{x \in M \mid xK = 0$ for some $K \in \mathscr{F}\}$, then $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, where $\mathsf{T} = \{M \mid t(M) = M\}$ and $\mathsf{F} = \{M \mid t(M) = 0\}$. It follows that there is a one-to-one correspondence between the hereditary torsion theories on $\mathbf{Mod}_R$ and the filters of right ideals of R.

If $\mathscr{F}$ is a filter of right ideals of R, then $\mathscr{F}$ will be called a *differential filter* if for each $K \in \mathscr{F}$, there is an $I \in \mathscr{F}$ such that $\delta(I) \subseteq K$. If τ is a hereditary torsion theory on $\mathbf{Mod}_R$ and $\mathscr{F}_\tau$ is a differential filter, then τ is said to be a *differential torsion theory*.

The following examples show that differential torsion theories do indeed exist.

Example 1.1. If R is a commutative ring, then every filter $\mathscr{F}$ of right ideals of R is a differential filter. Indeed if $I \in \mathscr{F}$, then $I^2 \in \mathscr{F}$, so if $a, b \in I$, then $\delta(ab) = \delta(a)b + a\delta(b) \in I$. It follows that $\delta(I^2) \subseteq I$. So the hereditary torsion theory determined by $\mathscr{F}$ is a differential torsion theory.

Example 1.2. Jans has shown in [10] that if $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion on $\mathbf{Mod}_R$ such that T is closed under direct products, then there is an idempotent ideal $I \in \mathscr{F}_\tau$ such that $I \subseteq K$ for each $K \in \mathscr{F}_\tau$. If $ab \in I^2 = I$, then $\delta(ab) = \delta(a)b + a\delta(b) \in I$ and from this we can conclude that $\delta(I) \subseteq K$. Thus τ is a differential torsion theory.

Example 1.3. If R is left perfect, then Alin and Armendariz [1] and Dlab [6] have independently proved that if $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, then T is closed under direct products. Thus, we see from the previous example that when R is left perfect every hereditary torsion theory on $\mathbf{Mod}_R$ is a differential torsion theory.

Example 1.4. Let S be a multiplicatively closed set of elements of R that is a right denominator set [11]. Then S satisfies:

1. If $(a, s) \in R \times S$, then there is a $(b, t) \in R \times S$ such that $at = sb$.
2. If $sa = 0$ with $s \in S$ and $a \in R$, then $at = 0$ for some $t \in S$.

The set $\mathscr{F} = \{K \mid K$ is a right ideal of R and $K \cap S \neq \emptyset\}$ is a filter of right ideals of R, If $K \in \mathscr{F}$, let $s \in K \cap S$. Since $(\delta(s), s) \in R \times S$, there is a $(b, t) \in R \times S$ such that $\delta(s)t = sb$. Now $\delta(st) = \delta(s)t + s\delta(t) = sb + s\delta(t) \in sR \subseteq K$, so if a $\in R$, then $\delta(sta) = \delta(st)a + st\delta(a) \in K$. *Hence* $\delta(stR) \subseteq K$. Therefore $\mathscr{F}$ is a differential filter, so the torsion theory determined by $\mathscr{F}$ is a differential torsion theory.

The following lemma will prove useful.

Lemma 1.5. *The following are equivalent for a hereditary torsion theory τ on* $\mathbf{Mod}_R$.

(1) $\mathscr{F}_\tau$ *is a differential filter.*
(2) *For every right R-module M and every $x \in t_\tau(M)$, there is an $I \in \mathscr{F}_\tau$ such that $\delta(I) \subseteq (0 : x)$.*
(3) *For every R -module M and every derivation d defined on M, $d(t_\tau(M)) \subseteq t_\tau(M)$.*

Proof. (1) $\Rightarrow$ (3): If $x \in t_\tau(M)$, then $(0 : x) \in \mathscr{F}_\tau$, so there is an $I \in \mathscr{F}_\tau$ such that $\delta(I) \subseteq (0 : x)$. If $a \in K = I \cap (0 : x) \in \mathscr{F}_\tau$, then $xa = 0$ and $x\delta(a) = 0$. Hence, $0 = d(xa) = d(x)a + x\delta(a) = d(x)a$ which shows that $d(x)K = 0$. Therefore $d(x) \in t_\tau(M)$, so $d(t_\tau(M)) \subseteq t_\tau(M)$.

