We develop some basic methods for calculating Morava K-theories of compact Lie groups, and compute certain pivotal examples. We show that $K(2)^*BP$ has odd elements, where $P$ is the 3-Sylow subgroup of $GL_n(\mathbb{Z}/3)$. This disproves a conjecture of Hopkins, Kuhn and Ravenel. We also calculate Morava K-theories of semidirect products of cyclic groups with elementary abelian groups, and prove a new theorem on complex-oriented cohomology of $BO(K)$. © 1997 Elsevier Science Ltd

1. INTRODUCTION

This paper is the product of about a year's worth of the author's effort to advance, slightly at least, our knowledge of Morava K-theories of the classifying spaces of finite (and compact Lie) groups. The motivation of this effort is two-fold: in a broader outlook, the geometry of Morava K-theories is interesting, since we can hope it may help us better understand — and eventually calculate — these mysterious building blocks of stable homotopy theory. One of the first things to learn about a geometry are the characteristic classes; the first examples are characteristic classes of representations.

In a more immediate outlook, there has been substantial recent interest in Morava K-theories of classifying spaces [2–5, 8–10]. The present paper provides examples for and adds force to some of the methods introduced in these papers.

Hopkins et al. [4] define a compact Lie group $G$ to be good if the Morava K-theory $K(n)^*BG$ is additively generated, as a $K(n)^*$-module, by transferred Euler classes of complex representations of subgroups of $G$. One of the highlights of the present paper is Section 5 where we give a counterexample to a conjecture of Hopkins et al. [4] that every finite group is good. The counterexample is the $p$-Sylow subgroup $P$ of $GL_n(\mathbb{Z}/p)$ — for computational simplicity we specialize to $p = 3$. We show that

$$K(2)^{odd}BP \neq 0.$$ 

This complements a recent result of Tanabe [8] that all finite Chevalley groups (in particular $GL_n(\mathbb{Z}/q)$) are good at primes $p \neq q$.

Ultimately, the most important feature of Section 5 may be the fact that it gives a practical example of a more or less general method for calculating Morava K-theories of good groups and detecting minimal "bad" groups.

This general method has two ingredients. The first ingredient is the Hochschild–Serre spectral sequence associated with an extension of the group $\mathbb{Z}/p$. This spectral sequence was originally used by Atiyah [1] to prove a completion theorem for the $K$-theory of finite groups, and was also suggested in [4] as a tool for computing Morava K-theories of finite
groups. In Section 2, we prove some general properties of this spectral sequence. In particular, we show that an extension $G$ of $\mathbb{Z}/p$ by a good group $H$ is good if and only if the relevant Hochschild–Serre spectral sequence collapses.

We also wish to consider the case when $G$ is known to be good but $H$ is not. In this case, the natural tool that comes to mind is the Eilenberg–Moore spectral sequence. But this sequence is hopelessly non-convergent. We observe, however, that the Eilenberg–Moore exact couple in this case is periodic, and thus leads to two spectral sequences, one homological and one cohomological. While (as rather because) the homological spectral sequence is non-convergent, the cohomological spectral sequence converges!

The “cohomological Eilenberg–Moore spectral sequence” is constructed and discussed in Section 3. Using this spectral sequence, we were able to deduce most results analogous to the results we prove for Hochschild–Serre spectral sequence, including the fact that $H$ is good if and only if the spectral sequence collapses. Eventually, this leads to a picture of curious duality/reciprocity between the two spectral sequences. In some sense, this reciprocity is our substitute for the representation-theoretical tools available in $K$-theory.

The second ingredient of our computational methods is Hopkins–Kuhn–Ravenel theory. The point is that if a group $G$ is good, then its (integral) Morava $K$-theory embeds into its Hopkins–Kuhn–Ravenel character theory [4], which is essentially the complex vector space of conjugation-invariant functions on $n$-tuples of pairwise commutative elements $(g_1, \ldots, g_n)$ of $G$. Our philosophy is that from the associated graded object produced by the Serre spectral sequence, we can always recover the extensions from the embedding into Hopkins–Kuhn–Ravenel character theory. This involves computing characters of representations, which uses basic methods of local class field theory. For a non-trivial example, see Section 5.

The present paper contains two additional calculations. In Section 4, we compute explicitly the Morava $K$-theories of semidirect products of cyclic groups with elementary abelian groups. In particular, we prove that these groups are always good; another proof of this fact was independently announced by Yagita. Some of the results of this section are needed in Section 5. The main effort in Section 4 is the calculational understanding of the Hochschild–Serre spectral sequence. This is not difficult, if one can see the right pattern. However, this pattern, while familiar elsewhere in algebraic topology, is quite substantially different from what one sees in $K$-theory. A bulk of the author’s time devoted to this paper was spent on eliminating errors in this calculation.

The other calculation is the complex oriented $E$-cohomology of $BO(k)$ and the Thom space if its canonical real $k$-bundle, with certain minimal calculational hypotheses on the spectrum $E$. This is done in Section 6. We introduce a spectral sequence, which replaces Milnor’s standard Gysin sequence method in the case of bundles without complex structure. In some sense, this spectral sequence, and its convergence properties, are analogous to the Eilenberg–Moore spectral sequence, discussed in Section 2. The formula for $E^*BO(k)$ has been known for a few complex oriented cohomology theories, including $K(n)$ and $BP$ \cite{11, 5} (see also \cite{7}).

**Conventions and notation**

In this article, Morava $K$-theory $\tilde{K}(n)$ is a commutative associative complex-oriented ring spectrum whose coefficient ring satisfies

$$\tilde{K}(n)_* = c_k[\nu_n, \nu_n^{-1}]$$
where $K$ is the degree $n$ unramified extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ is its integer ring. We also assume that the associated formal group law $F$ satisfies
\[
[-p]_F x = -px + x^{p^n}.
\] (1)

This completely determines the formal group law. By Lubin–Tate theory, $\mathcal{O}_K$ maps into the automorphism group of $F$. In fact, one can show that if $a$ is a $(p^n - 1)$th root of 1, then
\[
[a]_F x = ax.
\] (2)

Thus, in particular, if $p$ is odd, $[-1]_F x = -x$, and (1) can be rewritten as
\[
[p]_F x = px - x^{p^n}.
\]

Let $K(n) = K(n)/p$. Note that, while the formal group law is not $p$-typical, the total degrees of the monomials of $F(x, y)$ are congruent to 1 modulo $p^n - 1$ (even for $p = 2$). To see this, note that $x + F y$ is equal to the average
\[
\frac{1}{p^n - 1} \sum_{x^{p^n - 1}} [x^{-1}]([x] + [F][x]y).
\]

By (2), the terms on the right-hand side in total degrees not congruent to 1 modulo $p^n - 1$ drop out.

By convention, in this article, for a spectrum $E$, $E$-reduced (co)homology will be denoted by $E_{\text{red}}$.

This is because $-$ is already used in a different meaning.

2. THE SERRE SPECTRAL SEQUENCE

We shall investigate extensions of finite groups of the form
\[
0 \to H \to G \to \mathbb{Z}/p \to 0.
\] (3)

We will deduce various relations between the Morava $K$-theories of $BH$, $BG$. In particular, we will explore the structural properties which the Morava $K$-theory of one of the spaces $BH$, $BG$ must satisfy in order for the Morava $K$-theory of the other to be concentrated in even degrees.

From the point of view of the Hopkins–Kuhn–Ravenel conjecture, the most important fact is that there is a non-zero element in $K(n)^{\text{odd}}BG$ if the $\mathbb{Z}[\mathbb{Z}/p]$ module $\tilde{K}(n)^*BH$ in (3) has non-trivial first cohomology group.

To this end, we shall use the Serre spectral sequence of the fibration of classifying spaces associated with the extension (3).

In fact, we can work in a slightly more general context. Let
\[
F \to E \to B\mathbb{Z}/p
\] (4)

be any fibration. Then there is a Hochschild–Serre spectral sequence converging to $\tilde{K}(n)^*E$. To identify the $E_2$ term, we assume that $\tilde{K}(n)^*F$ is finitely generated as a $\tilde{K}(n)^*$-module. Then the spectral sequence has the form
\[
E_2^{p,q} = H^p(\mathbb{Z}/p, \tilde{K}(n)^qF) \Rightarrow \tilde{K}(n)^{p+q}E
\] (5)

where the cohomology of $\mathbb{Z}/p$ is with respect to the action of $\mathbb{Z}/p$ on $\tilde{K}(n)^*F$ induced by the action (up to homotopy) of $\mathbb{Z}/p$ on $F$. This is the main result of the present section:

**Theorem 2.1.** Suppose $K(n)^*E$ is concentrated in even dimensions. Then
\[
H^1(\mathbb{Z}/p, \tilde{K}(n)^{p+q}F) = 0.
\]
To interpret this theorem, recall the following standard result:

**Proposition 2.2.** A torsion free finitely generated \( \mathbb{Z}/p[\mathbb{Z}/p] \)-module \( M \) satisfies

\[
H^1(\mathbb{Z}/p, M) = 0;
\]

if and only if \( M \) is a permutation module. If \( p > 2 \), this happens if and only if \( M/p \) is a permutation module.

