On the behavior of the solutions to periodic linear delay differential and difference equations

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Abstract

Some new results on the behavior of the solutions to periodic linear delay differential equations as well as to periodic linear delay difference equations are given. These results are obtained by the use of two distinct roots of the corresponding (so called) characteristic equation.

Keywords: Delay differential equation; Delay difference equation; Periodic coefficients; Characteristic equation; Behavior of solutions

1. Introduction

Driver [6] obtained some significant results on the asymptotic behavior, the nonoscillation and the stability for first order linear autonomous differential equations with infinitely many distributed delays; some important results of the same type for a first order linear delay differential equation with constant coefficients and one constant delay have been given by Driver, Sasser and Slater [9]. Also, Driver’s paper [7] contains some significant asymptotic results for linear differential systems with small delays. Motivated by these old but excellent papers, a number of relevant papers has appeared in the literature during the last few years. See Arino and Pituk [1], Dix et al. [4,5], Frasson and Verduyn Lunel [10], Graef and Qian [11], Kordonis and Philos [17,18], Kordonis et al. [16,19,20], Philos [22], Philos and Purnaras [23–34], and Pituk [37]; * Corresponding author.

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the results in these papers concern delay differential equations, neutral delay differential equations and (neutral or non-neutral) integro-differential equations with unbounded delay, or delay difference equations, neutral delay difference equations and (neutral or non-neutral) Volterra difference equations with infinite delay. For some related results, see de Bruijn [2], Driver et al. [8], Győri [12,13], Norris [21], and Pituk [35,36].

In [30], the authors, motivated by Theorem 2 in Driver’s paper [6], obtained some results on the behavior of the solutions for autonomous linear delay differential equations as well as for autonomous linear neutral delay differential equations. The study in [30] was continued by the authors in [31] to a wide class of autonomous linear neutral delay differential equations (and, especially, delay differential equations) with distributed type delays. The discrete analogues of the results in [30] for autonomous linear delay difference equations as well as for autonomous linear neutral delay difference equations have been presented by the authors in [28]; this paper contains also a result of the same type for autonomous linear delay difference equations with continuous variable. Continuing the study in [28,30,31], the authors established in [33] some further results on the behavior of solutions of linear neutral integro-differential equations with unbounded delay, and, especially, of linear (non-neutral) integro-differential equations with unbounded delay; in [34], the authors gave the discrete analogues of the results in [33] to linear neutral (and, especially, non-neutral) Volterra difference equations with infinite delay. Note that the results in [28,30,31,33,34] are derived by the use of two distinct roots of the corresponding characteristic equation.

In the present paper, we continue the study in [28,30,31,33,34], and we establish some new results on the behavior of the solutions to periodic linear delay differential equations as well as to periodic linear delay difference equations. The case of differential equations is treated in Section 2, while Section 3 is devoted to the case of difference equations.

It is an open question if the results of this paper can be extended to more general periodic linear delay differential equations with several delays (such as treated by the first author in [22]) and to more general periodic linear delay difference equations with several delays (such as studied by Kordonis and the authors in [19]). Such an extension seems to exhibit considerable difficulty.

It will be the subject of a future work to present a study analogous to the one in this paper for periodic linear neutral delay differential and difference equations.

2. Periodic linear delay differential equations

Consider the delay differential equation

\[ x'(t) = a(t)x(t) + b(t)x(t - \tau), \]

(2.1)

where \( a \) and \( b \) are continuous real-valued functions on the interval \([0, \infty)\), and \( \tau \) is a positive real number. The function \( b \) is assumed to be not identically zero on \([0, \infty)\). Moreover, it will be supposed that the coefficients \( a \) and \( b \) are periodic functions with a common period \( T > 0 \) and that there exists a positive integer \( m \) such that

\[ \tau = mT. \]

By a solution of the delay differential equation (2.1), we mean a continuous real-valued function \( x \) defined on the interval \([-\tau, \infty)\), which is continuously differentiable on \([0, \infty)\) and satisfies (2.1) for all \( t \geq 0 \).

In the sequel, by \( C([-\tau, 0], \mathbb{R}) \) we will denote the set of all continuous real-valued functions on the interval \([-\tau, 0]\). It is well known (see, for example, Diekmann et al. [3], Hale [14], or
Hale and Verduyn Lunel [15]) that, for any given initial function $\Phi$ in $C([-\tau, 0], \mathbb{R})$, there exists a unique solution $x$ of the delay differential equation (2.1) which satisfies the initial condition
\[ x(t) = \Phi(t) \quad \text{for} -\tau \leq t \leq 0; \tag{2.2} \]
we shall call this function $x$ the solution of the initial value problem (2.1), (2.2) or, more briefly, the solution of the IVP (2.1), (2.2).

Throughout this section, we will use the notation
\[ A = \frac{1}{T} \int_{0}^{T} a(t) \, dt \quad \text{and} \quad B = \frac{1}{T} \int_{0}^{T} b(t) \, dt. \]
Clearly, $A$ and $B$ are real constants. Note that $B \neq 0$ in the case where the coefficient $b$ is assumed to be of constant sign on the interval $[0, \infty)$.

With the delay differential equation (2.1) we associate the equation
\[ \lambda = A + Be^{-\lambda \tau}, \tag{2.3} \]
which will be called the characteristic equation of (2.1). This terminology comes from the autonomous case where each one of the coefficients $a$ and $b$ is identically equal to a real constant.