(3) $\Rightarrow$ (2): If $x \in t_\tau(M)$, then $d(x) \in t_\tau(M)$, so $(0 : x)$ and $(0 : d(x))$ are in $\mathscr{F}_\tau$. Therefore $I = (0 : x) \cap (0 : d(x)) \in \mathscr{F}_\tau$. If $a \in I$, then $xa = d(x)a = 0$, so $0 = d(xa) = d(x)a + x\delta(a) = x\delta(a)$. Thus $\delta(a) \in (0 : x)$ and we have that $\delta(I) \subseteq (0 : x)$.

(2) $\Rightarrow$ (1): If $K \in \mathscr{F}_\tau$, then $1+K$ is a τ-torsion element of R/K. Thus $(0 : 1+K) \in \mathscr{F}_\tau$, which indicates there is an $I \in \mathscr{F}_\tau$ such that $\delta(I) \subseteq (0 : 1 + K) = K$. □

2. Derivations and modules of quotients

If τ is a torsion theory on $\mathbf{Mod}_R$, then an R-module $Q_\tau(M)$ together with an R-homomorphism $\varphi : M \to Q_\tau(M)$ is said to be a *localization* of M at τ provided that ker φ and coker φ are τ-torsion and $Q_\tau(M)$ is τ-injective and τ-torsion free. An R-module M is said to be *τ-injective* if $Hom_R(-, M)$ preserves short exact sequences $0 \to N_1 \to N \to N_2 \to 0$ in $\mathbf{Mod}_R$, where N_2 is a τ-torsion R-module. The module $Q_\tau(M)$, called the *module of quotients* of M, is unique up to isomorphism whenever it can be shown to exist. Ohtake [13] has shown that a localization $\varphi : M \to Q_\tau(M)$ exists for every R-module M if and only if the torsion theory is hereditary. It is well known that if τ is hereditary, then we can set $Q_\tau(M) = E_\tau(M/t_\tau(M))$, where $E_\tau(M/t_\tau(M))$ is the *τ-injective envelope* [4,9] of $M/t_\tau(M)$. In this case, if $\eta : M \to M/t_\tau(M)$ is the natural mapping and $\mu : M/t_\tau(M) \to Q_\tau(M)$ is the canonical injection, then $\varphi = \mu\eta$.

When the torsion theory is hereditary, Golan has shown in [8] that if a derivation d defined on an R-module M is such that $d(t_\tau(M)) \subseteq t_\tau(M)$, then d can be extended to a derivation d_τ on $Q_\tau(M)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & Q_\tau(M) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_\tau} \\ M & \xrightarrow{\varphi} & Q_\tau(M) \end{array}$$

is commutative. The question of uniqueness of the extension d_τ was not addressed by Golan other than to point out that a derivation δ on R has a unique extension to the ring of quotients $Q_\tau(R)$ of R provided that the hereditary torsion theory is faithful, i.e. if R is τ-torsion free. This observation is subsumed by the following more general proposition and corollary.

Proposition 2.1. *Let τ be a hereditary torsion theory on $\mathbf{Mod}_R$. If a derivation d on an R-module M extends to a derivation d_τ on the module $Q_\tau(M)$ of quotients of M, then d_τ is unique.*

Proof. Let $x \in Q_\tau(M)$. If $\bar{d}$ also extends d to $Q_\tau(M)$, then $(d_\tau - \bar{d})\varphi(M) = 0$ gives $(d_\tau - \bar{d})(x(\varphi(M) : x)) = 0$, since $x(\varphi(M) : x) \subseteq \varphi(M)$. But $d_\tau - \bar{d}$ is an R-linear mapping, so we have $(d_\tau - \bar{d})(x)(\varphi(M) : x) = 0$. Now $x \in Q_\tau(M)$ implies that $(\varphi(M) : x) \in \mathscr{F}_\tau$ and so $(d_\tau - \bar{d})(x) \in t_\tau(Q_\tau(M)) = 0$. Consequently $d_\tau = \bar{d}$. □

Corollary 2.2. *If τ is a hereditary torsion theory on $\mathbf{Mod}_R$, then any derivation d defined on a τ-torsion free R-module M has a unique extension to $Q_\tau(M)$.*

Proof. Since $t_\tau(M) = 0$ and $d(0) = 0$, we have $d(t_\tau(M)) \subseteq t_\tau(M)$. Thus, Golan's result shows that an extension d_τ of d to $Q_\tau(M)$ exists and the proposition shows that d_τ is unique. □

We can now show that d can always be extended uniquely to $Q_\tau(M)$ if and only if τ is a differential torsion theory.

