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LINEAR ALGEBRA AND ITS APPLICATIONS

Generalized exponents of non-primitive graphs

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Abstract

The exponent of a primitive digraph is the smallest integer k such that for each ordered pair of (not necessarily distinct) vertices x and y there is a walk of length k from x to y. As a generalization of exponent, Brualdi and Liu (Linear Algebra Appl. 14 (1990) 483–499) introduced three types of generalized exponents for primitive digraphs in 1990. In this paper we extend their definitions of generalized exponents from primitive digraphs to general digraphs which are not necessarily primitive. We give necessary and sufficient conditions for the finiteness of these generalized exponents for graphs (undirected, corresponding to symmetric digraphs) and completely determine the largest finite values and the exponent sets of generalized exponents for the class of non-primitive graphs of order n, the class of connected bipartite graphs of order n and the class of trees of order n.

Keywords: Graph; Digraph; Exponent

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1. Introduction

In 1990, R.A. Brualdi and Bolian Liu [1] introduced three types of the generalized exponents for non-negative primitive matrices and primitive digraphs. These notions of generalized exponents are natural extensions of the concepts of the primitive exponents for non-negative primitive matrices and the ergodic indices for the transition matrices of finite homogeneous Markov chains (see the comments following Definition 1.2). Primitive exponents and ergodic indices have been extensively studied.

Generalized exponents also have an interpretation in a model of "memoryless communication networks". For details, see [1]. Generalized exponents can be defined both for matrices and digraphs. In this paper, we adopt the graph-theoretic version to define the generalized exponents and use graph-theoretic methods to prove our main results (see [1] for matrix versions of the definitions of generalized exponents).

A digraph D is called primitive if there exists a positive integer k such that for each ordered pair of vertices x and y (not necessarily distinct), there is a walk of length k from x to y. The smallest such k is called the primitive exponent of D, denoted by $\gamma(D)$. It is well known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1. In order to define the three types of the generalized exponents, we first define the "set exponent" for the vertex subsets of digraphs.

Definition 1.1 [1]. Let D be a digraph and X be a vertex subset of D. The "set exponent" $\exp_D(X)$ is defined to be the smallest positive integer p such that for each vertex y of D, there exists a walk of length p from at least one vertex in X to y. If no such p exists, then we define $\exp_D(X) = \infty$.

If x is a vertex of D, then the "vertex exponent" $\gamma_D(x)$ is defined to be

$$\gamma_D(x) = \exp_D(\{x\}). \tag{1.1}$$

Definition 1.2 [1]. Let D be a digraph of order n. If we choose to order the n vertices $v_1, v_2, \dots, v_n$ of D in such a way that

$$\gamma_D(v_1) \leqslant \gamma_D(v_2) \leqslant \cdots \leqslant \gamma_D(v_n),$$

then we call $\gamma_D(v_k)$ "the kth first type generalized exponent of D", denoted by $\exp(D, k)$.

It is obvious that

$$\exp(D, 1) \leqslant \exp(D, 2) \leqslant \cdots \leqslant \exp(D, n).$$

In particular, if D is primitive, then $\exp(D, n)$ equals the primitive exponent $\gamma(D)$ of D. Also if D is the associated digraph of the transition matrix of a finite homogeneous Markov chain, then $\exp(D, 1)$ is just the ergodic index of the chain. Thus the classical primitive exponent and ergodic index are special cases of $k = n$ and $k = 1$ of our (first type) generalized exponent $\exp(D, k)$.

Now we define the second and third types of generalized exponents of digraphs.

Definition 1.3 [1]. Let D be a digraph of order n and let k be an integer with $1 \leqslant k \leqslant n$. Then we define

$$f(D, k) = \min\{\exp_D(X) \mid X \subseteq V(D) \text{ and } |X| = k\} \tag{1.2}$$

and

$$F(D, k) = \max\{\exp_D(X) \mid X \subseteq V(D) \text{ and } |X| = k\}, \tag{1.3}$$

$f(D, k)$ and $F(D, k)$ are called "the kth second type and the kth third type generalized exponents of D", respectively.

Note that the case $k = n$ for $f(D, k)$ and $F(D, k)$ is trivial, so in the study of $f(D, k)$ and $F(D, k)$ in this paper (Sections 3 and 4), we will only consider the case $1 \leqslant k \leqslant n - 1$.

It is easy to see from (Eq. (1.1)) that

$$f(D, 1) = \exp(D, 1), \qquad F(D, 1) = \exp(D, n) \tag{1.4}$$

We would like to mention here that the original definitions of the three types of generalized exponents in [1] were given only for primitive digraphs, while here we extend the definitions to all digraphs. Therefore it might happen that the values of some generalized exponents of some digraphs are infinity. It is not difficult to see that if a digraph D is primitive, then all the generalized exponents of D are finite. It has also been shown in [2] that primitivity is only a sufficient condition, but not (in general) a necessary condition for the finiteness of these generalized exponents. This means that there do exist non-primitive digraphs with finite values of generalized exponents. So it is natural for us to study generalized exponents for (more general) non-primitive digraphs.

A symmetric digraph D is a digraph where for any vertices x and y in D, (y, x) is an arc if and only if (x, y) is an arc. An (undirected) graph G can naturally correspond to a symmetric digraph D_G by replacing each (undirected) edge $[x, y]$ of G by a pair of (directed) arcs (x, y) and (y, x). In this paper we will identify the graph G with the corresponding symmetric digraph D_G. Thus the primitivity of G and the three types of generalized exponents of G are defined to be the same as that of the corresponding digraph D_G.

Since any edge of G corresponds to a directed cycle of length 2 in D_G, we deduce from the primitivity criterion for digraphs that an (undirected) graph G is primitive if and only if G is connected and contains at least one odd cycle, namely, G is a connected non-bipartite graph.

