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Modular bootstrap in Liouville field theory

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ABSTRACT

The modular matrix for the generic 1-point conformal blocks on the torus is expressed in terms of the fusion matrix for the 4-point blocks on the sphere. The modular invariance of the toric 1-point functions in the Liouville field theory with DOZZ structure constants is proved.

1. Introduction

The basic consistency conditions for any CFT on closed surfaces are the crossing symmetry of the 4-point function on the sphere and the modular invariance of the 1-point function on the torus [1]. In the case of the Liouville field theory defined by the DOZZ structure constants [2,3] the first issue was addressed by Ponsot and Teschner [4,5]. They derived a system of functional equations for the braiding and the fusion matrices and constructed its explicit solutions. The problem of crossing symmetry in the Liouville field theory can then be reduced to a certain orthogonality relations satisfied by the Barnes functions [6]. The exact form of the braiding and the fusion matrices can be also derived by direct calculations of the exchange relation of chiral vertex operators in the free field representation [7,8] (see also [9] for an earlier construction). Up to our knowledge the second consistency condition has not yet been analyzed in the Liouville field theory.

Although derived in the context of the Liouville field theory the results of [4,5] and [7,8] are more universal. From the point of view of the Moore–Seiberg approach [10] to classification of rational CFT models the braiding and the fusion matrices found in [4,5] are two of the generators of the duality grupoid describing the chiral structure of any CFT with the Liouville continuous spectrum. The only missing generator is the modular matrix relating 1-point conformal blocks on tori with modular parameters τ and $-\frac{1}{\tau}$.

Our aim in the present Letter is to derive an explicit form of the modular matrix in the case of Liouville spectrum and to prove the modular invariance of the Liouville 1-point functions on the torus. The first result is based on recently discovered relations [11,12] between 1-point conformal blocks on the torus and 4-point conformal blocks on the sphere inspired by a corresponding relation between Liouville correlation functions first proposed by Fateev, Litvinov, Neveu and Onofri in [13]. The second follows from the relation between DOZZ structure constants also suggested by the FLNO relation.

There are at least three problems which are natural continuation of the present work. The first one is a more detailed analysis of the Liouville modular grupoid. Since the Liouville spectrum is continuous the generators of the modular grupoid can be analytically continued well outside the spectrum. For instance in the case of a degenerate weight the integral over continuous spectrum localizes giving rise to a finite dimensional fusion matrix [14,15]. The question arises whether any (irreducible) modular grupoid for Virasoro conformal blocks can be obtained by an analytic continuation of the Liouville one. The second is to extend the results of the present work to the H_3^+ WZNW model [16,17]. Finally, the third problem is to complete the verification of the consistency conditions [18] for the Liouville field theory on bordered surfaces.

2. Conformal blocks

The 1-point toric and the 4-point spherical conformal blocks are defined by

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$$\mathcal{F}^{\lambda}_{c,\Delta}(q) = q^{\Delta-\frac{c}{24}} \sum_{n=0}^{\infty} q^n F^{\lambda,n}_{c,\Delta}, \tag{1}$$

$$F^{\lambda,n}_{c,\Delta} = \sum_{n=|M|=|N|} \rho(\nu_{\Delta,N}, \nu_\lambda, \nu_{\Delta,M}) \big[B^n_{c,\Delta}\big]^{MN}, \tag{2}$$

and

$$\mathcal{F}_{c,\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(z) = z^{\Delta-\Delta_2-\Delta_1}\left(1 + \sum_{n\in\mathbb{N}} z^n F^n_{c,\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}\right), \tag{3}$$

$$F^n_{c,\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} = \sum_{n=|M|=|N|} \rho(\nu_4, \nu_3, \nu_{\Delta,M}) \big[B^n_{c,\Delta}\big]^{MN} \rho(\nu_{\Delta,N}, \nu_2, \nu_1), \tag{4}$$

respectively. $\rho(\xi_3, \xi_2, \xi_1)$ denotes the 3-point spherical conformal block and $[B^n_{c,\Delta}]^{MN}$ is the inverse of the Gram matrix

$$\big[B^n_{c,\Delta}\big]_{MN} = \langle \nu_{\Delta,N} | \nu_{\Delta,M}\rangle, \qquad |M| = |N| = n,$$

calculated in the standard basis of the Verma module $\mathcal{V}_\Delta$:

$$\nu_{\Delta,M} = L_{-M}\nu_\Delta \equiv L_{-m_j} \cdots L_{-m_1}\nu_\Delta,$$

with $M = \{m_1, m_2, \ldots, m_j\} \subset \mathbb{N}$ standing for an arbitrary ordered set of indices $m_j \leqslant \cdots \leqslant m_2 \leqslant m_1$ and $\nu_\Delta \in \mathcal{V}_\Delta$ being the highest weight state. In the case of torus the 1-point elliptic conformal block $\mathcal{H}^{\lambda}_{c,\Delta}(\tilde{q})$ is defined by