**Proof of Theorem 2.1.** First of all, let

\[
N : \kappa(n)^*F \to H^0(\mathbb{Z}/p, \kappa(n)^*F)
\]

denote the norm map. Then, for \( x \in \kappa(n)^*F \), \( Nx \) is a permanent cycle in the spectral sequence (5) represented by \( \tau x \) where

\[
\tau : \kappa(n)^*F \to \kappa(n)^*E
\]

is the transfer (note that \( F \to E \) is a regular covering with fiber \( \mathbb{Z}/p \)). Thus, the spectral sequence (5) gives rise to a spectral sequence

\[
E_r = E_r/\text{Im} \, N.
\]

Further, we would like to claim that

\[
E_r \text{ converges to } \kappa(n)^*E/\tau\kappa(n)^*F.
\]

Note that this will follow, if we can prove

\[
\text{For } v \in \kappa(n)^*F, \text{ if } Nv = 0, \text{ then } \tau v = 0.
\]

But assume that \( Nv = 0 \) and \( \tau v \neq 0 \). Then certainly \( pv \in \text{Im}(1 - \alpha) \), since the odd cohomology of \( \mathbb{Z}/p \) is \( p \)-torsion. Thus, \( ptv = \tau(pv) = 0 \), so \( tv \in \kappa(n)^*E \) is \( p \)-torsion. But if there is torsion in \( \kappa(n)^*E \), then there are odd elements in \( \kappa(n)^*E \). So we may assume (8), and hence (7).

Now (6) is a spectral sequence of rings. For \( E = B\mathbb{Z}/p, F = * \),

\[
E_2 = (\kappa(n)^*/p)[x], \quad x \in E_2^{2,0}
\]

and the resulting spectral sequence collapses, so (6) is a spectral sequence of \( (\kappa(n)^*/p)[x]\)-modules by mapping the fibration

\[
F \to E \to B\mathbb{Z}/p
\]

to the fibration

\[
* \to B\mathbb{Z}/p \to B\mathbb{Z}/p.
\]

Now for an element \( v \in \kappa(n)^*E \), denote by \([v]\) its representative in \( E_\infty \). In particular, we require \( v \neq 0 \Rightarrow [v] \neq 0 \). To avoid confusion, say that \( \zeta \in E_\infty \) weakly represents \( v \in \kappa(n)^*E \) if \([v] = \zeta \) or \( \zeta = 0 \) and the filtration degree of \([v]\) is greater than the filtration degree of \( \zeta \).

Next, we claim that

For every \( \zeta \in E_\infty \) in the spectral sequence (5) and every \( z \in \kappa(n)^*E \) represented by \( \zeta \in E_\infty \), there exists a \( w \in \kappa(n)^*F \) such that \( pz + tw \in \kappa(n)^*E \) is weakly represented by \( \eta x^{\zeta - 1} \zeta \in E_\infty, \eta \in \mathbb{Z}/p^* \), in the spectral sequence (5).
To this end, we must recall several basic facts. First of all, specialize to the case $F = \ast (E = B\mathbb{Z}/p)$, $\zeta = 1$. Now,

$$\tilde{K}(n)^* B\mathbb{Z}/p = \tilde{K}(n)^* [[x]]/[p] x$$

where $[[x]]$ is the $i$-series of the formal group law $F$ and $x$ is the Euler class of the representation $\lambda$ of $\mathbb{Z}/p$ which sends the generator $\lambda$ of $\mathbb{Z}/p$ to $e^{2\pi i/p}$.

Next, recall the Frobenius reciprocity law (a direct consequence of naturality of transfer) which asserts that, for any commutative associative ring spectrum $K$, the transfer $\tau: K* F \to K* E$ is a map of $K* E$-modules, where $K* E$ acts on $K* F$ via the map $F \to E$. Thus, we see that

$$x.\tau(1) = \tau(x.1) = \tau(0) = 0.$$  \hfill (10)

On the other hand, the spectral sequence (5) for $F \to E \to B\mathbb{Z}/p$

implies that

$$\tau(1) = p \text{ mod } (x).$$

This together with (10) forces

$$\tau(1) = [p] x / x$$

since the kernel of $x: \tilde{K}(n)^* [[x]]/[p] x \to \tilde{K}(n)^* [[x]]/[p] x$ is the image of $[p] x / x: \tilde{K}(n)^* [[x]]/[p] x \to \tilde{K}(n)^* [[x]]/[p] x$. Now (9) follows.

Finally, for general $E, F$, (9) follows from the case $E = B\mathbb{Z}/p, F = \ast$ by the ring structure, naturality and the Frobenius reciprocity law.

In fact, (10) has another interesting consequence: Interpreting $x$ as the Euler class of the representation $\lambda$, we can consider the map

$$x: \tilde{K}(n)^* E \to \tilde{K}(n)^* E$$

which induces the obvious self-map of the spectral sequence (5) (by its ring structure). However, by (10), the map $x$ annihilates the image of $\tau$, so we obtain a map

$$x: \tilde{K}(n)^* E / \tau \tilde{K}(n)^* F \to \tilde{K}(n)^* E$$

which induces a map of spectral sequences

$$x: \tilde{E}_r \to \tilde{E}_r.$$  \hfill (11)

We shall next investigate the spectral sequence $\tilde{E}_r$. Unless specified otherwise, by the degree of an element $z \in \tilde{E}_r$, we shall mean the filtration degree of $z$; the corresponding notation will be $\deg(z)$.

**Lemma 2.3.** (1) As a $(\tilde{K}(n)^*/p)[[x]]$-module, $\tilde{E}_q$ has a presentation with homogeneous generators in degrees 0, 1 and homogeneous relations in degrees $\leq q$.

(2) If $d_r y \neq 0$ for some $r < q$ where $y \in \tilde{E}_{r}^{odd, even}$, then there is a non-zero element $z \in \tilde{E}_q$ of even total dimension such that $q > \deg(z) \geq 1$, and $xz = 0 \in \tilde{E}_q$.

**Proof.** An induction on $q$. For $q = 2$, (1) follows directly from periodicity of the cohomology of cyclic groups and (2) holds by default. Suppose (1), (2) hold for a given
$q = m$. Then

$$x: \tilde{E}^i_{m+\star} \to \tilde{E}^{i+2}_{m+\star}$$

is an isomorphism for $i \geq m$. (12)

Thus, Ker($d_m$) is generated, as a $\tilde{K}(n)^*/p[x]$-module, by elements in degrees 0, 1. Together with (12), this implies that

Ker($d_m$) has a presentation as a ($\tilde{K}(n)^*/p[x]$)-module with generators in degrees 0, 1 and homogeneous relations in degrees $< m$. (13)

Hence, $\bar{E}_{m+1} = \text{Ker}(d_m)/\text{Im}(d_m)$ has the same presentation as Ker($d_m$), except of added relations of the form $d_m y$ where $\deg(y) \leq 1$. This implies (1) for $q = m + 1$.

To prove (2) for $q = m + 1$, we distinguish two cases. First assume that $d_m y \neq 0$ for some $r < m$. Then, by the induction hypothesis, there is a non-zero element $z \in \tilde{E}_m$ such that $m > \deg(z) \geq 2$ and $xz = 0 \in \tilde{E}_m$. Then

$$0 = dm(xz) = xdm(z)$$

and hence $dm(z) = 0$ by (12). We conclude that $z \in \bar{E}_{m+1}$ satisfies the statement of (2) for $q = m + 1$.

Next, assume that $d_m y \neq 0$ for some $y$ of odd degree. By (1) of the induction hypothesis, we may assume that $\deg(y) = 1$. By the same induction hypothesis,

$$d_m y = xz$$

where $\deg(z) = m - 1$.

Now

$$0 = d_m d_m y = d_m(xz) = xdm(z)$$

so $d_m z = 0$ by (1) for $m$. Hence, $z \in \bar{E}_{m+1}$ satisfies the statement of (2) for $q = m + 1$. $\square$

We now conclude the proof of Theorem 2.1. Let

$$0 \neq \mu \in H^1(\mathbb{Z}/p, \tilde{K}(n)^*F).$$

If $\mu$ is a permanent cycle in the spectral sequence $E_r$, then $\tilde{K}(n)^{\text{odd}} \neq 0$, and we are done by the universal coefficient theorem. Thus, assume

$$d_r \mu \neq 0 \text{ for some } r \geq 2. \quad (14)$$

Note that the same differential applies in the spectral sequence $E_r$, since it is identical to $E_r$ in degrees $> 0$. Now note that $E^{0}_{r+1}$ is finitely generated as a $\tilde{K}(n)^*/p$-module for $i \leq 1$, and hence by Lemma 2.3 the spectral sequence collapses to a certain $E_q = E_{\infty}$. By Lemma 2.3(2) and by (14) there exists a non-zero element $z \in E_q = E_{\infty}$ such that

$$xz = 0, \quad \deg(z) \geq 1 \quad \text{and the total dimension of } z \text{ is even.} \quad (15)$$

**Lemma 2.4.** If $t \in \tilde{K}(n)^*E$ satisfies $\deg(t) = 2p^n - 2 + i$, then $t = pq + t(\sigma)$ for some $q \in \tilde{K}(n)^*E$, $\sigma \in \tilde{K}(n)^*F$ with $\deg(\sigma) \geq i$.