In [1,3,4], Brualdi and Liu derived various upper bounds of the generalized exponents for primitive digraphs and for primitive digraphs of special types such as: symmetric digraphs, tournaments and minimally strong digraphs. In this paper, we study the generalized exponents for non-primitive (undirected) graphs (including connected bipartite graphs and trees). We will give the finiteness criterion for the three types of the generalized exponents for graphs (in Section 2). Then (in Sections 3 and 4) we will completely determine the largest finite values and the entire exponent sets (the set of those numbers attainable as exponents) of these generalized exponents for the class of non-primitive graphs of order n, the class of connected bipartite graphs of order n and the class of trees of order n.

2. The finiteness conditions for the generalized exponents of undirected graphs

In order to study the three types of the generalized exponents for non-primitive undirected graphs, the first question is when these generalized exponents $\exp(G,k)$, $f(G,k)$ and $F(G,k)$ are finite for a non-primitive graph G. The necessary and sufficient conditions for the finiteness of $\exp(D,k)$, $f(D,k)$ and $F(D,k)$ for a general digraph D were given by Shao and Wu [2]. But for undirected graphs (i.e., symmetric digraphs), we can give more explicit formulations and direct proofs of the finiteness conditions of their generalized exponents.

Theorem 2.1. *Let G be an undirected graph of order n, $1 \leqslant k \leqslant n$, then $\exp(G,k) < \infty$ if and only if G is primitive.*

Proof. The sufficiency part is obvious. So we only consider the necessity part. Suppose G is not primitive. Then either G is not connected or G is connected and bipartite. If G is not connected, then the vertex exponent of any vertex of G is infinite, so $\exp(G,k) = \infty$. If G is a connected bipartite graph, let v be any vertex of G and m be any positive integer. If m is odd, then there is no walk of length m from v to v; If m is even, then there is no walk of length m from v to its adjacent vertices. This shows that $\gamma_D(v) = \infty$ for any vertex v of G, so $\exp(G,k) = \infty$. This proves the necessity part and completes the proof of the theorem. □

From Theorem 2.1 we see that there is no non-primitive graph G such that $\exp(G,k)$ is finite. Thus, there is no need to study the first type generalized exponents for non-primitive graphs, and so we will restrict ourselves to the study of the second and third type generalized exponents for non-primitive graphs.

Theorem 2.2. *Let G be an undirected graph of order $n, G_1, \ldots, G_r$ be the connected components of G which are primitive, $B_1, \ldots, B_s$ be the connected components of G which are not primitive (namely bipartite). Let $V(B_i) = X_i \cup Y_i$ be the bipartition of the vertex set of the connected bipartite graph B_i $(1 \leqslant i \leqslant s)$.*

Suppose X is any vertex subset of G, then the set exponent $\exp_G(X) < \infty$ *if and only if the following three conditions are satisfied*:

$$(1)\ X \cap V(G_j) \neq \emptyset \quad (j = 1, \ldots, r), \tag{2.1}$$

$$(2)\ X \cap X_i \neq \emptyset \quad (i = 1, \ldots, s), \tag{2.2}$$

$$(3)\ X \cap Y_i \neq \emptyset \quad (i = 1, \ldots, s),. \tag{2.3}$$

Proof. Necessity. If (1) is not satisfied, then $X \cap V(G_j) = \emptyset$ for some j, and so there is no walk from any vertex of X to any vertex of G_j, and thus $\exp_G(X) = \infty$. If (2) is not satisfied, then $X \cap X_i = \emptyset$ for some i, so there is no walk of even length from any vertex of X to any vertex of X_i, and there is no walk of odd length from any vertex of X to any vertex of Y_i, and thus we also have $\exp_G(X) = \infty$. A similar argument can be applied to the case when (3) is not satisfied.

Sufficiency. Suppose (1), (2) and (3) are all satisfied. Take

$$z_j \in X \cap V(G_j) \quad (j = 1, \ldots, r),$$
$$x_i \in X \cap X_i \quad (i = 1, \ldots, s),$$
$$y_i \in X \cap Y_i \quad (i = 1, \ldots, s).$$

Let $\gamma(G_j)$ be the primitive exponent of the primitive graph G_j $(j = 1, \ldots, r)$ and $d(B_i)$ be the diameter of the connected graph B_i, $(i = 1, \ldots, s)$. Let m be a positive integer such that

$$m \geqslant \max\{\gamma(G_1), \ldots, \gamma(G_r);\ d(B_1), \ldots, d(B_s)\}.$$

We will show that $\exp_G(X) \leqslant m < \infty$. Suppose y is an arbitrary vertex of G. If $y \in V(G_j)$ for some j, then there is a walk of length m from the vertex z_j of X to y; If $y \in X_i$ for some i, then there is a walk of length m from the vertex x_i of X to y in case m is even, and there is a walk of length m from the vertex y_i of X to y in case m is odd (see Remark 2.1). A similar argument can be applied to the case $y \in Y_i$ for some i. Thus we have shown that there is a walk of length m from some vertex of X to y. Since y is arbitrary, this implies $\exp_G(X) \leqslant m < \infty$, and thus completes the proof of the theorem. □

Remark 2.1. In the above proof we have used the following basic property for undirected graphs: If there is a walk of length b from a non-isolated vertex x to a vertex y in a graph G, then there is a walk of length $b + 2t$ from x to y for all integers $t \geqslant 0$.

This basic property is easy to prove and will be used several times later in this paper.

Theorem 2.3. *Let* $G, G_1, \ldots, G_r$ *and* $B_1, \ldots, B_s$ *be as in Theorem* 2.2, *where* $r, s \geqslant 0$ *and* $r + s \geqslant 1$. *Let k be an integer with* $1 \leqslant k \leqslant n - 1$. *Then:*

(1) $f(G,k) < \infty$ *if and only if* $k \geqslant r + 2s$;
(2) $F(G,k) < \infty$ *if and only if*

$$n - k < \min\{|V(G_1)|, \ldots, |V(G_r)|,\ |X_1|, \ldots, |X_s|,\ |Y_1|, \ldots, |Y_s|\}. \tag{2.4}$$

Proof. (1) If $k < r + 2s$, then any k-vertex subset X of G cannot satisfy all the three conditions (2.1), (2.2) and (2.3), so $\exp_G(X) = \infty$ for any k-vertex subset X of G (by Theorem 2.2), and thus $f(G,k) = \infty$. If $k \geqslant r + 2s$, then it is easy to obtain a k-vertex subset X_0 satisfying Eqs. (2.1)–(2.3), thus $f(G,k) \leqslant \exp_G(X_0) < \infty$.