$$\mathcal{F}^{\lambda}_{c,\Delta}(\tilde{q}) = \tilde{q}^{\Delta-\frac{c-1}{24}} \eta(\tilde{q})^{-1} \mathcal{H}^{\lambda}_{c,\Delta}(\tilde{q}), \tag{5}$$

where the elliptic variable $\tilde{q}$ is related to the torus moduli parameter τ by $\tilde{q} = \mathrm{e}^{2\pi i\tau}$ and $\eta(\tilde{q})$ is the Dedekind eta function.

The 4-point elliptic conformal block on the sphere $\mathcal{H}_\Delta\big[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\big](q)$ is given by [19]:

$$\mathcal{F}_{\Delta}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(z) = (16q)^{\Delta-\frac{c-1}{24}} z^{\frac{c-1}{24}-\Delta_1-\Delta_2}(1-z)^{\frac{c-1}{24}-\Delta_2-\Delta_3}\theta_3^{\frac{c-1}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\mathcal{H}_\Delta\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(q). \tag{6}$$

The variable q is related to the moduli parameter z of the 4-punctured sphere by

$$q(z) = \mathrm{e}^{i\pi\tau}, \quad \tau(z) = i\frac{K(1-z)}{K(z)}, \tag{7}$$

where $K(z)$ is the complete elliptic integral of the first kind.

3. Modular matrix

Our starting point are the identities conjectured in [11] and proved in our previous paper [12]:

$$\mathcal{H}^{\lambda}_{c,\Delta_\alpha}\big(q^2\big) = \mathcal{H}_{c',\Delta'_{\alpha'}}\begin{bmatrix} \frac{1}{2b'} & \frac{\lambda}{\sqrt{2}} \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}(q), \quad b' = \frac{b}{\sqrt{2}},\ \alpha' = \sqrt{2}\alpha, \tag{8}$$

and

$$\mathcal{H}^{\lambda}_{c,\Delta_\alpha}\big(q^2\big) = \mathcal{H}_{c',\Delta'_{\alpha'}}\begin{bmatrix} \frac{b'}{2} & \frac{\lambda}{\sqrt{2}} \\ \frac{b'}{2} & \frac{b'}{2} \end{bmatrix}(q), \quad b' = \sqrt{2}b,\ \alpha' = \sqrt{2}\alpha, \tag{9}$$

where

$$c = 1 + 6\left(b + \frac{1}{b}\right)^2, \qquad \Delta_\alpha = \frac{1}{4}\left(b + \frac{1}{b}\right)^2 - \frac{1}{4}\alpha^2.$$

Let us observe that the crossing symmetry transformation $z \to 1 - z$ on the sphere implies the modular transformation $\tau \to -\frac{1}{\tau}$ for $\tau(z)$ given by (7) and therefore the modular transformation of the elliptic variable $\tilde{q} = q^2$ of the torus. It follows that the crossing symmetry on the sphere on the r.h.s. of (8) and (9) can be interpreted as the modular transformation of the toric 1-point function on the l.h.s. of these equations. This yields the relation between the modular matrix for the 1-point blocks on the torus defined by

$$\mathcal{F}^{\lambda}_{c,\Delta_s}\big(q(\tau)\big) = (-i\tau)^{-\Delta_\lambda} \int_{i\mathbb{R}_+} \frac{\mathrm{d}\lambda_t}{2i} \mathsf{S}^{c,\lambda}_{\lambda_s\lambda_t} \mathcal{F}^{\lambda}_{c,\Delta_t}\left(q\left(-\frac{1}{\tau}\right)\right) \tag{10}$$

and the fusion matrix for the spherical 4-point blocks

$$\mathcal{F}_{c,\Delta_s}\begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(z) = \int_{i\mathbb{R}_+} \frac{\mathrm{d}\lambda_t}{2i} \mathsf{F}^{c}_{\lambda_s\lambda_t}\begin{bmatrix} \lambda_3 & \lambda_2 \\ \lambda_4 & \lambda_1 \end{bmatrix} \mathcal{F}_{c,\Delta_t}\begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_4 & \Delta_3 \end{bmatrix}(1-z). \tag{11}$$

