On Variational Principles for Sublinear Boundary Value Problems

CHARLES V. COFFMAN

Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

Received September 28, 1973

1. INTRODUCTION

The purpose of this paper is to demonstrate the equivalence of several variational principles relating to the existence of multiple solutions for a boundary value problem of the form

\[ [a(x)y']' + yb(x, y^2) = 0, \quad x \in (0, 1), \]
\[ y(0) = y(1) = 0. \tag{1.2} \]

Our assumptions are as follows:

(i) \( a \) is positive and continuously differentiable in \([0, 1] \);
(ii) \( b \) is continuous on \( \{(x, t) : x \in [0, 1], 0 < t < \infty \} \), and satisfies
\[ b(x, 0) > 0, \quad x \in [0, 1]; \tag{1.3} \]
(iii) \( b \) is a strictly decreasing function of its second argument, i.e.,
\[ b(x, t_1) > b(x, t_2) \quad \text{for} \quad 0 < t_1 < t_2, \quad x \in [0, 1]; \tag{1.4} \]
(iv) for each \( x \in [0, 1] \)
\[ \lim_{t \to \infty} b(x, t) \leq 0. \tag{1.5} \]

It is because of the imposition of condition (iii) that, following a fairly standard terminology, the problem is called sublinear.

Some studies of problems similar to the one under consideration here or the related eigenvalue problem

\[ (a(x)y')' + \lambda b(x, y^2) = 0, \quad x \in (0, 1), \]
\[ y(0) = y(1) = 0. \tag{1.7} \]

* This work was supported by NSF Grant GP 28377A1.
are [3, 9, 13, 14]; in these works however variational methods play little or no part. A numerical study of a similar problem is [12]. The most complete results on the structure of the eigenfunction branches of (1.6), (1.7) are contained in Rabinowitz [15] or Hartman [7]; these results contain the existence theory for (1.1), (1.2). The paper of Rabinowitz deals with general nonlinear eigenvalue problems and the results are obtained by application of the theory of topological degree. In [7] similar results are obtained directly for second order ordinary differential equations (but for problems of a more general form than (1.6), (1.7)) by more elementary methods.

The existence of multiple solutions of (1.1), (1.2) can be established by a variational argument based on the Lyusternik–Schnirelman theory. This variational approach is more significant however for the elliptic problem

\begin{equation}
\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial}{\partial x_j} u \right] + ub(x, u^2) = 0, \text{ in } \Omega,
\end{equation}

\begin{equation}
u(x) = 0 \quad \text{in } \varepsilon \Omega,
\end{equation}

where \( \Omega \) is a bounded region in \( \mathbb{R}^N \), for here when \( N \geq 2 \) the variational method gives results not otherwise obtainable. This approach to (1.8), (1.9) is taken in [1, 8, 16]; the treatment in [16] makes use of the work of Clark [2]. The results in the sources just quoted apply, when specialized to \( N = 1 \), to (1.1), (1.2). Since no information concerning nodal properties of solutions of (1.1), (1.2) is obtained in this way an alternative variational approach to (1.1), (1.2) which does give such information is also developed in [8]. This method, which is applicable only to ordinary differential equations, is the analogue of a method used in [10, 11, 17] in the study of superlinear boundary value problems.

Here we show that these two approaches are equivalent in the sense that they both determine the same sequence of critical values of the functional

\[ \int_{0}^{1} \left[ \int_{0}^{u(x)} b(x, t) dt - a(x) u^2 \right] dx, \]

of which (1.1) is the Euler equation. It will follow that if to the \( n \)th Lyusternik–Schnirelman critical value there corresponds a nontrivial solution of (1.1), (1.2) then there corresponds one which has exactly \( n - 1 \) zeros in \( (0, 1) \). As a by-product of this demonstration we obtain a third representation of these critical values by a formula analogous to that in the Courant–Weyl principle. The construction used in the proof of inequality (4.17), which proof is the essential step in the proof of the main theorem, is closely related to a construction used in [14] in conjunction with a fixed point argument.

