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Conditions for Solution of a Linear First-Order Differential Equation in the Hardy–Lebesgue Space and Applications

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INTRODUCTION

The singular differential equation

$$z^2 \frac{dy}{dz} + (\alpha_0 + \alpha_1 z) y = h(z), \tag{1}$$

where $h(z) = \sum_{n=1}^{\infty} h_n z^{n-1}$ is analytic in some neighbourhood of zero, was the subject of several investigations and generalizations [1]. A basic result is a necessary and sufficient condition for Eq. (1) to have analytic solutions in some neighbourhood of zero. This condition is the following:

$$h(0) = 0 \quad \text{for } \alpha_0 = 0 \qquad \text{and} \qquad \alpha_1 \neq -k \tag{2α}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \alpha_0^{n-1} \frac{h_n}{\Gamma(\alpha_1 + n - 1)} = 0 \quad \text{for } \alpha_0 \neq 0. \tag{2β}$$

In this paper we are interested in solutions of Eq. (1) which belong to the Hardy–Lebesgue space, i.e., the Hilbert space of functions $y(z) = \sum_{n=1}^{\infty} y_n z^{n-1}$ which are analytic in $\Delta = \{z : |z| < 1\}$ and satisfy the condition $\sum_{n=1}^{\infty} |y_n|^2 < \infty \Leftrightarrow \sup_{0<r<1} \int_0^{2\pi} |y(re^{i\theta})|^2 \, d\theta < \infty$.

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It is not difficult to find functions $h(z)$ which satisfy the condition (2α) such that the solutions of (1) do not belong to the space $H_2(\Delta)$. The equation $z^2y' + zy = h(z)$ with $h(z) = z/(1-z)^2$ provides such an example. This shows that condition (2α) fails in general to hold for solutions of Eq. (1) in $H_2(\Delta)$. A basic result of this paper is that under the assumption that $h(z)$ belongs to $H_2(\Delta)$ the condition (2) is necessary and sufficient for Eq. (1) to have solutions in $H_2(\Delta)$. Moreover we observe that the solutions of Eq. (1) belong to the class of functions $y(z) = \sum_{n=1}^{\infty} y_n z^{n-1}$ in $H_2(\Delta)$ which satisfy the condition $\sum_{n=1}^{\infty} n^2 |y_n|^2 < \infty$. This result can be easily generalized for a class of differential equations:

$$z^2 \frac{d\varphi}{dz} + \alpha(z)\,\varphi(z) = b(z) \tag{3}$$

which can be transformed to Eq. (1).

We observe that for $\alpha_0 = -\rho/2$, $\alpha_1 = \mu + 1$ and

$$h(z) = -\frac{\rho}{2} \exp\left(-\frac{\rho}{2} z\right) \qquad \left(h_n = \frac{(-1)^n}{(n-1)!}\left(\frac{\rho}{2}\right)^n\right)$$

the left-hand side of Eq. (2β) is the ordinary Bessel function $J_\mu(\rho)$. Thus it follows from the above result that $\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if the equation

$$z^2y'(z) + \left(-\frac{\rho}{2} + (\mu+1)\,z\right)y(z) = -\frac{\rho}{2}\exp\left(-\frac{\rho}{2}z\right), \qquad y(0) = 1$$

has a solution in $H_2(\Delta)$. On the other hand we know [3] that the study of Eq. (1) in $H_2(\Delta)$ is equivalent to the study of an operator equation in an abstract Hilbert space H with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. This equation has the form $(V+K)f = h$, $h \in H$, where V is the shift operator ($Ve_n = e_{n+1}$, $n = 1, 2, \ldots$) and K is compact. In the case of Bessel functions the above equation can be transformed to an eigenvalue equation of the form $A_\mu g = (2/\rho)\, g$, $g \in H$, where A_μ is a compact operator on H. This result leads us to an operator approach in an abstract separable Hilbert space for the study of the zeros of the Bessel functions $J_\mu(z)$. In case $\mu > -1$ the operator A_μ is similar to a self-adjoint compact operator S_μ. Using some properties of the operators A_μ and S_μ we are led easily to some alternative proofs of well-known properties of Bessel functions and some results which are not presented as new although we were unable to find them in the literature.