Indeed using Eqs. (5), (6), (8), (11) and the relations:

$$\eta\big(\mathrm{e}^{-\frac{2\pi i}{\tau}}\big) = \sqrt{-i\tau}\,\eta\big(\mathrm{e}^{2\pi i\tau}\big), \qquad \theta_3\big(\mathrm{e}^{-\frac{\pi i}{\tau}}\big) = \sqrt{-i\tau}\,\theta_3\big(\mathrm{e}^{\pi i\tau}\big), \tag{12}$$

one gets

$$\mathsf{S}^{c,\lambda}_{\lambda_s\lambda_t} = 2^{2(\lambda_s^2-\lambda_t^2)+\frac{1}{2}}\,\mathsf{F}^{c'}_{\sqrt{2}\lambda_s\sqrt{2}\lambda_t}\begin{bmatrix} \frac{1}{2b'} & \frac{\lambda}{\sqrt{2}} \\ \frac{1}{2b'} & \frac{1}{2b'} \end{bmatrix}, \quad b' = \frac{b}{\sqrt{2}}, \tag{13}$$

or (using the relation (9))

$$\mathsf{S}^{c,\lambda}_{\lambda_s\lambda_t} = 2^{2(\lambda_s^2-\lambda_t^2)+\frac{1}{2}}\,\mathsf{F}^{c'}_{\sqrt{2}\lambda_s\sqrt{2}\lambda_t}\begin{bmatrix} \frac{b'}{2} & \frac{\lambda}{\sqrt{2}} \\ \frac{b'}{2} & \frac{b'}{2} \end{bmatrix}, \quad b' = \sqrt{2}b. \tag{14}$$

Some remarks concerning the application of formula (11) in the derivation above are in order. Let us consider the fusion matrix for $\lambda_1 = \lambda_3 = \lambda_4 = \eta$, $\lambda_2 = \lambda$. In the present parametrization of conformal weights it reads[1] [4,5]:

$$\begin{aligned} \mathsf{F}^{c}_{\lambda_s\lambda_t}\begin{bmatrix} \eta & \lambda \\ \eta & \eta \end{bmatrix} &= \frac{\Gamma_b(\frac{Q}{2}-\eta-\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}-\eta+\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}-\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda_t}{2})}{\Gamma_b(\frac{Q}{2}-\eta-\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}-\eta+\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}-\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda_s}{2})} \\ &\quad\times \frac{\Gamma_b(\frac{Q}{2}-\frac{\lambda}{2}-\frac{\eta}{2}-\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda}{2}-\frac{\eta}{2}-\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}-\frac{\lambda}{2}-\frac{\eta}{2}+\frac{\lambda_t}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda}{2}-\frac{\eta}{2}+\frac{\lambda_t}{2})}{\Gamma_b(\frac{Q}{2}-\frac{\lambda}{2}-\frac{\eta}{2}-\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda}{2}-\frac{\eta}{2}-\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}-\frac{\lambda}{2}-\frac{\eta}{2}+\frac{\lambda_s}{2})\Gamma_b(\frac{Q}{2}+\frac{\lambda}{2}-\frac{\eta}{2}+\frac{\lambda_s}{2})} \\ &\quad\times \frac{\Gamma_b(Q+\lambda_s)\Gamma_b(Q-\lambda_s)}{\Gamma_b(\lambda_t)\Gamma_b(-\lambda_t)}\,I^{c}_{\lambda_s\lambda_t}\begin{bmatrix} \eta & \lambda \\ \eta & \eta \end{bmatrix}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} I^{c}_{\lambda_s\lambda_t}\begin{bmatrix} \eta & \lambda \\ \eta & \eta \end{bmatrix} &= \int\limits_{i\mathbb{R}} \frac{d\tau}{i}\Bigg[\frac{S_b(\frac{Q}{2}-\frac{\lambda}{2}+\tau)S_b(\frac{Q}{2}+\frac{\lambda}{2}+\tau)}{S_b(Q-\frac{\lambda_s}{2}+\frac{\eta}{2}+\tau-0^+)S_b(Q+\frac{\lambda_s}{2}+\frac{\eta}{2}+\tau-0^+)} \\ &\quad\times \frac{S_b(\frac{Q}{2}-\frac{\eta}{2}+\tau)S_b(\frac{Q}{2}+\frac{\eta}{2}+\tau)}{S_b(Q-\frac{\lambda_t}{2}-\frac{\eta}{2}+\tau-0^+)S_b(Q+\frac{\lambda_t}{2}-\frac{\eta}{2}+\tau-0^+)}\Bigg]. \end{aligned} \tag{16}$$

The relations (11), (15) and (16) were derived for conformal weights from the spectrum of the Liouville field theory, $\lambda_s, \lambda_t, \lambda, \eta \in i\mathbb{R}$, while in our derivation analytic continuations to $\eta = \frac{1}{2b}$ and $\eta = \frac{b}{2}$ are required.