In what follows, by $\tilde{a}$ and $\tilde{b}$ we shall denote the $T$-periodic extensions of the coefficients $a$ and $b$, respectively, on the interval $[-\tau, \infty)$. Moreover, for any real number $\lambda$, by $f_\lambda$ we will denote the continuous real-valued function defined on the interval $[-\tau, \infty)$ by
\[ f_\lambda(t) = \tilde{a}(t) + \tilde{b}(t)e^{-\lambda \tau} \quad \text{for} \ t \geq -\tau. \]

Theorem 2.1 below constitutes a basic asymptotic criterion for the solutions of the delay differential equation (2.1), which is closely related to the main result (Theorem 2.3 below) of this section. Theorem 2.1 has been proved by the first author [22], under the additional hypothesis that $b$ is of one sign on the interval $[0, \infty)$. But, as it has been shown by the authors [23], this additional hypothesis is not needed for this theorem to hold. Note that the results in [22] (respectively, in [23]) concern more general periodic linear delay differential equations (respectively, periodic linear neutral delay differential equations) with several delays.

To formulate Theorem 2.1, we introduce the notation
\[ \widehat{B} = \frac{1}{T} \int_{0}^{T} \left| b(t) \right| \, dt. \]
It is clear that $\widehat{B}$ is a positive constant. We obviously have $|B| \leq \widehat{B}$. Moreover, we have $|B| = \widehat{B}$ in the case where the coefficient $b$ is assumed to be of one sign on the interval $[0, \infty)$.

**Theorem 2.1.** Let $\lambda_0$ be a real root of the characteristic equation (2.3) with the property:
\[ \widehat{B} \tau e^{-\lambda_0 \tau} < 1. \]
(Note that this property guarantees that $1 + B \tau e^{-\lambda_0 \tau} > 0$.)
Then, for any $\Phi$ in $C([-\tau, 0], \mathbb{R})$, the solution $x$ of the IVP (2.1), (2.2) satisfies
\[ \lim_{t \to \infty} \left\{ x(t) \exp \left[ -\int_{0}^{t} f_{\lambda_0}(r) \, dr \right] \right\} = \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}}, \tag{2.4} \]
where

\[ L_{\lambda_0}(\Phi) = \Phi(0) + e^{-\lambda_0 \tau} \int_{-\tau}^{0} \tilde{b}(s) \Phi(s) \exp\left[ - \int_{0}^{s} f_{\lambda_0}(r) \, dr \right] \, ds. \] (2.5)

In the main result (Theorem 2.3 below) of this section, two distinct real roots of the characteristic equation (2.3) are used, under the hypothesis that the coefficient \( b \) is nonpositive on the interval \([0, \infty)\). This hypothesis together with the assumptions that \( b \) is not identically zero on \([0, \infty)\) and that \( b \) is a \( T \)-periodic function imply that the constant \( B \) is always negative. Before stating and proving Theorem 2.3, we give a lemma (Lemma 2.2 below) concerning the real roots of the characteristic equation (2.3) in the case where \( B \) is negative. This lemma is a special case of a more general lemma given by the authors in [30].

**Lemma 2.2.** Suppose that \( B \) is negative.

(I) Let \( \lambda_0 \) be a real root of the characteristic equation (2.3). Then

\[ 1 + B \tau e^{-\lambda_0 \tau} > 0 \]

if (2.3) has another real root less than \( \lambda_0 \), and

\[ 1 + B \tau e^{-\lambda_0 \tau} < 0 \]

if (2.3) has another real root greater than \( \lambda_0 \).

(II) In the interval \([A, \infty)\), the characteristic equation (2.3) has no roots.

(III) Assume that

\[ (-B) \tau e^{-A \tau} < \frac{1}{e}. \]

Then:

(i) \( \lambda = A - \frac{1}{\tau} \) is not a root of the characteristic equation (2.3).

(ii) In the interval \((A - \frac{1}{\tau}, A)\), (2.3) has a unique root.

(iii) In the interval \((-\infty, A - \frac{1}{\tau})\), (2.3) has a unique root.

Now, we proceed to present the main result of this section, i.e., Theorem 2.3 below.

**Theorem 2.3.** Suppose that \( b \) is nonpositive on the interval \([0, \infty)\), and let \( \lambda_0 \) and \( \lambda_1 \), \( \lambda_0 \neq \lambda_1 \), be two real roots of the characteristic equation (2.3).

Then, for any \( \Phi \) in \( C([-\tau, 0], \mathbb{R}) \), the solution \( x \) of the IVP (2.1), (2.2) satisfies

\[ U_1(\lambda_0, \lambda_1; \Phi) \leq \left\{ x(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}} \exp\left[ \int_{0}^{t} f_{\lambda_0}(r) \, dr \right] \right\} \exp\left[ - \int_{0}^{t} f_{\lambda_1}(r) \, dr \right] \]

\[ \leq U_2(\lambda_0, \lambda_1; \Phi) \quad \text{for all} \ t \geq 0, \] (2.6)

where \( L_{\lambda_0}(\Phi) \) is defined by (2.5), and
\[ U_1(\lambda_0, \lambda_1; \Phi) = \min_{-\tau \leq t \leq 0} \left( \Phi(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B\tau e^{-\lambda_0 \tau}} \exp \left[ \int_0^t f_{\lambda_0}(r) \, dr \right] \right) \exp \left[ -\int_0^t f_{\lambda_1}(r) \, dr \right], \]

\[ U_2(\lambda_0, \lambda_1; \Phi) = \max_{-\tau \leq t \leq 0} \left( \Phi(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B\tau e^{-\lambda_0 \tau}} \exp \left[ \int_0^t f_{\lambda_0}(r) \, dr \right] \right) \exp \left[ -\int_0^t f_{\lambda_1}(r) \, dr \right]. \]

**Note.** The constant \( B \) is always negative and so, by part (I) of Lemma 2.2, we have
\[ 1 + B\tau e^{-\lambda_0 \tau} \neq 0. \]

It must be noted that (2.6) can equivalently be written in the form
\[ U_1(\lambda_0, \lambda_1; \Phi) \exp \left\{ \int_0^t \left[ f_{\lambda_1}(r) - f_{\lambda_0}(r) \right] \, dr \right\} \]
\[ \leq x(t) \exp \left[ -\int_0^t f_{\lambda_0}(r) \, dr \right] - \frac{L_{\lambda_0}(\Phi)}{1 + B\tau e^{-\lambda_0 \tau}} \]
\[ \leq U_2(\lambda_0, \lambda_1; \Phi) \exp \left\{ \int_0^t \left[ f_{\lambda_1}(r) - f_{\lambda_0}(r) \right] \, dr \right\} \quad \text{for all } t \geq 0 \]
and consequently (2.4) holds, if \( \lambda_0 \) and \( \lambda_1 \) are such that
\[ \lim_{t \to \infty} \int_0^t \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr = \infty. \]