1. The Solvability of Eq. (1) in the Space $H_2(\Delta)$

We write the differential equation (3) as

$$z^2 \frac{d\varphi}{dz} + [\alpha_0 + \alpha_1 z + z^2(\alpha_2 + \alpha_3 z + \cdots)]\, \varphi(z) = b(z) \tag{4}$$

and set $\varphi(z) = e^{-p(z)} \cdot y(z)$, where $p(z) = \alpha_2 z + \alpha_3(z^2/2) + \alpha_4(z^3/3) + \cdots$, and Eq. (3) is thus transformed to Eq. (1) where $h(z) = e^{p(z)} b(z)$. We assume that

$$h(z) = \sum_{n=1}^{\infty} h_n z^{n-1} \tag{5}$$

belongs to the space $H_2(\Delta)$. Then (see Ref. [3]) Eq. (1) has a solution in $H_2(\Delta)$ if and only if the operator equation

$$(V^2 C_0 V^* + \bar{\alpha}_0 I + \bar{\alpha}_1 V) f = h \tag{6}$$

has a solution f in an abstract separable Hilbert space H with an orthonormal basis $e_1, e_2, \ldots, e_n, \ldots$. Here $h = \sum_{n=1}^{\infty} \bar{h}_n e_n$, V is the shift operator on $H(Ve_n = e_{n+1})$, V^* its adjoint, C_0 is the diagonal operator $C_0 e_n = n e_n$, $n = 1, 2, \ldots$, and bars denote complex conjugation. The operator C_0 is densely defined in H with a compact inverse B:

$$B\colon Be_n = (1/n) e_n, \qquad n = 1, 2, \ldots \tag{7}$$

(see Proposition 2 in Ref. [3]). Taking into account the relations $V^2 C_0 V^* = V(C_0 - I)$ and $VC_0 - C_0 V = -V$, Eq. (6) can be written as

$$(C_0 V + \bar{\alpha}_0 I + (\bar{\alpha}_1 - 2) V) f = h \tag{8$_1$}$$

and since C_0 is invertible we have

$$(V + K) f = Bh, \tag{8$_2$}$$

where

$$K = \bar{\alpha}_0 B + (\bar{\alpha}_1 - 2) BV. \tag{9}$$

The function $y(z) \in H_2(\Delta)$ and the element $f \in H$ are connected by the representation $y(z) = (f_z, f)$, where $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$[3], $|z| < 1$.

Remark 1. The definition domain of the operator C_0 is the range of the operator B[3]. Thus if f is a solution of Eq. (8_2) then the element $g = Vf = -Kf + Bh = -(\bar{\alpha}_0 B + (\bar{\alpha}_1 - 2) BV) f + Bh = B[(-\bar{\alpha}_0 + (2 - \bar{\alpha}_1) V) f + h]$ belongs to the range of B and therefore to the definition domain of C_0.

This means that $\sum_{n=1}^{\infty} n^2 |(g, e_n)|^2 < \infty$. Thus $\sum_{n=1}^{\infty} n^2 |(V^*g, e_n)|^2 = \sum_{n=1}^{\infty} n^2 |(g, e_{n+1})|^2 \leqslant \sum_{n=1}^{\infty} n^2 |(g, e_n)|^2 < \infty$. But $V^*g = V^*Vf = f$ and this shows that f belongs to the definition domain of C_0, i.e., it satisfies the condition $\sum_{n=1}^{\infty} n^2 |(f, e_n)|^2 < \infty$. This is the reason for which Eq. (8_1) follows from Eq. (8_2) and therefore Eqs. (8_1) and (8_2) are equivalent.

According to the representation $y(z) = (f_z, f)$ we obtain the following result: If f is a solution of Eq. (8_2) then $y(z) = \sum_{n=1}^{\infty} y_n z^{n-1}$ is a solution of Eq. (1) which satisfies the condition $\sum_{n=1}^{\infty} n^2 |y_n|^2 < \infty$. Since B is a compact operator on H and since V is bounded the operator K is a compact operator on H and since V is a Fredholm operator $V + K$ is also a bounded Fredholm operator. Thus Eq. (8_2) and therefore Eq. (6) has a solution in H if and only if Bh is orthonormal to the null space of the operator $V^* + K^*$, the adjoint of $V + K$, i.e.,

$$(f, Bh) = 0 \qquad \forall f \in \operatorname{Ker}(V^* + K^*) \tag{10}$$

where

$$K^* = \alpha_0 B + (\alpha_1 - 2) V^*B. \tag{11}$$

We give below some results concerning the dimension of the null space of $V^* + K^*(\dim \ker(V^* + K^*))$ which we shall use later.

