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Linear Algebra and its Applications 429 (2008) 2663–2669

LINEAR ALGEBRA AND ITS APPLICATIONS

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The character values of commutative quasi-thin schemes

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Received 11 June 2007; accepted 10 August 2007

Available online 22 October 2007

Submitted by Rowlinson

Abstract

The main result of this article shows that all character values of commutative quasi-thin schemes are cyclotomic algebraic integers. In particular, all of the eigenvalues of the adjacency matrices corresponding to relations in these association schemes are cyclotomic algebraic integers.

AMS classification: 05E30; 20C15

Keywords: Association scheme; Character; Table algebra

1. Introduction

A well-known question in the book by Bannai and Ito [2, p. 123] asks whether or not the character values of commutative association schemes have to be cyclotomic algebraic integers; that is, elements of $\mathbb{Z}(\zeta_n)$, where ζ_n is a primitive complex nth root of unity for some positive integer n. Even among the noncommutative association schemes (homogeneous coherent configurations), the authors are not aware of an example of a relation in a finite association scheme for which the eigenvalues of its adjacency matrix are not cyclotomic algebraic integers.

This cyclotomic character value property is known to hold for several classes of association schemes, including thin schemes (via the character theory of finite groups), Cayley schemes

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[1] Herman acknowledges the support of NSERC.

[2] Rahnamai Barghi was visiting the University of Regina during the time a part of this paper was written and he thanks University of Regina for its hospitality.

doi:10.1016/j.laa.2007.08.023

[2, Theorem 7.3(iii)], P- and Q-polynomial schemes (for which the character values are rational) [2, Section 3.7], and commutative Schurian schemes via the corresponding result for Hecke algebras [3, (11.24)]. A result of Munemasa shows that commutative schemes whose Krein parameters are all rational have cyclotomic character values [8]. From Munemasa's arguments, it can also be observed that this will be the case as long as the field obtained by adjoining the Krein parameters for the scheme to $\mathbb{Q}$ is a cyclic Galois extension of $\mathbb{Q}$. On the other hand, the adjacency matrix of a relation in an association scheme is also the adjacency matrix of a regular digraph, and a result of Godsil shows that any algebraic integer can occur as the eigenvalue of a regular digraph [4]. So there is still a huge gap in understanding that remains to be filled.

The next natural situation to consider is the case of quasi-thin association schemes, those for which the valency n_g of any basis relation g is at most 2. It is known that quasi-thin association schemes (not necessarily commutative) in which each basis relation g satisfies $n_{g^*g} \neq 2$ are always Schurian (see [10, Theorem 6.4.4]). However, for arbitrary quasi-thin association schemes the problem is still open. In this article we will show that commutative quasi-thin schemes have the cyclotomic character value property.

2. Background

Here we will give a brief introduction to the basic definitions and notation in the theory of association schemes that we will use, all of which the reader can find in [9] or [10].

Let (X, G) be an association scheme, with X an ordered index set of size n and $G = \{1_X = g_0, g_1, \ldots, g_d\}$ be the set of scheme relations on the set X. Let A_i be the $n \times n$ adjacency matrix for the corresponding scheme relation g_i. Each A_i is an $n \times n$ (0,1)-matrix for which the number of 1's in each row and column is a fixed positive integer n_{g_i}. This number is called the *valency* of the scheme relation g_i because it is the valency of the regular digraph whose adjacency matrix is precisely A_i. An association scheme is called *quasi-thin* if $n_{g_i} \leqslant 2$ for all $i = 0, 1, \ldots, d$.

It follows from (and in fact it is equivalent to) the definition of a finite association scheme that the set of $n \times n$ matrices $\mathscr{B} = \{I = A_0, A_1, \ldots, A_d\}$ satisfies the following conditions:

(1) the transpose map restricts to a one-to-one correspondence from $\mathscr{B}$ to itself;
(2) the sum of the elements of $\mathscr{B}$ is the $n \times n$ matrix J with every entry equal to 1;
(3) $\mathscr{B}$ is a basis for a $(d + 1)$-dimensional semisimple subalgebra of $M_n(\mathbb{C})$ with nonnegative integer structure constants.

The subalgebra of $M_n(\mathbb{C})$ defined by the span of $\mathscr{B}$ is called the *complex adjacency algebra* (or *Bose-Mesner algebra*) of the association scheme (X, G), and is denoted by $\mathbb{C}G$. If the complex adjacency algebra $\mathbb{C}G$ is commutative, then the association scheme (X, G) is said to be commutative.