Furthermore, we notice that another equivalent form of (2.6) is the following one:
\[ U_1(\lambda_0, \lambda_1; \Phi) \exp \left[ \int_0^t f_{\lambda_1}(r) \, dr \right] + \frac{L_{\lambda_0}(\Phi)}{1 + B\tau e^{-\lambda_0 \tau}} \exp \left[ \int_0^t f_{\lambda_0}(r) \, dr \right] \]
\[ \leq x(t) \leq U_2(\lambda_0, \lambda_1; \Phi) \exp \left[ \int_0^t f_{\lambda_1}(r) \, dr \right] + \frac{L_{\lambda_0}(\Phi)}{1 + B\tau e^{-\lambda_0 \tau}} \exp \left[ \int_0^t f_{\lambda_0}(r) \, dr \right] \]
\[ \quad \text{for all } t \geq 0. \]

**Proof of Theorem 2.3.** Consider an arbitrary initial function \( \Phi \) in \( C([-\tau, 0], \mathbb{R}) \), and let \( x \) be the solution of the IVP (2.1), (2.2). Set
\[ y(t) = x(t) \exp \left[ -\int_0^t f_{\lambda_0}(r) \, dr \right] \quad \text{for } t \geq -\tau. \]
Furthermore, we define
\[ z(t) = y(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}} \quad \text{for } t \geq -\tau. \]

As it has been shown by the first author in [22] for more general periodic delay differential equations (see also the authors’ paper [23] for the even more general case of periodic neutral delay differential equations), the fact that \( x \) satisfies (2.1) for \( t \geq 0 \) is equivalent to the fact that \( z \) satisfies
\[ z(t) = -e^{-\lambda_0 \tau} \int_{t-\tau}^{t} \tilde{b}(s)z(s) \, ds \quad \text{for } t \geq 0. \] (2.7)

Moreover, the initial condition (2.2) can equivalently be written in the form
\[ z(t) = \Phi(t) \exp \left[ -\int_{0}^{t} \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr \right] - \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}} \quad \text{for } -\tau \leq t \leq 0. \] (2.8)

Next, we consider the function \( w \) defined by
\[ w(t) = z(t) \exp \left\{ \int_{0}^{t} \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr \right\} \quad \text{for } t \geq -\tau. \]

Then we can easily see that (2.7) is equivalent to
\[ w(t) = -e^{-\lambda_0 \tau} \int_{t-\tau}^{t} \tilde{b}(s)w(s) \exp \left\{ \int_{s}^{t} \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr \right\} ds \quad \text{for } t \geq 0, \] (2.9)

and that (2.8) becomes
\[ w(t) = \left\{ \Phi(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}} \exp \left[ \int_{0}^{t} f_{\lambda_0}(r) \, dr \right] \right\} \exp \left[ -\int_{0}^{t} f_{\lambda_1}(r) \, dr \right] \quad \text{for } -\tau \leq t \leq 0. \] (2.10)

Also, from the definitions of \( y, z \) and \( w \) it follows immediately that
\[ w(t) = \left\{ x(t) - \frac{L_{\lambda_0}(\Phi)}{1 + B \tau e^{-\lambda_0 \tau}} \exp \left[ \int_{0}^{t} f_{\lambda_0}(r) \, dr \right] \right\} \exp \left[ -\int_{0}^{t} f_{\lambda_1}(r) \, dr \right] \quad \text{for } t \geq -\tau. \] (2.11)

Furthermore, by the definitions of \( U_1(\lambda_0, \lambda_1; \Phi) \) and \( U_2(\lambda_0, \lambda_1; \Phi) \) and because of (2.10), we have
\[ U_1(\lambda_0, \lambda_1; \Phi) = \min_{-\tau \leq t \leq 0} w(t) \quad \text{and} \quad U_2(\lambda_0, \lambda_1; \Phi) = \max_{-\tau \leq t \leq 0} w(t). \]

Thus, in view of (2.11), inequalities (2.6) take the form
\[ \min_{-\tau \leq s \leq 0} w(s) \leq w(t) \leq \max_{-\tau \leq s \leq 0} w(s) \quad \text{for all } t \geq 0. \]
We will restrict ourselves to prove the left-hand part of the above double inequality. The right-hand part of this double inequality can be shown by an analogous procedure. So, it remains to establish that

\[ w(t) \geq \min_{-\tau \leq s \leq 0} w(s) \quad \text{for every } t \geq 0. \]  

(2.12)

In order to prove (2.12), let us consider an arbitrary real number \( K \) such that \( K < \min_{-\tau \leq s \leq 0} w(s) \). Then

\[ w(t) > K \quad \text{for } -\tau \leq t \leq 0. \]  

(2.13)

We claim that

\[ w(t) > K \quad \text{for all } t \geq 0. \]  

(2.14)

Otherwise, by (2.13), there exists a point \( t_0 > 0 \) so that

\[ w(t) > K \quad \text{for } -\tau \leq t < t_0, \quad \text{and } \quad w(t_0) = K. \]  

(2.15)

Since the function \( b \) is \( T \)-periodic and \( \tau = mT \), the assumption that \( b \) is not identically zero on \([0, \infty)\) means that \( \tilde{b} \) is not identically zero on the interval \([t_0 - \tau, t_0)\), while the hypothesis that \( b \) is nonpositive on \([0, \infty)\) means that \( \tilde{b} \) is nonpositive, but not identically zero, on the interval \([t_0 - \tau, t_0)\). Hence, the function \( \tilde{b} \) is non-positive, but not identically zero, on the interval \([t_0 - \tau, t_0)\), and

\[ \int_{t_0 - \tau}^{t_0} \tilde{b}(s) \, ds = B\tau. \]