(2) It is not difficult to prove that

$F(G,k) < \infty$
$\iff \exp_G(X) < \infty$ for every k-vertex subset X of G
$\iff$ every k-vertex subset X of G satisfies Eqs. (2.1)–(2.3)
$\iff$ no $(n-k)$-vertex subset Y of G can contain any of $V(G_j)$ and X_i, Y_i ($j = 1, \ldots, r$; $i = 1, \ldots, s$)
$\iff n - k < |V(G_j)|$ for all $j = 1, \ldots, r$ and $n - k < |X_i|$ for all $i = 1, \ldots, s$ and $n - k < |Y_i|$ for all $i = 1, \ldots, s$
$\iff$ Eq. (2.4) is satisfied. □

3. The second type generalized exponent $f(G,k)$

In this section we study the second type generalized exponents $f(G,k)$ for non-primitive graphs and some special classes of non-primitive graphs such as connected non-primitive graphs (i.e., connected bipartite graphs) and trees. It is natural that we only consider those graphs G and integers k for which $f(G,k)$ is finite. From Theorem 2.3 we know that if G is a non-primitive graph with $f(G,k) < \infty$, then we must have $k \geqslant 2$, while for connected bipartite graphs (including trees), $f(G,k) < \infty$ if and only if $k \geqslant 2$. Also the case $k = n$ for $f(G,k)$ is trivial. So in this section we consider only the case $2 \leqslant k \leqslant n-1$.

Let n, k be integers with $2 \leqslant k \leqslant n-1$. Let $B(n)$ be the class of all connected bipartite graphs of order n (here $k \geqslant 2$ implies $f(G,k) < \infty$ for $G \in B(n)$), $T(n)$ be the class of all trees of order n (also $k \geqslant 2$ implies $f(G,k) < \infty$ for $G \in T(n)$), and $N(n,k)$ be the class of all non-primitive graphs of order n such that $f(G,k) < \infty$. Let

$$E_2(B(n),k) = \{f(G,k) \mid G \in B(n)\} \quad (2 \leqslant k \leqslant n-1), \tag{3.1}$$

$$E_2(T(n),k) = \{f(G,k) \mid G \in T(n)\} \quad (2 \leqslant k \leqslant n-1), \tag{3.2}$$

$$E_2(N(n,k),k) = \{f(G,k) \mid G \in N(n,k)\} \quad (2 \leqslant k \leqslant n-1) \tag{3.3}$$

be the exponent sets of the second type generalized exponents of the classes $B(n)$, $T(n)$ and $N(n,k)$. Let $e_2(B(n),k)$, $e_2(T(n),k)$ and $e_2(N(n,k),k)$ be the largest numbers of the exponent sets $E_2(B(n),k)$, $E_2(T(n),k)$ and $E_2(N(n,k),k)$. In this section we give explicit expressions for the numbers $e_2(B(n),k)$, $e_2(T(n),k)$, $e_2(N(n,k),k)$ and for the sets $E_2(B(n),k)$, $E_2(T(n),k)$ and $E_2(N(n,k),k)$.

Let X be a vertex subset of a graph G. We say that X "d-covers" G, if the set exponent $\exp_G(X) \leqslant d$, i.e., if for any vertex v of G, there exists a walk of length d from at least one vertex of X to the vertex v.

In the following, $[a]$ denotes the largest integer not exceeding a and $\lceil b \rceil$ denotes the smallest integer not less than b.

We first consider the generalized exponent $f(T,k)$ for trees T of order n.

Lemma 3.1. *Let T be a tree of order n. Let k, d be positive integers with $k \geqslant 2$ and*

$$n \leqslant \left\lceil \frac{k+2}{2} \right\rceil d + k. \tag{3.4}$$

Then there exists a vertex subset X of T with $|X| \leqslant k$ which can d-cover T.

Proof. We proceed by induction on k. First assume $k = 2$. Then $n \leqslant 2d + 2$. Let Q be a longest path of T with length $|Q|$. Then $|Q| \leqslant n - 1 \leqslant 2d + 1$. If $|Q|$ is odd, we take the two central vertices u and v of Q; If $|Q|$ is even, take the central vertex u of Q and a vertex v in Q adjacent to u. Then for any vertex y of T, one of the distances $d(u,y)$ and $d(v,y)$ does not exceed d since Q is a longest path of T with length $|Q| \leqslant 2d + 1$. Without loss of generality, we assume $d(u,y) \leqslant d$. If $d(u,y)$ and d have the same parity, then there is a walk of length d from u to y by Remark 2.1. If $d(u,y)$ and d have different parity, then there is a walk of length d from v to y since v is adjacent to u. This shows that the vertex subset $X = \{u,v\}$ can d-cover T, where $|X| = 2 = k$.

Now we assume $k \geqslant 3$ and use induction on k. Let Q be a longest path of T with length $|Q|$. If $|Q| \leqslant 2d + 1$, then using the same arguments as in the case $k = 2$, we can show that there exists a vertex subset $X = \{u,v\}$ with $|X| = 2 \leqslant k$ which can d-cover T. So in the following we may assume that $|Q| \geqslant 2d + 2$.

Let x and y be two end vertices of the path Q, let z be a vertex in Q such that the distance $d(z,y)$ (the length of the unique path between z and y in the tree T) is d, let u be a vertex in Q adjacent to z with $d(u,y) = d + 1$, let B be the connected component containing u in the spanning subgraph $T - E(Q)$. let $B_1, B_2, \ldots, B_m$ be all the connected components of the graph $B - u$. Then there is exactly one vertex (say u_i) in B_i adjacent to u (see Fig. 1).