The analogous questions for superlinear problems have been answered in [4 and 5]. In the superlinear case the third representation of the critical
values is by a formula analogous to that in Poincaré's principle, which is complementary to the Courant–Weyl principle. For a complete discussion of the Poincaré and Courant–Weyl principles see [18].

In order to be able to rely on the results proved in [8], we have made exactly the same assumptions concerning (1.1), (1.2) as are made there. Two possible directions of generalization are through relaxation of (1.5) and through allowing (1.1) to be quasi-linear, or more specifically to have the form of the one-dimensional case of the equation considered in [6]. Similar arguments also apply to the sublinear case of the generalized Emden–Fowler equation

\[ y'' + p(x) |y|^\gamma \text{sgn } y = 0, \quad x \in (0, 1), \]

where \( p \) is continuous and positive on \([0, 1]\) and \( 0 < \gamma < 1 \). The variational approach to the boundary value problem (1.2) for this equation can be made via the study of the Rayleigh quotient

\[ \left( \int_0^1 [y'(x)]^2 \, dx \right)^{(\gamma+1)/2} \int_0^1 p(x) |y(x)|^{\gamma+1} \, dx. \]

Variational methods are valuable for the study of this problem since the problem cannot be "linearized at 0" and hence does not yield to bifurcation methods.

2. Summary of Known Results

This section will consist of a brief summary of certain of the results of [8]. First we introduce some notation and terminology. For a closed interval \([\alpha, \beta] \subseteq [0, 1]\), \( \mathcal{H}[\alpha, \beta] \) will denote the space of real-valued functions \( u \) that are absolutely continuous on \([\alpha, \beta]\), possess square integrable derivatives and vanish at \( \alpha \) and \( \beta \). \( \mathcal{H}[\alpha, \beta] \) will be assigned a Hilbert space structure through the introduction of the inner product

\[ \langle u, v \rangle = \int_\alpha^\beta a(x) u'v' \, dx; \]

it is easily seen that the resulting Hilbert space structure is equivalent to that which arises from the inner product

\[ [u, v] = \int_\alpha^\beta u'v' \, dx. \]

By identifying \( u \in \mathcal{H}[\alpha, \beta] \) with the function on \([0, 1]\) that agrees with \( u \) on
VARIATIONAL PRINCIPLES

[α, β] and vanishes outside [α, β] we shall regard $\tilde{H}[\alpha, \beta]$ as a subspace of $\dot{H}[0, 1]$. We note that if $v \in \dot{H}[0, 1]$ and $v$ vanishes outside of $[\alpha, \beta] \subset [0, 1]$ then, with the above identification, $v \in \tilde{H}[\alpha, \beta]$.

As is customary we shall denote by $C[0, 1]$ the Banach space that consists of real valued functions continuous on $[0, 1]$ and is provided with the sup norm. The functions in $\tilde{H}[0, 1]$ belong to $C[0, 1]$ and the inclusion

$$\tilde{H}[0, 1] \rightarrow C[0, 1]$$

is compact. We note finally that, because of condition (ii), the functional

$$\int_0^1 \int_0^1 b(x, t) \, dt \, dx$$

is continuous on $C[0, 1]$.

The possibility of nonuniqueness of solutions to the initial value problem for (1.1) will not concern us here. It is however important to note that the particular initial value problem

$$y(x_0) = y'(x_0) = 0$$

for (1.1) does have a unique solution. Hereafter we shall refer to this property as "uniqueness of the zero solution." This uniqueness can be shown by the simple device of regarding a solution $y$ of (1.1) as a solution of the linear equation

$$[a(x)u']' + ub(x, y^2) = 0$$

and using the fact that the initial value problem for the linear equation has a unique solution.