Proposition 2.3. *If τ is a hereditary torsion theory on* $\mathbf{Mod}_R$, *then every derivation d defined on an R-module M has a unique extension d_τ to the module of quotients of M if and only if τ is a differential torsion theory*.

Proof. Let τ be a hereditary torsion theory on $\mathbf{Mod}_R$ and let $\varphi : M \to Q_\tau(M)$ be a localization at τ of an arbitrary R-module M. Suppose also that d is a derivation defined on M. If τ is a differential torsion theory, then $\mathscr{F}_\tau$ is a differential filter, so it follows from Lemma 1.5 that $d(t_\tau(M)) \subseteq t_\tau(M)$. It is now immediate from Golan's result that d can be extended to a derivation d_τ defined on $Q_\tau(M)$. Uniqueness follows from Proposition 2.1.

Conversely, suppose that every derivation d defined on M can be extended uniquely to a derivation d_τ on $Q_\tau(M)$. Since $\varphi d = d_\tau \varphi$, we see that if $x \in t_\tau(M) = \ker \varphi$, then $\varphi d(x) = 0$. This gives $d(x) \in t_\tau(M)$ and so we have $d(t_\tau(M)) \subseteq t_\tau(M)$. By invoking Lemma 1.5 again we see that τ is a differential torsion theory. □

One important consequence of the proposition above is that for a hereditary torsion theory τ on $\mathbf{Mod}_R$, the right ideals of the filter $\mathscr{F}_\tau$ can be tested with δ to determine if all δ-derivations defined on R-modules can be extended to their modules of quotients.

3. Derivations and modules of coquotients

We now show that a result similar to Proposition 2.3 holds for colocalizations of modules whenever they universally exist. Colocalizations have been investigated under various approaches by several authors, for example see [3,5,12].

An R-module $C_\tau(M)$ together with an R-linear mapping $\varphi : C_\tau(M) \to M$ is said to be a *colocalization* of M at τ provided that $\ker \varphi$ and $\operatorname{coker} \varphi$ are τ-torsion free and $C_\tau(M)$ is τ-torsion and τ-projective. We call $C_\tau(M)$ the *module of coquotients* of M. An R-module M is *τ-projective* if $Hom_R(M, -)$ preserves short exact sequences $0 \to N_1 \to N \to N_2 \to 0$ in $\mathbf{Mod}_R$, where N_1 is a τ-torsion free R-module. Ohtake was also able to show in [13] that a torsion theory τ is cohereditary if and only if every R-module M has a colocalization at τ. If $\varphi : C_\tau(M) \to M$ is a colocalization of M at τ, then there is an R-epimorphism $\pi : C_\tau(M) \to t_\tau(M)$ such that if $\mu : t_\tau(M) \to M$ is the canonical injection, then $\varphi = \mu\pi$. Furthermore, a module of coquotients is unique up to isomorphism whenever it can be shown to exist.

If $\varphi : C_\tau(M) \to M$ is a colocalization of M at τ and d is a derivation defined on M, then we will say that a derivation d_τ defined on $C_\tau(M)$ lifts d to $C_\tau(M)$ provided that the diagram

$$\begin{array}{ccc} C_\tau(M) & \xrightarrow{\varphi} & M \\ {\scriptstyle d_\tau}\downarrow & & \downarrow{\scriptstyle d} \\ C_\tau(M) & \xrightarrow{\varphi} & M \end{array}$$

is commutative.