Now, by using (2.15) and taking into account the above observations, from (2.9) we obtain

\[ K = w(t_0) = -e^{-\lambda_0 \tau} \int_{t_0 - \tau}^{t_0} \tilde{b}(s) w(s) \exp \left\{ \int_{s}^{t_0} \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr \right\} \, ds \]

\[ > -Ke^{-\lambda_0 \tau} \int_{t_0 - \tau}^{t_0} \tilde{b}(s) \exp \left\{ \int_{s}^{t_0} \left[ f_{\lambda_0}(r) - f_{\lambda_1}(r) \right] \, dr \right\} \, ds \]

\[ = -Ke^{-\lambda_0 \tau} \int_{t_0 - \tau}^{t_0} \tilde{b}(s) \exp \left[ \left( e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau} \right) \int_{s}^{t_0} \tilde{b}(r) \, dr \right] \, ds \]

\[ = Ke^{-\lambda_0 \tau} \int_{t_0 - \tau}^{t_0} \exp \left[ \left( e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau} \right) \int_{s}^{t_0} \tilde{b}(r) \, dr \right] d \left[ \int_{s}^{t_0} \tilde{b}(r) \, dr \right] \]

\[ = Ke^{-\lambda_0 \tau} \frac{e^{-\lambda_0 \tau}}{e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau}} \left\{ \exp \left( e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau} \right) \int_{t_0 - \tau}^{t_0} \tilde{b}(r) \, dr \right\} \]

\[ = K \frac{e^{-\lambda_0 \tau}}{e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau}} \left\{ 1 - \exp \left( e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau} \right) \int_{t_0 - \tau}^{t_0} \tilde{b}(r) \, dr \right\} \]
\[ K \frac{e^{-\lambda_0 \tau}}{e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau}} \{ 1 - \exp \left[ (e^{-\lambda_0 \tau} - e^{-\lambda_1 \tau}) B \tau \right] \} = K e^{-\lambda_0 \tau} - K e^{-\lambda_0 \tau} e^{-\lambda_1 \tau} \{ 1 - \exp \left[ B e^{-\lambda_0 \tau} - B e^{-\lambda_1 \tau} \tau \right] \} = K e^{-\lambda_0 \tau} - K e^{-\lambda_0 \tau} \{ 1 - e^{(\lambda_0 - \lambda_1) \tau} \} = K. \]

We have thus arrived at a contradiction, which establishes our claim, i.e., (2.14) holds true. Finally, since (2.14) is fulfilled for each real number \( K \) such that \( K < \min_{-\tau \leq s \leq 0} w(s) \), we conclude that (2.12) is always satisfied.

The proof of the theorem is complete. \( \square \)

Before closing this section, let us consider the special case of the autonomous linear delay differential equation
\[ x'(t) = ax(t) + bx(t - \tau), \tag{2.16} \]
where \( a \) and \( b \neq 0 \) are real constants, and \( \tau \) is a positive real number. The characteristic equation of (2.16) is
\[ \lambda = a + be^{-\lambda \tau}. \]

The constant coefficients \( a \) and \( b \) of the delay differential equation (2.16) can be considered as \( T \)-periodic functions, for each real number \( T > 0 \). Moreover, as it concerns the autonomous delay differential equation (2.16), the hypothesis that there exists a positive integer \( m \) such that \( \tau = mT \) holds by itself. After these observations, it is not difficult to apply the main result of this section, i.e., Theorem 2.3, to the special case of the autonomous linear delay differential equation (2.16). The result obtained by such an application is a special case of a more general result given by the authors in [30] for autonomous linear delay differential equations with several delays.

### 3. Periodic linear delay difference equations

Consider the delay difference equation
\[ \Delta x_n = a(n)x_n + b(n)x_{n-\tau}, \tag{3.1} \]
where \( (a(n))_{n \geq 0} \) and \( (b(n))_{n \geq 0} \) are sequences of real numbers, and \( \tau \) is a positive integer. It is supposed that the sequence \( (b(n))_{n \geq 0} \) is not identically zero. Moreover, it will be assumed that the coefficients \( (a(n))_{n \geq 0} \) and \( (b(n))_{n \geq 0} \) are periodic sequences with a common period \( T \) (where \( T \) is a positive integer) and that there exists a positive integer \( m \) such that \( \tau = mT \).

As usual, a solution of the delay difference equation (3.1) is a sequence of real numbers \( (x_n)_{n \geq -\tau} \), which satisfies (3.1) for all \( n \geq 0 \).

For convenience, let us introduce the set \( S \) defined by
\[ S = \{ \Phi = (\Phi_n)_{n=\tau}^0 : \Phi_n \in \mathbb{R} \text{ for } n = -\tau, \ldots, 0 \}. \]
It is clear that, for any given \( \Phi = (\Phi_n)_{n=-\tau}^0 \) in \( S \), there exists exactly one solution \( (x_n)_{n\geq-\tau} \) of the delay difference equation (3.1) which satisfies the initial condition

\[
x_n = \Phi_n \quad \text{for } n = -\tau, \ldots, 0; \tag{3.2}
\]

this unique solution \( (x_n)_{n\geq-\tau} \) is said to be the solution of the initial value problem (3.1), (3.2) or, more briefly, the solution of the IVP (3.1), (3.2).

The equation

\[
\lambda^T = \prod_{r=0}^{T-1} \left(1 + a(r) + b(r)\lambda^{-\tau}\right) \tag{3.3}
\]

is associated with the delay difference equation (3.1) and will be called the characteristic equation of (3.1). This terminology comes from the autonomous case where the coefficients \( (a(n))_{n\geq0} \) and \( (b(n))_{n\geq0} \) are constant sequences.

