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Why are all dualities conformal? Theory and practical consequences

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Abstract

We relate duality mappings to the "Babbage equation" $F(F(z)) = z$, with F a map linking weak- to strong-coupling theories and demonstrate that, under fairly general conditions, F may only be a specific conformal transformation of the fractional linear type. This general result has enormous practical consequences. For example, one can establish that weak- and strong-coupling series expansions of arbitrarily large finite size systems are trivially related, i.e., after generating one of those series the other is automatically determined through a set of linear constraints between the series coefficients. This latter relation *partially solves* or, equivalently, localizes the computational complexity of evaluating the series expansion to a simple fraction of those coefficients. As a bonus, those relations also encode non-trivial equalities between different geometric constructions in general dimensions, and connect derived coefficients to polytope volumes. We illustrate our findings by examining various models including, but not limited to, ferromagnetic and spin-glass Ising, and Ising gauge type theories on hypercubic lattices in $1 < D < 9$ dimensions.

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1. Introduction

The utility of weak- and strong-coupling expansions and of dualities in nearly all branches of physics can hardly be overestimated. This article is devoted to several inter-related fundamental questions. Mainly:

(1) What information does the existence of finite order complementary weak- and strong-coupling series expansion of given physical quantities (e.g., partition functions, matrix elements, etc.) provide?
(2) To what extent can dualities be employed to *partially solve* those various problems? By *partial solvability* we mean the ability to compute a specific physical quantity with complexity polynomial in the size of the system, given partial information that is determined by other means.

As we will demonstrate in this work, a universal problem deeply binds to the above two inquiries, and raises the critical question:

(3) Why do numerous dualities in very different fields always turn out to be *conformal transformations*?

To set the stage, we briefly recall general notions concerning dualities. Consider a theory of (dimensionless) coupling strength g for which weak- and strong-coupling expansions may, respectively, be performed in powers of g and $1/g$ or in other monotonically increasing/decreasing functions $f_+(g)/f_-(g)$. Common wisdom asserts that as ordinary expansion parameters (e.g., g and $1/g$) behave very differently, weak- and strong-coupling series cannot, generally, be simply compared. On a deeper level, if these expansions describe different phases (as they generally do) then the series must become non-analytic (in the thermodynamic limit) at finite values of g (where transitions occur) and thus render any equality between them void. A duality may offer insightful *information* on a strong coupling theory by relating it to a system at weak coupling that may be perturbatively examined. As is well known, when they are present, self-dualities are manifest as an equivalence of the coefficients in the two different series; this leads to an invariance under an inversion that is qualitatively (and in standard field theories, e.g., QED/Electroweak/QCD, is exactly) of the canonical form "$g \leftrightarrow 1/g$" (or, more generally, $f_+(g) \leftrightarrow f_-(g)$). For example, in vacuum QED with Lagrangian density $\mathcal{L} = [\epsilon_0 \vec{E}^2/2 - \vec{B}^2/(2\mu_0)]$, the ratio $g = \epsilon_0\mu_0$ of the couplings in front of the $\vec{E}^2$ and $\vec{B}^2$ terms relates to a $g \leftrightarrow 1/g$ reciprocity. This reciprocity is evident from the invariance of Maxwell's equations in vacuum under the exchange of electric and magnetic fields [1], $\vec{E} \to \vec{B}$; $\vec{B} \to -\vec{E}$, and the Lagrangian density that results. In Yang-Mills (YM) theories, such an exchange between dual fields has led to profound insights from analogies between the Meissner effect and the behavior of vortices in superconductors to confinement and flux tubes – a hallmark of QCD [2–5]. Abstractions of dualities in electromagnetism and in YM theories produced powerful tools such as those in Hodge and Donaldson theories [6].

In both classical and quantum models, dualities (and the $f_+(g) \leftrightarrow f_-(g)$ inversion) are generated by *linear transformations* (appearing, e.g., as unitary transformations or more general isometries relating one local theory to another in fundamental "bond-algebraic" [7–13] incarnations or, in the standard case, Fourier transformations [14–18]). Such linear transformations lead to an *effective* inversion of the coupling constant g. Dual models share, for instance, their partition functions (and thus the same series expansion). As realized by Kramers and Wannier

(KW) [19–25], self-dualities provide structure that enables additional information allowing, for instance, the exact computation of phase transition points. This does not imply that the full partition function is determined with complexity polynomial in the size of the system, that is, it is *solvable* via self-dualities *alone* (and indeed as we illustrate in this work, self-dualities do not suffice).

Now here is a main point – that concerning question (3) – which we wish to highlight in this article. In diverse arenas, the weak- and strong-coupling expansion parameters $f_+(g)$ and $f_-(g)$ *are related to one another via conformal transformations* that are of the fractional linear type. Amongst many others, prevalent examples are afforded by $SL(2,\mathbb{Z})$ dualities in YM theories as well as those in Ising models and Ising lattice gauge theories. In all of these examples, the transformations linking $z \equiv f_+(g)$ to $w \equiv f_-(g) \equiv F(z)$ are particular special cases of conformal (or fractional linear (Möbius)) transformations. That is, in these,

$$z \to F(z) = w = \frac{az+b}{cz+d}, \tag{1}$$

with a, b, c, and d complex coefficients, and determinant

$$\Delta = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0. \tag{2}$$

A well known mathematical property of fractional linear maps is their composition property: Given any two fractional linear functions $F_k = (a_k z + b_k)/(c_k z + d_k)$ (with $k = 1, 2$), direct substitution demonstrates that $F_1(F_2(z)) = (a'z+b')/(c'z+d')$ (i.e., yet another fractional linear transformation) where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{3}$$

This group multiplication property will be of great utility in our analysis of dualities. Fractional linear maps, as is commonly known by virtue of the trivial equality (valid when $c \neq 0$)

$$F(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{\Delta}{c(cz+d)}, \tag{4}$$

which may be expressed as compositions of transformations of the (formal) forms: translation ($z \to z + b$), scaling/rotation ($z \to az$), and inversion ($z \to 1/z$). As each of these individual operations generally map circles and lines onto themselves so do the general transformations of Eq. (4). This may be understood as a consequence of a projective transformation from the Riemann sphere onto the complex plane. Relating Lorentz transformations to Möbius transformations is one of the principal ideas underlying twistor theory [26]. Envisioning *standard dualities*[1] as particular induced maps on the Euclidean S^2 sphere will be an outcome of the current work.

The set of all conformal self-mappings of the upper half complex plane forms a group, with $SL(2,\mathbb{Z})$ a subgroup ("full modular group") that consists of all the fractional linear transformations with a, b, c, and d integers, and determinant $\Delta = 1$. In the aforementioned YM theories, e.g., [1,27], an $SL(2,\mathbb{Z})$ structure follows from a canonical invariance of the form $z \to (z + 1)$

[1] In the notation of what will follow in the current article, by "standard dualities" we are alluding to the typical "two-"dualities with one coupling constant g. For these, $F(F(z)) = z$ (the parameter z defining the theory, $z = f_+(g)$, is mapped back onto itself following two consecutive applications of F).

(stemming from charge quantization). As we will detail in the current work, in Ising models and Ising gauge theories, a canonical form of the duality is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Delta = -2. \tag{5}$$

The transformation of Eq. (5) may trivially be associated to one with $\Delta = 1$ by[2] a uniform scaling $(a, b, c, d) = (-1, 1, 1, 1) \to 2^{-1/2}\,\mathrm{i}(-1, 1, 1, 1)$ which does not change the ratio in Eq. (1). More widely, any fractional linear transformation of the form of Eq. (1) with a finite determinant may similarly be related to one with $\Delta = 1$ by a uniform scaling of all four elements of the matrix. In general, we are interested in duality mappings as applied to matrix elements, partition functions or path integrals, while the typical scenario in YM theories focuses on mappings of the action (or Hamiltonian).

In what will follow, we will first address question (3) and illustrate that disparate duality transformations must be of the form of Eq. (1). When applied to the expansion parameters, we will then demonstrate that these *fractional linear maps lead to linear constraints between the strong- and weak-coupling series coefficients*. A main message of this work is that these conformal transformations of Eq. (1), leading to linear relations among series coefficients, will allow a broad investigation of questions (1) and (2) above. Specifically, we will examine arbitrarily large yet *finite size* systems for which *no* phase transitions appear. As is well known, *analyticity enables a full determination of functions* over entire domains given their values in only a far more restricted regime (even if only of vanishing measure). For a finite size system, the weak-coupling (W-C) and strong-coupling (S-C) expansions describe the same analytic function and are everywhere convergent and may thus be equated to one another. Thus, a trivial yet practical consequence is, contrary to some lore, that *the naturally perturbative W-C and the seemingly more involved S-C expansions are equally hard*. We will apply this approach to the largest Ising model systems for which the exact expansions are known to data on both finite size cubic and square lattices. We further test other aspects of our methods on Ising and generalized Wegner models. The substitution of Eq. (1) relates the W-C and S-C expansion parameters in general dual models. We will more generally: (1′) Equate the W-C and S-C expansions to find *linear constraints* on the expansion coefficients, and (2′) When possible, invoke self-duality to obtain yet further linear equations that those coefficients need to satisfy. This analysis will lead to the concept of *partial solvability*: The linear equations that we will obtain will enable us to *localize NP hardness* of finding the exact partition function coefficients (or other quantities) to that of evaluating only a fraction of these coefficients. The remainder of these coefficients can be then trivially found by the linear relations that are derived from the duality of Eq. (1).