When $\tau = (\mathsf{T}, \mathsf{F})$ is cohereditary, the class F of τ is both a torsion and a torsion-free class, and the class F generates a hereditary torsion theory $\sigma = (\mathsf{F}, \mathsf{D})$ on $\mathbf{Mod}_R$. The pair (τ, σ) is often referred to as a TTF theory. Jans has shown in [10] that there is a one-to-one correspondence between TTF theories and idempotent ideals I of R. If (τ, σ) is a TTF theory with corresponding idempotent ideal I, then the filter determined by σ is given by $\mathscr{F}_\sigma = \{K \subseteq R \mid K \supseteq I, K \text{ a right ideal of } R\}$. In this setting, $t_\tau(R) = I$ and $t_\tau(M) = MI$ for each R-module M. We have seen in Example 1.2 that σ is a differential torsion theory although this condition on σ is not a factor in lifting derivations on M to the module $C_\tau(M)$ of coquotients of M. Sato has shown in [14] that if τ is cohereditary, then $I \otimes_R I \xrightarrow{\pi} I \xrightarrow{\mu} R$ is a colocalization of R, where the map $\pi : I \otimes_R I \to I$ is given by $\Sigma_{i=1}^n (a_i \otimes b_i) \mapsto \Sigma_{i=1}^n a_i b_i$. Furthermore $I \otimes_R I$ is a ring, possibly without an identity, and an (R, R)-bimodule. Sato also shows in [14] that $M \otimes_R I \otimes_R I \xrightarrow{\pi} MI \xrightarrow{\mu} M$ is a colocalization of M at τ. In this case, the map $\pi : M \otimes_R I \otimes_R I \to MI$ is such that $\Sigma_{i-1}^n (x_i \otimes a_i \otimes b_i) \mapsto \Sigma_{i=1}^n x_i a_i b_i$. Since I is an idempotent ideal, $\delta(I) \subseteq I$ and $d(MI) \subseteq MI$ for each derivation d defined on M. Hence, δ and d restricted to I and MI produce derivations on I and MI, respectively, and we denote these also by δ and d.

We need the following lemma in order to show that if τ is a cohereditary torsion theory on $\mathbf{Mod}_R$, then every derivation on an R-module M can be lifted to the module of coquotients of M.

Lemma 3.1. *If I is an idempotent ideal of R and d is a derivation on M, then the map $\rho' : M \times I \times I \to M \otimes_R I \otimes_R I$ given by*

$$\rho'(x, a, b) = d(x) \otimes a \otimes b + x \otimes \delta(a) \otimes b + x \otimes a \otimes \delta(b)$$

is R-balanced. That is, ρ' is additive in each variable and such that $\rho'(xr, a, b) = \rho'(x, ra, b)$ and $\rho'(x, ar, b) = \rho'(x, a, rb)$ for all $(x, a, b) \in M \times I \times I$ and all $r \in R$.

Proof. Since d and δ are additive, it is easy to see that ρ' is additive in each variable. We show $\rho'(xr, a, b) = \rho'(x, ra, b)$ and a similar proof holds for $\rho'(x, ar, b) = \rho'(x, a, rb)$. If $(x, a, b) \in M \times I \times I$ and $r \in R$, then

$$\begin{aligned} \rho'(xr, a, b) &= d(xr) \otimes a \otimes b + xr \otimes \delta(a) \otimes b + xr \otimes a \otimes \delta(b) \\ &= d(x)r \otimes a \otimes b + x\delta(r) \otimes a \otimes b + xr \otimes \delta(a) \otimes b + xr \otimes a \otimes \delta(b) \\ &= d(x) \otimes ra \otimes b + x \otimes [\delta(r)a + r\delta(a)] \otimes b + x \otimes ra \otimes \delta(b) \\ &= d(x) \otimes ra \otimes b + x \otimes \delta(ra) \otimes b + x \otimes ra \otimes \delta(b) \\ &= \rho'(x, ra, b), \end{aligned}$$

so we are done. □

Proposition 3.2. *If τ is a cohereditary torsion theory on $\mathbf{Mod}_R$, then every derivation defined on an R-module M lifts uniquely to a derivation defined on the module of coquotients of M.*

Proof. If $\tau = (\mathsf{T}, \mathsf{F})$ and $\sigma = (\mathsf{F}, \mathsf{D})$ is the torsion theory generated by F, let I be the idempotent ideal corresponding to the TTF theory (τ, σ). If d is a derivation on M, then we have a commutative diagram

$$\begin{array}{ccc} M \times I \times I & \xrightarrow{\rho} & M \otimes_R I \otimes_R I \\ & \searrow^{\rho'} & \downarrow d_\tau \\ & & M \otimes_R I \otimes_R I \end{array}$$

where $\rho : M \times I \times I \to M \otimes_R I \otimes_R I$ is the canonical R-balanced map given by $\rho(x, a, b) = x \otimes a \otimes b$, ρ' is the R-balanced map of Lemma 3.1 and d_τ is the group homomorphism produced by the tensor product $M \otimes_R I \otimes_R I$. Now consider the diagram

$$\begin{array}{ccccc} M \otimes_R I \otimes_R I & \xrightarrow{\pi} & MI & \xrightarrow{\mu} & M \\ d_\tau \downarrow & & \downarrow d & & \downarrow d \\ M \otimes_R I \otimes_R I & \xrightarrow{\pi} & MI & \xrightarrow{\mu} & M \end{array}$$