PROPOSITION 1. *If* $\alpha_0 \neq 0$, *or* $\alpha_0 = 0$, $\alpha_1 \neq -k$, $k = 0, 1, 2, \ldots$, *then* $\dim \ker(V + K) = 0$, $\dim \ker(V^* + K^*) = 1$. *If* $\alpha_0 = 0$, $\alpha_1 = -k$ *then* $\dim \ker(V + K) = 1$, $\dim \ker(V^* + K^*) = 2$.

Proof. Let $b = \sum_{n=1}^{\infty} b_n e_n$ and $c = \sum_{n=1}^{\infty} c_n e_n$ be two elements belonging to the null space of $V + K$ and $V^* + K^*$, respectively, i.e., $(V + K)\, b = 0$, $(V^* + K^*)\, c = 0$. Multiplying both sides of these equations by $e_1, e_2, \ldots$ we are led to the following conclusions:

(1) If $\alpha_0 \neq 0$, then $b_i = 0$, $i = 1, 2, \ldots$. Hence $b = 0$, $\dim \ker(V + K) = 0$ and all c_i's are expressed linearly in terms of c_1, so the space is at most one dimensional. On the other hand if we write $V^* + K^* = V^*(I + VK^*)$ then the Fredholm alternative for the compact operator $-VK^*$ implies that the null space of $V^* + K^*$ is nontrivial thus $\dim \ker(V^* + K^*) = 1$.

(2) If $\alpha_0 = 0$, $\alpha_1 \neq -k$, $k = 0, 1, 2, \ldots$, then $b_i = 0$, $i = 1, 2, \ldots$, hence $b = 0$, $\dim \ker(V + K) = 0$, $c_1 \neq 0$, $c_i = 0$, $i = 2, 3, \ldots$, hence $c = c_1 e_1$, $\dim \ker(V^* + K^*) = 1$.

(3) If $\alpha_0 = 0$, $\alpha_1 = -k$, $k = 0, 1, 2, \ldots$, then $b_i = 0$ for $i \neq k + 1$, $b_{k+1} \neq 0$, $b = b_{k+1} e_{k+1}$, $\dim \ker(V + K) = 1$ and $c_1 \neq 0$, $c_{k+2} \neq 0$, $c_i = 0$, $i \neq 1, k + 2$, hence $c = c_1 e_1 + c_{k+2} e_{k+2}$, $\dim \ker(V^* + K^*) = 2$.

Remark 2. Proposition 1 proves the well-known index theorem: $\text{Index}(V+K) = \text{Index } V = 1$ in a very special case.

Now from (10) we obtain the following necessary and sufficient condition in order that (1) has a solution in $H_2(\Delta)$.

Condition 1. If $\alpha_0 = 0$ then the necessary and sufficient condition for Eq. (1) to have a solution in $H_2(\Delta)$ is the following:

$$h(0) = 0 \qquad \text{and} \qquad \left.\frac{d^{k+1}h(z)}{dz^{k+1}}\right|_{z=0} = h^{(k+1)}(0) = 0 \tag{12}$$

in case $\alpha_1 = -k$, $k = 0, 1, 2,\ldots$, and

$$h(0) = 0 \tag{13}$$

in case $\alpha_1 \neq -k$, $k = 0, 1, 2,\ldots$.

Proof. From (10) and the above proposition we have $\langle e_1, Bh\rangle = \langle e_{k+2}, Bh\rangle = 0$ which means $h_1 = h_{k+2} = 0$ or

$$h(0) = h^{(k+1)}(0) = 0.$$

PROPOSITION 2. *Let $\alpha_0 \neq 0$. Then the null space of $V^* + K^*$ is spanned by the element*

$$f_0 = \sum_{n=1}^{\infty} \alpha_0^{n-1} n(-1)^{n-1} \cdot \frac{1}{\Gamma(\alpha_1 + n - 1)} e_n \tag{14}$$

in case $\alpha_1 \neq -k$, $k = 0, 1, 2,\ldots$, and by the element

$$f_0 = \sum_{n=k+2}^{\infty} \alpha_0^{n-k-2}(-1)^{n-k-1} \cdot n \frac{1}{\Gamma(n-k-1)} e_n \tag{15}$$

in case $\alpha_1 = -k$, $k = -1, 0, 1, 2,\ldots$.