A highly non-trivial consequence of our work is that of *relating mathematical identities to dualities* such as those broadly generated by Eq. (1). Specifically, as a concrete example in this work, we will illustrate how the relations that we obtain connecting the W-C and S-C expansions lead to *new combinatorial geometry equalities* in general dimensions. As a particular example we will do this by noting that, in Ising and generalized Wegner models, the expansion coefficients are equal to the number of geometrical shapes of a given magnitude of the d-dimensional surface

[2] As a curiosity, we remark that up to a trivial permutation and rescaling, the matrix of Eq. (5) embodying a duality transformation, diagonalizes the transfer matrix of the one-dimensional Ising chain of Eq. (45) with nearest neighbor $J_{\mathsf{ab}} = J$ (and for which we define $K \equiv \beta J$ with β the inverse temperature). That is,

$$\begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2} & -2^{-1/2} \end{pmatrix} \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} \begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2} & -2^{-1/2} \end{pmatrix} = 2 \begin{pmatrix} \cosh K & 0 \\ 0 & \sinh K \end{pmatrix}. \tag{6}$$

areas. The equality between the W-C and S-C expansions then lead to identities connecting these numbers.

2. General constraints on duality transformations

For the Ising, Ising gauge, and several other theories that we study in this work, the mapping between the W-C and S-C coupling expansion parameters is afforded by the particular Möbius transformation

$$F(z) = \frac{1-z}{1+z} \tag{7}$$

associated with Eq. (5). This transformation trivially satisfies Babbage's equation

$$F\big(F(z)\big) = z \tag{8}$$

for all z. For self-dual models, such as the $D = 2$ Ising model or $D = 4$ Ising gauge theories, we can easily find the critical (self-dual) point, z^*, by solving the equation $F(z^*) = z^*$ (fixed-point of the transformation). We will term theories obeying Eq. (8) as those that exhibit a "one-" duality. In general, one may find such transformations, represented by a function $F(z)$, in terms of some parameter z (a coupling constant which can be complex-valued). Richer transformations appear in diverse arenas including Renormalization Group (RG) calculations. Based on these considerations we may have

$$\begin{cases} F\big(z^*\big) = z^*, & \text{self-dual fixed point,} \\ F\big(F(z)\big) = z, & \text{self-duality/duality,} \\ F\big(\cdots F\big(F\big(z^*\big)\big)\cdots\big) = z^*, & \text{RG fixed points.} \end{cases} \tag{9}$$

More general transformations $F_1(F_2(\cdots F_n(z)\cdots))$ may yield linear equations in a manner identical to those appearing for the Ising theories studied in the current work. Expansion parameters z in self-dual theories satisfy $F(F(z)) = z$; this yields a *constraint on all possible self-dualities*. Solutions are afforded by fractional linear (conformal) maps

$$F(z) = \frac{az+b}{cz-a}, \tag{10}$$

with the determinant of Eq. (2) being non-zero, $a^2 + bc \neq 0$. As we will further expand on elsewhere, another related duality appearing in Ising and all Potts models is given by

$$F_1(z) = \frac{az+b}{cz+d}, \qquad F_2(z) = \frac{-dz+b}{cz-a}, \tag{11}$$

with determinant $ad - bc \neq 0$ such that

$$F_1\big(F_2(z)\big) = z \tag{12}$$

is satisfied. In fact, as we will next establish in Section 3, all "two-"dualities satisfying Eq. (12) must be of the form of Eqs. (11). Specifically, all duality mappings that can be made meromorphic by a change of variables, *can only be of the fractional linear type*. This *uniqueness* may rationalize the appearance of fractional linear (dual) maps in disparate arenas ranging from statistical mechanics models, such as the ones that we study here, to S-dualities in, e.g., YM theories.

Thus far, we focused on "one-" and "two-"dualities for which the coupling constants satisfy either Eq. (8) or Eq. (12), respectively. Our calculations may be extended to "n-"duality transformations for which

$$F_1\big(F_2\big(\cdots F_n(z)\cdots\big)\big) = z. \tag{13}$$

As the reader may verify, replicating the considerations invoked in the next section leads to the conclusion that if they are meromorphic each of the functions F_k (with $1 \le k \le n$) in Eq. (13) must be of the fractional linear (conformal) form

$$F_k(z) = \frac{a_k z + b_k}{c_k z + d_k}, \tag{14}$$

with a_k, b_k, c_k and d_k being constants.

In general, whether a function F solving Eq. (8) for all z is meromorphic in appropriate coordinates or not, it is impossible that any such function $F(z)$ obeying Eq. (8) will map the entire complex plane (or Riemann sphere) onto a subset $\mathcal{M}$ of the complex plane (or Riemann sphere). This subset $\mathcal{M}$ could be a disk or strip or any other subset of the complex plane. That is, it is impossible that a solution to Eq. (8) will be afforded by a function F which for all complex z, will map $z \to F(z) \in \mathcal{M}$. The proof of this latter assertion is trivial and will be performed by contradiction: Consider a point $z' \notin \mathcal{M}$, then a single application of F on z' leads to an image $F(z') \in \mathcal{M}$. As for all points z (including those that lie in $\mathcal{M}$) the image $F(z)$ is in $\mathcal{M}$, we have $F(F(z')) \in \mathcal{M}$. However, as stated in the beginning of our proof, $z' \notin \mathcal{M}$. This thus shows that $F(F(z')) \neq z'$. In other words, Eq. (8) cannot be satisfied by such a function. Thus, if we regard the map $z \to F(z)$ as a finite "time evolution" (or "flow" in the parlance of RG), the function $F(z)$ must "evolve" z as an "*incompressible fluid*" with area preserving dynamics in the complex plane (or Riemann sphere). This flow must be of period two in order to satisfy Eq. (8).

3. All meromorphic duality transformations must be conformal

Charles Babbage, "the father of the computer", [28] and others since, e.g, [29,30], have shown that the functional equation problem of Eq. (8) enjoys an infinite number of solutions. This observation can be summarized as follows: Given a particular solution f to Babbage's equation, $f(f(x)) = x$, a very general class of solutions can be written as

$$F(x) = \phi^{-1}\big(f\big(\phi(x)\big)\big), \tag{15}$$

where ϕ is an *arbitrary* (or in a physics type nomenclature,"gauge like") function with a well defined inverse ϕ^{-1}. In other words, if we have a particular solution we can find other solutions using a function ϕ with and inverse defined in a specific domain. That is,

$$\begin{aligned} F\big(F(x)\big) &= \phi^{-1}\big(f\big(\phi\big(\phi^{-1}\big(f\big(\phi(x)\big)\big)\big)\big)\big) = \phi^{-1}\big(f\big(f\big(\phi(x)\big)\big)\big) = \phi^{-1}\big(\phi(x)\big) \\ &= x. \end{aligned} \tag{16}$$

To make Babbage's observation clear, we note that if, as an example, we examine the Möbius transformation (Fig. 1) of Eq. (7), $f(x) = (1-x)/(1+x)$, and consider $\phi(x) = x^2$ and a particular branch $\phi^{-1}(x) = \sqrt{x}$ for complex x (or the standard $\sqrt{x}$ function for real $x \ge 0$) then it is clearly seen that $F = \sqrt{(1-x^2)/(1+x^2)}$ is also a solution to the equation $F(F(x)) = x$. Similarly, if we choose $\phi(x) = e^{-2x}$ then $\phi^{-1}(f(\phi(x))) = -\frac{1}{2}\ln((1-e^{-2x})/(1+e^{-2x}))$ which the astute reader will recognize as the transformation of Eq. (50).

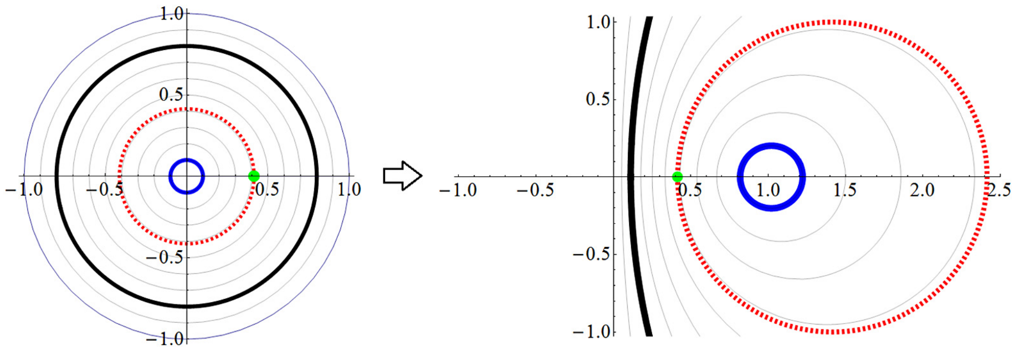

Fig. 1. The Möbius transformation of Eq. (7) embodying the duality of the Ising model, with $|z| \leq 1$, as a conformal map in the complex plane that maps circles onto new shifted circles with a different radius (see Eq. (4)). Let us consider a circle of radius r with its center at the origin. Using the transformation above, it would be mapped to a new circle of radius $2r/(1-r^2)$ with its center shifted to the point $(1+r^2)/(1-r^2)$ (on the real axis). Three of such circles with different colors are shown in the figure above on the left-hand side. On the right-hand side we see these three circles (with the same color as on the left-hand side) after transformation. The green dot represents the self-dual point ($z^* = \sqrt{2}-1$).

We now turn to a rather trivial yet as far as we are aware *new result* concerning this old equation that we establish here. We assert that if there exists a transformation ϕ that maps complex numbers z on the Riemann sphere, $z \to \phi(z)$, such that the resulting function F is meromorphic then *any such function F solving Eq. (8) must be of the fractional linear form (a particular conformal map)* of Eq. (10). Of course, a broad class of functions of the form of Eq. (15) may be generated by choosing arbitrary ϕ that have an inverse yet all possible rational functions will be of the fractional linear form. For instance, the function $F = \sqrt{(1-x^2)/(1+x^2)}$ discussed in the example above is, obviously, not of a fractional linear form.