Since $\varphi = \mu\pi$, where is $\pi : M \otimes_R I \otimes_R I \to M$ is such that $\pi(\Sigma_{i=1}^{n}(x_i \otimes a_i \otimes b_i)) = \Sigma_{i=1}^{n} x_i a_i b_i$ and $\mu : MI \to M$ is the canonical injection, we see that $\varphi(\Sigma_{i=1}^{n}(x_i \otimes a_i \otimes b_i)) = \Sigma_{i=1}^{n} x_i a_i b_i$ for each $\Sigma_{i=1}^{n}(x_i \otimes a_i \otimes b_i) \in M \otimes_R I \otimes_R I$. So if $x \otimes a \otimes b$ is a generator of $M \otimes_R I \otimes_R I$, then

$$\begin{aligned} \varphi d_\tau(x \otimes a \otimes b) &= \varphi\rho'(x, a, b) \\ &= \varphi(d(x) \otimes a \otimes b) + x \otimes \delta(a) \otimes b + x \otimes a \otimes \delta(b)) \\ &= d(x)ab + x\delta(a)b + xa\delta(b) \\ &= d(x)ab + x[\delta(a)b + a\delta(b)] \\ &= d(x)ab + x\delta(ab) \\ &= d(xab) \\ &= d\varphi(x \otimes a \otimes b). \end{aligned}$$

Since φd_τ and $d\varphi$ are additive functions, this suffices to show that $\varphi d_\tau = d\varphi$, so the diagram

$$\begin{array}{ccc} M \otimes_R I \otimes_R I & \xrightarrow{\varphi} & M \\ d_\tau \downarrow & & \downarrow d \\ M \otimes_R I \otimes_R I & \xrightarrow{\varphi} & M \end{array}$$

is commutative. Finally if $x \otimes a \otimes b \in M \otimes_R I \otimes_R I$ and $r \in R$, then

$$\begin{aligned} d_\tau((x \otimes a \otimes b)r) &= d_\tau(x \otimes a \otimes br) \\ &= d(x) \otimes a \otimes br + x \otimes \delta(a) \otimes br + x \otimes a \otimes \delta(br) \\ &= d(x) \otimes a \otimes br + x \otimes \delta(a) \otimes br + x \otimes a \otimes \delta(b)r + x \otimes a \otimes b\delta(r) \\ &= [d(x) \otimes a \otimes b + x \otimes \delta(a) \otimes b + a \otimes \delta(b)]r + (x \otimes a \otimes b)\delta(r) \\ &= d_\tau(x \otimes a \otimes b)r + (x \otimes a \otimes b)\delta(r). \end{aligned}$$

Since d_τ is additive, this last result shows that d_τ is a derivation that lifts d to the module of coquotients of M. Finally, if $\bar{d}$ also lifts d to $M \otimes_R I \otimes_R I$, then $\varphi(d_\tau - \bar{d}) = 0$, so $\operatorname{Im}(d_\tau - \bar{d}) \subseteq \ker \varphi$. Thus, $\operatorname{Im}(d_\tau - \bar{d})$ is τ-torsion free. But $M \otimes_R I \otimes_R I$ is τ-torsion, so since $d_\tau - \bar{d}$ is an R-linear mapping, $\operatorname{Im}(d_\tau - \bar{d})$ is also τ-torsion. Hence, $\operatorname{Im}(d_\tau - \bar{d}) = 0$ and we have $d_\tau = \bar{d}$. Therefore d_τ is unique. □

Corollary 3.3. *If τ is cohereditary, then a derivation δ defined on R lifts uniquely to a derivation δ_τ defined on the ring of coquotients of R.*

Proof. This follows from the observation that $R \otimes_R I \otimes_R I \cong I \otimes_R I$ and $RI = I$. □

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