We are interested in the values that the irreducible characters of the complex adjacency algebra take on elements of the defining basis $\mathscr{B}$. These are known as the *character values* of the association scheme (X, G). The character values of a commutative scheme are precisely the eigenvalues of the adjacency matrices of the scheme.

A subset H of the set G of relations of the association scheme (X, G) is said to be a *closed* subset of the scheme if, for every pair $g_i, g_j \in H$, the product $A_i A_j$ of the corresponding adjacency matrices can be written as a linear combination of the adjacency matrices of relations belonging to H.

Remark 2.1. Let (X, G) be an association scheme with $|X| = n$. Suppose A is an adjacency matrix for a relation g in G with valency n_g. Then by arguing exactly as in the proof of Birkhoff's theorem in [7, (8.7.1)], we can show that A is the sum of exactly n_g distinct permutation matrices. The list of permutation matrices appearing in such a sum does not need to be unique.

Lemma 2.2. *Let (X, G) be an association scheme with $|X| = n$. Let $g \in G$ be a symmetric relation of valency 2. Then every eigenvalue of its adjacency matrix A is of the form $\zeta + \zeta^{-1}$ for some mth root of unity ζ with $m \in \{1, \dots, n\} \setminus \{2, n-2, n-1\}$. In particular, every eigenvalue of A is a real cyclotomic algebraic integer.*

Proof. This follows easily from the fact that the matrix A is the adjacency matrix of a 2-regular graph on n vertices that is the union of disjoint cycles. Since the adjacency matrix of a 2-regular graph containing 2-cycles will have some entries equal to 2, the cycles must have length at least 3. Therefore, m cannot be 2. Furthermore, the graph cannot contain an $(n-1)$-cycle or an $(n-2)$-cycle, because either of these situations would prevent it from being regular. Therefore, m can be restricted as indicated. □

Remark 2.3. It is in fact the case that the m of the previous lemma is always a divisor of n. This observation is not needed for the remainder of the paper, but we will previde a short argument here. We can assume $m \geqslant 3$. By Remark 2.1 there is a permutation σ of order m such that the adjacency matrix A can be written as $P_\sigma + P_{\sigma^{-1}}$. The permutation σ can be decomposed as the product of disjoint cycles τ_j, $j = 1, \dots, \ell$. Since $A \neq I$, σ is fixed-point-free, and so the subsets $X_j = \{x \in X \colon \tau_j(x) \neq x\}$, $j = 1, \dots, \ell$, form a partition of X. It suffices to show that $|X_j| = m$ for each $j = 1, \dots, \ell$. To do this, let E be the equivalence relation corresponding to this partition, which means $X/E = \{X_1, \dots, X_\ell\}$ and $E = \bigcup_{j=1}^{\ell}(X_j \times X_j)$. It is easy to see that $(A^k)_{x,y} = 1$ if and only if $\sigma^k(x) = y$ or $\sigma^k(y) = x$ for all $k = 0, 1, \dots, m-1$. For $h \in G$, let $A(h)$ be the adjacency matrix of h. Define H to be the subset of G consisting of all $h \in G$ for which $A(h)$ appears in the decomposition of A^k for some integer $k = 0, \dots, m-1$. Then $E = \bigcup_{h \in H} h$. Considering E as a graph, its adjacency matrix is $A(E) = \sum_{h \in H} A(h)$. On the other hand, $A(E) = \sum_{X_j \in X/E} J_{X_j}$, where J_{X_j} is the matrix with entries equal to 1 in the positions (x, y) when $x, y \in X_j$, and all other entries equal to 0. Therefore, $A(E)^2 = \sum_{X_j \in X/E} |X_j| J_{X_j}$. It follows that the coefficient of $A(g_0) = I$ in $A(E)^2$ is equal to $|X_j|$ for each $j = 1, \dots, \ell$. This shows that $|X_j|$ does not depend on j, and thus each τ_j occurring in the decomposition of σ has the same order, which must be m.