Proof. The proof below is done by contradiction. A general meromorphic function $F(Z)$ on the Riemann sphere is a rational function, i.e.,

$$F(z) = \frac{P(z)}{Q(z)}, \tag{17}$$

with $P(z)$ and $Q(z)$ relatively prime polynomials. (If the polynomials P and Q are not relatively prime then we can obviously divide both by any common factors that they share to make them relatively prime in the ratio appearing in Eq. (17).) As a first step, we may find the solution(s) w to the equation

$$F(w) = z. \tag{18}$$

Unless both $P(w)$ and $Q(w)$ are linear in w, there generally will be (by the fundamental theorem of algebra) more than one solution to this equation (or, alternatively, a single solution may be multiply degenerate). That is, unless P and Q are both linear in w, the polynomial

$$W_z(w) = P(w) - zQ(w) \tag{19}$$

will be of order higher than one ($m > 1$) in w and will, for general z, have more than one different (non-degenerate) zero. When varying z over all possible complex values, it is impossible that the polynomial $W_z(w)$ will always have only degenerate zero(s) for the relatively prime $P(w)$ and $Q(w)$ (we prove this in the rather simple *(Multiplicity) Lemma* below).

We denote the general zeros of the polynomial $W_z(w)$ by $w_1, w_2, \cdots, w_m$. That is,

$$W_z(w_1) = W_z(w_2) = \cdots = W_z(w_m) = 0. \tag{20}$$

Now if $F(F(z)) = z$, then all solutions $\{z_{ji}\}$ to the equations $F(z_{ji}) = w_i$ (for which the polynomial (in z), $W_{w_i}(z) \equiv P(z) - w_i Q(z)$ vanishes) will, for all i, solve the equation

$$F\big(F(z_{ji})\big) = z. \tag{21}$$

In the last equation above, on the right-hand side there is a single (arbitrary) complex number z whereas on the left-hand side there are *multiple* (see, again, the (Multiplicity) Lemma) viable different solutions z_{ji}. Thus, at least one of the solutions in this set $z_{ji} \neq z$. We denote one such solution by Z. Putting all of the pieces together, the equation $F(F(z)) = z$ cannot be satisfied for all complex z (in particular, it is not satisfied for $z = Z$). Thus, both $P(z)$ and $Q(z)$ must be linear in z, and the fractional linear form of Eq. (10) follows once it is restricted to this class. □

Replicating the above steps *mutatis mutandis* for "two-"dualities satisfying Eq. (12) similarly leads to the conclusion that if the *transformations are meromorphic they must be given by ratios of linear functions (and thus conformal)*. In this case, F_1 can be a general fractional linear transformation with a finite determinant and further constraints on F_2 are afforded by the requirement that Eq. (12) is indeed obeyed. The calculation then leads to the result of Eq. (11). We will elaborate on this restriction in Section 4.

(Multiplicity) Lemma.

We prove (by contradiction) that it is impossible for $W_z(w)$ (Eq. (19)) to have an m-th order ($m > 1$) degenerate root for all z. Assume, on the contrary, that

$$W_z(w) = A(z)\big(w - B(z)\big)^m = P(w) - zQ(w), \tag{22}$$

with $A(z)$ and $B(z)$ functions of z, $m > 1$, and $P(w)$, $Q(w)$, relatively prime polynomials of w. At $z + \delta z$ (with infinitesimal δz), the degenerate root is given by

$$w = B(z + \delta z) \equiv B(z) + \delta B. \tag{23}$$

That is, by definition,

$$0 = W_{z+\delta z}\big(B(z) + \delta B\big). \tag{24}$$

We next use the Taylor expansion

$$0 = W_z\big(B(z)\big) + \delta B \frac{\partial W_z(w)}{\partial w}\bigg|_{w=B(z),z} + \delta z \frac{\partial W_z(w)}{\partial z}\bigg|_{w=B(z),z}. \tag{25}$$

Given the above form of $W_z(w)$, its partial derivative $\partial W_z/\partial w = 0$ at $w = B(z)$, for $m > 1$. Similarly, $W_z(w = B(z)) = 0$. Lastly, from Eq. (19)

$$\frac{\partial W_z(w)}{\partial z}\bigg|_{w=B(z),z} = -Q\big(B(z)\big). \tag{26}$$

Putting all of the pieces together,

$$0 = -\delta z Q\big(B(z)\big). \tag{27}$$

Therefore, $w = B(z)$ is a root of $Q(w)$. As the root of $Q(w)$ is independent of z, this implies that the assumed multiply degenerate root (i.e., $B(z)$) of $W_z(w)$ is independent of z, i.e. $B(z) = B$. Recall (Eq. (19)) that $W_z(w) = P(w) - zQ(w)$. As $w = B$ is (for all z) a root of both $W_z(w)$ and $Q(w)$, it follows that $w = B$ is also a root of $P(w)$. It follows that both $P(w)$ and $Q(w)$ share a root (and a factor of $(w - B)$ when factorized to their zeros), e.g., when written as

$$P(w) = C\prod_a (w - p_a), \qquad Q(w) = D\prod_b (w - q_b), \tag{28}$$

with C and D constants and with $\{p_a\}$ and $\{q_b\}$ the roots of $P(w)$ and $Q(w)$ respectively, at least one of the zeros ($\{p_a\}$) of $P(w)$ must be equal to one of the zeros ($\{q_b\}$) of $Q(w)$. Thus, $P(w)$ and $Q(w)$ are not relatively prime if $m > 1$. This, however, is a contradiction and therefore establishes our assertion and proves this lemma. □

4. Most general meromorphic "*n*-"dualities

Thus far, we largely focused on "two-"dualities satisfying Eq. (8). The ideas underlying our proof in Section 3 illustrated that all meromorphic dualities must be of the fractional linear form, Eq. (1). As elaborated, when applied to "two-"dualities satisfying Eq. (8), the most general meromorphic solution is that of Eq. (10). Similarly, more general dualities for which Eq. (12) is obeyed enjoy more solutions (such as those afforded by Eq. (11)).

We now explicitly solve the general case of Eq. (13). As proven, the fractional linear transformations, Eq. (14), are the only possible meromorphic solutions. We thus confine our attention to these. In what follows, we will invoke the composition property of Eq. (3). On the right-hand side of Eq. (13), the function z may be expressed in matrix form as

$$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \tag{29}$$

with γ an arbitrary complex number. This is so as the matrix elements ($a = \gamma,\ b = 0 = c,\ d = \gamma$) are such that, rather trivially, the associated fractional linear function of Eq. (1) is $(\gamma \cdot z + 0 \cdot 1)/(0 \cdot z + \gamma \cdot 1) = z$. If all functions F_k, in Eq. (13) are of the same form of Eq. (1), then when the representation of Eq. (29) is inserted we will trivially have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \equiv M^n = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \tag{30}$$

whose solutions are straightforward. When diagonalized by a unitary transformation, the matrix M must only have n-th roots of γ. Thus,

$$M = \gamma^{1/n}\mathcal{U}^\dagger \begin{pmatrix} e^{2\pi \mathrm{i} k_1/n} & 0 \\ 0 & e^{2\pi \mathrm{i} k_2/n} \end{pmatrix} \mathcal{U} \equiv \gamma^{1/n}\tilde{\mathsf{M}}, \tag{31}$$

with $k_{1,2}$ arbitrary integers and $\mathcal{U}$ any 2×2 unitary matrix. The latter may, of course, most generally be written as $\mathcal{U} = \exp[-\mathrm{i}\theta\vec{\sigma} \cdot \hat{n}/2]$ with $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, the triad of Pauli matrices, θ an arbitrary real number and $\hat{n} = ((\hat{n})_1, (\hat{n})_2, (\hat{n})_3)$ a unit vector. The factorization of $\gamma^{1/n}$ was performed in Eq. (31) because, as we briefly remarked earlier, a uniform scaling of all four elements of the general 2×2 matrix does not alter the fractional linear transformation of Eq. (1). All possible dualities are exhausted by *the space spanned by all of the matrices* $\tilde{\mathsf{M}}$ of the form of Eq. (31), and a duality with real $\hat{n}$ can then be interpreted as an induced map on the Euclidean S^2 sphere (or, more precisely, one of its hemispheres as we will explain shortly).

In the case of $n=2$ (i.e., that of Eq. (8)), the only non-trivial (i.e., non-identity matrix) solution of the form of Eq. (31) is formed by having $(k_2-k_1)\equiv 1 \pmod 2$. When this occurs, Eq. (31) becomes

$$\tilde{\mathsf{M}} = \mathcal{U}^\dagger \sigma^3 \mathcal{U} = \vec{\sigma}\cdot\hat{n}. \tag{32}$$

When $n=2$, the solution of Eq. (32) is, of course, identical to that of Eq. (10) once we set $\gamma^{1/n}\hat{n} = ((b+c)/2, \mathrm{i}(b-c)/2, a)$. For example, the Ising model duality of Eq. (7) is associated with the unit vector $\hat{n} = 2^{-1/2}(1,0,-1)$. We thus see how the particular solutions that we obtained earlier are a particular case of this more general approach. For "two-"dualities with real $\hat{n}$, any point on the southern hemisphere (i.e., one with $(\hat{n})_3<0$) is associated with a different transformation. This is so as scaling the global multiplication of the matrix by (-1) (associated with $\hat{n}\to-\hat{n}$) does not alter the fractional linear transformation of Eq. (1). This space spanned by the hemisphere is, of course, identical to that of the $\mathcal{R}P^2$ group associated with nematic liquid crystals having a two-fold homotopy group, $\Pi_1(\mathcal{R}P^2)=\mathbb{Z}_2$ and two associated possible defect charges. Geometrically, we may thus understand dualities by thinking of the space spanned by these group elements.