Let $h(u,B_i)$ be the largest distance between u and the vertices of B_i. Since Q is a longest path of T we must have

$$h(u,B_i) \leqslant d + 1 \quad (1 \leqslant i \leqslant m).$$

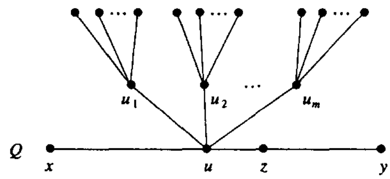

Fig. 1.

Without loss of generality we may assume that there exists an index r with $0 \leqslant r \leqslant m$ such that

$$h(u, B_i) = d + 1 \quad (1 \leqslant i \leqslant r) \tag{3.5}$$

and

$$h(u, B_j) \leqslant d \quad (r + 1 \leqslant j \leqslant m).$$

(Here we agree that if there is no B_i such that $h(u, B_i) = d + 1$, then the index r is considered to be equal to zero.)

Then from Eq. (3.5) we can deduce that

$$|V(B_i)| \geqslant d + 1 \quad (1 \leqslant i \leqslant r). \tag{3.6}$$

Now let V_1 be the set of vertices of T such that $v \in V_1$ if and only if the (unique) path between x and v passes through u, let T_1 be the subgraph induced by the vertex subset V_1. Then T_1 is a subtree of T, and T_1 can be d-covered by the following set of $(r + 2)$ vertices:

$$X_1 = \{z, u, u_1, u_2, \ldots, u_r\}.$$

Also from Eq. (3.6) we have that

$$|V(T_1)| = |V_1| \geqslant r(d + 1) + d + 2. \tag{3.7}$$

Let $V_2 = V(T) \setminus V_1$ and T_2 be the subgraph of T induced by the vertex subset V_2. Then T_2 is also a subtree of T. By the assumption that $|Q| \geqslant 2d + 2$ we have

$$|V(T_2)| = |V_2| \geqslant (2d + 3) - (d + 2) = d + 1. \tag{3.8}$$

On the other hand, from Eqs. (3.7) and (3.4) we also have

$$|V(T_2)| = n - |V(T_1)| \leqslant \left\lceil \frac{k + 2}{2} \right\rceil d + k - (r(d + 1) + d + 2). \tag{3.9}$$

Combining Eqs. (3.8) and (3.9) we have

$$\left\lceil \frac{k + 2}{2} \right\rceil d + k - (r + 2)d - (r + 3) \geqslant 0.$$

So

$$\left[\frac{k-2r-2}{2}\right]d+(k-r-3)\geqslant 0, \tag{3.10}$$

which implies that $k \geqslant r+3$.

Case 1. Suppose $k=r+3$. Then from Eq. (3.10) we have $[(1-r)/2]d \geqslant 0$. So $[(1-r)/2]d=0$ and equality holds in Eq. (3.10), and thus equality also holds in Eqs. (3.9) and (3.8). So $|V(T_2)|=d+1$ and T_2 is a subgraph of the path Q. Now let w be the vertex in Q adjacent to u and different from z, and take the following set of $r+3$ vertices:

$$X=X_1\cup\{w\}=\{w,z,u,u_1,u_2,\dots,u_r\}. \tag{3.11}$$

Then $|X|=r+3=k$ and X can d-cover T.

Case 2. Suppose $k\geqslant r+4$. Let $k_1=k-r-2$. Then $k_1\geqslant 2$. From Eq. (3.9) we have

$$|V(T_2)|\leqslant\left[\frac{k-2r}{2}\right]d+k_1=\left[\frac{k_1-r+2}{2}\right]d+k_1\leqslant\left[\frac{k_1+2}{2}\right]d+k_1. \tag{3.12}$$

Using induction on $k_1\geqslant 2$ for the tree T_2 we know that there exists a vertex subset X_2 of T_2 with $|X_2|\leqslant k_1$ which can d-cover T_2. Now take

$$X=X_1\cup X_2,$$

then $|X|=|X_1|+|X_2|\leqslant r+2+k_1=k$ and X can d-cover the original tree T. This completes the proof of the lemma. □

Remark 3.1. Lemma 3.1 is also true if T is a primitive graph of order n, but it is more difficult to use an inductive proof in the primitive case because the subgraph obtained in the inductive process does not necessarily remain primitive. Actually, the primitive case can be proved as an easy consequence of our tree case by simply taking a spanning tree of the primitive graph. This is an example to show the advantages of considering the non-primitive case in the study of the generalized exponents.

Lemma 3.2. *Let G be any connected graph (primitive or non-primitive) of order n and* $2\leqslant k\leqslant n-1$. *Then we have*

$$f(G,k)\leqslant\left\lceil (n-k)\Big/\left[\frac{k+2}{2}\right]\right\rceil. \tag{3.13}$$

Proof. Let T be a spanning tree of G. Then $f(G,k)\leqslant f(T,k)$. Let

$$d=\left\lceil (n-k)\Big/\left[\frac{k+2}{2}\right]\right\rceil, \tag{3.14}$$

then we have $n \leqslant [(k+2)/2]d + k$. By Lemma 3.1 there exists a vertex subset X of T with $|X| \leqslant k$ which can d-cover T, so we have

$$f(G,k) \leqslant f(T,k) \leqslant f(T,|X|) \leqslant \exp_T(X) \leqslant d. \tag{3.15}$$

Combining Eqs. (3.15) and (3.14) we obtain Eq. (3.13). □

It has been proved in [1] (Lemma 6.6) that there exists a primitive graph $\Gamma^*_{n,k}$ of order n satisfying the equality case of Eq. (3.13). By taking a spanning tree T^*_n of $\Gamma^*_{n,k}$, we have

$$\left\lceil (n-k) \Big/ \left[\frac{k+2}{2}\right] \right\rceil = f(\Gamma^*_{n,k}, k) \leqslant f(T^*_n, k) \leqslant \left\lceil (n-k) / \left[\frac{k+2}{2}\right] \right\rceil, \tag{3.16}$$

so

$$f(T^*_n, k) = \left\lceil (n-k) \Big/ \left[\frac{k+2}{2}\right] \right\rceil. \tag{3.17}$$