Proof. From (11) we have

$$V^* + K^* = V^*(I + \delta B) + \alpha_0 B, \qquad \delta = \alpha_1 - 2. \tag{16}$$

We know from Proposition 1 that $\dim\ker(V^* + K^*) = 1$. We can find the element f_0: $(V^* + K^*) f_0 = 0$ or $V^*(I + \delta B) f_0 = -\alpha_0 B f_0$ recursively by setting $\langle f_0, e_1\rangle = 1/\Gamma(\delta + 2)$ in case $\delta \neq -k - 2$, $k = 0, 1,\ldots$, and $\langle f_0, e_{k+2}\rangle = -k - 2$ in case $\delta = -k - 2$, $k = -1, 0, 1, 2,\ldots$. Note that for $k = -1$ both formulas (14), (15) give the null space element and coincide apart from a multiplicative constant.

THEOREM 1 (Condition 2). *Let $\alpha_0 \neq 0$ and $h(z)$ belong to $H_2(\Delta)$. Then the necessary and sufficient condition for Eq.* (1) *to have a solution in $H_2(\Delta)$ is the following*:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \alpha_0^{n-1} \frac{h_n}{\Gamma(\alpha_1 + n - 1)} = 0. \tag{17}$$

Proof. From (10) and (14) we have in case $\alpha_1 \neq -k$, $k = 0, 1, 2, \ldots$,

$$\begin{aligned}\langle f_0, Bh\rangle = \langle Bf_0, h\rangle &= \left\langle \sum_{n=1}^{\infty} \alpha_0^{n-1}(-1)^{n-1} \frac{1}{\Gamma(\alpha_1 + n - 1)} e_n, h\right\rangle \\ &= \left\langle \sum_{n-1}^{\infty} \alpha_0^{n-1}(-1)^{n-1} \frac{1}{\Gamma(\alpha_1 + n - 1)} e_n, \sum_{n=1}^{\infty} \bar{h}_n e_n\right\rangle \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \alpha_0^{n-1} \frac{h_n}{\Gamma(\alpha_1 + n - 1)} = 0,\end{aligned}$$

and from (10) and (15)

$$\sum_{n=k+2}^{\infty} \alpha_0^{n-k-2}(-1)^{n-k-1} \cdot \frac{h_n}{\Gamma(n-k-1)} = 0$$

in case $\alpha_1 = -k$, $k = -1, 0, 1, 2, \ldots$. The last relation follows from (17) for $\alpha_1 = -k$ because $1/\Gamma(n-k-1) = 0$ for $n < k+2$.

Remark 3. Taking into account Remark 1 we observe that the solutions $y(z) = \sum_{n=1}^{\infty} y_n z^{n-1}$ of Eq. (1) satisfy the condition $\sum_{n=1}^{\infty} n^2 |y_n|^2 < \infty$.

Remark 4. Equation (17) in the special case that $\alpha_0 = 1$ was found by Grimm and Hall [1] by a different method as the necessary and sufficient condition for Eq. (1) to have a solution in A_1 if $h(z)$ belongs to A_0. (A_0 is the class of functions analytic in Δ and continuous on $\bar{\Delta}$, and A_1 the class of functions analytic in Δ and continuously differentiable once on $\bar{\Delta}$.) In a special case ($\alpha_0 = \alpha_1 = 1$) it was found for the same class of functions by a different method by Hall [2]. It was also found in a special case by a different method by Turrittin [5].

2. CONNECTION TO THE ZEROS OF BESSEL FUNCTIONS

We set in Eq. (1) $\alpha_0 = -\rho/2$, $\alpha_1 = \mu + 1$ and

$$h(z) = -\frac{\rho}{2} \exp\left(-\frac{\rho}{2} z\right) = -\frac{\rho}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{\rho}{2} z\right)^{n-1}.$$

Then we have from (17)

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{\rho}{2}\right)^{2n-1}}{\Gamma(n)\,\Gamma(n+\mu)} = 0$$

or

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{\rho}{2}\right)^{2n+\mu} \frac{1}{\Gamma(n+1)\,\Gamma(n+\mu+1)} = 0, \qquad \text{i.e., } J_\mu(\rho) = 0.$$

It follows from Theorem 1 that $\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if the equation

$$z^2 \frac{dy}{dz} + \left(-\frac{\rho}{2} + (\mu+1)\,z\right) y(z) = -\frac{\rho}{2} \exp\left(-\frac{\rho}{2} z\right), \qquad y(0) = 1 \quad (18)$$

has a solution in $H_2(\Delta)$.