In a similar vein, in the "n-"duality solution of Eq. (31), the eigenvalues of M are any two roots of the identity (or stated equivalently, any two elements of the cyclic group $\mathbb{Z}_n$ (which, on its own, form the center of the group $SU(n)$)) multiplying $\gamma^{1/n}$. We now return to the general problem posed by Eq. (13). Repeating our arguments thus far, it is readily seen that the most general meromorphic solution is afforded by the fractional linear maps of Eq. (14) with the n-th 2×2 matrix (associated with the fractional linear map F_n) set by the inverse of all others. That is, rather explicitly,

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\cdot\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\cdots\begin{pmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{pmatrix}\right]^{-1}. \tag{33}$$

5. Multiple coupling constants

The considerations of Sections 3 and 4 can be extended to not only two but also to many coupling constants (or their parameterization) $\vec{z}=(z_1,z_2,\ldots,z_{\mathsf{q}})$, $\mathsf{q}\geq 1\in\mathbb{N}$. In the particular case of two coupling constants ($\mathsf{q}=2$), the duality mapping will be of the form $\vec{z}=(z_1,z_2)\to\vec{w}\equiv(F_1(z_1,z_2), F_2(z_1,z_2))\equiv\vec{F}(\vec{z})$, where the functions $F_1(z_1,z_2)$ and $F_2(z_1,z_2)$ must be fractional linear maps of two complex variables.[3]

[3] That all meromorphic duality transformations must generally be of the form of Eq. (34) is seen by extending the arguments of Section 3. Here, we briefly provide simple details explaining this assertion. If each of the functions $F_1(z_1,z_2)$ and $F_2(z_1,z_2)$ is meromorphic on Riemann spheres associated with both z_1 and z_2, then (similar to Eq. (17)) they must be rational, $F_{1,2}=P_{1,2}(z_1,z_2)/Q_{1,2}(z_1,z_2)$. For any fixed $z_2=\mathsf{const.}$, the proof of Section 3 demonstrates that the two functions $F_{1,2}(z_1,z_2=\mathsf{const.})$ are both functions of the form of Eq. (1) in the variable z_1. Similarly, if z_1 is held constant, $F_{1,2}$ must become fractional linear functions in z_2. Thus the four binomials $P_1(z_1,z_2)$, $P_2(z_1,z_2)$, $Q_1(z_1,z_2)$ and $Q_2(z_1,z_2)$ are of order no higher than linear in both z_1 and z_2. Similarly, for any number ($\mathsf{q}\geq 1$) of variables z_i, the most general functions $F_i(\vec{z})$ will be found to be ratios of two q-th order multinomials which are linear in each of the variables z_i. In what follows, we return to the case of two complex variables and show how this general form is further restricted. If bilinear (i.e., z_1z_2) terms are present in any of the functions $P_{1,2}$ and $Q_{1,2}$ then recursive applications of the duality transformations (i.e., $\vec{F}(\cdots(\vec{F}(\vec{z})\cdots)))$ generally lead to rational functions in which increasing powers of z_1 and z_2 appear. Thus, closure under applications by $\vec{F}$ generally appears only when all four functions $P_{1,2}$ and $Q_{1,2}$ are linear in both z_1 and z_2 with no bilinear terms allowed. This then only allows for the generalized fractional linear transformations of Eq. (34).

To obtain the proper fractional linear map in several variables, one has to remember that it is important to preserve the composition property of these maps, that is, the application of two of these maps should generate another fractional linear map. Consider a fractional linear map $\vec{F}^{(1)}(\vec{z})$ involving two complex variables

$$w_1 = F_1^{(1)}(z_1, z_2) = \frac{a_1^{(1)} z_1 + a_2^{(1)} z_2 + a_3^{(1)}}{c_1^{(1)} z_1 + c_2^{(1)} z_2 + c_3^{(1)}}, \tag{34}$$

$$w_2 = F_2^{(1)}(z_1, z_2) = \frac{b_1^{(1)} z_1 + b_2^{(1)} z_2 + b_3^{(1)}}{c_1^{(1)} z_1 + c_2^{(1)} z_2 + c_3^{(1)}}, \tag{35}$$

where all the coefficients $a_j^{(1)}, b_j^{(1)}, c_j^{(1)}$ $(j = 1, 2, 3)$ are complex numbers. Then, it is straightforward to verify that the composition of these generalized fractional linear maps, $\vec{F}^{(2)}(\vec{F}^{(1)}(\vec{z}))$, generates another fractional linear map and induces a, non-Abelian in general, group structure. That is, we may associate with each fractional linear map a 3×3 matrix $M^{(1)}$ given by

$$M^{(1)} = \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} \\ b_1^{(1)} & b_2^{(1)} & b_3^{(1)} \\ c_1^{(1)} & c_2^{(1)} & c_3^{(1)} \end{pmatrix}, \tag{36}$$

with a determinant $\Delta \neq 0$. As can be explicitly verified, the composition of maps corresponds to matrix multiplication. Moreover, we can re-scale all coefficients by the (in general, complex) factor $1/\sqrt[3]{\Delta}$ without affecting the map, so that the re-scaled (associated) matrix has a determinant equal to unity. The subset of 3×3 complex matrices with determinant 1 forms a group denoted $SL(3, \mathbb{C})$.

The fixed points of the transformation, $\vec{z}^* = (z_1^*, z_2^*)$, solve the equations

$$z_1^* = \frac{a_1^{(1)} z_1^* + a_2^{(1)} z_2^* + a_3^{(1)}}{c_1^{(1)} z_1^* + c_2^{(1)} z_2^* + c_3^{(1)}}, \tag{37}$$

$$z_2^* = \frac{b_1^{(1)} z_1^* + b_2^{(1)} z_2^* + b_3^{(1)}}{c_1^{(1)} z_1^* + c_2^{(1)} z_2^* + c_3^{(1)}}. \tag{38}$$

When these are satisfied

$$z_2^* = \frac{c_1^{(1)} (z_1^*)^2 + (c_3^{(1)} - a_1^{(1)}) z_1^* - a_3^{(1)}}{a_2^{(1)} - c_2^{(1)} z_1^*}, \tag{39}$$

and z_1^* is the solution of a cubic equation (there are obviously three such cubic equation solutions). Armed with the above, we now investigate the extension Babbage's equation of Eq. (8) with two variables $z_{1,2}$. That is, we now explicitly solve the equation

$$\vec{z} = \vec{F}^{(1)}\big(\vec{F}^{(1)}(\vec{z})\big). \tag{40}$$

There are several solutions to this equation. An important class, characterized by non-zero values of $b_1^{(1)}$ and $c_1^{(1)}$ is given by

$$M^{(1)} = \begin{pmatrix} a_1^{(1)} & \frac{(1+a_1^{(1)})(1+b_2^{(1)})}{b_1^{(1)}} & -\frac{(1+a_1^{(1)})(a_1^{(1)}+b_2^{(1)})}{c_1^{(1)}} \\ b_1^{(1)} & b_2^{(1)} & -\frac{b_1^{(1)}(a_1^{(1)}+b_2^{(1)})}{c_1^{(1)}} \\ c_1^{(1)} & \frac{c_1^{(1)}(1+b_2^{(1)})}{b_1^{(1)}} & -1-a_1^{(1)}-b_2^{(1)} \end{pmatrix}. \tag{41}$$

This solution constitutes a generalization of Eq. (10) to the case of $\mathsf{q} = 2$ complex variables for "two-"dualities.

The generalization of these ideas to more than two coupling constants (or their parameterization), i.e., $\mathsf{q} > 2$, is formally straightforward leading to the $SL(\mathsf{q}+1, \mathbb{C})$ group structure. The geometry of these mappings is a very interesting mathematical problem beyond the scope of the current paper.

6. "Partial solvability" – a non-trivial practical outcome of dualities

We will now examine *constraints that stem from the fractional linear maps* that we found, i.e., a particular set of conformal transformations. A highlight of the remainder of this work is that the results of Eqs. (10), (11), (14), (31), (33) allow for the *partial solvability* of many different theories. How this is done in practice will be best illustrated by detailed calculations. To make the concepts concrete and relatively simple to follow, we will employ, in Sections 7 and thereafter, as lucid examples some of the best studied statistical mechanics models, Ising models and generalized Ising-type lattice gauge theories and focus on $n = 2$ dualities with a single coupling constant ($\mathsf{q} = 1$). In this section, we wish to sketch the central idea behind this technique.

Let us consider an arbitrarily large yet *finite size* system for which no phase transition occurs and thus the partition function $\mathcal{Z}$ (or any other function) is an *analytic* function of all couplings and/or temperature. For such a finite size system, the W-C and S-C expansions (or, correspondingly, high- and low-temperature expansions) of $\mathcal{Z}$, can often be written as finite order series (i.e., polynomials) in the respective expansion parameters $z \equiv f_+(g)$ and $w \equiv f_-(g)$. That is, we consider the general finite order W-C and S-C series for the partition function $\mathcal{Z}$ (or any other analytic function)

$$\mathcal{Z}_{\text{W-C}} = Y_+(z)\sum_n C_n z^n, \qquad \mathcal{Z}_{\text{S-C}} = Y_-(w)\sum_{n'} C'_{n'} w^{n'}, \tag{42}$$

where $Y_\pm$ are analytic functions and $w = F_2(z)$ (for which, according to Eq. (12), $z = F_1(w)$). As in Eq. (42), the two expansions converge to the very same function $\mathcal{Z}$, we trivially have, by the transitive axiom of algebra, two equivalent relations,

$$\begin{aligned} Y_+(z)\sum_n C_n z^n &= Y_-\big(F_2(z)\big)\sum_{n'} C'_{n'}\big(F_2(z)\big)^{n'}, \\ Y_-(z)\sum_{n'} C'_{n'} z^{n'} &= Y_+\big(F_1(z)\big)\sum_n C_n\big(F_1(z)\big)^n, \end{aligned} \tag{43}$$

for the finite number of series coefficients $\{C_n\}$ and $\{C'_{n'}\}$. According to the simple results of Section 3, the functions $F_{1,2}$ appearing in the arguments of $Y_\pm$ and in the expansion itself are of the fractional linear type, i.e., functions of the form of Eq. (11). Now, here is the crux of our argument: When the functions of Eq. (11) are inserted, Eqs. (43) may give rise to constraints amongst the coefficients $\{C_n\}$ and $\{C'_{n'}\}$ and thus *partially solve* for the function $\mathcal{Z}$ with *no additional input*.

