Physics Letters B 760 (2016) 494–497

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Mellin transforming the minimal model CFTs: AdS/CFT at strong curvature

David A. Lowe

Physics Department, Brown University, Providence, RI, 02912, USA

ARTICLE INFO

Article history:
Received 24 May 2016
Accepted 6 July 2016
Available online 14 July 2016
Editor: M. Cvetič

ABSTRACT

Mack has conjectured that all conformal field theories are equivalent to string theories. We explore the example of the two-dimensional minimal model CFTs and confirm that the Mellin transformed amplitudes have the desired properties of string theory in three-dimensional anti-de Sitter spacetime.

1. Introduction

Mack [1,2] has long advocated that Mellin transforming CFT correlators is useful for efficiently understanding the structure of these theories. The examples that have primarily been studied in the past have been higher dimensional CFTs where some kind of perturbative expansion or partial wave expansion is needed to solve for the nontrivial CFT correlation functions.

In the present paper, we consider a two-dimensional CFT where the nontrivial 4-point correlation functions have analytic expressions. These expressions may be written as sums over products of chiral conformal blocks. The Mellin transform of these conformal blocks maps naturally into Koba–Nielsen open string amplitudes [3], for special values of the kinematic variables. This leads us to conjecture that the string theory dual to the CFT is equivalent to an open string description, with many features similar to the KLT construction [4]. The analogs of the Mandelstam kinematic invariants of the boundary S-matrix of the string theory provide coordinates for Mellin space. The analytic expressions for the 4-point functions allow us to study the transform without making perturbative expansions, or partial wave expansions, avoiding subtle issues of convergence.

For the minimal model CFTs in two dimensions, we find that the Mellin representation of the conformal blocks has simple poles along a set of Regge trajectories, with residues polynomial in the kinematic variables. By construction the amplitude satisfies unitarity, factorization and crossing symmetry. These are the necessary and sufficient requirements for the theory to have a dual interpretation as a dual resonance model. This construction should be viewed as a generalization of the Veneziano approach [5] to constructing open string theory from the same basic physical requirements.

The most economical interpretation in terms of on-shell string scattering comes from string theory in a three-dimensional AdS geometry. The general mapping of CFT amplitudes to AdS bulk amplitudes is described in [2]. Here the geometry is curved on scales of order the string length, so there is no low-energy gravity approximation. Nor do we have a perturbative string coupling to control a genus expansion. Nevertheless, the polynomial structure of the residues in the Mellin representation indicates the string representation is local in this three-dimensional spacetime.

2. Mellin transforming the CFT

For a rational two-dimensional conformal field theory, the four-point functions can be written

$$\langle \phi_1(0,0)\phi_2(z,\bar{z})\phi_3(1,1)\phi_4(\infty,\infty)\rangle = \sum_i X_i I_i(z) I_i(\bar{z}) \tag{1}$$

where the sum is over the full set of primary operators, and the I_i are conformal blocks. Here we have used global conformal symmetry to fix the positions of three of the operators.

For the minimal models there exist explicit integral expressions for the $I_i(z)$ derived in [6,7] using the Coulomb gas representation of the conformal field theory. These take the form of chiral correlators

$$I_i(z) = \oint_{C_1} du_1 \cdots \oint_{C_n} du_n \oint_{D_1} dv_1 \cdots \oint_{D_m} dv_m \Big\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) \times V_{\alpha_4}(\infty) \prod_{i=1}^{n} J_-(u_i) \prod_{j=1}^{m} J_+(v_j) \Big\rangle$$

E-mail address: lowe@brown.edu.

http://dx.doi.org/10.1016/j.physletb.2016.07.029

$$= N z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_3\alpha_2} \oint_{C_1} du_1 \cdots \oint_{C_n} du_n \oint_{D_1} dv_1 \cdots \oint_{D_m} dv_m$$
$$\times \prod_{i=1}^{n} u_i^{2\alpha_- \alpha_1}(u_i-1)^{2\alpha_-\alpha_3}(u_i-z)^{2\alpha_-\alpha_2}$$
$$\times \prod_{j=1}^{m} v_j^{2\alpha_+\alpha_1}(v_j-1)^{2\alpha_+\alpha_3}(v_j-z)^{2\alpha_+\alpha_2}$$
$$\times \prod_{i<j}^{n}(u_i-u_j)^{2\alpha_-^2}\prod_{i<j}^{m}(v_i-v_j)^{2\alpha_+^2}\prod_{i,j}(u_i-v_j)^{-2} \qquad (2)$$