Let us start with the analytic continuation of $I^{c}_{\lambda_s\lambda_t}\big[\begin{smallmatrix} \eta & \lambda \\ \eta & \eta \end{smallmatrix}\big]$. For $\lambda_s, \lambda_t, \lambda, \eta \in i\mathbb{R}$ the integrand in (16) has poles (coming from the poles of the functions S_b in the numerator) located at $\Re\tau < 0$ (to the left from the integration contour) and poles coming from the zeroes of the S_b functions in the denominator, located at $\Re\tau > 0$ (to the right from the integration contour). Some of these poles move when we analytically continue in η. If they cross the imaginary axis the process of analytic continuation requires an appropriate smooth deformation of the contour of integration in (16). As was discussed in [5] such deformation is possible unless there are some poles with locations coinciding at the terminal value of η, which "pinch" the τ integration contour in between. This happens for instance when the terminal value of η corresponds to a degenerate weight, $\eta = mb + \frac{n}{b}$, $m, n \in \mathbb{N}$, but neither for $\eta = \frac{1}{2b}$ nor for $\eta = \frac{b}{2}$. Thus $I^{c}_{\lambda_s\lambda_t}\big[\begin{smallmatrix} \eta & \lambda \\ \eta & \eta \end{smallmatrix}\big]$ remains regular for $\lambda_t \in i\mathbb{R}$ while $\eta \to \frac{1}{2b}$ or $\eta \to \frac{b}{2}$.

The product of Γ_b functions appearing in (15) has poles moving with η on both sides of the contour:

$$\lambda_t = \pm 2\left(\frac{Q}{2} - \eta + mb + nb^{-1}\right), \qquad \lambda_t = \pm 2\left(\frac{Q}{2} - \frac{\eta}{2} - \frac{\lambda}{2} + mb + nb^{-1}\right), \qquad \lambda_t = \pm 2\left(\frac{Q}{2} - \frac{\eta}{2} + \frac{\lambda}{2} + mb + nb^{-1}\right).$$

For $\eta \to \frac{1}{2b}$ and for $\eta \to \frac{b}{2}$ none of these poles crosses the imaginary axis. Thus the analytic continuation of the fusion formula (11) from imaginary values $\eta \in i\mathbb{R}$ to $\eta = \frac{1}{2b}$ or to $\eta = \frac{b}{2}$ does not change the integration contour. This justifies our definition of the modular matrix (10). It also implies that the fusion matrices on the right-hand side of Eqs. (13), (14) are just analytic continuation of the fusion matrices from the Liouville physical weights to $\eta = \frac{1}{2b}$ and to $\eta = \frac{b}{2}$.

Let us finally note that parallel reasoning with respect to the λ_s variable shows that the fusion matrix $\mathsf{F}^{c}_{\lambda_s\lambda_t}\big[\begin{smallmatrix} \eta & \lambda \\ \eta & \eta \end{smallmatrix}\big]$, multiplied by its conjugation and integrated over λ_s enjoys the usual orthogonality properties, which ensure the crossing symmetry

$$\big\langle \phi_\eta\phi_\eta\phi_\lambda(z)\phi_\eta\big\rangle^c = \big\langle \phi_\eta\phi_\eta\phi_\lambda(1-z)\phi_\eta\big\rangle^c \tag{17}$$

of the corresponding four-point Liouville correlation function:

[1] Definitions and discussion of some basic properties of the functions Γ_b and S_b and Υ_b appearing below can be found in [4,5]; see also Appendix A. For the detailed discussion of the Barnes special functions the reader may consult the papers [20,21].

$$\langle\phi_\eta\phi_\eta\phi_\lambda(z)\phi_\eta\rangle^c = \left|\left(z(1-z)\right)^{-\frac{Q^2}{4}+\frac{\eta^2}{4}+\frac{\lambda^2}{4}}\left(\theta_3(q)\right)^{-Q^2+3\eta^2+\lambda^2}\right|^2 \times \int\limits_{i\mathbb{R}^+}\frac{d\lambda_s}{2i}\left|(16q)^{-\frac{\lambda_s^2}{4}}H_{c,\Delta}\begin{bmatrix}\eta & \lambda\\ \eta & \eta\end{bmatrix}(q)\right|^2 C_c(-\eta,\eta,\lambda_s)C_c(-\lambda_s,\lambda,\eta). \tag{18}$$

4. Modular invariance

In this section we shall prove that for $\lambda \in i\mathbb{R}$ the Liouville 1-point functions on the torus satisfy the modular invariance condition [1]:

$$\langle\phi_\lambda\rangle_{-\frac{1}{\tau}} = |\tau|^{2\Delta_\lambda}\langle\phi_\lambda\rangle_\tau. \tag{19}$$