Similarly to the "n-"duality mappings of Section 2, the general methods of partial solvability introduced above may be trivially extended to this more general case. This, in particular, may also enable the examination of not only W-C and S-C series but also the matching of partition functions on finite size systems which in the thermodynamic limit will have multiple phases (and associated series for thermodynamic quantities and partition functions). If Eq. (13) applies in systems having a certain number of such regimes in each of which the partition function may be expressed as a different finite order series of the form of Eq. (42), i.e.,

$$\mathcal{Z}_h = Y_h(z) \sum_n C_n z^n, \tag{44}$$

with $1 \leq h \leq m$, where m is the number of finite order representations of the partition function $\mathcal{Z}$, then we will be able to find analogs of Eqs. (43). These, as before, will lead to partial solvability.

As the discussion above is admittedly abstract, we will now turn to concrete examples in the next few sections. One of the most pragmatic consequences of our approach, detailed in Section 10 and [31] is that the complexity of determining the W-C and S-C series expansions may be trivially identical. This lies diametrically opposite to the maxim that S-C series expansions are in many instances far harder to determine than perturbative W-C expansions [32].

7. Series expansions of Ising models

To demonstrate our concept, we will first use standard expansions [22–24,33,34] of the Ising models of Eq. (45) and their generalizations. The Hamiltonian

$$H = -\sum_{\langle \mathsf{ab} \rangle} J_{\mathsf{ab}} s_{\mathsf{a}} s_{\mathsf{b}}, \tag{45}$$

$s_{\mathsf{a}} = \pm 1$. In the remainder of this work, we will consider this and various other models on hypercubic lattices Λ of $N = L^D$ sites in D dimensions (with even length L), endowed with periodic boundary conditions. Unless stated otherwise, we will focus on uniform ferromagnetic systems ($J_{\mathsf{ab}} = J > 0$ for all lattice links $\langle \mathsf{ab} \rangle$). In [31] we consider other boundary conditions, system sizes and lattice aspect ratios, and show that our results are essentially unchanged for large systems with random $J_{\mathsf{ab}} = \pm J$.

In the notation of earlier sections, the coupling constant is $(g \equiv) K \equiv \beta J$ with β the inverse temperature. Defining $\tilde{T} \equiv \tanh K (\equiv f_+(K))$, the identity $\exp[K s_{\mathsf{a}} s_{\mathsf{b}}] = \cosh K [1 + (s_{\mathsf{a}} s_{\mathsf{b}}) \tilde{T}]$ leads to a high-temperature (H-T), or W-C, expansion for the partition function

$$\mathcal{Z}_{\text{H-T}} = (\cosh K)^{DN} \sum_{\{s\}} \prod_{\langle \mathsf{ab} \rangle} \big[1 + (s_{\mathsf{a}} s_{\mathsf{b}}) \tilde{T}\big]. \tag{46}$$

The sum $\sum_{\{s\}} (s_{\mathsf{a}} s_{\mathsf{b}}) \cdots (s_{\mathsf{m}} s_{\mathsf{n}}) = 2^N$ if s_{k} at each site k appears an even number of times and vanishes otherwise. Thus,

$$\mathcal{Z}_{\text{H-T}} = 2^N (\cosh K)^{DN} \sum_{l=0}^{DN/2} C_{2l} \tilde{T}^{2l}, \tag{47}$$

where $C_{l'}$ is the number of (not necessarily connected) *loops* of total perimeter $l' = 2l$ ($l = 1, 2, \cdots$) that can be drawn on the lattice and $C_0 = 1$. For each such loop, i.e., $\Gamma = (\mathsf{ab}) \cdots (\mathsf{mn})$

formed by the bonds (nearest neighbor pair products $\{(s_{\mathsf{a}}s_{\mathsf{b}})\}$) appearing in Eq. (47), there is a complementary loop $\overline{\Gamma} = \Lambda - \Gamma$ for which the sum of Eq. (47) remains unchanged. Consequently, the H-T series coefficients are trivially symmetric, $C_{DN-l'} = C_{l'}$.

We next briefly review the low-temperature (L-T), or S-C, expansion. There are two degenerate ground states (with $s_{\mathsf{a}} = +1$ for all sites a or $s_{\mathsf{a}} = -1$) of energy $E_0 = -JDN$. All excited states can be obtained by drawing closed surfaces marking domain wall boundaries. The domain walls have a total $(D-1)$-dimensional surface area s', the energy of which is $E = E_0 + 2s'J$. Taking into account the two-fold degeneracy, the L-T expansion of the partition function in powers of $(f_-(K) \equiv) e^{-2K}$ is

$$\mathcal{Z}_{\text{L-T}} = 2e^{KDN} \sum_{l=0}^{DN/2} C'_{2l} e^{-4Kl}, \tag{48}$$

with $C'_{s'}$ the number of (not necessarily connected) closed *surfaces* of total area $s' = 2l$ ($C'_0 = 1$). That is, the L-T expansion is in terms of $(D-1)$-dimensional "surface areas" enclosing D-dimensional droplets. Geometrically, there are no closed surfaces of too low areas s'. Thus, in the L-T expansion of Ising ferromagnets,

$$C'_{s'} = 0, \;\; s' = 2i, \tag{49}$$

where $1 \leq i \leq D-1$. The L-T coefficients exhibit a trivial *complementarity symmetry* akin to that in the H-T series. Given any spin configuration $\{s_{\mathsf{a}}\}$, there is a unique correspondence with a staggered spin configuration $s'_{\mathsf{a}} = (-1)^{\sum_{\alpha=1}^{D} a_\alpha} s_{\mathsf{a}}$ where a_α are the (integer) Cartesian components of the hypercubic lattice site a (i.e., $\mathsf{a} = (a_1, a_2, \cdots, a_D)$). Domain walls associated with such staggered configuration are inverted relative to those in the original spin configuration s_{a}. That is, if a particular domain wall appears in s_{a} then it will not appear in s'_{a} and vice versa. As a result, $C'_{DN-s'} = C'_{s'}$ (for the even L hypercubic lattices that we consider).

8. Equating weak (H-T) and strong (L-T) coupling series

We will now follow the program outlined in Section 6. Our approach is to compare H-T and L-T series expansions of the Ising (and other arbitrary) models by means of a duality mapping. In the Ising model, the Möbius transformation (that satisfies the "one-"duality condition of Eq. (8))

$$\tilde{T} = \frac{1 - e^{-2K}}{1 + e^{-2K}}, \quad e^{-2K} = \frac{1 - \tilde{T}}{1 + \tilde{T}}, \tag{50}$$

relates expansions in $\tilde{T}$ to those in e^{-2K}. In either of the expansion parameters $f_\pm(K)$ (i.e., $\tilde{T}$ or e^{-2K}), Eqs. (50) are examples of the fractional linear transformations discussed above. $\tilde{T}$ is the magnetization of a single Ising spin immersed in an external magnetic field of strength $h = K/\beta$ when there is a minimal coupling (a Zeeman coupling) between the dual fields: the Ising spin and the external field. This transformation may be applied to Ising models in *all dimensions D* – not only to the $D = 2$ model for which the KW correspondence holds. These transformations emulate, yet are importantly *different from*, a $g \leftrightarrow 1/g$ correspondence (the latter never enables an equality of two finite order polynomials in the respective expansion parameters). Employing the second of Eqs. (50),

$$\mathcal{Z}_{\text{L-T}} = 2\left(\frac{1+\tilde{T}}{1-\tilde{T}}\right)^{DN/2} \left[1 + \sum_{l=1}^{DN/2} C'_{2l} \left(\frac{1-\tilde{T}}{1+\tilde{T}}\right)^{2l}\right]. \tag{51}$$

By virtue of Eq. (47), this can be cast as a finite order series in $\tilde{T}$ multiplying $(\cosh K)^{DN}$. Indeed, by invoking $1-\tilde{T}^2 = \frac{1}{(\cosh K)^2}$ and the binomial theorem,

$$\mathcal{Z}_{\text{L-T}} = 2(\cosh K)^{DN} \sum_{m=0}^{DN} \tilde{T}^m \left[\binom{DN}{m} + \sum_{l=1}^{DN/2} C'_{2l} A^D_{\frac{m}{2},l} \right] \tag{52}$$

where

$$A^D_{k,l} = \sum_{i=0}^{2l} (-1)^i \binom{2l}{i} \binom{DN-2l}{2k-i}. \tag{53}$$

Analogously,

$$\mathcal{Z}_{\text{H-T}} = \frac{e^{KDN}}{2^{(D-1)N}} \sum_{m=0}^{DN} e^{-2Km} \left[\binom{DN}{m} + \sum_{l=1}^{DN/2} C_{2l}\, A^D_{\frac{m}{2},l} \right] \tag{54}$$

Equating Eqs. (47) and (52) and Eqs. (48) and (54) and invoking Eq. (49) leads to a linear relation among expansion coefficients,

$$W^D V + P = 0, \tag{55}$$

where V and P are, respectively, DN-component and $(DN+D-1)$-component vectors defined by

$$V_i = \begin{cases} C_{2i} & \text{when } i \le \frac{DN}{2}, \\ C'_{2(i-\frac{DN}{2})} & \text{when } i > \frac{DN}{2}, \end{cases}$$
$$P_i = \begin{cases} \binom{DN}{2i} & \text{when } i \le \frac{DN}{2}, \\ \binom{DN}{2(i-\frac{DN}{2})} & \text{when } \frac{DN}{2} < i \le DN, \\ 0 & \text{when } i > DN. \end{cases} \tag{56}$$

