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A Strict Inequality for Projection Constants

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It is shown that the projection constant of an n-dimensional space E is strictly less than $n^{1/2}$. Also, if the projection constant of E is close to $n^{1/2}$, so is the 1-summing norm of the identity on E. © 1987 Academic Press, Inc.

Let E be a closed subspace of a Banach space F. The *relative projection constant* of E in F is defined as

$$\lambda(E, F) = \inf\{\|u\|: u: F \to E \text{ a projection}\},$$

and the *projection constant* of E is

$$\lambda(E) = \sup\{\lambda(E, F): E \subset F \text{ isometrically}\}.$$

A well-known result of Kadec–Snobar [2] is that $\lambda(E) \leqslant n^{1/2}$ for every space E of dimension n. We show that a strict inequality holds.

THEOREM 1. *For $n \geqslant 2$ there is an $\varepsilon_n > 0$ with*

$$\lambda(E) \leqslant n^{1/2} - \varepsilon_n$$

for every space E of dimension n.

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The results of [3] show there are complex n-dimensional spaces with $\lambda(E) \geqslant n^{1/2} - n^{-1/2}$, and real spaces with $\lambda(E) \geqslant n^{1/2} - 1$. Thus $\varepsilon_n \leqslant n^{-1/2}$ and we conjecture that $cn^{-1/2} \leqslant \varepsilon_n$, that is:

Conjecture. There is an absolute constant c with $\lambda(E) \leqslant n^{1/2} - cn^{-1/2}$ for every E with $\dim E = n \geqslant 2$.

We use standard Banach space notation and terminology as given in the books of Lindenstrauss–Tzafriri [4] and Pietsch [5]. In particular the *Banach–Mazur distance* between spaces E and F is

$$d(E, F) = \inf\{\|u\| \, \|u^{-1}\|: u: E \to F \text{ an isomorphism}\},$$

and $\operatorname{tr}(u)$ denotes the trace of the finite rank operator u. Pietsch's book [5] contains the facts about the nuclear norm, integral norm i_1 and p-summing norms π_p which are needed here.

The *singular numbers* of an operator u on a Hilbert space H are defined for $n \geqslant 1$ by

$$s_n(u) = \inf\{\|u - v\|: v \in L(H) \text{ and } \operatorname{rank}(v) < n\}.$$

For $u \in L(H)$ compact, $\lambda_i(u)$ denotes the sequence of eigenvalues of u, ordered with non-increasing modulus and counted according to their multiplicities. *Weyl's Inequality* [6] imply that, for all n and $p \in (0, \infty)$,

$$\sum_{i \leqslant n} |\lambda_i(u)|^p \leqslant \sum_{i \leqslant n} s_i(u)^p.$$

For $u \in L(H)$ one has $s_n(u) = \lambda_i(u^*u)^{1/2}$ (cf. [5, 11.3]). A key step in the proof of Theorem 1 is a duality argument. Write $\pi_1(E)$ for the 1-summing norm of 1_E, the identity map on E. By Garling–Gordon [1]

$$n \leqslant \lambda(E)\, \pi_1(E)$$

whenever $\dim E = n$, with equality holding for spaces E with enough symmetries. Although equality need not hold for arbitrary E (cf. the examples of [1]) there is near equality for spaces with large projection constants.

THEOREM 2. *Let* $0 \leqslant \varepsilon < (2n)^{-1}$. *If* $\dim E = n$ *and* $\lambda(E) \geqslant (1 - \varepsilon)\, n^{1/2}$, *then*

$$\pi_1(E) \leqslant n^{1/2}[1 + 2\sqrt{2n\varepsilon}(1 - \sqrt{2n\varepsilon})^{-1}].$$

Proof of Theorem 2. By trace duality there is a $u \in L(E)$ with $\lambda(E) = \operatorname{tr}(u)/\pi_1(u)$. Scaling if necessary assume $\pi_2(u) = n^{1/2}$, so that

$$\operatorname{tr}(u) \geqslant \pi_1(u)(1 - \varepsilon)\, n^{1/2} \geqslant n(1 - \varepsilon). \tag{1}$$

Since $\pi_2(1_E)=n^{1/2}$,

$$\operatorname{tr}(u)\leqslant i_1(u)\leqslant \pi_2(1_E)\,\pi_2(u)=n$$

which with (1) yields

$$\pi_1(u)\leqslant n^{1/2}(1-\varepsilon)^{-1}. \tag{2}$$

It remains to estimate $\pi_1(1_E-u)$.