**Proof.** By (9), we have

For every $\zeta \in E_{\infty}$, there is a $v \in \tilde{K}(n)^*E$ and $w \in \tilde{K}(n)^*F$ such that $[v] = \zeta$ and $x^{p^n - 1}\zeta = [pv + tw]$ or $x^{p^n - 1}\zeta = 0$ and $\deg[pv + tw] > 2p^n - 2 + i$. (Thus, $x^{p^n - 1}\zeta$ weakly represents $pv + tw$.)
Thus, by Proposition 2.2, 
\[ E^{i,j} \supseteq \{ [pv + tw] | v \in \tilde{K}(n)^*E, w \in \tilde{K}(n)^*F, \deg[v] \geq i \}. \] (17)
By (17), if \( \deg[t] \geq 2p^n - 2 + i \), \( t \) can be approximated by \( pv + tw \) with \( \deg[v] \geq i \) in the sense that 
\[ \deg[t - pv - tw] \]
can be made arbitrarily large. The elements \( v, w \) have limit points \( v_0, w_0 \) with respect to the \( p \)-adic topology. By convergence, then \( t = pv_0 + tw_0 \).

Now recall (15). By (9), (15) and (16), there are \( v \in \tilde{K}(n)^*E, w \in \tilde{K}(n)^*F \) with 
\[ \deg[pv + tw] > 2p^n - 2 + \deg(z). \]
By Lemma 2.4 applied for \( t = pv + tw \), 
\[ p(u - Q) = pQ + z \tau \]
for some \( Q \in \tilde{K}(n)^*E, \tau \in \tilde{K}(n)^*F \) with 
\[ \deg[\tau] > \deg[u]. \] (19)
In particular, by (19), \( v - Q \neq 0 \), while by (18), 
\[ p(v - Q) \in \tau \tilde{K}(n)^*F. \] (20)
But now in the spectral sequence (5), elements in \( \tau \tilde{K}(n)^*F \) are represented in filtration degree 0 (by (7)), while \( p(v - Q) \) is in filtration degree > 0. Thus, 
\[ p(v - Q) = 0 \in \tilde{K}(n)^*E \]
while 
\[ v - Q \neq 0 \in \tilde{K}(n)^{even}E. \]
Thus, \( \tilde{K}(n)^*E \) has \( p \)-torsion, and hence, by the universal coefficient theorem, 
\[ \tilde{K}(n)^{odd}E \neq 0 \]
as claimed. We have finished the proof of Theorem 2.1.

3. THE EILENBERG–MOORE SPECTRAL SEQUENCE

In this section, we shall introduce a Morava \( K \)-theory Eilenberg–Moore spectral sequence of the fibration (4). The basic idea is that this allows us to obtain analogues of many of the results of the previous section with the roles of \( G \) and \( H \) interchanged. In practical calculations (including the counterexample to the Hopkins–Kuhn–Ravenel conjecture), we need these results to better understand Morava \( K \)-theory of good groups, in particular to show that certain modules are torsion free.

The very concept of an Eilenberg–Moore spectral sequence in this situation is not entirely trivial, since it is easy to see that the classical Eilenberg–Moore spectral sequence in this situation almost always diverges. However, we shall take advantage of the obvious fact that a 2-periodic resolution gives rise to two exact couples: one of homological, one of cohomological type. The “total divergence” of one is, in some sense, equivalent to the
convergence of the other (suggesting, perhaps, the possibility of "Tate phenomena" in Eilenberg–Moore spectral sequences). Thus, we have the following result:

**Theorem 3.1.** There is an Eilenberg–Moore spectral sequence of the fibration (4)

$$E^p_{2q} = \operatorname{Ext}^p_{\mathbb{Z}/p}([\tilde{K}(n)^*], \mathbb{Z}/p^q) \to \tilde{K}(n)^* \mathbb{Z}/p^q.$$  \hspace{1cm} (21)

Moreover, this spectral sequence converges if $\tilde{K}(n)^* \mathbb{Z}/p$ is a finitely generated $\tilde{K}(n)^*$-module.

First note that we have the following free $\tilde{K}(n)^* \mathbb{Z}/p$-resolution of $\tilde{K}(n)^*$:

$$[px\mathbb{K}(n)^* + \mathbb{K}(n)^*/[p] \times \mathbb{A} \mathbb{K}(n)^*/[p] \times \cdots $$ \hspace{1cm} (22)

Thus, for $k \geq 1$, 

$$\operatorname{Ext}^{k, \bullet}_{\mathbb{K}(n)^* \mathbb{Z}/p} \mathbb{Z}/p \simeq \operatorname{Ext}^{k+2, \bullet}_{\mathbb{K}(n)^* \mathbb{Z}/p} \mathbb{Z}/p$$

for $k \geq 1$.

**Proof of Theorem 3.1.** The idea is to construct a geometrical realization of the resolution (22). To this end, consider the following based cofibration sequences:

$$F_+ \to E_+ \to F_+ \wedge \mathbb{Z}/p \tilde{Z}/p$$ \hspace{1cm} (23)

$$F_+ \wedge \mathbb{Z}/p \tilde{Z}/p \to E_+ \to \Sigma F_+.$$ \hspace{1cm} (24)

Here $\gamma$ denotes the bundle induced from the canonical complex line bundle (one-dimensional representation) on $\mathbb{Z}/p$ and $\tilde{Z}/p$ denotes the unbased suspension of $\mathbb{Z}/p$ with one of the two special points selected as base point. The map (24) is obtained from the obvious inclusion $F_+ \to \Sigma F_+$.

Applying $\tilde{K}(n)^*$ to (23), (24), we can obtain two different exact couples: one homological, one cohomological. This is because the resolution (23), (24) is 2-periodic. In both cases, the $D$-terms are the terms related to the fiber $F$, while the $E$-terms are the terms related to the total space $E$. The homological exact couple has $E_2$ term

$$E^p_{2q} = \operatorname{Tor}^{\tilde{K}(n)^* \mathbb{Z}/p}_{p, q}(\tilde{K}(n)^* E, \tilde{K}(n)^*),$$

which is the classical Eilenberg–Moore $E_2$-term. However, as remarked above, this spectral sequence virtually always diverges. On the other hand, the spectral sequence corresponding to the cohomological exact couple has $E_2$-term (21).

To establish convergence of the cohomological spectral sequence, assemble the connecting maps of the cofibrations (23), (24) into a sequence

$$F_+ \leftarrow F_+ \wedge \mathbb{Z}/p \tilde{Z}/p \leftarrow F_+ \leftarrow F_+ \leftarrow \cdots $$ \hspace{1cm} (25)

We need to prove that after applying $\tilde{K}(n)^*$, to (25), the inverse-limit of the resulting sequence is 0. But to this end, note that the composition

$$\tilde{K}(n)^* F \to \tilde{K}(n)^* \mathbb{Z}/p(F_+ \wedge \mathbb{Z}/p \tilde{Z}/p) \to \tilde{K}(n)^* F$$

is the multiplication by $1 + \alpha + \cdots + \alpha^{p-1}$ where $\alpha$ is the generator of $\mathbb{Z}/p$. Since this element is in the maximal ideal which is the kernel of mod $p$ reduction of the augmentation, we are done by Nakayama's lemma. \hfill $\Box$

We now state the main result of this subsection.
Theorem 3.2. Assume that $K(n)^* F$ is finitely generated as a $K(n)^*$ module and concentrated in even degrees. Then

$$0 = \text{Ext}_{K(n)^* BZ/p}^1(\tilde{K}(n)^*, \tilde{K}(n)^* E).$$

Proof. Since the proof is completely analogous to the proof of Theorem 2.1, we will proceed in somewhat less detail. First, there is a natural diagram

$$
\begin{array}{ccc}
\mathbb{Z}/p & \to & \mathbb{Z}/p \\
\downarrow & & \downarrow \\
F & \to & E \\
\end{array}
$$

By the resolution (22), the top Eilenberg–Moore spectral sequence collapses to the $E_2$-term, which is concentrated in even dimensions. Further, the generator $T$ of $\text{Ext}^1$ is represented by $1 - x$ in $\tilde{K}(n)^*(\mathbb{Z}/p) = \tilde{K}(n)^* [\mathbb{Z}/p]$. (This last formula is funny: the right-hand side indicates a group algebra—recall that $x$ is the generator of $\mathbb{Z}/p$.)

Anyway, we have shown that the spectral sequence (21) is a spectral sequence of $\tilde{K}(n)^* T$-modules. Next, consider the map

$$x^{p-1} - p : \text{Ext}_{K(n)^* BZ/p}^0(\tilde{K}(n)^*, \tilde{K}(n)^* E) \to \text{Ext}_{K(n)^* BZ/p}^0(\tilde{K}(n)^*, \tilde{K}(n)^* E).$$

Now note that the second map of (24) is the transfer and that $r = x^{p-1} - p : \tilde{K}(n)^* E \to \tilde{K}(n)^* E$.