We consider the transformation

$$y(z) = \exp\left(-\frac{\rho}{2} z\right) g(z) \quad (19)$$

which transforms Eq. (18) to

$$z^2 g'(z) - \left(\frac{\rho}{2} z^2 - (\mu+1)\,z + \frac{\rho}{2}\right) g(z) = -\frac{\rho}{2}, \qquad g(0) = 1. \quad (20)$$

Since $\exp(\pm(\rho/2)\,z)$ is an entire function it follows that $y(z) \in H_2(\Delta) \Rightarrow g(z) = \exp((\rho/2)\,z)\,y(z) \in H_2(\Delta)$ and $g(z) \in H_2(\Delta) \Rightarrow y(z) \in H_2(\Delta)$. Thus solvability of Eq. (18) in $H_2(\Delta)$ implies solvability of Eq. (20) and conversely.

Now we consider the equivalent operator equation of Eq. (20) by the representation $g(z) = (f_z, g)$, $g \in H$. This equation is the following:

$$V^2 C_0 V^* g - \left(\frac{\bar{\rho}}{2} V^2 - (\bar{\mu}+1)\,V + \frac{\bar{\rho}}{2}\right) g = -\frac{\bar{\rho}}{2} e_1, \qquad (g, e_1) = 1$$

or, since $V^2 C_0 V^* = V(C_0 - I)$,

$$\left[V(C_0 + \bar{\mu})\,g - \frac{\bar{\rho}}{2} V^2 - \frac{\bar{\rho}}{2}\right] g = -\frac{\bar{\rho}}{2} e_1, \qquad (g, e_1) = 1. \quad (21)$$

Thus we have proved the following:

LEMMA 1. *$\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if Eq. (21) has a solution in the abstract Hilbert space H.*

Moreover we can prove the following:

LEMMA 2. *The solvability of Eq. (21) implies the solvability of the operator equation*

$$(C_0 + \bar{\mu}) g = \frac{\bar{\rho}}{2} (V + V^*) g \tag{22}$$

and conversely.

Proof. Equation (21) implies Eq. (22) because $V^*V = I$ and $V^*e_1 = 0$. Suppose that Eq. (22) has a solution $g \neq 0$ in H. We observe that $(g, e_1) \neq 0$ because otherwise from (22) it follows that $(g, e_2) = (g, e_3) = \cdots = 0$, i.e., $g = 0$. We normalize g by setting $(g, e_1) = 1$ and write Eq. (22) as

$$V^* \left(V(C_0 + \bar{\mu}) - \frac{\bar{\rho}}{2} V^2 - \frac{\bar{\rho}}{2} \right) g = 0. \tag{23}$$

Since the null space of V^* is one dimensional we have

$$V(C_0 + \bar{\mu}) g - \frac{\bar{\rho}}{2} V^2 g - \frac{\bar{\rho}}{2} g = c e_1, \qquad c = \text{constant}$$

and since $(g, e_1) = 1$ we obtain $c = -\bar{\rho}/2$, i.e., Eq. (21). From Lemmas 1 and 2 we get the following:

COROLLARY 1. *$\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if Eq. (22) has a solution in H.*

Now we consider the diagonal operator

$$L_\mu e_n = \frac{1}{n + \mu} e_n, \qquad n = 1, 2, \ldots \tag{24}$$

which is the inverse of the operator $C_0 + \mu I$, $\mu \neq -n$, $n = 1, 2, \ldots$. The operator L_μ is defined on H for $\mu \neq -n$, $n = 1, 2, \ldots$. This is not a restriction with respect to the zeros of $J_{-n}(z)$, $n = 1, 2, \ldots$ because due to the relation $J_{-n}(z) = (-1)^n J_n(z)$ the functions $J_{-n}(z)$ and $J_n(z)$ have the same zeros.