From Eq. (3.17) and Lemma 3.2 we already have

$$e_2(T(n),k) = \left\lceil (n-k) \Big/ \left[\frac{k+2}{2}\right] \right\rceil \quad (2 \leqslant k \leqslant n-1), \tag{3.18}$$

where $e_2(T(n),k)$ is the largest number of the exponent set $E_2(T(n),k)$. Since $E_2(T(n),k) \subseteq E_2(B(n),k)$, we also have

$$e_2(T(n),k) \leqslant e_2(B(n),k). \tag{3.19}$$

So by Eqs. (3.18) and (3.19) and Lemma 3.2 we also have

$$e_2(B(n),k) = \left\lceil (n-k) \Big/ \left[\frac{k+2}{2}\right] \right\rceil. \tag{3.20}$$

Now we want to determine the exponent sets $E_2(T(n),k)$ and $E_2(B(n),k)$. Firstly we need to prove the following "ascending property" of the exponent set $E_2(T(n),k)$.

Lemma 3.3. *Let n, k be integers with $2 \leqslant k \leqslant n-1$. Then we have*

$$E_2(T(n),k) \subseteq E_2(T(n+1),k). \tag{3.21}$$

Proof. Let $m \in E_2(T(n),k)$ and suppose $m = f(T,k)$ for some tree T of order n. Let v be a pendant vertex (i.e. a vertex of degree one) of T and u be the unique vertex adjacent to v in T. Construct a new tree T^* of order $n+1$ as follows:

$$V(T^*) = V(T) \cup \{v^*\} \quad (\text{where } v^* \notin V(T)),$$
$$E(T^*) = E(T) \cup \{[u, v^*]\}.$$

We want to show $f(T^*,k) = f(T,k)$.

Since $f(T,k) = m$, there exists a k-vertex subset $X_0 \subseteq V(T)$ which can m-cover T. It is easy to see that X_0 can also m-cover T^*, so

$$f(T^*,k) \leqslant m = f(T,k). \tag{3.22}$$

On the other hand, suppose $f(T^*,k) = p$. Then there exists a k-vertex subset $Y_0 \subseteq V(T^*)$ which can p-cover T^*. Now let

$$X_1 = \begin{cases} Y_0 & \text{if } v^* \notin Y_0, \\ (Y_0 \setminus \{v^*\}) \cup \{v\} & \text{if } v^* \in Y_0, \end{cases} \tag{3.23}$$

then X_1 is a vertex subset of T and $|X_1| \leqslant k$. It is easy to see that X_1 can p-cover T, so

$$f(T,k) \leqslant f(T,|X_1|) \leqslant \exp_T(X_1) \leqslant p = f(T^*,k). \tag{3.24}$$

Combining Eqs. (3.22) and (3.24), we have

$$f(T^*,k) = f(T,k) = m. \tag{3.25}$$

Since T^* is a tree of order $n+1$, Eq. (3.25) implies $m \in E_2(T(n+1),k)$ and thus $E_2(T(n),k) \subseteq E_2(T(n+1),k)$, as desired. □

Now we can give explicit expressions for the exponent sets $E_2(B(n),k)$ and $E_2(T(n),k)$.

Theorem 3.1. *Let n, k be integers with $2 \leqslant k \leqslant n-1$, then*

$$E_2(B(n),k) = E_2(T(n),k) = \left\{1,2,\ldots,\left\lceil (n-k) \Big/ \left[\frac{k+2}{2}\right]\right\rceil\right\}. \tag{3.26}$$

Proof. By definition we know that $e_2(T(n),k) \in E_2(T(n),k)$. Using Lemma 3.3 we also have for $k+1 \leqslant m \leqslant n$ that

$$e_2(T(m),k) \in E_2(T(m),k) \subseteq E_2(T(n),k) \quad (k+1 \leqslant m \leqslant n). \tag{3.27}$$

Thus

$$\{e_2(T(m),k) \mid k+1 \leqslant m \leqslant n\} \subseteq E_2(T(n),k). \tag{3.28}$$

Noticing that

$$e_2(T(m),k) = \left\lceil (m-k) \Big/ \left[\frac{k+2}{2}\right]\right\rceil,$$

we see that the left-hand side of Eq. (3.28) is a set of consequtive integers from 1 to $e_2(T(n),k)$, so Eq. (3.28) can be rewritten as

$$\{1,2,\ldots,e_2(T(n),k)\} = \{e_2(T(m),k) \mid k+1 \leqslant m \leqslant n\} \subseteq E_2(T(n),k). \tag{3.29}$$

Also it is obvious that

$$E_2(T(n),k) \subseteq E_2(B(n),k) \subseteq \left\{1,2,\ldots,\left\lceil (n-k)\Big/\left[\frac{k+2}{2}\right]\right\rceil\right\}.$$

Combining this with Eqs. (3.29) and (3.18) we have

$$E_2(T(n),k) = E_2(B(n),k) = \left\{1,2,\ldots,\left\lceil (n-k)\Big/\left[\frac{k+2}{2}\right]\right\rceil\right\}.$$

This proves the theorem. □

Now we consider the general non-primitive case, and determine the exponent set $E_2(N(n,k),k)$ and the number $e_2(N(n,k),k)$. First we prove the following upper bound of $f(G,k)$ for $G \in N(n,k)$.