Thus, for $v \in \tilde{K}(n)^* E$, the element $(x^{p-1} - p)v$ is a permanent cycle in (22) and is represented by $r(x)$. Thus, we can consider the spectral sequence

$$E_r = E_r \text{ Im}(x^{p-1} - p)$$

Further, we would like to claim

$$E_r \text{ converges to } \tilde{K}(n)^* F/\text{Im}(r).$$

The proof is completely analogous to the proof of (7); We need to prove that, for $v \in \tilde{K}(n)^* E$,

$$(x^{p-1} - p)v = 0 \implies rv = 0.$$  

But if this were false for $v$, then, since $\text{Ext}_{K(n)^* BZ/p}^1(\tilde{K}(n)^*, ?)$ is $p$-torsion, $pv \in \text{Im} x$, but $rx = 0$, so $prv = 0$, and $rv$ is $p$-torsion, which contradicts $K(n)^* F$ being concentrated in even degrees. Thus, (29) is proved.

Now here is the Eilenberg–Moore analogue of Lemma 2.3:

Lemma 3.3. (1) As a $(\tilde{K}(n)^*/p)[T]$-modules, $E_q$ has a presentation with homogeneous generators in degrees $0, 1$ and homogeneous relations in degrees $\leq q$.

(2) If $d, y \neq 0$ for some $r < q$ where $y \in E_{r, even}$ then there is a non-zero element $z \in E_q$ of even total dimension such that $q > \deg(z) \geq 1$ and $xz = 0 \in E_q$.

Proof. Word by word analogous to the proof of Lemma 2.3.

So, in fact, is most of the rest of the proof of the Theorem. The one ingredient we have not discussed yet is the analogue to (9):

For every $\zeta \in E_{x, r}$, in the spectral sequence (22) and every $z \in \tilde{K}(n)^* F$ represented by $\zeta \in E_{x, r}$, there exists a $w \in \tilde{K}(n)^* E$ such that $pz + rw \in \tilde{K}(n)^* E$ is weakly represented by $\eta(1 - x)^{p-1}\zeta \in E_{x, r}$, $\eta \in \mathcal{L}_r^*$, in the spectral sequence (22).
To this end, note that it suffices to show that \( N = 1 + \alpha + \cdots + \alpha^{p-1} \) is an Eisenstein polynomial in \( 1 - \alpha \). But this, of course is standard: it is the cyclotomic polynomial (substitute \( T \) for \( 1 - \alpha \)).

Next, the analogue of Lemma 2.4 reads:

**Lemma 3.4.** If \( t \in \overline{K}(n)^*F \) satisfies \( \deg[t] = 2p - 2 + i \), then \( t = p\sigma + r(\sigma) \) for some \( \sigma \in K(n)^*F, \sigma \in \overline{K}(n)^*F \) with \( \deg[\sigma] \geq i \).

Now we can put it together: Similarly as in the proof of Theorem 2.1, we obtain an element \( z \in E_\infty \) such that

\[
Tz = 0 \in E_\infty, \deg[z] \geq 1 \quad \text{and the total dimension of } z \text{ is even.}
\]

By (30), and Lemma 3.4, there are \( v \in \overline{K}(n)^*F, w \in \overline{K}(n)^*E \) with

\[
[v] = z, \quad \deg[pv + rw] > 2p - 2 + \deg(z).
\]

By Lemma 3.4 applied for \( t = pv + rw \),

\[
pv + rw = p\sigma + r(\sigma)
\]

for some \( \sigma \in \overline{K}(n)^*F, \sigma \in \overline{K}(n)^*E \) with

\[
\deg[\sigma] > \deg[v].
\]

In particular, by (32), \( v - \sigma \neq 0 \), while by (31),

\[
p(v - \sigma) \in r\overline{K}(n)^*F.
\]

But now in the spectral sequence (22), elements in \( r\overline{K}(n)^*E \) are represented in filtration degree 0 (by (29)), while \( p(v - \sigma) \) is in filtration degree > 0. Thus,

\[
p(v - \sigma) = 0 \in \overline{K}(n)^*F
\]

while

\[
v - \sigma \neq 0 \in \overline{K}(n)^{even}F.
\]

Thus, \( \overline{K}(n)^*F \) has \( p \)-torsion, and hence, by the universal coefficient theorem,

\[
K(n)^{odd}F \neq 0
\]

as claimed.

**Reciprocity.** We can compress the information of the previous two sections into the following statement.

**Theorem 3.5.** The diagram

\[
\begin{array}{c}
\overline{K}(n)^*F \xrightarrow{1 - \alpha} \overline{K}(n)^*F \\
\downarrow \quad \uparrow \\
\overline{K}(n)^*E \xrightarrow{\alpha} \overline{K}(n)^*E
\end{array}
\]

is a chain complex (the composition of any two consecutive arrows is 0). Moreover, if \( \overline{K}(n)^*E \) (resp. \( \overline{K}(n)^*F \)) is torsion free and concentrated in even dimensions, then the diagram (34) is exact at the top two terms (resp. the bottom two terms) in even dimensions.
Proof. The fact that (34) is a chain complex is obvious: \( \tau(1 - \alpha) = 0 \) by naturality of transfer, \( \tau t = 0 \) by Frobenius reciprocity, \( \tau_\mathbb{Z}/p = 0 \) because \( \tau_\mathbb{Z}/p \) is the Euler class of a trivial bundle, and \( (1 - \alpha)r = 0 \) because \( \tau_\mathbb{Z}/p = r \) by the definition of (the action of) \( \alpha \) as the translation in a regular covering space.

To see the exactness, recall that we have commutative diagrams:

\[
\begin{array}{ccc}
\tilde{\mathcal{K}}(n)^*F & \xrightarrow{\tau} & \tilde{\mathcal{K}}(n)^*F \\
\downarrow N & & \downarrow N \\
\tilde{\mathcal{K}}(n)^*E & \xrightarrow{r} & \tilde{\mathcal{K}}(n)^*F
\end{array}
\]

Thus, in particular, \( \text{Ker}(N) \subseteq \text{Ker}(\tau) \) and \( \text{Ker}(\tau(x^{p^r - 1} - p)) \subseteq \text{Ker}(\tau) \) and therefore non-exactness at the upper left or lower right corner of (34) produces non-zero elements in the corresponding odd cohomology groups.

On the other hand, by definition, non-exactness at the upper right or lower left corner of (34) means that certain degree 0 elements of the respective spectral sequences (5), (21) are not permanent cycles, and hence support differentials. However, by Theorems 2.1 and 3.2 non triviality of differentials implies that the target cannot be torsion free and concentrated in even dimensions.

**Corollary 3.6.** Assume \( \tilde{\mathcal{K}}(n)^*E \) (resp. \( \tilde{\mathcal{K}}(n)^*F \)) is torsion free and in even dimensions. Then

\[
\tilde{\mathcal{K}}(n)^{\text{even}E}, \tilde{\mathcal{K}}(n)^{\text{even}F}/\text{Im}(1 - \alpha), \tilde{\mathcal{K}}(n)^{\text{even}F}/\text{Im}(\tau)
\]

(resp. \( \tilde{\mathcal{K}}(n)^{\text{even}E}, \tilde{\mathcal{K}}(n)^{\text{even}E}/\text{Im}(\alpha), \tilde{\mathcal{K}}(n)^{\text{even}E}/\text{Im}(\tau) \))

are torsion free.

**Proof.** Assume that \( \tilde{\mathcal{K}}(n)^*E \) is torsion free and in even dimensions. Then, by Theorem 2.1, \( H^1(\mathbb{Z}/p, \tilde{\mathcal{K}}(n)^{\text{even}F}) = 0 \), but then \( \tilde{\mathcal{K}}(n)^{\text{even}F} \) must be torsion free. If, for \( z \in \tilde{\mathcal{K}}(n)^{\text{even}F}, \ p^z \in \text{Im}(1 - \alpha) \), then certainly \( \tau(z) = 0 \), because \( \tilde{\mathcal{K}}(n)^*E \) is torsion free. Thus, \( z \in \text{Im}(1 - \alpha) \) by Theorem 3.5. The other statements are proved analogously.

**Remark 3.7.** The theorems of this section all fit into a heuristic pattern. Consider the following table:

<table>
<thead>
<tr>
<th>( \tilde{\mathcal{K}}(n)^*E )</th>
<th>( \tilde{\mathcal{K}}(n)^*F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Serre spectral sequence</td>
<td>Eilenberg-Moore spectral sequence</td>
</tr>
<tr>
<td>( 1 - \alpha )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( 1 + \alpha + \cdots + \alpha^{p^r - 1} )</td>
<td>( \alpha^{p^r - 1} - p )</td>
</tr>
<tr>
<td>( H^*(\mathbb{Z}/p, ?) )</td>
<td>( \text{Ext}_{\mathcal{R}(\mathbb{Z}/p)}(\tilde{\mathcal{K}}(n)^*, ?) )</td>
</tr>
</tbody>
</table>
Now for any statement which is true and involves the entries of the table, the statement remains true if each entry is replaced by the opposite entry in the same row.

While by now we have collected plenty of experimental evidence, I have no idea how to make this "duality" rigorous, or how to prove it.

**Remark 3.8.** In the special case when the fibration (4) is of the form

$$BH \to BG \to B\mathbb{Z}/p$$

and arises from the group extension (3), the above results can be further refined by applying Hopkins–Kuhn–Ravenel character theory [4, Theorem 3.3]. In particular, we have the following result.

**Theorem 3.9.** Suppose the fibration (4) arises from the extension (3). Suppose, further, that $\tilde{K}(n)^*BH$ is torsion free and concentrated in even dimensions, and that in the spectral sequence (5), $0 \neq z \in H^1(\mathbb{Z}/p, \tilde{K}(n)^*BH)$. Then, for some $k \geq 1$, $z^k$ is the target of a differential.