Let $[α, β] \subseteq [0, 1]$. A weak solution of

$$[a(x)y']' + yb(x, y^2) = 0, \quad x \in (α, β)$$

$$y(α) = y(β) = 0$$

is a function $u \in \tilde{H}[α, β]$ which satisfies

$$\int_α^β [a(x)y'v' + vyb(x, y^2)] \, dx = 0$$

for every continuously differentiable function $v$ on $[α, β]$ which vanishes outside of a compact subinterval of $(α, β)$. It can be shown by a standard argument that a weak solution of (2.1), (2.2) is actually a solution in the ordinary sense.
For \( u \in \mathcal{H}[\alpha, \beta] \) let
\[
\Gamma(u) = \int_0^1 \left[ a(x) u^2 - \int_0^x b(x, t) \, dt \right] \, dx,
\]
and define \( A_1[\alpha, \beta] \) by
\[
A_1[\alpha, \beta] = \sup \{-\Gamma(u) : u \in \mathcal{H}[\alpha, \beta]\}.
\]

We observe that, as is almost immediate, \( A_1[\alpha, \beta] \) is a continuous function of \((\alpha, \beta)\). For \( n = 1, 2, \ldots \), let \( A_n = A_n[0, 1] \) be defined by
\[
A_n = \inf \left\{ \sum_{i=1}^n A_1[\xi_{i-1}, \xi_i] : 0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_n = 1 \right\}. \quad (2.3)
\]

Following [8] we shall say that the interval \([\alpha, \beta] \subseteq [0, 1]\) is a critical interval if the lowest eigenvalue \( \mu_1 = \mu_1[\alpha, \beta] \) of the problem
\[
[a(x)u']' + \mu b(x, 0) = 0 \quad \text{in} \quad (\alpha, \beta),
\]
\[
u(\alpha) = u(\beta) = 0
\]
satisfies
\[
\mu_1[\alpha, \beta] \geq 1;
\]
\([\alpha, \beta]\) is said to be noncritical otherwise.

The results from [8] which we shall require here are summarized in (*) and (**) below.

(*) The following are equivalent:

(a) the interval \([\alpha, \beta] \subseteq [0, 1]\) is noncritical,

(b) \( A_1[\alpha, \beta] > 0 \),

(c) (2.1), (2.2) has a solution which is positive in \((\alpha, \beta)\). In the case that (c) holds, this positive solution is unique.

(**) Let the eigenvalues \( \{\mu_n\} \) of the linear problem
\[
[a(x)u']' + \mu b(x, 0) = 0 \quad \text{in} \quad (0, 1),
\]
\[
u(0) = u(1) = 0 \quad (2.4)
\]
satisfy
\[
0 < \mu_1 < \mu_2 < \cdots < \mu_p < 1 \leq \mu_{p+1} < \cdots
\]
for some integer \( p \). Then
\[
A_1 > A_2 > \cdots > A_p > 0 \quad (2.6)
\]
VARIATIONAL PRINCIPLES

and

\[ A_n = 0 \quad \text{for} \quad n > p. \]

For \( n = 1, \ldots, p \), the problem (1.1), (1.2) has a solution \( y_n \) with exactly \( n - 1 \) zeros in \((0, 1)\) and which satisfies

\[ -\Gamma(y_n) = A_n. \]

There exists no nontrivial solution of (1.1), (1.2) having more than \( p - 1 \) zeros in \((0, 1)\).

3. VARIATIONAL INEQUALITIES

From (1.4) and (1.5) it follows by an elementary argument (given in detail in [6]) that if \( \varepsilon > 0 \) is preassigned there exist \( f, g \in L^2[0, 1] \) with

\[ \int_0^1 |f|^2 \, dx < \varepsilon \]

(3.1)

and such that

\[ \int_0^1 b(x, s) \, ds \leq f(x)|t| + g(x). \]

(3.2)

Since the inclusion mapping

\[ \tilde{H}[0, 1] \to C[0, 1] \]

(3.3)

is continuous it readily follows from (3.1) and (3.2), with \( \varepsilon \) sufficiently small, that for \( u \in \tilde{H}[0, 1] \),

\[ \Gamma(u) \geq \varepsilon \int_0^1 a(x) u'^2 \, dx - \gamma, \]

(3.4)

where \( \varepsilon \) and \( \gamma \) are positive constants.