The association scheme for which the elements of $\mathscr{B}$ are defined by

$$\sum_{i=0}^{8} i A_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 & 7 & 8 & 8 \\ 1 & 0 & 4 & 5 & 2 & 3 & 6 & 6 & 7 & 7 & 8 & 8 \\ 3 & 5 & 0 & 2 & 1 & 4 & 7 & 7 & 8 & 8 & 6 & 6 \\ 2 & 4 & 3 & 0 & 5 & 1 & 8 & 8 & 6 & 6 & 7 & 7 \\ 5 & 3 & 1 & 4 & 0 & 2 & 7 & 7 & 8 & 8 & 6 & 6 \\ 4 & 2 & 5 & 1 & 3 & 0 & 8 & 8 & 6 & 6 & 7 & 7 \\ 6 & 6 & 7 & 8 & 7 & 8 & 0 & 1 & 2 & 4 & 3 & 5 \\ 6 & 6 & 7 & 8 & 7 & 8 & 1 & 0 & 4 & 2 & 5 & 3 \\ 7 & 7 & 8 & 6 & 8 & 6 & 3 & 5 & 0 & 1 & 2 & 4 \\ 7 & 7 & 8 & 6 & 8 & 6 & 5 & 3 & 1 & 0 & 4 & 2 \\ 8 & 8 & 6 & 7 & 6 & 7 & 2 & 4 & 3 & 5 & 0 & 1 \\ 8 & 8 & 6 & 7 & 6 & 7 & 4 & 2 & 5 & 3 & 1 & 0 \end{bmatrix}$$

provides an example of a nonsymmetric association scheme of rank 9 and order 12 which is quasi-thin and each basis relation of valency 2 is symmetric. (This example is [6, as12, No. 53].) Therefore, a quasi-thin association scheme satisfying the conditions of the next result is not necessary symmetric.

Theorem 2.4. *Let (X, G) be a quasi-thin association scheme. Suppose that each relation with valency 2 is symmetric. Then all character values of (X, G) are cyclotomic algebraic integers.*

Proof. This is an easy consequence of Lemma 2.2. □

3. Commutative quasi-thin schemes

Let (X, G) be a commutative association scheme. Then $(\mathbb{C}G, \mathscr{B})$ is an integral table algebra with respect to the algebra automorphism $A \to \overline{A}^{\mathrm{t}}$, where $\overline{A}^{\mathrm{t}}$ is the conjugate transpose of A as a matrix. (For the definition and the basic properties of integral table algebras used here, see [1].) We will need some basic properties of $(\mathbb{C}G, \mathscr{B})$ that come from its natural integral table algebra structure. First of all, the map $A_i \mapsto n_i, 0 \leqslant i \leqslant d$ can be extended to an algebra homomorphism $f: \mathscr{A} \to \mathbb{C}$ satisfying $f(A_i) = f(\overline{A_i}^{\mathrm{t}}) \in \mathbb{R}$ for $0 \leqslant i \leqslant d$. Second, there is a Hermitian form $[\ ,\ ]$ on $\mathbb{C}G$ given by

$$[D, E] := \mathrm{trace}(D\overline{E}^{\mathrm{t}})/n$$

for all $D, E \in \mathbb{C}G$. Finally, for all basis elements $A, B, C \in \mathscr{B}$, this form satisfies

$$[AB, C] = [A, CB^{\mathrm{t}}] = [B, A^{\mathrm{t}}C] \quad \text{and} \quad [A, B] = \delta_{A,B} f(A),$$

where $\delta_{A,B}$ is the Kronecker delta.

Since $\mathbb{C}G$ is a commutative semisimple subalgebra of $M_n(\mathbb{C})$ with basis $\mathscr{B}$, all of the matrices in $\mathbb{C}G$ can be simultaneously diagonalized. Therefore, there is a basis of $\mathbb{C}^n$ for which all vectors in this basis are eigenvectors for all elements of $\mathscr{B}$ at the same time. Fix one element v of this basis. If $A \in \mathscr{B}$, then let λ_A be the eigenvalue of A corresponding to v. If we can show that λ_A lies in a cyclotomic extension of $\mathbb{Q}$, then it will follow that every eigenvalue of A is a cyclotomic algebraic integer. We use this argument in the proof of our main theorem.

Theorem 3.1. *Every character value of a commutative quasi-thin scheme is a cyclotomic algebraic integer.*

Proof. Let (X, G) be a finite commutative quasi-thin scheme, and let $\mathscr{B}$ be the set of adjacency matrices for the relations in G.

By [5, Lemma 3.2], the possibilities for the product of two adjacency matrices A, B in $\mathscr{B}$ of degree (or valency) 2 are:

- $2C + 2D$ for some $C, D \in \mathscr{B}$ with $f(C) = f(D) = 1$;
- $2C$ for some $C \in \mathscr{B}$ with $f(C) = 2$;
- $2C + D$ for some $C, D \in \mathscr{B}$ with $f(C) = 1$ and $f(D) = 2$;
- $C + D$ for some $C, D \in \mathscr{B}$ with $f(C) = f(D) = 2$.