Similarly, suppose that $\tilde{K}(n)^*BG$ is torsion free and concentrated in even dimensions and that, in the spectral sequence (21),

$$0 \neq z \in \text{Ext}_{K(n)^*BG}^1(\tilde{K}(n)^*, \tilde{K}(n)^*BG).$$

Then, for some $k \geq 1$, $z(1 - x)^k$ is the target of a differential.

**Proof.** We shall prove the first statement: the second statement is analogous. Thus, assume that $0 \neq z \in H^1(\mathbb{Z}/p, \tilde{K}(n)^*BH)$. We first claim

$$z$$

is a permanent cycle. (35)

Indeed, if (35) were false, then, by the proof of Theorem 2.1, there would be even torsion in $\tilde{K}(n)^*BG$. But then, by Theorem 3.2, $\tilde{K}(n)^*BH$ cannot be torsion free and concentrated in even dimensions. Thus, (35) is proved.

Therefore, if the statement of the Theorem is false, then $0 \neq z^k \in E_2$ for every $k$. But then $\tilde{K}(n)^{odd}BG$ has a non-torsion element in odd dimension, contradicting Hopkins–Kuhn–Ravenel character theory [4, Theorem 3.3].

4. **SEMINDUCT PRODUCTS**

In this section, we compute the Morava $K$-theory of semidirect products of cyclic $p$-groups with elementary abelian $p$-groups. In particular, we show that these groups are good in the sense of Hopkins–Kuhn–Ravenel.

**Theorem 4.1.** Every $p$-group which is a semidirect product of a cyclic group with an elementary abelian group is good in the sense of Hopkins–Kuhn–Ravenel.

Let $H_s = \mathbb{Z}/p[T]/(T^s), p^s - 1 < s \leq p^s$ and let $\mathbb{Z}/p^s$ act on $H_s$ by $1 - x = T$ where $x$ is the generator of $\mathbb{Z}/p^s$. Then every $\mathbb{Z}/p[\mathbb{Z}/p^s]$-module is a direct sum of the modules $H_s$. These modules will be called $s$-cycles.

Next, put

$$M_s := K(n)^pBH_s = \mathbb{Z}/p[x_1, \ldots, x_s]/(x_1^{p^s}).$$
Then \( \mathbb{Z}/p^k \) acts on \( M_s \) by
\[
\alpha : x_i \mapsto x_i + \xi x_{i+1}, \quad x_{s+1} := 0.
\]
For a \( \mathbb{Z}/p[\mathbb{Z}/p^k] \)-module \( M \), the module of 1-cycles of \( M \) can be canonically identified with
\[
T_1(M) = \ker(1 - \alpha)/(\text{im}(1 - \alpha) \cap (\ker(1 - \alpha))).
\]
Then Theorem 4.1 follows from the following result:

**Theorem 4.2.** The module \( M_s \) is a permutation \( \mathbb{Z}/p^s \)-module. Further, the module of 1-cycles \( T_1(M_s) \) has a \( \mathbb{Z}/p \)-basis
\[
\{1, \lambda_s, \ldots, \lambda_s^{p^s-1}\}
\]
if \( p^k - 1 < s < p^k \)
\[
\{1, \lambda_s, \ldots, \lambda_s^{p^s-1}\}
\]
if \( s = p^k \)
(36)

where
\[
\lambda_s = \prod_{i=0}^{s^s-1} \alpha'(x_i), \quad \lambda_s = \prod_{i=0}^{s^s-1} \alpha'(x_{s+1} - \xi x_i).
\]

Finally, if \( s = p^k \), the following identity holds in \( T_k(p^k) \):
\[
\lambda_p = \eta \xi^{p^s-1}, \quad \eta \in \mathbb{Z}/p^s.
\]
(37)

We first treat the case \( s < p \).

**Lemma 4.3.** Theorem 4.2 holds for \( s = p \).

**Proof.** It turns out slightly more convenient to work with the integral Morava K-theory \( \tilde{K}(n) \). Thus, we will consider the \( \mathbb{Z}/p[\mathbb{Z}/p] \)-module
\[
\tilde{M}_s := \tilde{K}(n)^0BH_s = \mathbb{Z}_p[x_1, \ldots, x_s]/([p], x_i).
\]
We will prove that \( \tilde{M}_s \) is a permutation \( \mathbb{Z}_p[\mathbb{Z}/p] \)-module where the basis of
\[
\tilde{R}_s^0(\mathbb{Z}/p, M_s)
\]
as indicated in (36). Note that \( \lambda_s = x_s \).

The proof is an induction on \( s \). For \( s = 1 \) the statement is certainly true. Suppose the statement is true for a given \( s < p \), we shall prove it for \( s + 1 \). Then filter \( \tilde{M}_{s+1} \) by powers of \( x_{s+1} \) (a decreasing filtration). There results a spectral sequence computing Tate cohomology. In fact, the \( E_1 \) term is the Tate cohomology of the associated graded object, which is
\[
E_1^{\text{even}} = \tilde{R}_s^0(\mathbb{Z}/p, \tilde{M}_s)[x_{s+1}]
\]
\[
E_1^{\text{odd}} = x_{s+1} \tilde{R}_s^0(\mathbb{Z}/p, \tilde{M}_s)[x_{s+1}].
\]
(We have \( E_0^{\tilde{M}_{s+1}} = \tilde{M}_s[x_{s+1}]/(px_{s+1}) \). To describe the differentials, we will denote differentials originating in even (resp. odd) cohomological degree by \( d^{\text{even}} \), resp. \( d^{\text{odd}} \). In this notation, the non-trivial differentials in our spectral sequence are the Kudo differentials:
\[
d_{1}^{\text{even}} : x_i x_i^{j-1} x_{i+1}^{j+1}, \quad i = 1, \ldots, p - 1, \quad j = 0, 1, \ldots
\]
\[
d_{p-1}^{\text{odd}} : x_s^{p^s-1} x_i + x_i^{j+p-1}, \quad i = 1, 2, \ldots
\]
(It is understood that each differential can be multiplied by powers of the permanent cycle \( \kappa_{s+1} \).) The spectral sequence collapses to \( E_p \). This immediately implies the induction statement for Tate cohomology. The pattern of the remaining \( p \)-cycles is easy to deduce.
In fact, we claim that a basis of the free part of $\tilde{M}_{s+1}$ can be described as follows: the union of

1. the $\mathbb{Z}/p[\mathbb{Z}/p]$-bases of the $\mathbb{Z}/p[\mathbb{Z}/p]$-free summands of the associated graded pieces of $\tilde{M}_{s+1}$ of filtration degrees $< p^n$ (reduced mod $p$ in filtration degree 0)
2. the set
$$\{x_i^s | 1 \leq i \leq p-2\} \cup \{x_i^{p-1}x_{i+1}^s | 0 \leq j \leq p^n - p - 1\} \otimes \{1, \kappa_{s+1}, \ldots, \kappa_{s+p}^{p-1}\}.$$  

It is not hard to see that this set, together with the 0th Tate cohomology of $\tilde{M}_{s+1}$, generate $M_{s+1}$. Independence is then established by a counting argument.

Finally, consider the case $s = p$. We have proved that $T_1(M_p)$ is spanned by
$$\{1, \kappa_p, \ldots, \kappa_p^{p-1}\} \otimes \{1, x_p, \ldots, x_p^{p-1}\}.$$  

But we also know that $H_p$ is isomorphic to $(\mathbb{Z}/p)^p$ with the permutation action of $\mathbb{Z}/p$. Thus, (see [4]), $T_1(M_p)$ is spanned by
$$\{1, \kappa_p, \ldots, \kappa_p^{p-1}\}.$$  

A comparison of total algebraic degrees of the elements involved (well defined modulo $(p^n - 1)$) completes the argument.

Proof of Theorem 4.1. An induction on $k$. Consider the short exact sequence
$$0 \rightarrow A \rightarrow \mathbb{Z}/p^k \rightarrow C \rightarrow 0, \quad A \cong \mathbb{Z}/p^{k-1}.$$  

First, as an $A = \mathbb{Z}/p^{k-1}$-module, $H_s$ is a direct sum
$$H_{s_1} \oplus \cdots \oplus H_{s_p}$$

where
$$s_j = \begin{cases} \left\lceil \frac{j}{p} \right\rceil & \text{for } j > i \mod p \\ \left\lceil \frac{j}{p} \right\rceil - 1 & \text{for } j \leq i \mod p. \end{cases}$$

where $i \mod p$ is the remainder of $i$ modulo $p$: $0 \leq (i \mod p) < p$. Thus, as an $A$-module,
$$M_s = \bigotimes_{i=1}^{p} M_i$$

and hence $M_s$ is a permutation $A$-module by the induction hypothesis. Thus, a $t$-cycle of the $\mathbb{Z}/p^k$-module $M_s$ with $t > p$ satisfies $t = p^j$ for some $j$, as required.