Since L_μ is a diagonal operator and $\lim(1/(n + \mu)) = 0$ $n \to \infty$, L_μ is a compact operator. (In our case the compactness of L_μ follows also from the fact that it possesses absolute norm $N(L_\mu) = \sqrt{\sum_{n=1}^\infty \|L_\mu e_n\|^2} = \sqrt{\sum_{n=1}^\infty 1/|n + \mu|^2} < \infty$.) Moreover, since the operator

$$T_0 = V + V^* \tag{25}$$

is bounded ($\|T_0\| = 2$), the operators

$$A_\mu = L_\mu T_0, \qquad B_\mu = T_0 L_\mu \tag{26}$$

are also compact non-self-adjoint operators, therefore $\sigma(A_\mu)$, the spectrum of A_μ, except for the point zero, is a purely point spectrum. The same holds for $\sigma(B_\mu)$. On the other hand we see that zero is not an eigenvalue of A_μ and B_μ. In fact $A_\mu f = 0$ (or $B_\mu f = 0$) implies $T_0 f = 0$ (or $L_\mu f = 0$) which also implies $f = 0$ because the spectrum of T_0 is purely continuous [4] (or the inverse of the operator L_μ exists). This can also be seen from the fact that $A_\mu f = 0$, with $f = \sum_{n=1}^{\infty} c_n e_n$ implies $c_2 = c_1 + c_3 = c_2 + c_4 = c_3 + c_5 = \cdots = 0$, i.e., $c_2 = c_4 = \cdots = 0$ hence $c_1 = c_2 = c_3 = \cdots = 0$. It is now easy to prove the following.

THEOREM 2. *$\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if $2/\rho$ is an eigenvalue of the operator A_μ or the operator B_μ.*

Proof. We give first the proof for the operator $B_\mu = T_0 L_\mu$. Let ρ be a zero of $J_\mu(z)$. Then from Corollary 1 we obtain that $2/\bar{\rho}$ is an eigenvalue of the operator $L_{\bar{\mu}} T_0$ and therefore $2/\rho$ is an eigenvalue of $(L_{\bar{\mu}} T_0)^* = T_0 L_\mu$ because $L_{\bar{\mu}} T_0$ is compact. Conversely if $2/\rho$ is an eigenvalue of $T_0 L_\mu$ then $2/\bar{\rho}$ is an eigenvalue of $(T_0 L_\mu)^* = L_{\bar{\mu}} T_0$, i.e., Eq. (22) holds for $g \neq 0$. Therefore ρ is a zero of the Bessel function $J_\mu(z)$ due to the same corollary. For the operator A_μ the theorem follows from the relation $T_0 L_\mu = L_\mu^{-1}(L_\mu T_0) L_\mu$ which shows that the operators $T_0 L_\mu$ and $L_\mu T_0$ have the same eigenvalues. It is easy to see that for $\mu > -1$ the operator L_μ is nonnegative ($(L_\mu f, f) \geqslant 0$), and therefore its square root $L_\mu^{1/2}$ exists and is the diagonal operator

$$L_\mu^{1/2} e_n = \frac{1}{\sqrt{n+\mu}} e_n, \qquad n = 1, 2, \ldots.$$

Defining the operator S_μ as

$$S_\mu = L_\mu^{1/2} T_0 L_\mu^{1/2} \tag{27}$$

we see from the relation

$$L_\mu^{-1/2} A_\mu L_\mu^{1/2} = S_\mu \tag{28}$$

that for $\mu > -1$ the operators A_μ and S_μ have the same spectrum. Thus we obtain from Theorem 2 the following:

COROLLARY 2. *For $\mu > -1$, $\rho \neq 0$ is a zero of $J_\mu(z)$ if and only if $2/\rho$ is an eigenvalue of the compact and self-adjoint operator S_μ.*

Remark 5. The part "only if" of Theorem 2 can be obtained easily if we observe that if $J_\mu(\rho)=0$ then $f_\mu=\sum_{n=1}^{\infty} J_{\mu+n}(\rho)\, e_n$ is the eigenelement of A_μ that corresponds to the eigenvalue $2/\rho$. Due to the recurrence relation $J_{\mu+n+1}(\rho)+J_{\mu+n-1}(\rho)=(2/\rho)(\mu+n)\, J_{\mu+n}(\rho)$ $(n=1, 2,...)$ the eigenelement f_μ corresponding to the eigenvalue $\lambda=2/\rho$ is uniquely determined up to a constant $\langle f_\mu, e_1\rangle = J_{\mu+1}(\rho)=\alpha\neq 0$. Since $J_\mu(\rho)=0$, $J_{\mu+1}(\rho)=0$ is impossible because in this case $J'_\mu(\rho)=0$ from $zJ'_\mu(z)=\mu J_\mu(z)-zJ_{\mu+1}(z)$ hence $J''_\mu(\rho)=0$ from Bessel's differential equation. Taking the successive derivatives of this equation we find $J^{(3)}_\mu(\rho)=J^{(4)}_\mu(\rho)=\cdots=0$ which is impossible for the function $J_\mu(z)$. This is a proof of Bourget's hypothesis in a very special case.