In the sequel, by \( (\tilde{a}(n))_{n\geq-\tau} \) and \( (\tilde{b}(n))_{n\geq-\tau} \) we will denote the \( T \)-periodic extensions of the coefficients \( (a(n))_{n\geq0} \) and \( (b(n))_{n\geq0} \), respectively. (Note that \( \tau \) is a multiple of the period \( T \).) Moreover, for any positive real number \( \lambda \), by \( (h_\lambda(n))_{n\geq-\tau} \) we shall denote the sequence of real numbers defined as follows

\[
h_\lambda(n) = 1 + \tilde{a}(n) + \tilde{b}(n)\lambda^{-\tau} \quad \text{for } n \geq -\tau.
\]

We observe that the characteristic equation (3.3) can be written in the form

\[
\lambda^T = \prod_{r=0}^{T-1} h_\lambda(r). \tag{3.3}
\]

This form of the characteristic equation (3.3) will be used quite frequently in the sequel.

Throughout this section, we consider positive roots \( \lambda \) of the characteristic equation (3.3) with the following property:

**If** \( T > 1 \), **then** \( h_\lambda(r) \equiv 1 + a(r) + b(r)\lambda^{-\tau} > 0 \quad (r = 1, \ldots, T - 1). \) \tag{p(\( \lambda \))}

(Clearly, \( p(\lambda) \)) holds by itself when \( T = 1 \).)

We give here a simple result, which we shall keep in mind in what follows; this result has been proved by Kordonis and the authors in [19] for the case of more general periodic delay difference equations (see also the authors’ paper [29] for the even more general case of periodic neutral delay difference equations).

If \( \lambda \) is a positive root of the characteristic equation (3.3) with the property (p(\( \lambda \))), then we have

\[
h_\lambda(n) > 0 \quad \text{for all } n \geq -\tau.
\]

In what follows, for any positive root \( \lambda \) of the characteristic equation (3.3) with the property \( (p(\lambda)) \), by \( (H_\lambda(n))_{n\geq-\tau} \) we shall denote the sequence of positive real numbers defined by

\[
H_\lambda(n) = \begin{cases} \prod_{r=0}^{n-1} h_\lambda(r) & \text{for } n \geq 0, \\ \left[\prod_{r=0}^{-n} h_\lambda(r)\right]^{-1} & \text{for } n = -\tau, \ldots, 0. \end{cases}
\]

Note that, in this section, we use the usual convention

\[
\prod_{r=0}^{-1} = 1.
\]
A fundamental asymptotic result for the solutions of the delay difference equation (3.1) is Theorem 3.1 below, which is closely related to the main result (Theorem 3.4 below) of this section. This asymptotic result has been established by Kordonis and the authors [19] (see also the authors’ paper [29] for the more general case of periodic neutral delay difference equations). It must be noted that in [19] (respectively, in [29]) more general periodic linear delay difference equations (respectively, periodic linear neutral delay difference equations) with several delays are treated.

**Theorem 3.1.** Let \( \lambda_0 \) be a positive root of the characteristic equation (3.3) with the property
\[
(p(\lambda_0))\quad \text{ and the additional property:}
\]
\[
\lambda_0^{-\tau} \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} |b(s)| < 1.
\]
Set
\[
\gamma_{\lambda_0} = \lambda_0^{-\tau} \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s).
\]
(Note that property (3.4) guarantees that \( 1 + \gamma_{\lambda_0} > 0 \).)
Then, for any \( \Phi = (\Phi_n)_{n=-\tau}^0 \) in \( S \), the solution \( (x_n)_{n=-\tau}^\infty \) of the IVP (3.1), (3.2) satisfies
\[
\lim_{n \to \infty} \frac{x_n}{H_{\lambda_0}(n)} = \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}},
\]
where
\[
M_{\lambda_0}(\Phi) = \Phi_0 + \lambda_0^{-\tau} \sum_{s=-\tau}^{\tau-1} \left[ \prod_{r=s+1}^{\tau} h_{\lambda_0}(r) \right] b(s) \Phi_s.
\]

The main result of this section is Theorem 3.4 below, in which two suitable distinct positive roots of the characteristic equation (3.3) will be used and the hypothesis that the coefficient sequence \( (b(n))_{n \geq 0} \) is nonpositive will be posed. Before presenting this theorem, we give two lemmas, i.e., Lemmas 3.2 and 3.3 below. Lemma 3.2 is rather technical and will be used in proving Lemma 3.3 as well as Theorem 3.4. Lemma 3.3 provides some useful information about two appropriate distinct positive roots of the characteristic equation (3.3); this information is necessary in establishing Theorem 3.4.

**Lemma 3.2.** Let \( \lambda_0 \) and \( \lambda_1, \lambda_0 \neq \lambda_1 \), be two positive roots of the characteristic equation (3.3) with the properties \( (p(\lambda_0)) \) and \( (p(\lambda_1)) \), respectively. Then
\[
-\lambda_0^{-\tau} \sum_{s=-\tau}^{n-\tau} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s) \prod_{r=s+1}^{n-\tau} h_{\lambda_0}(r) h_{\lambda_1}(r) = 1 \quad \text{for all } n \geq 0.
\]

**Proof.** It follows immediately that the \( T \)-periodicity of the coefficients \( (a(n))_{n \geq 0} \) and \( (b(n))_{n \geq 0} \) implies that, for any positive real number \( \lambda \), the sequence \( (h_{\lambda}(n))_{n \geq -\tau} \) is also \( T \)-periodic. By using this fact as well as the fact that \( \tau = mT \), we get for \( n \geq 0 \)
\[
\prod_{r=n-\tau}^{n-1} h_{\lambda_0}(r) = \prod_{r=0}^{\tau-1} h_{\lambda_0}(r) = \left[ \prod_{r=0}^{T-1} h_{\lambda_0}(r) \right]^m = (\lambda_0^T)^m = \lambda_0^m T = \lambda_0^{-\tau}.
\]
That is,
\[ \prod_{r=n^{-\tau}}^{n-1} h_{\lambda_0}(r) = \lambda_1^\tau \quad \text{for every } n \geq 0. \] (3.9)