In Eq. (55), the rectangular matrix

$$W^D = \begin{pmatrix} M^D_{DN\times DN} \\ T^D_{(D-1)\times DN} \end{pmatrix}, \tag{57}$$

where the $DN \times DN$ matrix M^D is equal to

$$M^D = \begin{pmatrix} -2^{N-1}\mathbb{1}_{\frac{DN}{2}\times\frac{DN}{2}} & A^D_{\frac{DN}{2}\times\frac{DN}{2}} \\ A^D_{\frac{DN}{2}\times\frac{DN}{2}} & -2^{(D-1)N+1}\mathbb{1}_{\frac{DN}{2}\times\frac{DN}{2}} \end{pmatrix}, \tag{58}$$

with a square matrix $A^D_{\frac{DN}{2}\times\frac{DN}{2}}$ whose elements $A^D_{k,l}$ $(1 \le k,l \le DN/2)$ are given by Eq. (53). Constraints (49) are captured by T^D in Eq. (57), $T^D = (O_{(D-1)\times\frac{DN}{2}}\ B^D_{(D-1)\times\frac{DN}{2}})$, where the matrix elements $B^D_{k,l} = 1$, if $k = l$, and $B^D_{k,l} = 0$ otherwise; O is a $(D-1)\times\frac{DN}{2}$ null matrix. Apart from the direct relations captured by Eq. (55) that relate the H-T and L-T series coefficients to each other, there are additional constraints including those (i) originating from equating coefficients of odd powers of $\tilde{T}$ and e^{-2K} to zero and (ii) of trivial symmetry related to complimentary loops/surfaces in the H-T and L-T expansion that we discussed earlier, $C_i = C_{DN-i}$ and

$C'_i = C'_{DN-i}$. It may be verified that these restrictions are already implicit in Eq. (55). Notably, as the substitutions $i \leftrightarrow (2k - i)$, $(2l) \leftrightarrow (DN - 2l)$ in Eq. (53) show, Eqs. (52) and (54) are, respectively, invariant under the two independent symmetries $C'_{2l} \leftrightarrow C'_{DN-2l}$ and $C_{2l} \leftrightarrow C_{DN-2l}$ and thus the linear relations of Eq. (55) adhere to these symmetries. Thus, the equalities between the lowest (small $2l$) and highest (i.e., $(DN - 2l)$) order coefficients are a consequence of the duality given by Eq. (50) that relates expansions in the W-C and S-C parameters.

The total number of unknowns (series coefficients) in Eq. (55) is $U = DN$ with $1/2$ of these unknowns being the H-T expansion coefficients and the other $1/2$ being the L-T coefficients (the components V_i). In [31] (in particular, Table 1 therein), we list the rank (R) of the matrix W^D appearing in Eq. (55) for different dimensions D and number of sites N. As seen therein, for the largest systems examined the ratio R/U tends to $3/4$ suggesting that in *all* D only $\sim 1/4$ of the combined L-T and H-T coupling series coefficients need to be computed by combinatorial means. The remaining $\sim 3/4$ are determined by Eq. (55). This fraction might seem trivial at first sight. If, for instance, the first $1/2$ of the H-T coefficients C_{2l} are known (i.e., those with $l \leq DN/4$) then the remaining H-T coefficients C_{2l} (with $l > DN/4$) can be determined by the symmetry relation $C_{DN-2l} = C_{2l}$ and once all of the H-T series coefficients are known (and thus the partition function fully determined), the partition function may be written in the form of Eq. (48) and the L-T coefficients $\{C'_{2l}\}$ extracted. Thus by the symmetry relations alone knowing a $1/4$ of the coefficients alone suffices. The symmetry relations are a rigorous consequence of the duality relations for any value of N. As the duality relations may include additional information apart from symmetries, it is clear that $R/U \geq 3/4$ for finite N (i.e., knowing more than a $1/4$ of the coefficients is not necessary in order to evaluate all of the remaining H-T and L-T coefficients with the use of duality). For a given aspect ratio, the smaller N is (and the smaller the number of unknowns U), the additional relations of Eqs. (49) carry larger relative weight and the ratio R/U may become larger. Thus, $3/4$ is its lower bound. Indeed, this is what we found numerically for all (non-self-dual) systems that we examined [31]. As D increases, the lowest non-vanishing orders in the L-T expansion become more separated and Eqs. (55) become more restrictive for small N systems.[4]

The H-T and L-T series are of the form of Eqs. (47) and (48) for all geometries that share the same minimal D-dimensional hypercube (i.e., of minimal size $L = 2$) of 2^D sites. Thus, equating the series gives rise to linear relations of the same form for both a hypercube of size $N = L^D$ (with general even L) as well as a tube of $N/2^{D-1}$ hypercubes stacked along one Cartesian direction. However, although the derived linear equations are the same, the partition functions for systems of different global lattice geometries are generally dissimilar (indicating that the linear equations can never fully specify the series). Thus, the set of coefficients not fixed by the linear relations depends on the global geometry.

Parity and boundary effects may influence the rank R of the matrix W^D in Eq. (55). As demonstrated in [31] for $D = 2$ lattices in which (at least) one of the Cartesian dimensions L is *odd*, as well as systems with non-periodic boundary conditions, $R/U \sim 2/3$. That is, in such cases $\sim 1/3$ of the coefficients need to be known before Eq. (55) can be used to compute the rest. As explained in [31], symmetry and duality arguments can be enacted to show that in these cases, $R/U \geq 2/3$ for finite N, i.e., its lower bound is $2/3$. A further restriction is that of discreteness –

[4] If, hypothetically, in equating the H-T and L-T expansions in similar systems, the non-vanishing coefficients in the L-T expansion remain far separated and only appeared at order $l' = 2D + 2(D - 1)n$ for $n = 0, 1, 2, \ldots$ then replicating our calculations leads to $R/U \sim 1 - 1/(2D)$.

the coefficients C_{2l}, C'_{2l} (counting the number of loops/surfaces of given perimeter/surface area) must be non-negative integers for the ferromagnetic Ising model.

Let us illustrate the concepts above with a minimal periodic 2×2 ferromagnetic ($J > 0$) system with Hamiltonian $H = -2J[s_1 s_2 + s_1 s_3 + s_2 s_4 + s_3 s_4]$. From Eqs. (47), (48)

$$\mathcal{Z}_{\text{H-T}} = 16 \cosh^8 K \big[1 + C_2 \tilde{T}^2 + C_4 \tilde{T}^4 + C_6 \tilde{T}^6 + C_8 \tilde{T}^8\big],$$
$$\mathcal{Z}_{\text{L-T}} = 2e^{8K} \big[1 + C'_2 e^{-4K} + C'_4 e^{-8K} + C'_6 e^{-12K} + C'_8 e^{-16K}\big]. \tag{59}$$

Invoking Eqs. (56), (57), $V^\dagger = (C_2, C_4, C_6, C_8, C'_2, C'_4, C'_6, C'_8)$, $P^\dagger = (28, 70, 28, 1, 28, 70, 28, 1, 0)$, and

$$W = \begin{pmatrix} -8 & 0 & 0 & 0 & 4 & -4 & 4 & 28 \\ 0 & -8 & 0 & 0 & -10 & 6 & -10 & 70 \\ 0 & 0 & -8 & 0 & 4 & -4 & 4 & 28 \\ 0 & 0 & 0 & -8 & 1 & 1 & 1 & 1 \\ 4 & -4 & 4 & 28 & -32 & 0 & 0 & 0 \\ -10 & 6 & -10 & 70 & 0 & -32 & 0 & 0 \\ 4 & -4 & 4 & 28 & 0 & 0 & -32 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -32 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

There are $U = 8$ unknown coefficients in Eq. (55); the rank (R) of the matrix W is eight. Thus, in this minimal finite system, Eqs. (55) are linearly independent ($R/U = 1$) and all coefficients may be determined ($C_2 = C_6 = 4$, $C_4 = 22$, $C_8 = 1$, $C'_2 = C'_6 = 0$, $C'_4 = 6$, $C'_8 = 1$).

Generally, not all coefficients may be determined by duality alone. As we discussed, in the large system limit, $R/U \to 3/4$. A 4×4 example appears in [31].

9. Partial solvability and binary spin glasses

If $J_{\text{ab}} = \pm J$ independently on each lattice link $\langle \text{ab} \rangle$, then Eqs. (49) need not hold. Instead of Eq. (55), we have [31]

$$S^D V + Q = 0. \tag{60}$$

This less restrictive equation (by comparison to Eq. (55)), valid for all $J_{\text{ab}} = \pm J$, is of course still satisfied by the ferromagnetic system. For the matrix S^D, a large system value of $R/U \sim 3/4$ is still obtained [31] (see Table 2 therein). The partition functions for different $J_{\text{ab}} = \pm J$ realizations will be obviously different. Nevertheless, all of these systems will share these linear relations.[5] Unlike the ferromagnetic system, the integers $C_{l'}, C'_{s'}$ may be negative. Computing the partition function of general binary spin glass $D = 2$ Ising models is a problem of *polynomial* complexity in the system size. When $D \geq 3$, *the complexity becomes that* of an NP complete problem [35,36]. Therefore, our equations partially solve and "*localize*" NP-hardness to only a fraction of these coefficients; the remaining coefficients are determined by linear equations. The complexity of computing $\binom{n}{m}$, required for each element of S^D, is $\mathcal{O}(n^2)$. Our equations enable a polynomial (in N) consistency checks of partition functions. In performing the expansions of

[5] Information regarding even a single coefficient might differentiate amongst different $\{J_{\text{ab}}\}$ realizations (all adhering to Eqs. (60)). For instance, $C_4 = N$ if and only if $J_{\text{ab}} = +J$ for all $\langle \text{ab} \rangle$.

Eqs. (47) and (48), the complexity of determining the number of loops (or surfaces) of given size l' (or s') (i.e., the coefficients $C_{l'}$ or $C'_{s'}$) increases rapidly with l' (or s').