We now state a basic variational inequality, namely, for \( u, v \in \tilde{H}[0, 1] \),

\[ \Gamma(v) - \Gamma(u) \geq \int_0^1 a(x) v'^2 \, dx - \int_0^1 \varphi^2 b(x, u^2) \, dx \]

\[ - \int_0^1 a(x) u'^2 \, dx + \int_0^1 u^2 b(x, u^2) \, dx. \]

(3.5)

This inequality follows immediately from the fact that \(-\int_0^t b(x, t) \, dt\) is convex in the argument \( u^2 \), and since, moreover, this convexity is strict, equality can hold in (3.5) only if \( u^2 = v^2 \).
Lemma 3.1. The functional $\Gamma(u)$ is weakly lower semicontinuous on $\tilde{H}[0, 1]$.

Proof. Since the imbedding (3.3) is compact and the functional

$$\int_0^1 \int_0^1 b(x, t) \, dt \, dx$$

is continuous on $C[0, 1]$ it follows that that functional is weakly continuous on $\tilde{H}[0, 1]$. The weak lower semicontinuity of $\Gamma$ on $\tilde{H}[0, 1]$ then follows immediately from that of the functional

$$\int_0^1 a(x) \, u^2 \, dx.$$ 

4. Main Result

Let $[\alpha, \beta] \subseteq [0, 1]$, we define $y(\cdot; \alpha, \beta) \in \tilde{H}[0, 1]$ as follows: if $[\alpha, \beta]$ is noncritical then $y(\cdot; \alpha, \beta)$ agrees in $[\alpha, \beta]$ with the unique positive solution of (2.1), (2.2) and $y(\cdot; \alpha, \beta)$ vanishes identically in $[0, 1]$ outside of $[\alpha, \beta]$; $y(\cdot; \alpha, \beta)$ is identically zero if $[\alpha, \beta]$ is critical.

Lemma 4.1. The mapping

$$(\alpha, \beta) \mapsto y(\cdot; \alpha, \beta)$$

of $[0, 1] \times [0, 1]$ into $\tilde{H}[0, 1]$ is continuous.

Proof. For any $\alpha, \beta$ with $0 < \alpha < \beta < 1$ we have

$$\Gamma[y(\cdot; \alpha, \beta)] = -A_1[\alpha, \beta].$$

Since $A_1[\alpha, \beta] \geq 0$ this implies that $y = y(\cdot; \alpha, \beta)$ satisfies

$$e \int_0^1 a(x) \, y'^2 \, dx \leq \gamma,$$

where $e$ and $\gamma$ are the constants in (3.4). This enables one to reduce the continuity assertion in the above lemma to the following.

(*) If $[\alpha_n, \beta_n] \subseteq [0, 1]$ for $n = 1, 2, \ldots$,

$$\lim_{n \to \infty} \alpha_n = \alpha, \quad \lim_{n \to \infty} \beta_n = \beta,$$

and $\{y(\cdot; \alpha_n, \beta_n)\}$ converges weakly to $u$ in $\tilde{H}[0, 1]$ then $u = y(\cdot; \alpha, \beta)$. 
To prove (†) we first note that since the inclusion (3.3) is compact, 
\{y(:, \alpha_n, \beta_n)\} converges uniformly to \(u\), thus
\[
u(x) = 0, \quad x \notin [\alpha, \beta], \tag{4.2}
\]
and
\[
u(x) \geq 0, \quad x \in [\alpha, \beta]. \tag{4.3}
\]
From (4.2) and the fact that \(\{y(:, \alpha_n, \beta_n)\}\) converges weakly to \(u\) it follows that \(u\) is a weak solution of (2.1), (2.2). Hence \(u\) is in fact a solution of (2.1), (2.2) in the ordinary sense. From the continuity of \(A_1\) and the weak lower semicontinuity of \(\Gamma\) we have
\[
\Gamma(u) \geq -\liminf_{n \to \infty} \Gamma(y(:, \alpha_n, \beta_n)) \\
\geq \lim_{n \to \infty} A_1[\alpha_n, \beta_n] \\
\geq A_1[\alpha, \beta]. \tag{4.4}
\]
If \([\alpha, \beta]\) is a critical interval then \(u\) must vanish identically in \([\alpha, \beta]\) and thus indeed \(u = y(:, \alpha, \beta)\). Otherwise, by (*) of Section 2, \(A_1[\alpha, \beta] > 0\) and thus, since \(\Gamma(0) = 0\), it follows from (4.4) that \(u \neq 0\) in \([\alpha, \beta]\). From (4.3) and the uniqueness of the zero solution of (1.1) it then follows that \(u\) is positive in \((\alpha, \beta)\), and hence \(u\) coincides in \((\alpha, \beta)\) with the unique positive solution of (2.1), (2.2). This completes the proof of (†), and the latter assertion, in view of (4.1), implies the continuity assertion of Lemma 4.1.