Similarly, we have
\[ \prod_{r=n^{-\tau}}^{n-1} h_{\lambda_1}(r) = \lambda_1^\tau \quad \text{for every } n \geq 0. \] (3.10)

Now, by taking into account (3.9) and (3.10), for every \( n \geq 0 \), we obtain

\[
-\lambda_0^{-\tau} \sum_{s=n^{-\tau}}^{n-1} \frac{1}{h_{\lambda_0}(s)} h_{\lambda_0}(s) \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} h_{\lambda_0}(r)
= \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n^{-\tau}}^{n-1} \frac{(\lambda_1^{-\tau} - \lambda_0^{-\tau}) \tilde{b}(s)}{h_{\lambda_0}(s)} \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} h_{\lambda_0}(r)
= \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n^{-\tau}}^{n-1} \frac{\tilde{b}(s) \lambda_1^{-\tau} - \tilde{b}(s) \lambda_0^{-\tau}}{h_{\lambda_0}(s)} \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} h_{\lambda_0}(r)
= \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n^{-\tau}}^{n-1} \left[ \prod_{r=s+1}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] - \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} 
= \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n^{-\tau}}^{n-1} \Delta \left[ \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] 
= \lambda_0^{-\tau} \sum_{s=n^{-\tau}}^{n-1} \left\{ \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right\}_{s=(n-1)+1} - \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} 
= \lambda_0^{-\tau} \left[ \prod_{r=n}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} - \prod_{r=n^{-\tau}}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] 
= \lambda_0^{-\tau} \left[ 1 - \prod_{r=n^{-\tau}}^{n-1} h_{\lambda_0}(r) \right] = \lambda_0^{-\tau} \left( 1 - \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \right) = 1.
\]

Thus, (3.8) has been shown. Note that we have used the usual convention

\[ \prod_{r=n}^{n-1} = 1. \]

The proof of our lemma is complete. \( \square \)
Lemma 3.3. Suppose that \((b(n))_{n \geq 0}\) is nonpositive, and let \(\lambda_0\) and \(\lambda_1, \lambda_0 \neq \lambda_1\), be two positive roots of the characteristic equation (3.3). Then we have:

(I) If \(\lambda_1 < \lambda_0\) and \(\lambda_1\) has the property \((p(\lambda_1))\), then \(\lambda_0\) has also the property \((p(\lambda_0))\) and

\[
1 + \gamma_{\lambda_0} > 0,
\]

where \(\gamma_{\lambda_0}\) is defined by (3.5).

(II) If \(\lambda_1 > \lambda_0\) and \(\lambda_0\) has the property \((p(\lambda_0))\), then \(\lambda_1\) has also the property \((p(\lambda_1))\) and

\[
1 + \gamma_{\lambda_0} < 0,
\]

where \(\gamma_{\lambda_0}\) is defined by (3.5).

Proof. (I) Assume that \(\lambda_1 < \lambda_0\) and that \(\lambda_1\) has the property \((p(\lambda_1))\). Then, since \((b(n))_{n \geq 0}\) is nonpositive, we have

\[
h_{\lambda_1}(n) \leq h_{\lambda_0}(n) \quad \text{for all } n \geq -\tau.
\]

So, \(\lambda_0\) has also the property \((p(\lambda_0))\). Furthermore, from (3.8) for \(n = \tau\) it follows that

\[
-\lambda_0^{-\tau} \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s) \prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} = 1. \tag{3.11}
\]

By combining (3.5) and (3.11), we find

\[
1 + \gamma_{\lambda_0} = \lambda_0^{-\tau} \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s) \left[ 1 - \prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right]. \tag{3.12}
\]

Since the coefficient sequence \((b(n))_{n \geq 0}\) is assumed to be a \(T\)-periodic sequence and \(\tau = mT\), the assumption that \((b(n))_{n \geq 0}\) is not identically zero is equivalent to the fact that \((b(n))_{n=0}^{\tau-1}\) is not identically zero, and the hypothesis that \((b(n))_{n \geq 0}\) is nonpositive is equivalent to the statement that \((b(n))_{n=0}^{\tau-1}\) is nonpositive. That is, \((b(n))_{n=0}^{\tau-1}\) is nonpositive, but not identically zero. Hence, we have

\[
b(s) \leq 0 \quad (s = 0, 1, \ldots, \tau - 1) \quad \text{and} \quad b(s_0) < 0, \tag{3.13}
\]

where \(s_0\) is some integer with \(0 \leq s_0 \leq \tau - 1\). As \(\lambda_1 < \lambda_0\), (3.13) gives

\[
h_{\lambda_1}(s) \leq h_{\lambda_0}(s) \quad (s = 0, 1, \ldots, \tau - 1) \quad \text{and} \quad h_{\lambda_1}(s_0) < h_{\lambda_0}(s_0)
\]

and consequently

\[
\prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \geq 1 \quad (s = 0, 1, \ldots, \tau - 1) \quad \text{and} \quad \prod_{r=s_0}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} > 1.
\]

Thus, by using again (3.13), we conclude that

\[
b(s) \left[ 1 - \prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] \geq 0 \quad (s = 0, 1, \ldots, \tau - 1) \quad \text{and} \quad b(s_0) \left[ 1 - \prod_{r=s_0}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] > 0,
\]

which implies that

\[
\sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s) \left[ 1 - \prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] > 0.
\]
Hence, it follows from (3.12) that \(1 + \gamma_{\lambda_0} > 0\).