Lemma 3.4. *Let n, k be positive integers with $2 \leqslant k \leqslant n-1$. Let G be a graph of order n such that $f(G,k) < \infty$. Then*

$$f(G,k) \leqslant n-k. \tag{3.30}$$

Proof. Suppose $G_1,\ldots,G_r,G_{r+1},\ldots,G_{r+s}$ are all the connected components of G where $G_1,\ldots,G_r$ are primitive and $G_{r+1},\ldots,G_{r+s}$ are bipartite. Then from Theorem 2.3 we know that $f(G,k) < \infty$ implies $k \geqslant r+2s$. Thus we can take positive integers $k_1,\ldots,k_r,k_{r+1},\ldots,k_{r+s}$ such that

$$k_j \geqslant 1 \quad (1 \leqslant j \leqslant r), \tag{3.31}$$

$$k_j \geqslant 2 \quad (r+1 \leqslant j \leqslant r+s) \tag{3.32}$$

and

$$k = \sum_{j=1}^{r+s} k_j. \tag{3.33}$$

Now for any $1 \leqslant j \leqslant r+s$, we take a vertex subset X_j in G_j with $|X_j| = k_j$ such that

$$\exp_{G_j}(X_j) = f(G_j,k_j) \quad (1 \leqslant j \leqslant r+s). \tag{3.34}$$

Let

$$X^* = \bigcup_{j=1}^{r+s} X_j, \tag{3.35}$$

then

$$|X^*| = \sum_{j=1}^{r+s} |X_j| = \sum_{j=1}^{r+s} k_j = k. \tag{3.36}$$

So X^* is a k-vertex subset of G and we have

$$f(G,k) \leqslant \exp_G(X^*) = \max_{1 \leqslant j \leqslant r+s} \exp_{G_j}(X_j) = \max_{1 \leqslant j \leqslant r+s} f(G_j, k_j). \tag{3.37}$$

Next we estimate those $f(G_j, k_j)$. We write $n_j = |V(G_j)|$ and consider the following two cases.

Case 1. $k_j = 1$. Then $1 \leqslant j \leqslant r$ and G_j is primitive. From Eq. (1.4) we have $f(G_j, 1) = \exp(G_j, 1)$. Using the upper bound $\exp(D, 1) \leqslant n - 1$ for a primitive digraph D of order n given in [1], Theorem 6.2 we have

$$\begin{aligned} f(G_j, k_j) &= f(G_j, 1) = \exp(G_j, 1) \leqslant n_j - 1 = n_j - k_j \\ &= (n-k) - \sum_{i \neq j} (n_i - k_i) \leqslant n - k. \end{aligned} \tag{3.38}$$

Case 2. $k_j \geqslant 2$. Then by Lemma 3.2 of this paper we also have

$$\begin{aligned} f(G_j, k_j) &\leqslant \left\lceil (n_j - k_j) \Big/ \left\lfloor \frac{k_j + 2}{2} \right\rfloor \right\rceil \leqslant n_j - k_j \\ &= (n-k) - \sum_{i \neq j} (n_i - k_i) \leqslant n - k. \end{aligned} \tag{3.39}$$

Combining Cases 1 and 2 we obtain $f(G_j, k_j) \leqslant n - k$ for any $1 \leqslant j \leqslant r + s$. Thus from Eq. (3.37) we obtain $f(G,k) \leqslant n - k$, completing the proof of the lemma. □

Now we give explicit expressions of the exponent set $E_2(N(n,k), k)$ and the number $e_2(N(n,k), k)$.

Theorem 3.2. *Let n, k be integers with* $2 \leqslant k \leqslant n - 1$. *Then we have*

$$E_2(N(n,k), k) = \{1, 2, \ldots, (n-k)\} \tag{3.40}$$

and thus

$$e_2(N(n,k), k) = n - k. \tag{3.41}$$

Proof. Take any integer m with $1 \leqslant m \leqslant n - k$. We construct a non-primitive graph $G(n,k,m)$ of order n as follows: $G(n,k,m)$ has k connected components $G_1, \ldots, G_k$. For $1 \leqslant j \leqslant k - 1$, each component G_j is a single loop vertex u_j, while the component G_k consists of a path $v_1, v_2, \ldots, v_m$ with a loop at its end vertex v_1 together with a star of order $n - k - m + 2$ centred at v_m (see Fig. 2).

It is not difficult to verify that

$$f(G(n,k,m), k) = \exp_{G(n,k,m)}(\{u_1, \ldots, u_{k-1}, v_1\}) = m. \tag{3.42}$$

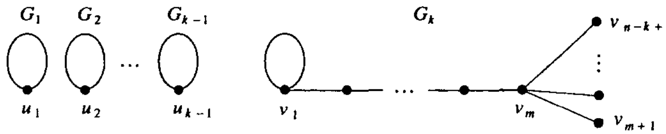

Fig. 2. The graph $G(n, k, m)$.

So $m \in E_2(N(n,k),k)$. This is true for any integer m with $1 \leqslant m \leqslant n-k$, and thus

$$\{1, 2, \cdot, (n-k)\} \subseteq E_2(N(n,k),k). \tag{3.43}$$

Combining Eq. (3.43) and Lemma 3.4 we obtain the desired result. □

4. The third type generalized exponent $F(G, k)$

In this section we study the third type generalized exponents $F(G,k)$ for the class of non-primitive graphs, the class of connected bipartite graphs and the class of trees of order n. As in the case for the second type generalized exponents $f(G,k)$, we only consider those graphs G and integers k for which $F(G,k)$ is finite.

Let

$$EN_3(n,k) = \{F(G,k) \mid G \text{ is a non-primitive graph of order } n \text{ with } F(G,k) < \infty\}, \tag{4.1}$$

$$EB_3(n,k) = \{F(G,k) \mid G \text{ is a connected bipartite graph of order } n \text{ with } F(G,k) < \infty\}, \tag{4.2}$$

$$ET_3(n,k) = \{F(G,k) \mid G \text{ is a tree of order } n \text{ with } F(G,k) < \infty\} \tag{4.3}$$

be the three exponent sets of the three related classes of non-primitive graphs. In this section, we will first give upper bounds of $F(G,k)$ for non-primitive graphs G and then completely determine these three exponent sets.