Now the submodule of $M_s$ consisting of $t$-cycles for $t \leq p$ is isomorphic to the $C$-module $T_{k-1}M_s$ (defined by identifying $\mathbb{Z}/p^{k-1} \cong A$). Now, by the induction hypothesis,
$$T_{k-1}M_s$$

has basis
$$\{1, h_1, \ldots, h_1^{p-1}\} \otimes \cdots \otimes \{1, h_p, \ldots, h_p^{p-1}\}$$
$$\{1, k_1, \ldots, k_1^{p-1}\} \otimes \cdots \otimes \{1, k_p, \ldots, k_p^{p-1}\}$$

where
$$h_j = \prod_{i=0}^{p^j-1} \alpha^i(x_j), \quad k_j = \prod_{i=0}^{p^{j-1}-1} \alpha^i(x_{s+j-p^j}).$$

In fact, in most cases we can put in (38)
$$h_i = \kappa_i, \quad k_i = \frac{i}{s_i}. $$
More precisely, this works if

\[ s \notin \{p^{k-1} + 1, \ldots, p^{k-1} + p\} \cup \{p^k - p + 1, \ldots, p^k\}. \]

If \( s = p^k + j - p, 1 \leq j \leq p \), put

\[ h_i = \kappa_{s_i}, \]
\[ k_i = \begin{cases} \kappa_{s_i}^{p^{-1}} & \text{if } i \leq j \text{ (hence } s_i = p^{k-j-1}) \\ \lambda_{s_i} & \text{if } i > j \end{cases} \]

If \( s = p^k + j - p, 1 \leq j \leq p \), put

\[ k_i = \lambda_{s_i}, \]
\[ h_i = \begin{cases} \lambda_{s_i}^{p^{-1}} & \text{if } i \leq j \text{ (hence } s_i = p^k-j) \\ \kappa_{s_i}^{p^{-1}} & \text{if } i > j \end{cases} \]

Now the \( C \)-action of \( T_{k-1}M_s \) is given by

\[ \alpha: h_j \mapsto h_j + p h_{j+1} \quad (39) \]
\[ \alpha: k_j \mapsto k_j + p k_{j+1} \quad (h_{p+1} = k_{p+1} = 0). \quad (40) \]

For example, to prove (40), fix \( j \) and put

\[ q_i = \alpha^p(x_{i+j-1}), \quad r_i = \alpha^p(x_{i+j+1-1}). \]

Then

\[ \alpha(k_j) = \prod_{i=0}^{p^{k-1}-1} (q_i + r_i). \quad (41) \]

Now consider (41) as a polynomial in indeterminates \( q_i, r_i \) with the \( \mathbb{Z}/p^{k-1} \)-action

\[ \alpha: q_i \mapsto q_{i+1}, \quad r_i \mapsto r_{i+1} \quad (q_{p^{k-1}} = q_1, r_{p^{k-1}} = r_1). \]

Then, writing (41) as a sum of monomials, the only \( \mathbb{Z}/p^{k-1} \)-invariant monomials are those on the right hand side of (40). The other monomials can be grouped into disjoint set of monomials

\[ \{m, \alpha m, \ldots, \alpha^p m\} \quad (42) \]

where \( m \) is a monomial invariant under the action of \( \mathbb{Z}/p^{k-1} \). But the sum of (41) is 0 in \( T_{k-1}(M_s) \), so (40) is proved. The proof of (39) is similar.

But now let

\[ h'_j = \alpha^{j-1} h_1, \quad k'_j = \alpha^{j-1} k_1. \]

Then \( C \cong \mathbb{Z}/p \) acts on these elements by permutation. Further, we claim that (38) remains valid with \( h_j, k_j \) replaced by \( h'_j, k'_j \). Note that this is not immediate, since (39), (40) do not completely determine the \( C \)-action on \( T_{k-1}M_s \), due to the fact that we did not choose to fully calculate the multiplicative structure of \( T_{k-1}M_s \).

However, note that we have a decreasing augmentation ideal filtration on \( M_s \), which determines a decreasing filtration on \( T_{k-1}M_s \) (the filtration degree of an element is the maximum filtration degree of its representative). But then, in the associated graded module \( E^0 T_{k-1}M_s \), (39) and (40) are valid with \( +p \) replaced by \(+\), thus completely determining the \( C \)-action. It is easily deduced that (38) remains valid with \( h_i, k_i \) replaced by \( h'_i, k'_i \) in \( E^0 T_{k-1}M_s \) and hence in \( T_{k-1}M_s \).

Thus, \( T_{k-1}M_s \) is a permutation \( C \)-module whose fixed points are \( h'_1 \ldots h'_p, k'_1 \ldots k'_p \), concluding the proof of the induction step.
Finally, we consider the case \( s = p^k \). We have proved that \( T_k M_p^s \) is spanned by 
\[
\{1, \kappa_{p^s}, \ldots, \kappa_{p^s}^{p^k-1}\} \otimes \{1, \lambda_{p^s}, \ldots, \lambda_{p^s}^{p^k-1}\}.
\]
But now, again \( M_p^s \) is isomorphic to \((\mathbb{Z}/p)^{p^k}\) with the permutation action of \( \mathbb{Z}/p^k \), so by [4], 
\( T_k M_p^s \) is spanned by 
\[
\{1, \kappa_{p^s}, \ldots, \kappa_{p^s}^{p^k-1}\}.
\]
Once again, an argument involving the total algebraic degrees of the elements involved (well defined modulo \( p^s - 1 \)) concludes the proof.

5. THE \( p \)-SYLOW SUBGROUP OF \( GL_4(\mathbb{Z}/p) \)

In this section, we show that the \( p \)-Sylow subgroup of \( GL_4(\mathbb{Z}/p) \) is not good for \( p = 3 \). This is a counterexample to the Hopkins-Kuhn-Ravenel conjecture.

The basic idea of this calculation is, of course, to break up our \( p \)-group into semidirect products with quotient \( \mathbb{Z}/p^k \): this allows us, in principle, to use the results on the Serre spectral sequence in Section 2. The difficulty is that, in this inductive calculation, we must consider at least one semidirect product where the kernel is non-abelian. This is also to be expected in view of the previous section (although technically we did not exclude the possibility of a bad group which would be a semidirect product of a cyclic groups with a non-elementary abelian group).

The reader is very strongly encouraged to contemplate for a few moments the difficulty involved here: in the induction, which we call Atiyah induction in view of [1], the Serre spectral sequences only give associated graded objects. However, the precise objects are needed for the next step of the induction.

In the absence of any notion of “\( n \)-representations”, which would form a natural basis for Morava K-theories of classifying spaces of good groups, we use the Hopkins-Kuhn-Ravenel theory of \( n \)-characters, which give, at least, a natural basis of \( \tilde{K}(n)^*BG \otimes \mathbb{Q} \) ([4]). Thus, for good groups, integral Morava K-theory, which is additively generated by Euler classes of representations, embeds into the Hopkins-Kuhn-Ravenel \( n \)-character space. Therefore, if we can calculate \( n \)-characters of representations, we can deduce the structure of integral Morava K-theory of classifying spaces of finite groups precisely, without resorting to associated graded objects.

The main content of this section is performing these \( n \)-character calculations of characteristic classes of representations for the concrete groups at hand. Once the relevant \( \mathbb{Z}[\mathbb{Z}/p] \)-module is calculated, we proceed to reason that it has odd cohomology.

Alternately to reasoning, one can choose to use brute force to conclude the argument. This, in fact, has also been done by the author (using Maple) and later in even greater detail by Neil Strickland and Doug Ravenel (using Mathematica).

Most of the material of this section is done for a general prime. However, we shall always assume that \( p \) is odd. First of all, let us write
\[
P = (\mathbb{Z}/p)^2 \ltimes (\mathbb{Z}/p)^4 = \mathbb{Z}/p\{c, d\} \ltimes \mathbb{Z}/p\{a_{11}, a_{12}, a_{21}, a_{22}\}.
\]

Comment. With notation ambiguous in the literature, we write semidirect products in the form \( A \ltimes N \) where \( A \) acts on \( N \). Thus, \( N \) is the normal subgroup.

In the case of (43), \( c \) commutes with \( a_{21} \), \( d \) commutes with \( a_{12} \), and
\[
c^{-1}a_{11}c = a_{11}a_{21}, \quad d^{-1}a_{11}d = a_{11}a_{12}.
\]
We first consider the groups

\[ H = \mathbb{Z}/p\langle c \rangle \rtimes \mathbb{Z}/p\{a, b\}, \quad c^{-1}ac = ab \]
\[ G = \mathbb{Z}/p\langle c \rangle \rtimes \mathbb{Z}/p\{a_1, a_2, a_21, a_{22}\} \]

and then the Hochschild–Serre spectral sequence of the fibration

\[ BG \to BP \to B\mathbb{Z}/p\{d\}. \]

The idea is that \( H, G \) are good by the results of Section 4 (see also \([9, 10]\)), and the complete action of \( \mathbb{Z}/p \) on the Morava \( K \)-theory of \( BG \) can be recovered using generalized character theory. In fact, the reason of introducing \( H \) is that it is smaller, but all of the relation in \( R(n)^*BG \) can be recovered from \( R(n)^*BH \) by means of maps \( G \to H \).