In the Liouville field theory the 1-point function can be expressed in terms of the elliptic blocks as follows:

$$\langle\phi_\lambda\rangle_\tau = \int\limits_{i\mathbb{R}^+}\frac{d\lambda_s}{2i}\left|\tilde{q}^{-\frac{\lambda_s^2}{4}}\eta(\tilde{q})^{-1}\mathcal{H}^\lambda_{c,\Delta_s}(\tilde{q})\right|^2 C_c(-\lambda_s,\lambda,\lambda_s), \tag{20}$$

where $\tilde{q} = q^2 = e^{2\pi i\tau}$ and the DOZZ structure constants are given by

$$C_c(\lambda_1,\lambda_2,\lambda_3) = \left[\pi\mu\gamma\left(b^2\right)b^{2-2b^2}\right]^{-\frac{1}{2b}(\lambda_3+\lambda_2+\lambda_1+Q)}\frac{\Upsilon_b(b)\Upsilon_b(Q+\lambda_3)\Upsilon_b(Q+\lambda_2)\Upsilon_b(Q+\lambda_1)}{\Upsilon_b(\frac{Q+\lambda_3+\lambda_2+\lambda_1}{2})\Upsilon_b(\frac{Q+\lambda_3+\lambda_2-\lambda_1}{2})\Upsilon_b(\frac{Q+\lambda_3-\lambda_2+\lambda_1}{2})\Upsilon_b(\frac{Q-\lambda_3+\lambda_2+\lambda_1}{2})}.$$

Using the explicit form of the modular matrix for the Liouville spectrum $\lambda, \lambda_s \in i\mathbb{R}$ (13) one could in principle analyze the behavior of the 1-loop function by direct calculations.

There is however a simpler derivation suggested by the relation between the 1-point Liouville function on the torus (20) and the 4-point Liouville function on the sphere (18) first proposed by Fateev, Litvinov, Neveu and Onofri in [13]. It should be stressed that the FLNO relation was the original inspiration for relations between conformal blocks (8), (9) [11,12]. So it was for the following relations between the Liouville structure constants:

$$C_c(-\lambda_s,\lambda,\lambda_s) = 16^{-\lambda_s^2}g_1(\lambda,b)C_{c'}\left(-\frac{1}{2b'},\frac{1}{2b'},\sqrt{2}\lambda_s\right)C_{c'}\left(-\sqrt{2}\lambda_s,\frac{\lambda}{\sqrt{2}},\frac{1}{2b'}\right),\quad b'=\frac{b}{\sqrt{2}}, \tag{21}$$

$$C_c(-\lambda_s,\lambda,\lambda_s) = 16^{-\lambda_s^2}g_2(\lambda,b)C_{c'}\left(-\frac{b'}{2},\frac{b'}{2},\sqrt{2}\lambda_s\right)C_{c'}\left(-\sqrt{2}\lambda_s,\frac{\lambda}{\sqrt{2}},\frac{b'}{2}\right),\quad b'=\sqrt{2}b, \tag{22}$$

where

$$g_1(\lambda,b) = \left[\pi\mu\gamma\left(b^2\right)b^{2-2b^2}\right]^{-\frac{1}{b}(\frac{Q}{2}+\frac{\lambda}{2})}\left[\pi\mu\gamma\left(b'^2\right)b'^{2-2b'^2}\right]^{\frac{1}{b'}(Q'+\frac{1}{4b'}+\frac{\lambda}{2\sqrt{2}})} \times 2^{\frac{b^2}{2}+\frac{2}{b^2}-\frac{3}{4}+\frac{3b}{4}\lambda+\frac{1}{2b}\lambda+\frac{1}{2}\lambda^2}b^{6-\frac{4}{b^2}}\gamma^{-2}\left(b^{-2}\right)\frac{\Upsilon_b(\frac{b}{2})}{\Upsilon_b(b)}\frac{\Upsilon_b(\frac{1}{2b}-\frac{\lambda}{2})}{\Upsilon_b(\frac{Q}{2}+\frac{\lambda}{2})},\quad b'=\frac{b}{\sqrt{2}},$$

$$g_2(\lambda,b) = \left[\pi\mu\gamma\left(b^2\right)b^{2-2b^2}\right]^{-\frac{1}{b}(\frac{Q}{2}+\frac{\lambda}{2})}\left[\pi\mu\gamma\left(b'^2\right)b'^{2-2b'^2}\right]^{\frac{1}{b'}(Q'+\frac{b'}{4}+\frac{\lambda}{2\sqrt{2}})} \times 2^{2b^2+\frac{1}{2b^2}-\frac{3}{4}+\frac{3}{4b}\lambda+\frac{b}{2}\lambda+\frac{\lambda^2}{2}}b^{4b^2-6}\gamma^{-2}\left(b^2\right)\frac{\Upsilon_b(\frac{1}{2b})}{\Upsilon_b(\frac{1}{b})}\frac{\Upsilon_b(\frac{b}{2}-\frac{\lambda}{2})}{\Upsilon_b(\frac{Q}{2}+\frac{\lambda}{2})},\quad b'=\sqrt{2}b.$$