(II) Suppose that \(\lambda_1 > \lambda_0\) and that \(\lambda_0\) has the property \((p(\lambda_0))\). Then, by the hypothesis that 

\[ (b(n))_{n \geq 0} \text{ is nonpositive, it follows that} \]

\[ h_{\lambda_1}(n) \geq h_{\lambda_0}(n) \quad \text{for all} \quad n \geq -\tau, \]

which guarantees that \(\lambda_1\) has also the property \((p(\lambda_1))\). Furthermore, as in part (I), we see that (3.12) holds. Also, by the arguments applied in part (I), we conclude that (3.13) is satisfied, where 

\[ s_0 \text{ is some integer with} \quad 0 \leq s_0 \leq \tau - 1. \]

Since \(\lambda_1 > \lambda_0\), from (3.13) it follows that 

\[ h_{\lambda_1}(s) \geq h_{\lambda_0}(s) \quad (s = 0, 1, \ldots, \tau - 1) \quad \text{and} \quad h_{\lambda_1}(s_0) > h_{\lambda_0}(s_0). \]

By an analogous procedure with that in part (I), we can obtain 

\[ \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s) \left[ 1 - \prod_{r=s}^{\tau-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] < 0, \]

which, because of (3.12), gives \(1 + \gamma_{\lambda_0} < 0\).

The proof of the lemma is finished. \(\Box\)

Now, we are in position to state and prove the main result of this section, i.e., the following theorem.

**Theorem 3.4.** Suppose that \((b(n))_{n \geq 0}\) is nonpositive. Let \(\lambda_0\) and \(\lambda_1\), \(\lambda_0 \neq \lambda_1\), be two positive roots of the characteristic equation (3.3), and assume that \(\lambda_1\) has the property \((p(\lambda_1))\) if \(\lambda_1 < \lambda_0\), or \(\lambda_0\) has the property \((p(\lambda_0))\) if \(\lambda_1 > \lambda_0\). Define \(\gamma_{\lambda_0}\) by (3.5).

Then, for any \(\Phi = (\Phi_n)_{n=-\tau}^0\) in \(S\), the solution \((x_n)_{n \geq -\tau}\) of the IVP (3.1), (3.2) satisfies

\[ V_1(\lambda_0, \lambda_1; \Phi) \leq \frac{1}{h_{\lambda_1}(n)} \left[ x_n - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}} h_{\lambda_0}(n) \right] \leq V_2(\lambda_0, \lambda_1; \Phi) \quad \text{for all} \quad n \geq 0, \quad (3.14) \]

where \(M_{\lambda_0}(\Phi)\) is defined by (3.7), and

\[ V_1(\lambda_0, \lambda_1; \Phi) = \min_{n=-\tau, \ldots, 0} \left\{ \frac{1}{h_{\lambda_1}(n)} \left[ \Phi_n - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}} h_{\lambda_0}(n) \right] \right\}, \]

\[ V_2(\lambda_0, \lambda_1; \Phi) = \max_{n=-\tau, \ldots, 0} \left\{ \frac{1}{h_{\lambda_1}(n)} \left[ \Phi_n - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}} h_{\lambda_0}(n) \right] \right\}. \]

**Note.** By Lemma 3.3, \(\lambda_0\) and \(\lambda_1\) have the properties \((p(\lambda_0))\) and \((p(\lambda_1))\), respectively, and we have

\[ 1 + \gamma_{\lambda_0} \neq 0. \]

We immediately observe that (3.14) can equivalently be written as follows

\[ V_1(\lambda_0, \lambda_1; \Phi) \frac{H_{\lambda_1}(n)}{H_{\lambda_0}(n)} \leq \frac{x_n}{H_{\lambda_0}(n)} - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}} \leq V_2(\lambda_0, \lambda_1; \Phi) \frac{H_{\lambda_1}(n)}{H_{\lambda_0}(n)} \quad \text{for all} \quad n \geq 0, \]

which implies (3.6), provided that \(\lambda_0\) and \(\lambda_1\) are such that

\[ \lim_{n \to \infty} \frac{H_{\lambda_0}(n)}{H_{\lambda_1}(n)} = \infty. \]
Also, we see that (3.14) is equivalent to
\[
V_1(\lambda_0, \lambda_1; \Phi)H_{\lambda_1}(n) + \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}}H_{\lambda_0}(n) \leq x_n \leq V_2(\lambda_0, \lambda_1; \Phi)H_{\lambda_1}(n) + \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}}H_{\lambda_0}(n) \quad \text{for all } n \geq 0.
\]

**Proof of Theorem 3.4.** Let \( \Phi = (\Phi_n)_{n=-\tau}^0 \) be an arbitrary element in \( S \), and let \( (x_n)_{n \geq -\tau} \) be the solution of the IVP (3.1), (3.2). Define
\[
y_n = \frac{x_n}{H_{\lambda_0}(n)} \quad \text{for } n \geq -\tau
\]
and, next, set
\[
z_n = y_n - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}} \quad \text{for } n \geq -\tau.
\]
It has been shown by Kordonis and the authors in [19] for more general periodic delay difference equations (see, also, the authors’ paper [29] for the even more general case of periodic neutral delay difference equations) that \( (x_n)_{n \geq -\tau} \) satisfies (3.1) for \( n \geq 0 \) if and only if \( (z_n)_{n \geq -\tau} \) satisfies
\[
z_n = -\lambda_0^{-\tau} \sum_{s=-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s)z_s \quad \text{for } n \geq 0.
\]
(3.15)

Now, let us introduce the sequence of real numbers \((w_n)_{n \geq -\tau}\) defined by
\[
w_n = \frac{H_{\lambda_0}(n)}{H_{\lambda_1}(n)}z_n \quad \text{for } n \geq -\tau.
\]
By the use of this sequence, (3.15) can equivalently be written as follows:
\[
w_n = -\lambda_0^{-\tau} \sum_{s=-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s) \frac{H_{\lambda_0}(n)}{H_{\lambda_0}(s)} \frac{H_{\lambda_0}(s)}{H_{\lambda_1}(n)} w_s \quad \text{for } n \geq 0.
\]
(3.16)