By Lemma 2.2, we may assume $A \in \mathscr{B}$ is a nonsymmetric element of valency 2. Thus by the argument in Remark 2.1 there exists two permutation matrices P_τ and P_γ such that $A = P_\tau + P_\gamma$. On the other hand, by the above possibilities for the product of A and A^{t} it follows that either

(a) $AA^{\mathrm{t}} = 2I + 2B$ for some basis element B of degree 1, or
(b) $AA^{\mathrm{t}} = 2I + B$ for some basis element B of degree 2.

Suppose that (a) holds. Then there is a permutation ϵ such that $B = P_\epsilon$, since $f(B) = 1$. Since our algebra is commutative, from (a) and the fact that B is symmetric we get

$$P_{\tau\gamma^{-1}} + P_{\gamma\tau^{-1}} = 2P_\epsilon = P_{\gamma^{-1}\tau} + P_{\tau^{-1}\gamma},$$

which implies that $\epsilon = \tau\gamma^{-1} = \gamma\tau^{-1} = \gamma^{-1}\tau = \tau^{-1}\gamma$. It follows that $\gamma\tau = \tau\gamma$. Therefore, A can be written as the sum of two commuting permutation matrices, and so all of its eigenvalues are cyclotomic, as desired.

We now may assume that (b) holds. Since B is symmetric, we have that $[AB, A] = [B, AA^{\mathrm{t}}] = 2$, which implies that the coefficient of A in the decomposition of AB is 1. So we have two possibilities:

(1) $AB = A + 2C$ for some basis element C of degree 1, or
(2) $AB = A + C$ for some basis element C of degree 2, with $C \neq A$.

If (1) occurs, then $BC = A$, because $f(C) = 1$ and $[BC, A] = [AB, C] = 2$. It follows that $\lambda_A = \lambda_B\lambda_C$. Since B is symmetric, it follows from Lemma 2.2 that λ_B is cyclotomic. Since C is a permutation matrix, λ_C is a root of unity. Therefore, λ_A is cyclotomic also, and we are done in this case.

So now we may assume that (b) and (2) occur. Since $A \neq A^{\mathrm{t}}$ there are two non-identity basis elements H and K such that one of the following holds:

(i) $A^2 = 2H + 2K$ if $f(H) = f(K) = 1$,
(ii) $A^2 = H + K$ if $f(H) = f(K) = 2$, or
(iii) $A^2 = 2H + K$ if $f(H) = 1, f(K) = 2$.

If (i) occurs, it follows that $A^{\mathrm{t}}H = A$ and $A^{\mathrm{t}}K = A$. These facts along with (2) yield

$$2A + AB = 3A + C = A(AA^{\mathrm{t}}) = A^2A^{\mathrm{t}} = 2A^{\mathrm{t}}H + 2A^2K = 4A,$$

which forces $A = C$, a contradiction. Hence (i) cannot occur.

Now suppose that (ii) occurs. Then

$$A^{\mathrm{t}}H + A^{\mathrm{t}}K = A^2A^{\mathrm{t}} = A(AA^{\mathrm{t}}) = 2A + AB = 3A + C.$$

By interchanging H and K if necessary, we can assume that $A^{\mathrm{t}}H = 2A$ and $A^{\mathrm{t}}K = A + C$. If $H \neq K$, then

$$2 = [H, H] = [H, H + K] = [H, A^2] = [A^{\mathrm{t}}H, A] = [2A, A] = 4$$

a contradiction. If $H = K$, then

$$2HA^{\mathrm{t}} = A^2A^{\mathrm{t}} = A(AA^{\mathrm{t}}) = 3A + C,$$

which is also impossible since the coefficient of C in $2HA^{\mathrm{t}}$ cannot be 1. Therefore, (ii) cannot occur either.