The Coulomb gas charges

$$\alpha_{n,m} = \frac{1}{2}(1-n)\alpha_- + \frac{1}{2}(1-m)\alpha_+$$

determine the operator conformal weights

$$\Delta_{n,m} = \frac{1}{4}\left((\alpha_- n - \alpha_+ m)^2 - (\alpha_- + \alpha_+)^2\right)$$

where

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1}$$

and α_0 is determined by the central charge

$$c = 1 - 24\alpha_0^2.$$

In the above N is a normalization constant. The specification of the contours is described in detail in [6,7]. The contours may be chosen as intervals along the real axis. For a given set of operators $V_i(z)$ there is a minimal choice for the set of screening operators $J_\pm$ which yield non-vanishing integrals. The number of independent contours depends on the set of $V_i(z)$.

The derivation of (1) and (2) [6,7] proceeds via an identical analytic continuation to the mapping of tree-level open string amplitudes into closed string amplitudes [4]. The mapping is guaranteed for worldsheets with disc topology, since the process may be viewed either as creation of a closed string from the vacuum (with other vertex operator insertions) or as an open string being created and then subsequently annihilated. This motivates the conjecture that the minimal model CFT can likewise be viewed as arising from a more fundamental chiral description.

Our interest then will be to study the Mellin transform of the basic chiral blocks of the minimal model CFTs in order to study the open string description. One may then reinterpret the two-dimensional CFT amplitudes as a definition of a dual holographic theory in three-dimensional anti-de Sitter spacetime following [2]. The Mellin transformed amplitudes are naturally interpreted as boundary S-matrix elements of the bulk theory, expressed in terms of kinematic variables. Our goal is to study the analytic properties of these Mellin amplitudes, and show that the expected structure of a dual resonance model emerges, in particular Regge trajectories with vanishing dispersion, and interactions polynomial in the kinematic variables.

To define the Mellin transform of $I_i(z)$ we follow Mack's suggestion [2], eqn (69), to consider the chiral transform

$$I_i(z) = \frac{1}{(2\pi i)^2}\int_{-i\infty+c}^{i\infty+c} d\beta_{12} \int_{-i\infty+c'}^{i\infty+c'} d\beta_{23}\, z^{-\beta_{12}}(1-z)^{-\beta_{23}} \hat{M}_i\left(\{\beta_{ij}\}\right) \qquad (3)$$

to define the reduced Mellin amplitude $\hat{M}_i$. Here c and c' are constants chosen so the integral converges. Further details of the interpretation of the Mellin amplitudes can be found in [2].

Our first task is to invert the formula (3) and solve for $\hat{M}_i$. Consider the integrand in (2),

$$J_i(z) = z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_3\alpha_2}\prod_{i=1}^{n} u_i^{2\alpha_-\alpha_1}(u_i-1)^{2\alpha_-\alpha_3}(u_i-z)^{2\alpha_-\alpha_2}$$
$$\times \prod_{j=1}^{m} v_j^{2\alpha_+\alpha_1}(v_j-1)^{2\alpha_+\alpha_3}(v_j-z)^{2\alpha_+\alpha_2}$$
$$\times \prod_{i<j}^{n}(u_i-u_j)^{2\alpha_-^2}\prod_{i<j}^{m}(v_i-v_j)^{2\alpha_+^2}\prod_{i,j}(u_i-v_j)^{-2}$$

This is a special case of the integrand that appears in the general Koba–Nielsen expression for the $(4+m+n)$-open string scattering amplitudes [3].

The Mellin transform of open string amplitudes has been studied by Stieberger and Taylor [8]. In order to write their expression in a more symmetric form, it is written as a distribution to be integrated over an overcomplete set of cross-ratios. Let us label the $N = (4+m+n)$ points by z_i. Defining

$$u_{i,j} = \frac{(z_i - z_j)(z_{i-1}-z_{j+1})}{(z_i - z_{j+1})(z_{i-1}-z_j)}$$

we obtain a basis for the $N(N-3)/2$ anharmonic ratios. In this formula (i,j) run over pairs conjugate to the kinematic channels, corresponding to the range $i=2, j=3,\cdots,N-1$ and $i=3,\cdots,N-1<j=4,\cdots,N$. The notation P denotes this set of channels which corresponds to the set of independent kinematic invariants of the string amplitude $s_{i,j}$ which are analogs of the usual flat spacetime Mandelstam variables $\left(k_i+\cdots+k_j\right)^2$.