We now define the Courant–Weyl numbers associated with (1.1), (1.2) as follows, for \(n = 1, 2, \ldots,\)
\[
C_n = \inf_{\text{co-dim} M \leq n - 1} \sup \{-\Gamma(u) : u \in M\}, \tag{4.5}
\]
where \(M\) denotes a subspace of \(H[0, 1]\) and co-dim \(M\) is the co-dimension of \(M\) in \(H[0, 1]\).

In what follows \(S\) will denote a compact subset of \(H[0, 1] / \{0\}\) which is symmetric through the origin. The genus of such a set \(S\), \(\gamma(S)\), is defined to be zero if \(S = \emptyset\) and otherwise to be the least integer \(n\) such that there exists an odd continuous map \(f : S \to R^n / \{0\}\). The Lyusternik–Schwierelman numbers associated with (1.1), (1.2) are defined as follows, for \(n = 1, 2, \ldots,\)
\[
c_n = \sup_{\gamma(S) \geq n} \inf \{-\Gamma(u) : u \in S\}. \tag{4.6}
\]

**Lemma 4.2.** The sequences \(\{C_n\}\) and \(\{c_n\}\) are nonincreasing sequences of nonnegative numbers. For each \(n,\)
\[
c_n \leq C_n. \tag{4.7}
\]
Proof. The nonincreasing character of these sequences is clear from the definitions. The inequality (4.7) also is clear when \( n = 1 \), in fact equality is clear in that case. To prove (4.7) for \( n > 1 \) let \( M \) be a subspace of \( \tilde{H}[0, 1] \) of co-dimension \( n - 1 \) and let \( \gamma(S) = n \). We claim that \( M \cap S \) is nonvoid. Indeed let \( P \) be the orthogonal projection onto the orthogonal complement of \( M \). Clearly then, since \( \gamma(S) = n \), \( P \) must annihilate an element of \( S \), which element therefore belongs to \( M \cap S \). Thus

\[
\inf\{-\Gamma(u) : u \in S\} \leq \sup\{-\Gamma(u) : u \in M\}
\]  

(4.8)

for any such \( S \) and \( M \). The inequality (4.7) follows immediately from (4.8) and Definitions (4.5) and (4.6).

To complete the proof it suffices to show that \( c_n \geq 0 \) for all \( n \). If \( n \geq 1 \) is given and \( L \) is any \( n \)-dimensional subspace of \( \tilde{H}[0, 1] \) then by the Borsuk Antipodal Mapping Theorem,

\[
S = \left\{ u \in L : \int_0^1 a(x) u^2 \, dx = \epsilon \right\}
\]

is a set of genus \( n \) for any \( \epsilon > 0 \) and clearly

\[
\inf\{-\Gamma(u) : u \in S\} \geq \epsilon.
\]

It follows that \( c_n \geq 0 \).