But, for any integers \( n \) and \( s \) with \( n \geq 0 \) and \(-\tau \leq s \leq n - 1\), we obtain
\[
\frac{H_{\lambda_0}(n)}{H_{\lambda_0}(s)} = \begin{cases} \prod_{r=0}^{s-1} h_{\lambda_0}(r) & \text{if } 0 \leq s \leq n - 1 \\ \prod_{r=0}^{n-1} h_{\lambda_0}(r) & \text{if } -\tau \leq s \leq 0 \\ \prod_{r=0}^{n-1} h_{\lambda_0}(r) \prod_{r=s}^{n-1} h_{\lambda_0}(r) & \text{if } -\tau \leq s \leq 0. \end{cases}
\]
(3.17)

So, we have
\[
\frac{H_{\lambda_0}(n)}{H_{\lambda_0}(s)} = \prod_{r=s}^{n-1} h_{\lambda_0}(r) \quad \text{for } n \geq 0 \text{ and } -\tau \leq s \leq n - 1.
\]
(3.18)

Similarly,
In view of (3.17) and (3.18), Eq. (3.16) becomes

\[ w_n = -\lambda_0^{-\tau} \sum_{s=-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s) \left[ \prod_{r=s}^{n-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] w_s \quad \text{for } n \geq 0. \]  

(3.19)

Furthermore, by the definitions of \((y_n)_{n \geq -\tau}\), \((z_n)_{n \geq -\tau}\) and \((w_n)_{n \geq -\tau}\), we can find

\[ w_n = \frac{1}{H_{\lambda_1}(n)} \left[ x_n - \frac{M_{\lambda_0}(\Phi)}{1 + \gamma_{\lambda_0}(n)} \right] H_{\lambda_0}(n) \quad \text{for } n \geq -\tau. \]  

(3.20)

By the definitions of \(V_1(\lambda_0, \lambda_1; \Phi)\) and \(V_2(\lambda_0, \lambda_1; \Phi)\), from (3.2) and (3.20) it follows that

\[ V_1(\lambda_0, \lambda_1; \Phi) = \min_{n=-\tau, \ldots, 0} w_n \quad \text{and} \quad V_2(\lambda_0, \lambda_1; \Phi) = \max_{n=-\tau, \ldots, 0} w_n. \]

Hence, in view of (3.20), inequalities (3.14) can be written in the form

\[ \min_{s=-\tau, \ldots, 0} w_s \leq w_n \leq \max_{s=-\tau, \ldots, 0} w_s \quad \text{for all } n \geq 0. \]

We shall show that

\[ w_n \geq \min_{s=-\tau, \ldots, 0} w_s \quad \text{for every } n \geq 0. \]  

(3.21)

In a similar way, one can prove that

\[ w_n \leq \max_{s=-\tau, \ldots, 0} w_s \quad \text{for every } n \geq 0. \]

In the rest of the proof we will establish (3.21). To this end, it suffices to show that, for any real number \(K\) with \(K < \min_{s=-\tau, \ldots, 0} w_s\), it holds

\[ w_n > K \quad \text{for all } n \geq 0. \]  

(3.22)

Let us consider an arbitrary real number \(K\) with \(K < \min_{s=-\tau, \ldots, 0} w_s\). Then

\[ w_n > K \quad \text{for } n = -\tau, \ldots, 0. \]  

(3.23)

Assume, for the sake of contradiction, that (3.22) fails to hold. Then, by taking into account (3.23), we conclude that there exists an integer \(n_0 > 0\) so that

\[ w_n > K \quad \text{for } n = -\tau, \ldots, n_0 - 1, \quad \text{and} \quad w_{n_0} \leq K. \]  

(3.24)

Since the sequence \((b(n))_{n \geq 0}\) is \(T\)-periodic and \(\tau = mT\), the assumption that \((b(n))_{n \geq 0}\) is not identically zero means that \((\tilde{b}(n))_{n=n_0-\tau}^{n_0-1}\) is not identically zero, while the hypothesis that \((b(n))_{n \geq 0}\) is nonpositive means that \((\tilde{b}(n))_{n=n_0-\tau}^{n_0-1}\) is nonpositive. So, \((\tilde{b}(n))_{n=n_0-\tau}^{n_0-1}\) is nonpositive, but not identically zero. Furthermore, by taking into account the above observation and using (3.24) as well as (3.8) for \(n = n_0\), from (3.19) we obtain

\[ K \geq w_{n_0} = -\lambda_0^{-\tau} \sum_{s=n_0-\tau}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s) \left[ \prod_{r=s}^{n_0-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] w_s \]

\[ > \left[ -\lambda_0^{-\tau} \sum_{s=n_0-\tau}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} \tilde{b}(s) \prod_{r=s}^{n_0-1} \frac{h_{\lambda_0}(r)}{h_{\lambda_1}(r)} \right] K = K. \]
This is a contradiction, which implies that (3.22) holds true. We have thus shown that (3.21) is satisfied.

The proof of the theorem is complete. □

Before closing this section and ending the paper, we will concentrate on the special case of the autonomous linear delay difference equation

\[ \Delta x_n = ax_n + bx_{n-\tau}, \quad (3.25) \]

where \( a \) and \( b \neq 0 \) are real constants, and \( \tau \) is a positive integer. The constant coefficients \( a \) and \( b \) of (3.25) can be considered as \( T \)-periodic sequences of real numbers with \( T = 1 \). The assumption that there exists a positive integer \( m \) such that \( \tau = mT \) holds by itself. Moreover, we notice that the characteristic equation of (3.25) is

\[ \lambda - 1 = a + b\lambda^{-\tau}. \]

By applying the main result of this section, i.e., Theorem 3.4, to the special case of the autonomous linear delay difference equation (3.25), we can easily be led to a particular case of a more general result obtained by the authors in [28] for autonomous linear delay difference equations with several delays.

References

[21] M.J. Norris, Unpublished notes on the delay differential equation \( x'(t) = bx(t-1) \) where \(-1/e \leq b < 0\), October 1967.