Suppose n, k are integers with $1 \leqslant k \leqslant n$. From Theorem 2.3 we can see that if $F(G,k) < \infty$ for some non-primitive graph G of order n, then we must have $n-k \leqslant (n/2)-1$, and so $k \leqslant (n/2)+1$. Also the case $k=n$ is trivial. So in this section we consider only the case $(n/2)+1 \leqslant k \leqslant n-1$.

We first give an upper bound of $F(G,k)$ for a connected bipartite graph G of order n with $F(G,k) < \infty$.

Lemma 4.1. *Let n, k be positive integers with $1 \leqslant k \leqslant n-1$. Let G be a connected bipartite graph of order n with $F(G,k) < \infty$. Then we have*

$$F(G,k) \leqslant 2(n-k). \tag{4.4}$$

Proof. Suppose $V(G) = X \cup Y$ is the bipartition of the vertices of G. Write $|X| = n_1$, $|Y| = n_2$, then $n_1 + n_2 = n$. By Theorem 2.3 we can see that $F(G,k) < \infty$ implies $k > n - n_2 = n_1$, and also $k > n_2$.

Let Z be any k-vertex subset of G. Write

$$|Z \cap X| = k_1, \qquad |Z \cap Y| = k_2, \tag{4.5}$$

then $k_1 + k_2 = k$ and

$$k_1 = k - k_2 \geqslant k - n_2 \geqslant 1, \tag{4.6}$$

$$k_2 = k - k_1 \geqslant k - n_1 \geqslant 1. \tag{4.7}$$

Let v_0 be any vertex of G. Without loss of generality, we may assume that $v_0 \in X$. Let z_0 be the vertex in $Z \cap X$ which is nearest to v_0 among all the vertices of $Z \cap X$. Let P be a shortest path between z_0 and v_0. Then P will not pass through the remaining $k_1 - 1$ vertices in $Z \cap X$ other than z_0, so P contains at most $n_1 - k_1 + 1$ vertices in X and thus contains at most $n_1 - k_1$ vertices in Y (by the property of bipartite graphs), so we have

$$d(z_0, v_0) = |P| \leqslant 2(n_1 - k_1) = 2(n - k) - 2(n_2 - k_2) \leqslant 2(n - k). \tag{4.8}$$

But z_0 and v_0 are both in X, so $d(z_0, v_0)$ is even, and thus by Remark 2.1 we know that there is a walk from z_0 to v_0 with length equal to $2(n-k)$. Since v_0 is an arbitrary vertex of G and $z_0 \in Z \cap X \subseteq Z$, this shows that

$$\exp_G(Z) \leqslant 2(n - k). \tag{4.9}$$

Now Eq. (4.9) holds for any k-vertex subset Z of G, so we have $F(G,k) \leqslant 2(n-k)$ as desired. □

Notice that the upper bound in Eq. (4.4) also holds if G is a primitive graph of order n ([1], Theorem 6.3), so we have seen that Eq. (4.4) holds for all connected graphs G of order n with $F(G,k) < \infty$. In the following lemma we will show that this is actually true for all graphs G of order n with $F(G,k) < \infty$.

Lemma 4.2. *Let n, k be integers with $1 \leqslant k \leqslant n-1$ and G be a graph of order n with $F(G,k) < \infty$, then*

$$F(G,k) \leqslant 2(n - k). \tag{4.10}$$

Proof. Let $G_1, \ldots, G_r$ be all the connected components of G and write

$$n_j = |V(G_j)| \quad (j = 1, \ldots, r). \tag{4.11}$$

Let X be any k-vertex subset of G and write

$$X_j = X \cap V(G_j) \quad (j = 1, \ldots, r),$$

$$|X_j| = k_j \quad (j = 1, \ldots, r). \tag{4.12}$$

Then $F(G, k) < \infty$ implies that

$$F(G_j, k_j) < \infty \quad (j = 1, \ldots, r). \tag{4.13}$$

Now each G_j is a connected graph, so by the arguments preceding this lemma we have

$$F(G_j, k_j) \leqslant 2(n_j - k_j) = 2(n-k) - 2\sum_{i \neq j}(n_i - k_i) \leqslant 2(n-k)$$
$$(j = 1, \ldots, r). \tag{4.14}$$

Thus

$$\exp_G(X) = \max_{1 \leqslant j \leqslant r} \exp_{G_j}(X_j) \leqslant \max_{1 \leqslant j \leqslant r} F(G_j, k_j) \leqslant 2(n-k). \tag{4.15}$$

Since Eq. (4.15) is true for any k-vertex subset X of G, we obtain

$$F(G, k) = \max\{\exp_G(X) \mid X \subseteq V(G),\ |X| = k\} \leqslant 2(n-k).$$

This completes the proof of the lemma. □

Next we will construct trees of order n to show that all the integers between 2 and $2(n-k)$ are in the exponent set $ET_3(n, k)$. We consider the cases for even numbers and odd numbers separately.

Lemma 4.3. *Let n, k be positive integers with $(n/2) + 1 \leqslant k \leqslant n - 1$, and let t be an integer with $1 \leqslant t \leqslant n - k$. Then we have*

$$2t \in ET_3(n, k). \tag{4.16}$$

Proof. Let T be a tree of order n as shown in Fig. 3. Let $X = \{x_1, \ldots, x_{n-k+1}\}$, $Y = \{y_1, \ldots, y_{k-1}\}$. Then $V(T) = X \cup Y$ is the bipartition of the vertices of the connected bipartite graph T. It is easy to see that the diameter $d(T)$ of T is

$$d(T) = d(x_1, y_{t+1}) = 2t + 1. \tag{4.17}$$

Take any k-vertex subset Z of T, and let v_0 be any vertex of T. Since $k \geqslant (n/2) + 1$, we have

$$|Y| = k - 1 \geqslant n - k + 1 = |X|$$

and so $|Z| > |Y| \geqslant |X|$. Thus, $Z \cap X \neq \emptyset$ *and* $Z \cap Y \neq \emptyset$. Therefore there exists a vertex z_0 in Z which is in the same part (X or Y) as v_0, and so the distance $d(z_0, v_0)$ is even. Also we have

$$d(z_0, v_0) \leqslant d(T) = 2t + 1.$$

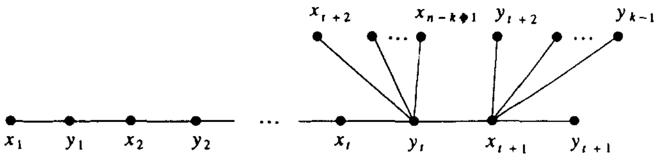

Fig. 3.