Now, in \( \tilde{R}(n)^*BH \), let \( z, w, \kappa \), respectively, denote the Euler classes of the representations \( \gamma : H \to \mathbb{Z}/p\{a\} \subset S^1 \), \( \delta : H \to \mathbb{Z}/p\{c\} \subset S^1 \), and the induced representation \( \nu \) by the inclusion \( \mathbb{Z}\{c, b\} \subset H \) of the representation \( \mu : \mathbb{Z}\{b, c\} \to \mathbb{Z}/p\{b\} \subset S^1 \). Here the convention on the maps is that the named generators in the source map into the generators with the same name in the target, or to 0 if there is no generator with the same name in the target. Let \( \tilde{R} \) be the dimension 0 homogeneous summand of \( \tilde{R}(n)^*BH/\tau \tilde{R}(n)^*(\mathbb{Z}/p\{a, b\}) \) where \( \tau \) denotes the transfer.

**Lemma 5.1.** In \( \tilde{R} \), we have

\[ z^p = w^{-1}z. \] (44)

**Proof.** We will use \( n \)-character theory. Because the group \( H \) is good, we have \( \tilde{R}(n)^*BH \subseteq \tilde{R}(n)^*BH \otimes \mathbb{Q} \). Now the image of \( \tau \) in the group of \( n \)-characters of \( H \) consists of all characters which are 0 on \( n \)-tuples which are not contained in \( \mathbb{Z}/p\{a, b\} \). Thus, by \( n \)-character theory, it suffices to check our equality after pullback to all abelian subgroups of \( H \) which are not subgroups of \( \mathbb{Z}/p\{a, b\} \).

But now it is easy to see that such subgroups \( A \) either project trivially by \( H \to \mathbb{Z}/p\{a\} \), in which case the pullback of \( \tilde{z} \) is 0, or are subgroups of groups of the form \( \mathbb{Z}/p\{b, c^i\} \), \( i \neq 0 \). In the latter case, choosing a non-trivial \( n \)-tuple \( g = (g_1, \ldots, g_n) \) in \( A \), the value of the element \( w \) on \( g \) will be a uniformizing parameter \( \pi \) in \( K[\pi]/(\pi^{p-1} - p) \), while the value of \( \tilde{z} \) on \( g \) will be \([i] \pi \) for some \( i \in \mathbb{Z}/p^* \). Thus, any polynomial \( p(x, y) \) such that

\[ p(\pi, [i] \pi) = 0 \quad \text{for } i \in \mathbb{Z}/p \]

will satisfy

\[ p(\tilde{z}, w) = 0 \in \tilde{R}. \]

In particular, we can put

\[ p(x, y) = y^p - x^{p-1}y. \]

This is because \([i] \pi = e_i \pi \) where \( e_i \in K \) run through the set consisting of \((p - 1)\)th roots of unity and 0. Thus, \( ([k] \pi)^p - \pi^{p-1}[k] \pi = \prod_{i=0}^{p-1} ([k] \pi - [i] \pi) = 0 \).

**Lemma 5.2.** In \( \tilde{R} \otimes \mathbb{Q} \), we have

\[ \left( \frac{K^{p-1}}{w^{p-1}} \right) + \left( \frac{K^{p-2}}{w^{p-2}} \right) + \cdots + \left( \frac{K^0}{w^0} \right) = 0. \] (45)
Comment. First, recall that $\bar{R}$ is torsion free by Corollary 3.6. Now note that, in $\bar{R}$, $w^{p^{-1}} = p$. (See the conclusion of the proof of (9) above.) Further, upon pullback to abelian subgroups, $v$ becomes a product of $p$ Euler classes of one-dimensional representations, and hence

$$\bar{R}^p = p^p \bar{R}.$$ 

Thus, we first conclude that the left-hand side of (45) converges in the $p$-adic topology.

Second, we consider the modulo $p$ case. If we multiply (45) by $w^{p^{-1}}$, it will be integral (i.e. will make sense in $\bar{R}$), by the above remarks. Further, recall from the Conventions and notation part of the Introduction that monomials of the formal group law $F$ are of total degree $1 \mod (p^n - 1)$. Therefore, the non-linear terms of the $+_F$ operations will all be divisible by $p$, and hence we may replace $+_F$ by $+$. We conclude that,

$$\bar{R}^p + w^{p^{-1}} \bar{R}^{p-1} + \cdots + w^{p^{-1}} \bar{R}^{p-1} = 0 \text{ in } \bar{R}/p.$$ 

(46)

Proof. Similar as the proof of Lemma 5.1. We must consider the pullbacks of $\bar{c}, w$ to abelian subgroups $A$ of $H$ which project non-trivially to $\mathbb{Z}/p\{c\}$. Further, in such abelian subgroups, we are only interested in $n$-tuples $g = (g_1, \ldots, g_n)$ at which the value of $w$ is $\pi$ (alternately, this value could be 0). But then the restriction of the representation $v$ to $A$ is $\mu + \mu \otimes \delta + \cdots + \mu \otimes \delta^{p-1}$, so the value of $\bar{c}$ on $g$ is

$$\prod_{i=0}^{p-1} [x + i] \pi \text{ for some } x \in \mathcal{C}_K/p\mathcal{C}_K.$$ 

Thus, we will have

$$p(\bar{c}, w) = 0$$

if we can prove

$$p \left( \prod_{i=0}^{p-1} [x + i] \pi, \pi \right) = 0 \text{ for all } x \in \mathcal{C}_K/p\mathcal{C}_K.$$ 

(48)

To find $p$, first recall that there is an injection of abelian groups $\psi: (\mathcal{C}_K/p\mathcal{C}_K)^* \hookrightarrow \mathcal{C}_K^*$. Thus, we can compute

$$\prod_{i=0}^{p-1} [x + i] \pi = \psi \left( \prod_{i=0}^{p-1} (x + i) \right) \pi = \psi(x^p - x) \pi^p.$$ 

But now $(x^p - x), x \in \mathcal{C}_K/p\mathcal{C}_K$ are roots in $\mathcal{C}_K/p\mathcal{C}_K$ of the polynomial

$$t^{p-1} + t^{p-1} + \cdots + t^{p-1}.$$ 

Next, we have

$$\psi(x) +_F \psi(\beta) = \psi(x + \beta) \pi.$$ 

We conclude that the power series

$$p(t, s) = \left( \frac{t^{p-1}}{\delta^{p-1}} \right) +_F \left( \frac{t^{p-1}}{\delta^{p-1} - 1} \right) +_F \cdots +_F \left( \frac{t^{p-1}}{\delta^{p-1} - (p-1)} \right)$$

satisfies the condition (48).

We now turn to the group $G$. We have

$$0 \to \mathbb{Z}/p\{a_i\} \to G \to \mathbb{Z}/p\{c\} \to 0.$$
When calculating $\tilde{K}(n)^*BG$ using the Hochschild–Serre spectral sequence, we find that the $\mathbb{Z}/p\{c\}$-module $\tilde{K}(n)^*B\mathbb{Z}/p\{a_{ij}\}$ is isomorphic to

$$\mathbb{Q}^n\tilde{B}Z/p\{a, b\}.$$  

Consequently, also the Tate cohomology of $\mathbb{Z}/p\{c\}$ with coefficients in $\tilde{K}(n)^*B\mathbb{Z}/p\{a_{ij}\}$ is isomorphic to

$$H^0(\mathbb{Z}/p\{c\}, \tilde{K}(n)^*B\mathbb{Z}/p\{a, b\}) \otimes \tilde{H}^0(\mathbb{Z}/p\{c\}, \tilde{K}(n)^*B\mathbb{Z}/p\{a, b\}).$$

On the other hand, the two factors of the expression above are isomorphic and have been calculated in Theorem 4.2 above. Now there are two distinguished maps of groups $G \rightarrow H$: one sends the generators $a_{12}$ to $a$, $b$ and the generators $a_{11}$ to $0$, the other map sends $a_{11}$ to $a$, $b$ and the generators $a_{12}$ to $0$. Denote the pullbacks of the classes $\kappa, z$ via these maps by $\kappa$, $z$, resp. $\lambda$, $t$. Thus, equations (44), (46) imply that the 0-dimensional homogeneous summand $S$ of $K(n)^*BG/\text{Im } \tau$ (where $\tau$ is the transfer from $\mathbb{Z}/p\{a_{ij}\}$) is isomorphic to

$$\mathbb{Z}/p[\kappa, \lambda, z, t, w]/I$$

where

$$I = (w^{p-1}, w^{p-1} + w^{p-1} - 1 + \cdots + w^{p-1} - 1),$$

(49)

(We are using the naturality of transfer to conclude that, when pulling back by the map $G \rightarrow H$, $\text{Im } \tau$ lands in $\text{Im } \tau$. Also, note that the compositions of the two maps $G \rightarrow H$ with the projection $H \rightarrow \mathbb{Z}/p\{c\}$ coincide, and hence so do the two pullbacks of the representation $\delta$, and its Euler class $w$.)