The relations above can be obtained using the following identities for the Υ-function [13]:

$$\Upsilon_b(2x) = 2^{4(x-Q/4)^2}\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)\Upsilon_b(x+\frac{1}{2}b^{-1})\Upsilon_b(x+\frac{1}{2}Q)}{\Upsilon_b^2(\frac{1}{4}Q)\Upsilon_b^2(\frac{1}{4}Q+\frac{1}{2}b)} = 2^{4x(x-\frac{1}{2}Q)+1}\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)\Upsilon_b(x+\frac{1}{2}b^{-1})\Upsilon_b(x+\frac{1}{2}Q)}{\Upsilon_b(\frac{1}{2}b)\Upsilon_b(\frac{1}{2}b^{-1})},$$

$$\Upsilon_{\frac{b}{\sqrt{2}}}(x\sqrt{2}) = 2^{x(x-\frac{1}{b}-\frac{1}{2}b)+\frac{1}{2}}\Upsilon_{\frac{b}{\sqrt{2}}}\left(\frac{b}{\sqrt{2}}\right)\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)}{\Upsilon_b(\frac{1}{2}b)\Upsilon_b(b)},$$

$$\Upsilon_{b\sqrt{2}}(x\sqrt{2}) = 2^{x(x-\frac{1}{2b}-b)+\frac{1}{2}}\Upsilon_{b\sqrt{2}}\left(\frac{b^{-1}}{\sqrt{2}}\right)\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b^{-1})}{\Upsilon_b(\frac{1}{2}b^{-1})\Upsilon_b(b^{-1})}. \tag{23}$$

For completeness we present a derivation of these formulae in Appendix A. Relations (8) and (21) imply:

$$\langle\phi_\lambda\rangle^c_\tau = f(\lambda,q,b)g_1(\lambda,b)\left\langle\phi_{\frac{1}{2b'}}\phi_{\frac{1}{2b'}}\phi_{\frac{\lambda}{\sqrt{2}}}(z)\phi_{\frac{1}{2b'}}\right\rangle^{c'},\quad b'=\frac{b}{\sqrt{2}}, \tag{24}$$

while (9) and (22) yield:

$$\langle\phi_\lambda\rangle^c_\tau = f\left(\lambda,q,b^{-1}\right)g_2(\lambda,b)\left\langle\phi_{\frac{b'}{2}}\phi_{\frac{b'}{2}}\phi_{\frac{\lambda}{\sqrt{2}}}(z)\phi_{\frac{b'}{2}}\right\rangle^{c'},\quad b'=\sqrt{2}b, \tag{25}$$

where

$$f(\lambda,q,b)=\left|\eta\left(q^2\right)\left(z(1-z)\right)^{-\frac{b^2}{8}-\frac{3}{8b^2}-\frac{1}{2}+\frac{\lambda^2}{8}}\left(\theta_3(q)\right)^{-\frac{b^2}{2}-\frac{1}{2b^2}-2+\frac{\lambda^2}{2}}\right|^{-2}=\left|\eta\left(q^2\right)\left(\theta_2(q)\theta_4(q)\right)^{-\frac{b^2}{2}-\frac{3}{2b^2}-2+\frac{\lambda^2}{2}}\left(\theta_3(q)\right)^{b^{-2}}\right|^{-2}. \tag{26}$$

Note that (25) is the original FLNO relation of [13]. Formulae (9) and (22) provide a simple proof of this relation. Relation (24) is new but of the same origin.

Using (24) and the crossing symmetry of the 4-point function (17) one can reduce the modular symmetry condition (19) to the relation

$$f\left(\lambda,\mathrm{e}^{-i\pi\frac{1}{\tau}},b\right)=|\tau|^{\frac{b^2}{2}+\frac{1}{2b^2}+1-\frac{\lambda^2}{2}}f\left(\lambda,\mathrm{e}^{i\pi\tau},b\right)$$

which can be easily verified using formulae (12). This completes our proof of the modular invariance in the Liouville field theory.