By Pietsch's theorem [5, 17.3], u has a factorization $u=\beta\alpha$ for some maps $\alpha\colon E\to l_2^n$ and $\beta\colon l_2^n\to E$ which satisfy $\pi_2(\alpha)\,\|\beta\|\leqslant \pi_2(u)=n^{1/2}$. Write $H=l_2^n$ and $w=\alpha\beta\in L(H)$. Note that

$$\pi_2(w)\leqslant \pi_2(\alpha)\,\|\beta\|=n^{1/2} \qquad \text{and} \qquad \operatorname{tr}(w)=\operatorname{tr}(u).$$

The 2-summing and Hilbert–Schmidt norm coincide on $L(H)$, so using (1) and the last two estimates,

$$\begin{aligned}\pi_2(1_H-w)^2&=\operatorname{tr}(1_H-w)(1_H-w)^*\\&=n-2\operatorname{tr}(w)+\operatorname{tr}(ww^*)\\&\leqslant n-2\operatorname{tr}(u)+\pi_2(w)\,\pi_2(w^*)\\&=2n-2\operatorname{tr}(u)\leqslant 2n\varepsilon.\end{aligned}$$

This shows both

$$\pi_2(1_H-w)\leqslant(2n\varepsilon)^{1/2} \qquad \text{and} \qquad \|w^{-1}\|\leqslant[1-\sqrt{2n\varepsilon}]^{-1}.$$

Since $1_E-u=\beta(1_H-w)\,w^{-1}\alpha$, the last two inequalities show

$$\begin{aligned}\pi_1(1_E-u)&\leqslant \pi_2(\beta(1_H-w))\,\pi_2(\omega^{-1}\alpha)\\&\leqslant \pi_2(\alpha)\,\|\beta\|\,\pi_2(1_H-w)\,\|w^{-1}\|\\&\leqslant n^{1/2}(2n\varepsilon)^{1/2}\,[1-\sqrt{2n\varepsilon}]^{-1}.\end{aligned}$$

This last inequality, (2) and the triangle inequality for π_1-norm establish the theorem.

Proof of Theorem 1. For n fixed the set $\mathscr{M}_n$ of all n-dimensional spaces is compact under the metric $\log d(E,F)$, and λ is continuous on $\mathscr{M}_n$ because of the inequality $\lambda(E)\leqslant\lambda(F)\,d(E,F)$. Taking into account the Kadec–Snobar Theorem $\lambda(E)\leqslant n^{1/2}$, we need only show that $\lambda(E)=n^{1/2}$ is impossible for $n\geqslant 2$. Assume to the contrary that $\lambda(E)=n^{1/2}$ for some E; by Theorem 1 $\pi_1(E)=n^{1/2}$ and we will show this forces $n=1$.

Assume $E\subset C(S)$ isometrically for S compact Hausdorff. Since $\pi_1(E)=n^{1/2}$, Pietsch's integral representation theorem [5] gives a probability measure μ on S satisfying $\|x\| \leqslant n^{1/2}\mu(|x|)$, all $x \in E$. Write $v: E \to L_\infty(\mu)$ for the natural map, so v is $1-1$. Write $\phi_{p,q}: L_p(\mu) \to L_q(\mu)$ for inclusion, $1 \leqslant q < p \leqslant \infty$. A standard argument from the integral representation produces an operator $u: L_1(\mu) \to L_\infty(\mu)$ with $\|u\| \leqslant n^{1/2}$ and $v = u\phi_{\infty,1} v$. The claim now is that $w_1 = u\phi_{\infty,1}$ is a rank n projection on $L_\infty(\mu)$. To see this write $H = L_2(\mu)$ and let $w \in L(H)$ be $w = \phi_{\infty,2} u \phi_{2,1}$. Then

$$\pi_2(w^*) = \pi_2(w) \leqslant \pi_2(\phi_{\infty,2}) \|u\| \leqslant n^{1/2}.$$