Remark 6. For $\mu>-1$ S_μ is a Hilbert–Schmidt operator with absolute norm

$$N(S_\mu)=\sqrt{\sum_{n=1}^{\infty} \|S_\mu e_n\|^2}=\sqrt{\frac{2}{1+\mu}}. \tag{29}$$

In fact we find easily

$$S_\mu e_n=\frac{e_{n-1}}{\sqrt{(\mu+n)(\mu+n-1)}}+\frac{e_{n+1}}{\sqrt{(\mu+n+1)(\mu+n)}} \qquad \text{for} \quad n>1 \tag{29a}$$

and

$$S_\mu e_1=\frac{e_2}{\sqrt{(\mu+1)(\mu+2)}} \qquad \text{for} \quad n=1. \tag{29b}$$

Hence

$$\|S_\mu e_1\|^2=\frac{1}{(\mu+1)(\mu+2)}=\frac{1}{\mu+1}-\frac{1}{\mu+2} \tag{30}$$

$$\begin{aligned}\|S_\mu e_n\|^2&=\frac{1}{(\mu+n+1)(\mu+n)}+\frac{1}{(\mu+n-1)(\mu+n)}\\ &=\frac{1}{n+\mu-1}-\frac{1}{n+\mu+1}.\end{aligned} \tag{31}$$

From (31) we observe

$$\sum_{n=2}^{\infty} \|S_\mu e_n\|^2=\frac{1}{1+\mu}+\frac{1}{2+\mu} \tag{32}$$

and from (32) and (30) follows (29).

3. Some Qualitative Results for the Zeros of $J_\mu(z)$

Some of the most important results which follow easily from Theorem 2 and its corollary are the following:

1. *Lommel–Hurwitz theorem* [6]. For $\mu > -1$ all the zeros of $J_\mu(z)$ are real.

This follows from the corollary to Theorem 2 because S_μ is self-adjoint.

2. *Rayleigh's formula* [6]. Since S_μ ($\mu > -1$) is self-adjoint and compact and since the point zero is not an eigenvalue, it possesses a complete orthonormal system of eigenelements f_n, $n = 1, 2, \ldots$,

$$S_\mu f_n = \lambda_n f_n, \qquad \|f_n\| = 1, n = 1, 2, \ldots$$

with λ_n the corresponding eigenvalues. Hence

$$\|S_\mu f_n\|^2 = \lambda_n^2, \qquad n = 1, 2, \ldots . \tag{33}$$

Since for every linear operator A the sum $\sum_{n=1}^{\infty} \|Ae_n\|^2$, if it exists, is independent of the complete orthonormal system e_n it follows from (29) and (33) that

$$\sum_{n=1}^{\infty} \lambda_n^2 = \frac{2}{1+\mu}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{\rho_n^2} = \frac{1}{2(1+\mu)}, \qquad \mu > -1. \tag{34}$$

Note that this result includes all zeros (positive and negative) and therefore the above sum is equal to $2\sigma_\mu^{(1)}$ where

$$\sigma_\mu^{(1)} = \frac{1}{2^2(1+\mu)}$$

as found by Rayleigh [6].

3. If $J_\mu(\rho) = 0$ and $J_{\mu+m}(\rho) = 0$ for μ rational and m natural numbers, then $J_{\mu+m} = 0$ for $m = 1, 2, 3, \ldots$. This follows from the recurrence relation $J_{\mu+1}(\rho) + J_{\mu-1}(\rho) = (2\mu/\rho) J_\mu(\rho)$ and a theorem of Siegel [6] which implies that if μ is rational then the zeros of $J_\mu(z)$ are not algebraic numbers. It therefore follows that the eigenelement $f_0 = \sum_{n=1}^{\infty} J_{\mu+n}(\rho) (\sqrt{\mu+n}) e_n$ of the operator S_μ is zero which is impossible because, as in the case of A_μ, the eigenvalues of the operator S_μ are simple. (See Remark 5.) This is an alternative proof of Bourget's hypothesis [6] in its generalized version.