Our relations may be applied to systematically simplify the calculation of these coefficients. As we now explain, the situation becomes exceedingly transparent in the Ising models discussed thus far. For these theories, the coefficients are symmetric: $C_{l'} = C_{DN-l'}$, $C'_{s'} = C'_{DN-s'}$. By virtue of these symmetries that are embodied in the duality relations of Eq. (60), it is clear that if the lower 1/2 of the H-T coefficients $\{C_{l' \leq DN/2}\}$ (or, similarly, the lower 1/2 of L-T coefficients. i.e., $\{C'_{s' \leq DN/2}\}$), i.e., a 1/4 of the combined H-T and L-T series coefficients, were known then the remaining H-T (or L-T) coefficients are trivially determined. Then, armed with either the full H-T (or L-T) series, the exact partition functions can be equated $\mathcal{Z}_{\text{H-T}} = \mathcal{Z}_{\text{L-T}}$ and written in the form of Eqs. (47) and (48) to determine the remaining unknown L-T (or H-T) coefficients. That is, once the partition functions are known, the series expansions (and thus coefficients) are uniquely determined. By construction, Eq. (60) incorporates, of course, the relation

$$\mathcal{Z}_{\text{H-T}} = \mathcal{Z}_{\text{L-T}} \tag{61}$$

which forms the core of our analysis. Thus, as the symmetry is a consequence of the duality relations, it is clear that knowing a 1/4 of the combined H-T and L-T coefficients suffices to determine all of them via Eq. (60), i.e., that the required fraction of coefficients to find all of the others via duality satisfies the inequality $(1 - R/U) \leq 1/4$. As the asymptotic ratio of $R/U \sim 3/4$ suggests, and as we have verified, knowing the first 1/4 of both the H-T and L-T coefficients (i.e., those with $l' \leq DN/4$ and $s' \leq DN/4$) instead of 1/2 of the H-T (or L-T) coefficients discussed above, suffices to completely determine all other coefficients. As the difficulty of evaluating coefficients increases rapidly with their order, systematically computing this 1/4 lowest order coefficients ($\{C_{l' \leq DN/4}\}$, $\{C'_{s' \leq DN/4}\}$) is less numerically demanding than computing the first 1/2 of all the H-T coefficients ($\{C_{l' \leq DN/2}\}$), or calculating the first 1/2 of all of the L-T coefficients ($\{C'_{s' \leq DN/2}\}$).

10. Generating "hard"' series expansions from their "easier" counterparts

The central idea underlying our approach is that, for finite size systems, the H-T and L-T series expansions are different representations of the very same partition function, Eq. (61). This equality followed from the analyticity of the partition function on any (arbitrary size yet) finite size system. As the astute reader noted throughout all earlier sections, this relation forms the nub of the current study. It is worthwhile to step back and ask what the practical implications of our results are for disparate H-T and L-T series expansions (or other W-C and S-C series). First and foremost, Eq. (61) implies, of course, that the generation of the H-T and L-T series on finite size lattice are equally hard, as obtaining one immediately yields the other.

As stated by certain insightful textbooks, e.g., [32,37–39], the H-T and L-T expansions differ in their conceptual premise. For instance, as [32] notes, "the derivation of a high-temperature expansion is, in principle, straightforward", since it just amounts to counting the number of closed loops, while, as befits the more meticulous examination of the ground states and myriad possible excitations about them, it may seem that "the generation of lengthy low-temperature series is a highly specialized art". Much work has been devoted to a finite lattice method that improves the bare H-T and L-T series (as in, e.g., the H-T loop tallying briefly reviewed in Section 7) [39–42]. Many specialized texts [37,38] laud the simplifying features of general H-T expansions vis a vis their L-T counterparts, including commending their features such as "smoothness" [37], the uniform sign of the H-T coefficients in disparate theories, and their applicability to gapless systems

[37,38]. In a more recent detailed exposition [39], it was noted that “while the high-temperature series are well-behaved the situation at low temperatures is less satisfactory, in particular above two dimensions”. In a related vein, we remark that the H-T series are well known to naturally relate to one of the oldest and simplest expansions – that of the virial coefficients [43] as well as large-n expansions [44]. Thus, with all of the above, it would generally seem that H-T and L-T qualitatively differ. However, as seen by Eq. (61) and the linear equations that we derived in earlier sections connecting the H-T and L-T expansions, the complexity of deriving either expansion on all general finite size lattices is the same. Thus for finite size lattices with finite order H-T and L-T series related by a transformation of their expansion parameter, the general maxim concerning the different intrinsic complexity of the H-T and L-T expansions does not hold.

Concretely, we may derive H-T coefficients from L-T coefficients and vice versa from the simple relation of Eq. (61). In the case of the Ising model that formed much of the focus of the current study, from Eq. (55) we have that

$$C_{2k} = \frac{1}{2^{N-1}}\left[\sum_{l=1}^{DN/2} A^D_{k,l} C'_{2l} + \binom{DN}{2k}\right],$$
$$C'_{2k} = \frac{1}{2^{(D-1)N+1}}\left[\sum_{l=1}^{DN/2} A^D_{k,l} C_{2l} + \binom{DN}{2k}\right]. \tag{62}$$

In [31], we apply our method to derive the H-T expansions from their L-T counterparts on finite size periodic two- and three-dimensional lattices [45].

It is notable that our method *applies to non-trivial systems such as the three-dimensional Ising model*. Our relations enable *a consistency check of proposed series solutions* and the derivation of the entire series from a knowledge of only a fraction of coefficients. Indeed, we verified that the L-T series provided in [45] satisfy the linear equations of Eq. (55) (and our derived H-T series adhere to the same relations). As we explained in Section 8 for regular uniform coupling systems, and in Section 9 for less constrained non-uniform systems, a partial knowledge of both the L-T and/or H-T series may enable a construction of the full partition function.

As we have reiterated earlier and do so once again here, our approach applies to arbitrarily large yet finite size lattices.

11. New combinatorial geometry relations from dualities

Mathematical identities are system independent and enable the general transformation of one set of objects into another. As such, they are reminiscent of dualities, i.e., isometries [9,10]. Symbolically, let us consider particular partition functions (or “generating functions”) $\{\mathcal{Z}_1\}$ that encode all quantities that we wish to determine in a particular set of systems. If certain identities universally apply, we may invoke these relations to transform each function into an equivalent dual, and formally rewrite

$$\{\mathcal{Z}_1\} = \{\mathcal{Z}_2\} \tag{63}$$

for the two sets of functions. In Eq. (63), $\{\mathcal{Z}_2\}$ can be interpreted as the set of generating functions of very different problems or physical systems. As such, dualities and, in particular, the universal relations that we obtained from conformal transformations linking dual systems may encode very general mathematical relations.

In what follows, we concretely demonstrate that dualities may lead to an extensive number of (new) mathematical relations such as those connecting the number of surfaces and volumes of a particular size. These relations are already contained in our previously derived Eqs. (60). The key conceptual point is that dualities between different types of partition functions (irrespective of the general coupling constants associated with a large set of such functions) can hold generally by virtue of mathematical identities.

Wegner's duality [46] relates interactions between $\{s_{\mathsf{a}}\}$ on the boundaries of "d-dimensional cells" to generalized Ising gauge type models with interactions between $\{s_{\mathsf{a}}\}$ on the boundaries of "$(D-d)$-dimensional cells". These generalized Ising lattice gauge theories are given by the Hamiltonian

$$H = -\sum_{\square_d} K_d \prod_{\mathsf{a}\in\partial\square_d} s_{\mathsf{a}}, \tag{64}$$

with $s_{\mathsf{a}} = \pm 1$ and K_d general coupling constants. Here, a "$(d=1)$-dimensional cell" $\square_{d=1}$ corresponds to a (one-dimensional) nearest neighbor edge (i.e., one whose boundary $\square_{d=1}$ is $\langle \mathsf{ab}\rangle$) associated with standard $s_{\mathsf{a}} s_{\mathsf{b}}$ interactions that we largely focused on thus far (i.e., the Ising model Hamiltonian of Eq. (45). The case of $d=2$ corresponds to a product of four s_{a}'s at the centers of the four edges which form the boundary of a two-dimensional plaquette (as in standard hypercubic lattice gauge theories). That is, $d=2$ corresponds to the lattice gauge Hamiltonian

$$H = -\sum_{\square_2} K_{d=2}\ (U_{\mathsf{ab}} U_{\mathsf{bc}} U_{\mathsf{cd}} U_{\mathsf{da}}), \tag{65}$$

where the link variables $U_{\mathsf{ab}} = \pm 1$, and with $\square_2$ being the standard "$(d=2)$ dimensional" cells (i.e., square plaquettes) whose one-dimensional boundary $\partial\square_2$ is formed by the nearest neighbor one-dimensional links (ab), (bc), (cd), and (da). The case of $d=3$ corresponds to the product of six s_{a}'s at the center of the six two-dimensional faces which form the boundary of a three-dimensional cube, etc. The Hamiltonian is the sum of products of $(2d)$ s_{a}'s on the boundaries of all of the d-dimensional hypercubes in the lattice (in a lattice of $\tilde{N}$ sites there are $N_c = \tilde{N}\binom{D}{d}$ such hypercubes and $N_s = \tilde{N}\binom{D}{d-1}$ Ising variables s_{a} at the centers of their faces). If the dimensionless interaction strength for a d-dimensional cell is K_d then the couplings in the two dual models will be related by Eqs. (50) or, equivalently, $\sinh 2K_d \sinh 2K_{D-d} = 1$. The $D=3, d=1$ duality corresponds to the duality between the $D=3$ Ising model and the $D=3$ Ising gauge theory. The $D=2, d=1$ case is that of the KW self-duality. For general d, Wegner derived his duality from an equivalence between the H-T and L-T coefficients.