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Appendix A. Some identities satisfied by the Barnes functions

For $\Re s>2$ the Barnes double zeta function can be defined as

$$\zeta_b(x;s)=\sum_{n,m=0}^{\infty}\left(x+mb+nb^{-1}\right)^{-s}. \tag{A.1}$$

Let us denote:

$$F(b,x;s)=\zeta_b(x;s)-\zeta_b(Q/2;s).$$

With a help of the Mellin transform $a^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\,t^{s-1}\mathrm{e}^{-at}$ we get

$$F(b,x;s)=\Gamma(1-s)\int\limits_{\mathcal{C}}\frac{\mathrm{d}t}{2\pi it}(-t)^s\frac{\mathrm{e}^{-tx}-\mathrm{e}^{-tQ/2}}{(1-\mathrm{e}^{-tb})(1-\mathrm{e}^{-t/b})},$$

where the integration contour $\mathcal{C}$ surrounds (in the positive direction) the cut of the $(-t)^s$ function which is chosen along the positive real semi-axis. The last expression is valid also for $\Re s<2$. Since

$$\Gamma(1-s)\int\limits_{\mathcal{C}}\frac{\mathrm{d}t}{2\pi it}(-t)^s\mathrm{e}^{-t}=1,\qquad \int\limits_{\mathcal{C}}\frac{\mathrm{d}t}{2\pi it}(-t)^{s-1}=0$$

(the last formula holds for $\Re s<1$) one has

$$F(b,x;s)=\frac{1}{2}\left(\frac{Q}{2}-x\right)^2+\Gamma(1-s)\int\limits_{\mathcal{C}}\frac{\mathrm{d}t}{2\pi it}(-t)^s\left[\frac{\mathrm{e}^{-tx}-\mathrm{e}^{-tQ/2}}{(1-\mathrm{e}^{-tb})(1-\mathrm{e}^{-t/b})}-\frac{Q/2-x}{t}-\frac{1}{2}(Q/2-x)^2\mathrm{e}^{-t}\right]. \tag{A.2}$$

Formula (A.2) is valid also for s close to 0 and gives

$$F(b,x;0)=\frac{1}{2}\left(\frac{Q}{2}-x\right)^2$$

together with

$$\log\Gamma_b(x)=\frac{\partial}{\partial s}F(b,x;s)\Big|_{s=0}=\int\limits_0^\infty\frac{\mathrm{d}t}{t}\left[\frac{\mathrm{e}^{-tx}-\mathrm{e}^{-tQ/2}}{(1-\mathrm{e}^{-tb})(1-\mathrm{e}^{-t/b})}-\frac{Q/2-x}{t}-\frac{1}{2}(Q/2-t)^2\mathrm{e}^{-t}\right]. \tag{A.3}$$

Separating the sum over integers m and n, appearing in the definition (A.1) of the Barnes zeta, onto sum of even m,n, even m and odd n, odd m and even n and odd m,n one gets

$$\zeta_b(2x;s)=2^{-s}\left[\zeta_b(x;s)+\zeta_b\left(x+\frac{1}{2}b;s\right)+\zeta_b\left(x+\frac{1}{2}b^{-1};s\right)+\zeta_b\left(x+\frac{1}{2}Q;s\right)\right]$$

and similarly

$$\zeta_b\left(\frac{1}{2}Q;s\right)=2^{-s}\left[\zeta_b\left(\frac{1}{4}Q;s\right)+\zeta_b\left(\frac{1}{4}Q+\frac{1}{2}b;s\right)+\zeta_b\left(\frac{1}{4}Q+\frac{1}{2}b^{-1};s\right)+\zeta_b\left(\frac{3}{4}Q;s\right)\right].$$

This gives

$$F(b,2x;s)=2^{-s}\left\{F(b,x;s)+F\left(b,x+\frac{1}{2}b;s\right)+F\left(b,x+\frac{1}{2}b^{-1};s\right)+F\left(b,x+\frac{Q}{2};s\right)\right.$$
$$\left.-F\left(b,\frac{Q}{4};s\right)-F\left(b,\frac{Q}{4}+\frac{1}{2}b;s\right)-F\left(b,\frac{Q}{4}+\frac{1}{2}b^{-1};s\right)-F\left(b,\frac{3}{4}Q;s\right)\right\}$$

and

$$\begin{aligned}\Gamma_b(2x)&=\exp\left\{\frac{\partial}{\partial s}F(b,2x;s)\Big|_{s=0}\right\}\\&=2^{-2(x-\frac{1}{4}Q)^2}\Upsilon_b\left(\frac{1}{4}Q\right)\Upsilon_b\left(\frac{1}{4}Q+\frac{1}{2}b\right)\Gamma_b(x)\Gamma_b\left(x+\frac{1}{2}b\right)\Gamma_b\left(x+\frac{1}{2}b^{-1}\right)\Gamma_b\left(x+\frac{1}{2}Q\right).\end{aligned}\tag{A.4}$$