The cross-ratios satisfy constraint equations, leaving only $N-3$ free variables to be integrated over. The main result found in [8] is

$$\prod_{(i,j)\in P} u_{i,j}^{n_{i,j}}\theta(1-u_{i,j})\delta\left(\{u_{k,l}\}\right)$$
$$= \frac{1}{(2\pi i)^m}\left(\prod_{(i,j)\in P}\int_{-i\infty+c}^{i\infty+c} ds_{i,j}u_{i,j}^{-s_{i,j}}\right)B_N\left(\{s_{k,l}\},\{n_{k,l}\}\right)$$

Here the $n_{i,j}$ are integers subject to the constraint $n_{i,j} = n_{j+1,i-1}$ and $n_{k,N} = n_{1,k-1}$. The constraint delta function is

$$\delta\left(\{u_{i,j}\}\right) = \prod_P{}'\delta\left(u_P - 1 + \prod_{\tilde{P}} u_{\tilde{P}}\right)$$

where the prime denotes the exclusion of the $(2,k)$ channels and $\tilde{P}$ denotes the set of channels incompatible with P. Incompatible channels cannot appear simultaneously with a given channel P and satisfy the condition that if $u_P = 0$ then $u_{\tilde{P}} = 1$. For more details on the definition of incompatible channels see [8] and references therein.

The $B_N\left(\{s_{n,l}\},\{n_{k,l}\}\right)$ are just the Koba–Nielsen open string amplitudes [3] written as a Mellin transform

$$B_N\left(\{s_{k,l}\},\{n_{k,l}\}\right) = \left(\prod_{i,j\in P}\int_0^\infty du_{i,j}u_{i,j}^{s_{i,j}-1+n_{i,j}}\theta(1-u_{i,j})\right)$$
$$\times\,\delta\left(\{u_{k,l}\}\right)$$

For the case at hand, considering the transform of a conformal block in a minimal model, only two of the $s_{i,j}$ are independent, corresponding to the β_{12} and β_{23} of (3). The remaining $s_{i,j}$ are fixed by the exponents in (2) (after performing the change of variables from z_i to $u_{i,j}$ [8]).

From the known properties of the Koba–Nielsen amplitudes, we conclude the general minimal model Mellin transforms have infinite towers of single poles in the independent kinematic variables. Moreover the residues at these poles are polynomial in the other variables.

3. Example: critical Ising model

To provide an explicit example of the above correspondence let us consider the conformal blocks that appear in the 4-spin correlator in the Ising model. The conformal blocks [9] are

$$\mathcal{F}_\pm(x) = \frac{1}{\sqrt{2}}\,(x(1-x))^{-1/8}\sqrt{1\pm\sqrt{1-x}}$$

To define the Mellin transformed open string amplitude we consider the integral

$$B_3^\pm(\beta_{12},\beta_{23}) = \int_0^1 dx\, x^{\beta_{12}-1}(1-x)^{\beta_{23}-1}\mathcal{F}_\pm(x)$$

This gives the expression

$$B_3^+ = \sqrt{2}\Gamma\left(\beta_{12}-\frac{1}{8}\right)\Gamma\left(2\beta_{23}-\frac{1}{4}\right) \times {}_2\tilde{F}_1\left(\frac{5}{8}-\beta_{12}, 2\beta_{23}-\frac{1}{4}; \beta_{12}+2\beta_{23}-\frac{3}{8}; -1\right)$$

The analytic structure is simple. The regularized hypergeometric functions ${}_2\tilde{F}_1$ have no poles as a function of β_{12} and β_{23}. Single poles appear along the two Regge trajectories (with m,n positive integers)

$$\beta_{12} = \frac{9}{8}-n$$
$$\beta_{23} = \frac{5}{8}-\frac{m}{2}$$

with polynomial residues in the other variables.