We now turn to *new, and rather universal, geometrical results* obtained by our approach that hold in general dimensions d and D. If the ground state degeneracy is 2^{N_g} (e.g., $N_g = 1$ for the standard $(d=1)$ Ising models, $N_g = \tilde{N} + 2$ in $D = d = 2$ Ising gauge theories with periodic boundary conditions), then we find[6] that, *irrespective of the coupling constants*, the H-T and L-T series for these models are given by Eqs. (47) and (48) with the following substitutions

$$N = \frac{N_c}{D}, \qquad C_{2l} = 2^{N_s - N} C_{2l}^{(d)}, \qquad C'_{2l} = 2^{N_g - 1} C'^{(d)}_{2l}. \tag{66}$$

Thus, Eq. (60) obtained for standard $(d=1)$ Ising models also holds for general d following this substitution. *In systems with d-dimensional cells, $C_{2l}^{(d)}$ and $C_{2l}^{\prime(d)}$ denote, respectively, the*

6 For general d, $\mathcal{Z}_{\text{H-T}} = 2^{N_s} (\cosh K)^{N_c} \sum_{l=0}^{N_c/2} C_{2l} \tilde{T}^{2l}$, $\mathcal{Z}_{\text{L-T}} = 2^{N_g} e^{K N_c} \sum_{l=0}^{N_c/2} C'_{2l} e^{-4Kl}$. Comparing these expressions with Eqs. (47), (48) leads to the substitution written in the main text.

number of closed surfaces of total d- and $(D-d)$-dimensional surface areas equal to $2l$. By building on our earlier results, we observe that, when the hypercubic lattice length L is even, *Eq.* (60) *universally relates, in any dimension D (and for any d)*, these numbers to each other leaving only $\sim 1/4$ of these undetermined. By comparison to Eq. (60), additional geometrical conditions that hold for $d=1$ (Eqs. (49)) produce the slightly more restrictive Eq. (55). Similar additional constraints appear for $d>1$. A KW type self-duality present for $D=2d$ leads to linear equations that relate $\{C_{2l}^{(d)}\}$ (the number of surfaces of total $(D/2)$-dimensional surface area $(2l)$) to themselves. We next explicitly discuss the $D=2$, $d=1$ case (i.e., the standard $D=2$ Ising model). Similar considerations hold for any $D=2d$ system. Dualities may, potentially, further provide aysmptotic scaling information concerning the areas of d dimensional closed surfaces that are embedded in D dimensions.

12. Dualities versus self-dualities

More information can be gleaned for self-dual systems, e.g., the KW self-duality of the $D=2$ Ising model. In this model, $C_{2l} \sim C'_{2l}$ (as C_{2l} and C'_{2l} are both the number of closed $(d=1)$-dimensional loops of length $2l$) when Eqs. (55) are applied to large systems ($L \gg l$), see [31]. Consequently, the number of coefficients that need to be explicitly evaluated is nearly 1/2 of those obtained by matching the H-T and L-T expansions without invoking self-duality [31]: $R/U \sim 7/8$ of the coefficients are determined by self-duality once $\sim 1/8$ of the coefficients are provided. We caution that the relation $C_{2l} \sim C'_{2l}$ is only asymptotically correct in the limit of large system sizes. Consequently, we find [31] that R/U asymptotically approaches 7/8 from below (and not from above as it would have if this relation were exact for finite size systems[7]) as N becomes larger.

13. Summary

We demonstrated that *all meromorphic duality transformations on the Riemann sphere (satisfying a generalized form of Babbage's equation) must be a conformal map of the fractional linear type* (and simple generalizations in the case of multiple coupling constants), in the appropriate coupling constants. The bulk of our analysis was focused on investigating the consequences of such general duality maps. As we demonstrated in this work, these maps may lead to linear constraints relating finite order series expansions of two dual models. We speculate that in models with numerous isometries (e.g., $N=4$ supersymmetric YM theories [47]), much of the theory might become encoded in relations analogous to the linear equations studied here. Employing Cramer's rule and noting that the determinants of the matrices appearing therein are volumes of polytopes spanned by vectors comprising the columns of these matrices, relates series amplitudes to *polyhedral volumes* [31]. In $N=4$ supersymmetric YM theories, polyhedral volume correspondences for scattering amplitudes led to a flurry of recent activity [48].

A main theme of our approach is that the analyticity of any quantity ensures that its different series expansions must match for all values of the coupling constants. Consequently, a main outcome of our study is that the *mere existence* of two or more such finite order series expansions of a theory, related by dualities (of the form of Eq. (1)), may "partially solve" that theory. By *partial*

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[7] If the H-T and L-T coefficients were exactly equal to each other for finite size systems N (i.e., if $C_{2l} = C'_{2l}$) then by virtue of the symmetries $C_{2l} = C_{DN-2l}$ and $C'_{2l} = C'_{DN-2l}$ (that are valid for all N), we would have $R/U > 7/8$.

solvability we allude to the ability to compute, with complexity polynomial in the system size, the full partition function $\mathcal{Z}$, for instance, given partial information (e.g., a finite fraction $(1 - R/U)$ of all series coefficients in the examples discussed in this work). Stated equivalently, we saw how to systematically exhaust all of the information that duality relations between disparate systems provides. This yields restrictive linear equations on the combined set of series coefficients of the dual systems. These equations allow for more than the computation of one set of (e.g., low-temperature (L-T) or strong-coupling (S-C)) coefficients in terms of the other half (e.g., high-temperature (H-T) or weak-coupling (W-C)). In Ising models and generalized Ising gauge (i.e., Wegner type) theories on even length hypercubic lattices in general dimensions D, only $\sim 1/4$ of the coefficients were needed as an input to fully determine the partition functions; in the self-dual planar Ising model only $\sim 1/8$ of the coefficients were needed as an input – the self-duality determined all of the remaining coefficients by linear relations. For an Ising chain, the H-T series expansion contains only one (two) term(s) for open (periodic) boundary conditions, i.e., $\mathcal{Z} = 2(2\cosh K)^{L-1}(\mathcal{Z} = [(2\cosh K)^L + (2\sinh K)^L])$, thus trivially all coefficients are determined. As Ising models on varied $D > 1$ lattices and random Ising spin glass systems all solve a common set of linear equations, our analysis demonstrates that properties such as *critical exponents cannot*, in general, be determined by dualities alone. To avoid confusion, we briefly elaborate on this point. Although all of the properties may, of course, be determined by the series coefficients (especially when investigated via powerful tools such as Padé approximants [49] and numerous others), the information supplied by the duality relations *on their own* does not suffice to establish the exact critical exponents – some direct calculations of the coefficients must be invoked. Our linear relations might nevertheless prove useful in evaluating critical exponents more efficiently as they allow for a double pincer approach in which the H-T and L-T series inform about each other.

For the even size hypercubic lattices with periodic boundary conditions studied in this work there are no closed loops (surfaces) of an odd length. Consequently, $C_{l'} = C'_{s'} = 0$ for odd l' or odd s' as we have invoked. If we were to formally allow for additional odd l' or s' coefficients then the ratio $R/U = 1/2$ instead of the values of R/U that we derived (see Table 1). However, when the conditions $C_{l'} = 0$ for odd l' are imposed for the H-T coefficients these lead (via duality) to non-trivial constraints on the L-T series coefficients $C_{l'} = f_{l'}(\{C'_{s'}\}) = 0$ with $f_{l'}$ linear functions. (Similarly, a vanishing of the L-T series coefficients leads to non-trivial relations amongst the H-T coefficients.) These constraints lead to $R/U > 1/2$ and to the universal geometric equalities discussed earlier. We earlier obtained lower bounds on R/U using a complementarity symmetry; the linear constraints may relate to the complementarity of the coefficients. From *a practical point of view*, we explained and showed how *S-C series expansions may be generated from their W-C counterparts* (and vice versa). Thus, we saw that seemingly easily perturbative W-C (or H-T) and more nontrivial S-C (or L-T) expansions are actually identically equally hard to generate. We applied these ideas [31] to concrete test cases for some of the *largest exactly known series for both two- and three-dimensional Ising models on finite size lattices* [45]. It is worth reiterating this and underscoring that this construct may be thus applied to general non-integrable systems (such as the three-dimensional Ising model, the general $D > 2$ models in Table 1), and numerous other theories.

Table 1 summarizes our findings for numerous models on even size lattices in D dimensions endowed with periodic boundary conditions.[8] In [31], we discuss other lattice sizes and boundary

[8] Asymptotically, by virtue of self-duality, for large systems, $C_{2l} \sim C'_{2l}$ for $l \ll L$ in Eq. (60) (we have not re-evaluated this fraction for small systems).

Table 1
Partial solvability of various models. A fraction R/U of the coefficients are simple functions of a fraction $(1 - R/U)$ of coefficients of the H-T(W-C)/L-T(S-C) series.

Model	D	R/U
Ising hypercubic	> 2	3/4
Ising hypercubic spin-glass	> 2	3/4
Wegner models	> 2	3/4
Spin-glass Wegner models	> 2	3/4
Self-dual Ising	2	7/8
Honeycomb and triangular Ising	2	3/4
Potts hypercubic ($q > 2$)	> 2	2/3
Self-dual Ising gauge	4	7/8

conditions. With the aid of our linear equations, the NP hardness of models such as the Ising spin glass in finite dimensions $D > 2$ is confined to a fraction $(1 - R/U)$ of determining all $\mathcal{O}(N)$ coefficients in these models. As we underscored, once these are computed, the remaining fraction R/U of the coefficients are given by rather trivial linear equations. A similar matching of series, performed in this work for the partition function, may be replicated for *any physical quantity*, such as matrix elements of operators, admitting a finite series expansion. Although the illustrative models shown in Table 1 are all classical, all of our proofs concerning the conformal character of general dualities and the restrictions that these imply are completely general and *hold for both classical and quantum systems*.

A highly nontrivial consequence of our work is the systematic derivation of new mathematical relations via dualities. In the test case of the Ising, Ising gauge, and generalized Wegner models explored in detail in this work, *we found an extensive set of previously unknown equalities in combinatorial geometry* given by substituting Eqs. (66) into Eq. (60).

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.nuclphysb.2014.12.026.

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