Eq. (A.4) and the definition $\Upsilon_b^{-1}(x)=\Gamma_b(x)\Gamma_b(Q-x)$ yield

$$\Upsilon_b(2x)=2^{4(x-Q/4)^2}\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)\Upsilon_b(x+\frac{1}{2}b^{-1})\Upsilon_b(x+\frac{1}{2}Q)}{\Upsilon_b^2(\frac{1}{4}Q)\Upsilon_b^2(\frac{1}{4}Q+\frac{1}{2}b)}.\tag{A.5}$$

For $x=\frac{1}{2}b$ the formula (A.5) gives

$$\Upsilon_b^2\left(\frac{1}{4}Q\right)\Upsilon_b^2\left(\frac{1}{4}Q+\frac{1}{2}b\right)=2^{\frac{1}{2}(b^{-1}-b)^2}\Upsilon_b\left(\frac{1}{2}b\right)\Upsilon_b\left(\frac{1}{2}b^{-1}\right).$$

Substituting this expression into (A.5) we arrive at the double argument formula of FLNO

$$\Upsilon_b(2x)=2^{4x(x-\frac{1}{2}Q)+1}\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)\Upsilon_b(x+\frac{1}{2}b^{-1})\Upsilon_b(x+\frac{1}{2}Q)}{\Upsilon_b(\frac{1}{2}b)\Upsilon_b(\frac{1}{2}b^{-1})}.\tag{A.6}$$

Proceeding in a similar way and splitting the sum over m appearing in (A.1) onto even and odd integers one gets

$$F\left(\frac{b}{\sqrt{2}},x\sqrt{2};s\right)-F\left(\frac{b}{\sqrt{2}},\frac{b}{\sqrt{2}};s\right)=2^{-\frac{s}{2}}\left\{F(b,x;s)+F\left(b,x+\frac{b}{2};s\right)-F\left(b,\frac{b}{2};s\right)-F(b,b;s)\right\}$$

and therefore:

$$\Gamma_{\frac{b}{\sqrt{2}}}(x\sqrt{2})=2^{-\frac{1}{2}x(x-\frac{1}{b}-\frac{1}{2}b)-\frac{1}{4}}\Gamma_{\frac{b}{\sqrt{2}}}\left(\frac{b}{\sqrt{2}}\right)\frac{\Gamma_b(x)\Gamma_b(x+\frac{1}{2}b)}{\Gamma_b(\frac{1}{2}b)\Gamma_b(b)}.\tag{A.7}$$

The function

$$H(b,x;s)=2\zeta_b\left(\frac{1}{2}Q;s\right)-\zeta_b(x;s)-\zeta_b(Q-x;s)$$

satisfies

$$\frac{\partial}{\partial s}H(b,x;s)\Big|_{s=0}=\log\Upsilon_b(x),\qquad H(b,x;0)=-\left(\frac{1}{2}Q-x\right)^2.$$

Repeating for H the previous calculation one obtains

$$H\left(\frac{b}{\sqrt{2}},x\sqrt{2};s\right)-H\left(\frac{b}{\sqrt{2}},\frac{b}{\sqrt{2}};s\right)=2^{-\frac{s}{2}}\left\{H(b,x;s)+H\left(b,x+\frac{1}{2}b;s\right)-H\left(b,\frac{1}{2}b;s\right)-H(b,b;s)\right\}.$$

This implies the FLNO shift formula:

$$\Upsilon_{\frac{b}{\sqrt{2}}}(x\sqrt{2})=2^{x(x-\frac{1}{b}-\frac{1}{2}b)+\frac{1}{2}}\Upsilon_{\frac{b}{\sqrt{2}}}\left(\frac{b}{\sqrt{2}}\right)\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b)}{\Upsilon_b(\frac{1}{2}b)\Upsilon_b(b)}.\tag{A.8}$$

Finally, replacing in (A.8) $b\to b^{-1}$ one gets the relation:

$$\Upsilon_{b\sqrt{2}}(x\sqrt{2})=2^{x(x-\frac{1}{2b}-b)+\frac{1}{2}}\Upsilon_{b\sqrt{2}}\left(\frac{b^{-1}}{\sqrt{2}}\right)\frac{\Upsilon_b(x)\Upsilon_b(x+\frac{1}{2}b^{-1})}{\Upsilon_b(\frac{1}{2}b^{-1})\Upsilon_b(b^{-1})}.\tag{A.9}$$

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