But $d(z_0, v_0)$ is even, so we must have $d(z_0, v_0) \leqslant 2t$. By Remark 2.1 we further know that there is a walk of length exactly $2t$ from z_0 to v_0. Since v_0 is an arbitrary vertex of T and $z_0 \in Z$, we have

$$\exp_T(Z) \leqslant 2t. \tag{4.18}$$

Now Eq. (4.18) holds for any k-vertex subset Z of T, so this gives us

$$F(T,k) \leqslant 2t. \tag{4.19}$$

On the other hand, take a special k-vertex subset $Z_0 = Y \cup \{x_1\}$, then it is not difficult to see that there is no walk of length $2t - 1$ from any vertex of Z_0 to the vertex y_{t+1}, so we have

$$F(T,k) \geqslant \exp_T(Z_0) \geqslant 2t. \tag{4.20}$$

Combining Eqs. (4.19) and (4.20) we have

$$2t = F(T,k) \in ET_3(n,k).$$

This proves the lemma. □

Lemma 4.4. *Let n, k be positive integers with $(n/2) + 1 \leqslant k \leqslant n - 1$, let t be an integer with $2 \leqslant t \leqslant n - k$, then we have*

$$2t - 1 \in ET_3(n,k). \tag{4.21}$$

Proof. Let T^* be a tree of order n as shown in Fig. 4. Let $X = \{x_1, \ldots, x_{n-k+1}\}$, $Y = \{y_1, \ldots, y_{k-1}\}$. Then $V(T^*) = X \cup Y$ is the bipartition of the vertices of the connected bipartite graph T^*. Since $t \geqslant 2$, we have $d(T^*) = d(x_1, x_{t+1}) = 2t$.

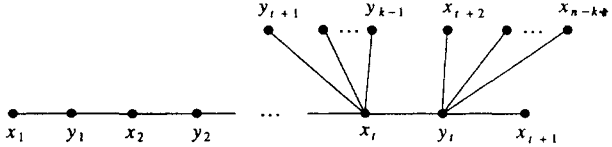

Fig. 4.

Let Z be any k-vertex subset of T^*. By the same arguments as in Lemma 4.3, we have $Z \cap X \neq \emptyset$ and $Z \cap Y \neq \emptyset$. Let v_0 be any vertex of T^*. Then there exists a vertex z_1 in Z which is in the part (X or Y) different from the part where v_0 is in, so $d(z_1, v_0)$ is odd. By arguments similar to that of Lemma 4.3 we know that there is a walk of length exactly $2t-1$ from z_1 to v_0. So $\exp_{T^*}(Z) \leqslant 2t-1$ and thus $F(T^*, k) \leqslant 2t-1$.

On the other hand, take a special k-vertex subset $Z_0 = Y \cup \{x_1\}$. Then there is no walk of length $2t-2$ from any vertex of Z_0 to the vertex x_{t+1}, so we have $F(T^*, k) \geqslant \exp_{T^*}(Z_0) \geqslant 2t-1$.

Combining the above two aspects we have

$$2t - 1 = F(T^*, k) \in ET_3(n, k)$$

and the lemma is proved. □

From Lemmas 4.2–4.4 we already have that

$$\begin{aligned} \{2, 3, \ldots, 2(n-k)\} &\subseteq ET_3(n,k) \subseteq EB_3(n,k) \subseteq EN_3(n,k) \\ &\subseteq \{1, 2, 3, \ldots, 2(n-k)\}. \end{aligned} \tag{4.22}$$

Lemma 4.5. *Let n, k be positive integers with* $(n/2) + 1 \leqslant k \leqslant n-1$. *Then*:

(1) $1 \in EB_3(n,k)$,

(2) $1 \notin ET_3(n,k)$.

Proof. (1) Take the complete bipartite graph $K_{(n-k+1),(k-1)}$, then it is easy to see that

$$1 = F(K_{(n-k+1),(k-1)}, k) \in EB_3(n,k).$$

(2) Let T be any tree of order n, v_0 be a vertex of degree one in T, and v_1 be the unique vertex adjacent to v_0. Since $k \leqslant n-1$, we can take a k-vertex subset Z_0 of T such that $v_1 \notin Z_0$. Then there is no walk of length 1 from any vertex of Z_0 to the vertex v_0. This shows that

$$F(T,k) \geqslant \exp_T(Z_0) \geqslant 2. \tag{4.23}$$

Now Eq. (4.23) holds for any tree T of order n, so we obtain $1 \notin ET_3(n,k)$. □

Combining Eq. (4.22) and Lemma 4.5, we finally obtain the following explicit expressions of the exponent sets $EN_3(n,k), EB_3(n,k)$ and $ET_3(n,k)$.

Theorem 4.1. *Let n, k be positive integers with* $(n/2) + 1 \leqslant k \leqslant n-1$. *Then*:

$$EN_3(n,k) = \{1, 2, \ldots, 2(n-k)\}, \tag{4.24}$$

$$EB_3(n,k) = \{1, 2, \ldots, 2(n-k)\}, \tag{4.25}$$

$$ET_3(n,k) = \{2, 3, \ldots, 2(n-k)\}, \tag{4.26}$$

From expressions (4.24), (4.25) and (4.26) we can also directly see that the largest numbers of the exponent sets $EN_3(n,k), EB_3(n,k)$ and $ET_3(n,k)$ are all $2(n-k)$. Thus the upper bounds given in Lemmas 4.1 and 4.2 are all the best possible upper bounds of the third type generalized exponents $F(G,k)$.

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