Lemma 4.3. Let \( u \) be a solution of (1.1), (1.2) with exactly \( n - 1 \) zeros in \((0, 1)\). Then

\[
-\Gamma(u) \geq C_n.
\]  

(4.9)

In particular

\[
A_n \geq C_n.
\]  

(4.10)

Proof. Let \( u \) be as above, then \( u \) is the \( n \)th eigenfunction of the linear problem

\[
(a(x)u')' + \mu vb(x, u^2) = 0 \quad \text{in} \quad (0, 1),
\]

\[
\tau(0) = \tau(1) = 0,
\]  

(4.11)

and the corresponding \( n \)th eigenvalue, \( \mu_n \), is one. Thus

\[
\int_0^1 a(x) u'^2 \, dx = \int_0^1 u^2b(x, u^2) \, dx,
\]  

(4.13)
and if \( u_1, u_2, \ldots, u_{n-1} \) are the first \( n - 1 \) eigenfunctions of (4.11), (4.12) then from the theory of the Sturm-Liouville problem,

\[
\int_0^1 a(x) v'' \, dx \geq \int_0^1 v^2 b(x, u^2) \, dx \tag{4.14}
\]

for any \( v \) satisfying

\[
\int_0^1 a(x) v' u_j' \, dx = 0, \quad j = 1, 2, \ldots, n - 1. \tag{4.15}
\]

If \( v \) satisfies (4.15) then it follows from (4.13), (4.14) and (3.5) that

\[
-\Gamma(v) \leq -\Gamma(u). \tag{4.16}
\]

Since (4.15) determines a subspace of co-dimension \( n - 1 \) in \( \dot{H}[0, 1] \), the inequality (4.9) follows from (4.16) and (4.5).

The inequality (4.10) follows immediately from (***) and what we have just proved provided \( \Lambda_n > 0 \). If on the other hand \( \Lambda_n = 0 \), then by (***) the \( n \)th eigenvalue \( \mu_n \) of (2.4), (2.5) satisfies

\[
\mu_n \geq 1.
\]

If we take \( u_1, \ldots, u_{n-1} \) to be the first \( n - 1 \) eigenfunctions of (2.4), (2.5) then for \( v \in \dot{H}[0, 1] \) satisfying (4.15), we have (4.14) for \( u = 0 \). Taking \( u = 0 \) in (3.5) then gives

\[
-\Gamma(v) \leq 0
\]

for all such \( v \) and thus it follows as before that

\[
C_n \leq \Lambda_n = 0.
\]

**Remark.** Note that when one applies the argument used above to prove (4.9) to the case \( n = 1 \), then of course condition (4.15) becomes void, (4.14) is strict unless \( v \) is proportional to \( u \) and (3.5) is strict unless \( v^2 = u^2 \). Thus we conclude

\[
-\Gamma(v) < -\Gamma(u)
\]

unless \( v = u \) or \( v = -u \). This gives an independent proof of the uniqueness assertion in (*). This argument is generalized and applied in [6] to a quasi-linear elliptic problem.

To establish the main result of this note it remains only to demonstrate that for each \( n = 1, 2, \ldots \),

\[
\Lambda_n \leq c_n. \tag{4.17}
\]
The inequality (4.17) clearly holds if \( n = 1 \) or if \( A_n = 0 \), thus it will be assumed in the remainder of the proof that \( n > 1 \) and \( A_n > 0 \). To demonstrate (4.17) in this case it suffices to show that there exists a set \( S \subseteq \mathcal{H}[0, 1] \) of genus not less than \( n \) such that

\[
A_n \leq \inf \{-\Gamma(u) : u \in S\}. \quad (4.18)
\]

We claim that such a set is the collection of all functions \( u \) such that there exist \( 0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_n = 1 \) with

\[
u(x) = \pm y(x; \xi_{i-1}, \xi_i), \quad \text{for } x \in [\xi_{i-1}, \xi_i], \quad i = 1, \ldots, n. \quad (4.19)
\]

If \( u \) satisfies (4.19) then

\[
-\Gamma(u) = \sum_{i=1}^{n} A_1(\xi_{i-1}, \xi_i),
\]

and thus the set \( S \) of all functions of this form satisfies (4.18). It remains to show that this set has genus not less than \( n \). This is done as follows. Let

\[
K_n = \{ \eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n : |\eta_1| + \cdots + |\eta_n| = 1 \}.
\]