So we have that (iii) holds. It follows that

$$2HA^{\mathrm{t}} + KA^{\mathrm{t}} = A^2A^{\mathrm{t}} = A(AA^{\mathrm{t}}) = 2A + AB = 3A + C,$$

which implies that $HA^{\mathrm{t}} = A$ and $KA^{\mathrm{t}} = A + C$. Therefore

$$2H + K = A^2 = A(A^{\mathrm{t}}H) = (AA^{\mathrm{t}})H = 2H + BH$$

and so $BH = K$. It follows from this that

$$(\lambda_A)^2 = \lambda_H(2 + \lambda_B).$$

Since λ_H is a root of unity, it suffices to show that every eigenvalue of $2I + B$ is cyclotomic. By Lemma 2.2, every eigenvalue of B has the form $\zeta + \zeta^{-1}$, for some root of unity ζ. Therefore, the eigenvalues of $2I + B$ are of the form

$$2 + \zeta + \zeta^{-1} = \zeta^{-1}(\zeta^2 + 2\zeta + 1) = \zeta^{-1}(\zeta + 1)^2.$$

The theorem now follows from the observation that the square root of every eigenvalue of $2I + B$ is cyclotomic. □

Remark 3.2. Let (X, G) be a commutative scheme and let $\mathscr{B}$ be the set of adjacency matrices for relations in G. Suppose H is a closed subset of G, and let $\mathscr{B}'$ be the subset of $\mathscr{B}$ consisting of the adjacency matrices for the elements of H. If $\mathbb{C}H = \mathrm{span}_{\mathbb{C}}(\mathscr{B}')$, then we have that $(\mathbb{C}H, \mathscr{B}')$ is also an integral table algebra with the same degree map as the integral table algebra $(\mathbb{C}G, \mathscr{B})$.

Note that the above proof of Theorem 3.1 does not require the assumption that $\sum_{i=0}^{d} A_i = J$. Therefore, if H is a closed subset of G consisting entirely of relations of valency at most 2, then it follows from proof of Theorem 3.1 that all eigenvalues of elements of $\mathscr{B}'$ are cyclotomic algebraic integers. The last result demonstrates the applicability of the main theorem to other classes of association schemes.

Theorem 3.3. *Let (X, G) be a commutative scheme. Suppose that*

(a) *there exists an integer $k > 4$ such that for all $g \in G$, $n_g \in \{1, 2, 3, k\}$,*
(b) *there is at most one relation in G of valency 3,*
(c) *there is at most one relation in G of valency k.*

Then every character value of (X, G) is a cyclotomic algebraic integer.

Proof. Let $\mathscr{B}$ be the set of adjacency matrices for relations in G. Since we are assuming $k > 4$, it follows that the set H of all relations in G which have valency 1 or 2 forms a closed subset of G. Let $\mathscr{B}'$ be the set of adjacency matrices corresponding to relations in H. Then $(\mathbb{C}H, \mathscr{B}')$ is an integral table algebra for which each basis element has degree 1 or 2. Now by Remark 3.2, all of the eigenvalues of every element of $\mathscr{B}'$ are cyclotomic algebraic integers.

Suppose A is a unique element of $\mathscr{B}$ of degree 3. (Note that the theorem does not require such a basis element to exist.) Let m be the multiplicity of some $C \in \mathscr{B}$ of degree 2 in the decomposition of A^2. Then $2m = [A^2, C] = [AC, A] = 3l$ where l is the multiplicity of A in AC. This shows that whenever $m \neq 0$ occurs, we have $m = 3$ and $l = 2$, and so $AC = A$. On the other hand, suppose that m' is the multiplicity of some $D \in \mathscr{B}$ of degree 1 in A^2. If $m' \neq 0$, then $m' = [A^2, D] = [AD, A] = 3$, and so $AD = A$.

This means that there are three possibilities for A^2: either

(a) $A^2 = 3.1 + 2A$; or
(b) $A^2 = 3.1 + 3D + A$, where D has degree 1; or
(c) $A^2 = 3.1 + 3C$, where C has degree 2.

If case (a) holds, then A satisfies in the polynomial $g(x) = x^2 - 2x - 3$. Therefore, all eigenvalues of A are in the set $\{-1, 3\}$, as desired. If case (b) holds, then $A^3 = 3A + 3DA + A^2 = 6A + A^2$, so the matrix A satisfies in the polynomial $g(x) = x(x - 3)(x + 2)$. Hence all eigenvalues of A are in the set $\{-2, 0, 3\}$ and we are done. Finally, if case (c) occurs, then $A^3 = 3A + 3CA = 6A$, so A satisfies in the polynomial $g(x) = x(x^2 - 6)$ and so all of its eigenvalues are in the set $\{0, \pm\sqrt{6}\}$.

Now let B be a unique element of $\mathscr{B}$ of degree k, assuming such a basis element exists. To complete the proof of the theorem we need only to show that all eigenvalues of B are in a cyclotomic field. But this is a direct consequence of the equality

$$B = J - \left(\sum_{C \in \mathscr{B} - \{B\}} C \right). \qquad \square$$

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