4. Consider the eigenvalue equation $L_\mu T_0 f = (2/\rho) f$ or

$$(\rho T_0 - 2C_0) f = 2\mu f \tag{35}$$

for some real number ρ. In Eq. (35) the operator ρT_0 is self-adjoint and is perturbed by a self-adjoint operator with compact resolvent hence the operator $\rho T_0 - 2C_0$ has a discrete spectrum in the sense that it consists of a sequence of real numbers $2\mu_n$, $n = 1, 2, \ldots$, such that $|\mu_n| \to \infty$. Thus we obtain the following result:

For every real ρ there exists a sequence of real numbers μ_n, $|\mu_n| \to \infty_{n\to\infty}$, such that $J\mu_n(\rho) = 0$, $n = 1, 2, \ldots$.

4. Lower Bounds for the Real and Imaginary Parts of the Complex Zeros of the Bessel Functions

We consider the case where $\mu = \mu_1 + i\mu_2$ with $\mu_2 \neq 0$ and suppose that ρ is a zero of $J_\mu(z)$, then according to Theorem 2 we have: $L_\mu T_0 f = (2/\rho) f$ with $\|f\| = 1$, or $\rho T_0 f = 2(C_0 + \mu) f$ and

$$\rho(T_0 f, f) = 2(C_0 f, f) + 2\mu_1 + 2i\mu_2. \tag{36}$$

Since T_0 and C_0 are self-adjoint $(T_0 f, f)$ and $(C_0 f, f)$ are real numbers. Thus it follows from (36) that for complex μ the function $J_\mu(z)$ has no real zeros.

Setting $\rho = \rho_1 + i\rho_2$, $\rho_2 \neq 0$, in (36) and comparing real and imaginary parts we obtain

$$\rho_2(T_0 f, f) = 2\mu_2 \tag{37}$$

$$\rho_1(T_0 f, f) = 2(C_0 f, f) + 2\mu_1 \tag{38}$$

or

$$\frac{\rho_1}{\rho_2}\mu_2 = (C_0 f, f) + \mu_1. \tag{39}$$

Since $|(T_0 f, f)| \leqslant \|T_0 f\| \leqslant \|T_0\| = 2$ for $\|f\| = 1$ we obtain from (37) a lower bound for ρ_2:

$$|\rho_2| \geqslant |\mu_2|. \tag{40}$$

Moreover since $(C_0 f, f) > 1$ for $\|f\| = 1$, it follows from (39) that

$$\frac{\rho_1}{\rho_2} > \frac{1 + \mu_1}{\mu_2} \qquad \text{for} \quad \mu_2 > 0 \tag{40α}$$

$$\frac{\rho_1}{\rho_2} < \frac{1 + \mu_1}{\mu_2} \qquad \text{for} \quad \mu_2 < 0 \tag{40β}$$

($(C_0f,f) \geqslant 1$ by the relation $(C_0f,f)=\sum_{n=1}^{\infty} n\,|(f,e_n)|^2 = 1+|(f,e_2)|^2 + 2\,|(f,e_3)|^2+\cdots$ and $(C_0f,f)>1$ because if $(C_0f,f)=1$, then $f=e_1$, which is not an eigenelement of A_μ).

It follows from (40α) and (40β) that for $\mu_1>-1$ and $\mu_2>0$ the parts ρ_1 and ρ_2 have the same sign while for $\mu_1>-1$ and $\mu_2<0$ they have different signs.

For $\mu_1>-1$ we have in both cases $\mu_2>0$ and $\mu_2<0$ so that

$$\frac{\rho_1^2}{\rho_2^2}>\frac{(1+\mu_1)^2}{\mu_2^2}, \qquad \mu_1>-1. \tag{41}$$

From (40) and (41) we obtain

$$\rho_1^2>(1+\mu_1)^2, \qquad \mu_1>-1. \tag{42}$$

Remark 7. From (40) and (42) we obtain a lower bound for $|\rho|^2 = \rho_1^2+\rho_2^2>(1+\mu_1)^2+\mu_2^2$, $\mu_1>-1$. This bound follows easily from a bound of the norm of $B_\mu = T_0L_\mu$ or A_μ.

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