For \( \eta \in K_n \) let \( \xi(\eta) = (\xi_0(\eta), \ldots, \xi_n(\eta)) \) be defined by

\[
\xi_i(\eta) = \sum_{j=1}^{i} |\eta_j|, \quad i = 0, 1, \ldots, n,
\]

and let \( T(\eta) \in \mathcal{H}[0, 1] \) be the function that is defined as follows

\[
T(\eta)(x) = (\text{sgn } \eta_i) y(x, \xi_{i-1}(\eta), \xi_i(\eta)), \xi_{i-1}(x) \leq x \leq \xi_i(x), \quad i = 1, \ldots, n.
\]

We then have, for \( \eta \in K_n \)

\[
-\Gamma(T(\eta)) = \sum_{i=1}^{n} A_1(\xi_{i-1}(\eta), \xi_i(\eta)) \geq A_n > 0,
\]

and thus \( T(\eta) \neq 0 \). The map \( T \) is continuous by Lemma 4.1; it is odd, i.e., \( T(-\eta) = -T(\eta) \), and \( T(K_n) \subseteq \mathcal{H}[0, 1]/\{0\} \). An odd continuous map of \( T(K_n) \) into \( \mathbb{R}^{n-1}/\{0\} \) would induce such a map of \( K_n \) into \( \mathbb{R}^{n-1}/\{0\} \), but in view of the Borsuk Antipodal Mapping Theorem such a map as the latter cannot exist. Thus \( \gamma(T(K_n)) \geq n \). It is clear that \( T(K_n) \) is just the set of all \( n \) of the form (4.19), thus that set has genus not less than \( n \). Thus (4.17) has been proved.

Summarizing what has been proved above and combining with that the
results of Hempel quoted in (***) of Section 2, we have the following; as before \( M \) will denote a closed subspace of \( \tilde{H}[0, 1] \) and \( S \) will denote a compact symmetric subset of \( \tilde{H}[0, 1]/\{0\} \).

**Theorem 4.1.** The critical values \( \lambda_1, \lambda_2, \ldots \) of \(-\Gamma\), defined by (2.3) can also be represented by the formula

\[
\lambda_n = \inf_{\dim M \leq n-1} \sup \{-\Gamma(u) : u \in M\}
\]

(4.20)

or by the formula

\[
\lambda_n = \sup_{\gamma(S) \geq n} \inf \{-\Gamma(u) : u \in S\}.
\]

(4.21)

If for a given \( n \), \( \lambda_n > 0 \) then there exists a pair \((u, -u)\) of nontrivial solutions of (1.1), (1.2) such that \( u \) has exactly \( n - 1 \) zeros in \((0, 1)\) and

\[
-\Gamma(u) = \lambda_n.
\]

(4.22)

**Remark.** If it is the case that to each \( n > 1 \) there corresponds at most one pair of solutions \((u, -u)\) of (1.1), (1.2) such that \( u \) has exactly \( n - 1 \) zeros in \((0, 1)\) (for brevity hereafter we shall say “solutions are uniquely determined by nodal properties” if this is the case) then it is clear that all of the critical values of \(-\Gamma\) will be given by (2.3), (4.20), or (4.21), i.e., every nontrivial solution of (1.1), (1.2) will satisfy (4.22) for some \( n \); moreover, to each \( \lambda_n > 0 \) there will correspond exactly one pair \((u, -u)\) of nontrivial solutions. Solutions are uniquely determined by nodal properties when (1.1) is autonomous and in certain other cases (see Section 5) but this is not so in general, as is shown by an example in [3]. When unique determination by nodal properties fails it can occur that there will be more than one pair of nontrivial solutions \((u, -u)\) satisfying (4.22) for a given \( n > 1 \), and/or that there will be nontrivial solutions \( u \) which do not satisfy (4.22) for any \( n \). It is generally true however, and follows from Lemma 4.3 and (2.6), that any nontrivial solution \( u \) satisfying (4.22) must have at least \( n - 1 \) zeros in \((0, 1)\).