For the other block we get

$$B_3^- = \sqrt{2}\Gamma\left(\beta_{12}+\frac{3}{8}\right)\Gamma\left(2\beta_{23}-\frac{1}{4}\right) \times {}_2\tilde{F}_1\left(\frac{9}{8}-\beta_{12}, 2\beta_{23}-\frac{1}{4}; \beta_{12}+2\beta_{23}+\frac{1}{8}; -1\right)$$

Single poles appear at

$$\beta_{12} = \frac{5}{8}-n$$
$$\beta_{23} = \frac{5}{8}-\frac{m}{2}$$

with polynomial residues in the other variables. These results indicate infinite towers of massive states contribute to the amplitude in the string theory dual.

4. Discussion

Our main result is that minimal model correlation functions can be viewed as a computed by a kind of open string theory with meromorphic Mellin amplitudes. This is in accord with Mack's conjecture that all conformal field theories are dual to string theories. In the present paper, the string theory is to be treated at the purely classical level, with no sum over topologies beyond the disk. This may be viewed as a limit where the string coupling goes to zero. Alternatively, the string worldsheet theory may be viewed as a chiral gravity theory.

One can nevertheless propose a duality between the minimal models and some theory containing gravity in three-dimensional anti-de Sitter spacetime. Each primary of the conformal group maps to a bulk field. For scalar operators, the mass of the field in the bulk is related to the conformal dimension via

$$\Delta = \frac{1}{2}\pm\sqrt{\frac{1}{4}+m^2R^2} \tag{4}$$

where R is the AdS radius of curvature in string units. Analogous formulas exist for general spin. The on-shell boundary S-matrix written in terms of kinematic variables for this bulk theory may then be identified with the Mellin amplitude of the boundary conformal field theory. This is described in detail in [2]. The infinite sequences of poles in the chiral Mellin amplitudes considered above leads to the prediction that the dual bulk string theory contains infinite towers of massive string states.

This leads to the dramatic conclusion that the bulk theory may be viewed as a local theory. The Mellin amplitudes are meromorphic with residues that are polynomial in the other kinematic variables, as is the case in the familiar critical string theory amplitudes. Of course the underlying CFT structure guarantees crossing symmetry and factorization. But bulk locality in the sense of this analytic structure comes as a complete surprise. In the past, locality would emerged only in a large N limit where the bulk correlators have a perturbative expansion [10–12].

The minimal model CFTs have only a finite number of primary fields. Each of these primary fields nevertheless has an infinite number of descendant fields, obtained by the action of the Virasoro algebra. Null states and their descendants may be truncated, but infinite towers of states remain. The Mellin amplitude contains a Regge trajectory associated with each primary, with the higher satellite poles corresponding to the descendant states.

It has been conjectured [13] that the simplest minimal model, the critical Ising model, is dual to Einstein gravity in three-dimensional anti-de Sitter spacetime. The correspondence there relied on a matching of partition functions. At first sight, this seems surprising in the present context, because the reduced Mellin amplitudes have poles along entire Regge trajectories. Since the Einstein gravity theory is to be studied at strong curvature and large Newton constant, the matching of the proposal is hard to check. It might be that the Einstein action is sufficient to describe black hole states with the properties of the Regge trajectories described above.

The present results should be viewed as complementary to the results of Mack [2,1], where an expansion in terms of Euclidean partial waves leads to similar conclusions regarding the analytic structure of the Mellin amplitudes. The new feature in the present work is the fact that a conformal block sums over an infinite number of Euclidean partial waves. Nevertheless, the analytic structure is preserved. We also note our conjecture that the minimal models have an open string interpretation in three-dimensional anti-de Sitter spacetime goes somewhat beyond the original conjecture of Mack [1], which the additional assumption that the boundary CFT descended from a higher dimensional CFT.

Starting with a local bulk theory with a perturbative expansion, it was shown in [14] that the Mellin transform provides a useful description of the bulk correlators. In the present work there is no perturbative expansion for the bulk correlators. In fact the entire bulk description is made largely at the level of mappings of operators according to representations of the conformal group. Given the AdS radius of curvature will be of order one in string units, for minimal model duals, there will not be a low energy limit where gravity decouples. However it seems likely a version of string field theory will be applicable, and these results indicate the interactions will have a local interpretation.

Acknowledgements

I thank Marcus Spradlin and Steven Avery for helpful discussions. The research of D.L. was supported in part by DOE grant DE-SC0010010.

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