The subspace $F = \phi_{\infty,2} v(E) \subset H$ has dimension n because v is 1-to-1 and $w \mid F = 1_F$, so $|\lambda_i(w)| \geqslant 1$ for $1 \leqslant i \leqslant n$. Weyl's Inequalities show

$$\begin{aligned} n &\leqslant \sum_{i \leqslant n} |\lambda_i(w)|^2 \leqslant \sum_{i \leqslant n} \lambda_i(w^* w) \\ &\leqslant \sum_{i \geqslant 1} \lambda_i(w^* w) \leqslant \pi_2(w^*)\, \pi_2(w) = n, \end{aligned}$$

which implies that $\lambda_i(w^*w) = 1$ for $1 \leqslant i \leqslant n$ and $\lambda_i(w^*w) = 0$ for $i > n$. The map w has polar decomposition $w = g(w^*w)^{1/2}$ for some partial isometry g. Hence both w and $w_1 = u\phi_{\infty,1}$ have rank at most n. But since $v(E)$ is an n-dimensional subspace of $L_\infty(\mu)$ and $w_1 \mid v(E) =$ identity, the claim is proven.

For w_1 a finite rank operator on $L_\infty(\mu)$, the spectral trace coincides with the trace of w_1 as a nuclear map, and the nuclear and integral norms of w_1 coincide too. Thus

$$n = \operatorname{tr}(w_1) \leqslant i_1(w_1) \leqslant i_1(\phi_{\infty,1}) \|u\| \leqslant n^{1/2}.$$

EXAMPLES. The spaces with the largest known projection constants and smallest known π_1-norms were constructed in [3]. For n a prime the subspace X_n of $l_\infty^{n^2}$ spanned by

$$x_j = \{\exp[2\pi i(s_1 j + s_2 j^2)\, n^{-1}\}_{1 \leqslant s_1, s_2 \leqslant n}, \qquad 1 \leqslant j \leqslant n,$$

has projection constant

$$\lambda(X_n) = n^{1/2}[1 - n^{-1} + n^{-3/2}].$$

By the results of [3] there is a canonical projection $u: l_\infty^{n^2} \to X_n$ with $\|u\| = \lambda(X_n)$ for which $w = u - (n^{-1} - n^{-3/2})\,1$ satisfies $i_1(w) = n^{1/2}$. This yields

$$\pi_1(w \mid X_n) \leqslant \pi_1(w) = i_1(w) = n^{1/2}.$$

However, $w \mid X_n = [1 - n^{-1} + n^{-3/2}]\, 1_{X_n}$ so that

$$[1 - n^{-1} + n^{-3/2}]\, \pi_1(X_n) = \pi_1(w \mid X_n) \leqslant n^{1/2}.$$

Thus

$$\pi_1(X_n) = n^{1/2}[1 - n^{-1} + n^{-3/2}]^{-1} \leqslant n^{1/2}[1 + n^{-1}]$$

and

$$\lambda(X_n)\,\pi_1(X_n) = n.$$

In the real case there are spaces X_n with at least $\pi_1(X_n) \leqslant n^{1/2} + 1$.

It is not clear whether there exist spaces E satisfying the hypothesis of Theorem 2. However, if one could show that $\pi_1(E) \geqslant n^{1/2} + \delta_n$ always holds for suitable δ_n, Theorem 2 would establish Theorem 1 for the associated (computable) ε_n.

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