5. **Examples**

We conclude with two examples. Actually each of these falls outside of the class of problems treated above but both are “sublinear” and the methods employed above can be adapted in a straightforward way to their treatment. Both of the examples below arise from physical problems and both contain a physically significant parameter so that they can be regarded as nonlinear eigenvalue problems. Here we shall view the parameter as fixed and discuss
the possibility of the existence of multiple solutions for a fixed value of the parameter. The result of varying the parameter is easily deduced; complete discussions of that aspect of the problem can be found in [9 and 13].

The first example is the problem studied by Kolodner in [9], namely

$$y'' + \omega^2(x^2 + y^2)^{-1/2}y = 0, \quad (5.1)$$
$$y(0) = y'(1) = 0. \quad (5.2)$$

This problem represents a mathematical model of a heavy rotating string; the parameter $\omega$ is the angular velocity. Because (5.2) involves a free end condition, in the analysis of this problem the space $\hat{H}[a, 1]$, for $0 \leq \alpha < 1$, is replaced by the space of functions which are absolutely continuous with square integrable derivative and vanish at $\alpha$; no real difficulty results from the mild singularity of the coefficient at $x = y = 0$. The functional $I'$ that corresponds to (5.1), (5.2) is

$$I'(u) = \int_0^1 \left\{u'^2 - 2\omega^2[(x^2 + u^2)^{1/2} - x]\right\} dx.$$ 

With the indicated substitution for the spaces $\hat{H}[a, 1]$ and the above $I'$ there holds the exact analogue of Theorem 4.1. It is shown in [9] that solutions of (5.1), (5.2) are uniquely determined by nodal properties.

The second example is the problem which arises from consideration of the bending of a non-homogeneous rod under longitudinal loading,

$$y''(1 - y'^2)^{1/2} + \lambda p(x)y = 0, \quad (5.3)$$
$$y(0) = y(1) = 0, \quad (5.4)$$

where $p(x)$ is continuous and positive on $[0, 1]$ and the parameter $\lambda$ is the load on the rod. The appropriate functional for (5.3), (5.4) is

$$I'(u) = \int_0^1 \left\{\int_0^u 2 \arcsin t \, dt - \lambda p(x) u^2\right\} dx.$$ 

Let

$$B = \{u \in \hat{H}[0, 1] : \text{ess sup}_{0 \leq x \leq 1} |u'| \leq 1\},$$

then $I'$ is defined on $B$. In place of the inequality (3.5) we have, for $u, v \in B$,

$$I'(v) - I'(u) \geq \int_0^1 \left[v'^2 \frac{\arcsin u'}{u'} - \lambda p(x) v^2\right] dx - \int_0^1 \left[u'^2 \frac{\arcsin u'}{u'} - \lambda p(x) u^2\right] dx,$$
which follows from the convexity of $\int_0^{1/\beta} \arcsin t \, dt$ as a function of $s$. The role of the linearized equation (4.11) is played in this case by the equation

$$\left[ \left( \frac{\arcsin u'}{u} \right) v' \right] + \lambda p(x)v = 0.$$ 

Corresponding to a given coefficient $p(x)$ there is an $L > 0$ such that the analogue for (5.3), (5.4) of Theorem 4.1 holds provided $0 < \lambda < L$. The formulas (4.20) and (4.21) are replaced respectively by

$$A_n = \inf_{\co-dim M \leq n-1} \sup \{-\Gamma(u) : u \in B \cap M\} \quad (5.5)$$

and

$$A_n = \sup_{\gamma(S) \geq n} \inf_{S \subseteq B} \{-\Gamma(u) : u \in S\}.$$ 

In the case of a homogeneous rod, i.e., $p(x) = \text{const.}$, solutions are uniquely determined by nodal properties.

REFERENCES

5. C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, Arch. Rational Mech. Anal. 46 (1972), 81–95.