In order to obtain information about $\tilde{K}(n)^*BP$, we need to reconstruct the action of $\mathbb{Z}/p\{d\}$ on $S$. First of all, it is obvious (from the action on representations) that the generator $d$ acts on $z$ by

$$z \mapsto z + t.$$  

**Lemma 5.3.** In the 0-dimensional homogeneous summand $S$ of $\tilde{K}(n)^*BG/\text{Im } \tau$ tensored with $Q$, $d$ acts on $\kappa$ by

$$\kappa \mapsto \kappa \frac{\lambda}{w^{p-1}}.$$  

(50)

**Proof.** Again, we shall use $n$-character theory. In fact, the situation is quite similar to the proof of Lemma 5.2. Consider an abelian subgroup $A \subset G$ with at most two generators, and such that $A$ surjects to $\mathbb{Z}/p\{c\}$. Then the images $A'$, $A''$ of $A$ under the two distinguished maps $G \rightarrow H$ are, of course, also abelian, and, therefore, the representation $\mu : \mathbb{Z}/p\{b, c\} \rightarrow S^1$ restricts to the subgroups $A', A''$. Let $s, q$ denote the pullbacks of the Euler classes of these extensions via the maps $A \rightarrow A'$, $A \rightarrow A''$. Then we have, in $\tilde{K}(n)^*BA$,

$$\kappa = \prod_{i=0}^{p-1} (s + \gamma \lceil \frac{i}{w}), \quad \lambda = \prod_{i=0}^{p-1} (q + \gamma \lceil \frac{i}{w}).$$  

(51)
Further, the generator $d$ acts by

$$s \mapsto s + q.$$  

(Note that, while the generators $s$, $q$ are not well defined, the products (51) are. Thus, the action of $d$ may be $s \mapsto s + q + w^j$ with a different choice of $s$, $q$ on a particular abelian subgroup $A$. However, since the choices differ only by a formal summand $+w$, the resulting action on the products (51) is the same.)

We conclude that the generator $d$ acts on $\kappa$ by

$$\kappa \mapsto \prod_{i=0}^{p-1} (s + q + w^i).$$

(52)

Consider a commuting $n$-tuple $g = (g_1, \ldots, g_n)$ on which the value of $w$ is $\pi$. For some $\alpha, \beta \in \mathcal{O}_K$, the value of $s$, $q$ is $[\alpha] \pi$, $[\beta] \pi$ (this, in effect, is true for arbitrary one-dimensional representations, since $\pi$ is a uniformizing parameter), so we are done if we can prove

$$\prod_{i=0}^{p-1} (\alpha + \beta + i)^{\pi} = \prod_{i=0}^{p-1} (\alpha + i)^{\pi} + \prod_{i=0}^{p-1} (\beta + i)^{\pi}$$

(53)

for all values of $\alpha, \beta \in \mathcal{O}_K/p\mathcal{O}_K$. But now (53) is equivalent to

$$\prod_{i=0}^{p-1} (\alpha + \beta + i)^{\pi} = \prod_{i=0}^{p-1} (\alpha + i)^{\pi} + \prod_{i=0}^{p-1} (\beta + i)^{\pi}$$

in $\mathcal{O}_K/p\mathcal{O}_K$

which is equivalent to

$$(\alpha + \beta)^p - (\alpha + \beta) = \alpha^p - \alpha + \beta^p - \beta$$

in $\mathcal{O}_K/p\mathcal{O}_K$

which certainly holds.

Now for the purpose of more concrete computations, we restrict to the case $n = 2$, $p = 3$. In this case, we obtain in $S$

$$\kappa^2 = -w^6 \kappa, \quad \lambda^3 = -w^6 \lambda$$

and consequently the action of $d$ on $S$ is

$$z \mapsto z + t + w^6 z t^2 + w^6 z^2 t, \quad k \mapsto k + \lambda - w^2 \kappa^2 \lambda - w^2 \kappa \lambda^2.$$  

(54)

Thus, we see that in $S$

$$(1 - d)(\kappa t - \kappa \lambda) = tw^3 \kappa \lambda^2 + tw^3 \kappa \lambda \mod w^6$$

but that implies that, in $\tilde{S}$,

$$(1 - d)(w(\kappa t - \kappa \lambda)) = tw^3 \kappa \lambda^2 + tw^3 \kappa \lambda \mod (w^2, 3w)$$

and since $3w = w^9$ in $\tilde{S}$, we can conclude that $(1 - d)(w(\kappa t - \kappa \lambda))$ is equal to $tw^3 \kappa \lambda^2 + tw^3 \kappa \lambda$ plus a sum $\Sigma$ of multiplies of higher powers of $w$ in $\tilde{S}$.

Thus,

$$(1 - d)(w(\kappa t - \kappa \lambda)) - (tw^3 \kappa \lambda^2 + tw^3 \kappa \lambda) + \Sigma \in \text{Im}(\kappa) \subset \tilde{K}(n)^* BG.$$  

(55)

But now note that $1 - d$ respects the $w$-filtration in $\tilde{K}(n)^* BG$. Thus, $(1 - d)(w(\kappa t - \kappa \lambda))$ is in filtration $\geq 1$, and hence the same holds for the entire left-hand side of (55). We claim that an element in $\text{Im}(\kappa)$ which also has $w$-filtration $\geq 0$, is zero.
To this end, recall that the $w$-filtration is the filtration on $\tilde{K}(n)^*BG$ associated with the Hochschild–Serre spectral sequence of the fibration

$$B\mathbb{L}/p\{a_{ij}\} \to BG \to B\mathbb{L}/p\{c\}.$$ 

The $E_2$-term of this spectral sequence is

$$H^*(\mathbb{Z}/p\{c\}, \tilde{K}(n)^*B\mathbb{L}/p\{a_{ij}\}).$$

In this $E_2$-term, an element $\tau(q)$ is weakly represented in filtration 0 by the element $(1 + c + \cdots + c^{p-1})q$ (for the meaning of "weakly represented", see Section 2). Since, in our case, $\tau(q)$ is in filtration $> 0$, we have

$$(1 + c + \cdots + c^{p-1})q = 0.$$ 

However, since $H^1(\mathbb{Z}/p\{c\}, \tilde{K}(n)^*B\mathbb{L}/p\{a_{ij}\}) = 0$, for any $q \in \tilde{K}(n)^*B\mathbb{L}/p\{a_{ij}\}$, $(1 + c + \cdots + c^{p-1})q = 0$ implies $q \in \text{Im}(1 - c)$, which implies $\tau(q) = 0$. Thus, an element in $\text{Im}(\tau)$ represented in filtration $\geq 1$ in $\tilde{K}(n)^*BG$ is zero, as claimed.

We conclude that

$$(1 - d)(w(kt - z)) - (tw^2c^2 + tw^3c^2) + C = 0 \in \tilde{K}(n)^*BG. \quad (56)$$

Thus, an element $\zeta$ of the form $twk\lambda^2 + twk^2\lambda$ plus multiples of higher powers of $w$ is in the kernel of $1 + d + \cdots + d^{p-1}$ in $\tilde{K}(n)^*BG$. Indeed, choose $\zeta$ so that

$$w^2\zeta = (tw^3k\lambda^2 + tw^3k^2\lambda) - \Sigma.$$ 

Then, by (56), and since $\zeta$ is a multiple of $w$,

$$3(1 + d + d^2)\zeta = w^3(1 + d + d^2)\zeta = w^6(1 + d + d^2)((tw^3k\lambda^2 + tw^3k^2\lambda - \Sigma) = 0 \in \tilde{K}(n)^*BG.$$ 

But $\tilde{K}(n)^*BG$ is torsion free. Hence, we have

$$\zeta \in H^1(\mathbb{Z}/p\{d\}, \tilde{K}(n)^*BG).$$

**Lemma 5.4.** $0 \neq \zeta \in H^1(\mathbb{Z}/p\{d\}, \tilde{K}(n)^*BG)$.

**Proof.** Obviously, it suffices to prove the statement with $\tilde{K}(n)^*BG$ replaced by $S$. Now consider the filtration in $S$ by powers of $w$. In the associated graded module $E_0S$, $\zeta$ is represented by an element in filtration degree 1. Now observe that, by (49), (54), the structure formulas for $S$ are the same as the structure formulas in $E_0S$, modulo error terms involving $w^m$ for $m \geq 2$. Thus, if an element in filtration degree 1 is in $\text{Im}(1 - d) \subset S$, then the same element must be in $\text{Im}(1 - d) \subset E_0S$.

In other words, consider the following spectral sequence associated with the (decreasing) $w$-filtration on $S$:

$$E_1 = H^*(\mathbb{Z}/p\{d\}, E_0S) \Rightarrow H^*(\mathbb{Z}/p\{d\}, S).$$

We have shown that, in this spectral sequence,

$$d_1 = 0.$$ 

Thus, it suffices to show that

$$0 \neq \zeta \in H^1(\mathbb{Z}/p\{d\}, E_0S).$$

But this follows from the following result. \(\square\)
Lemma 5.5. Put $T = 1 - d$. We have

$$E_0S \cong \ \mathbb{Z}/p[w]$$

$$\otimes (\mathbb{Z}/p[T]/(T) \otimes \{1, \kappa^2, \lambda\} \oplus \mathbb{Z}/p[T]/(T^2) \otimes \{\kappa, \kappa^2, \lambda\} \otimes \mathbb{Z}/p[T]/(T^3) \otimes \{\kappa^2\})$$

$$\otimes (\mathbb{Z}/p[T]/(T) \otimes \{1, z^2t^2\} \oplus \mathbb{Z}/p[T]/(T^2) \otimes \{z, z^2t\} \oplus \mathbb{Z}/p[T]/(T^3) \otimes \{z^2\}).$$

**Remark.** Neil Strickland points out that in the case $p = 3$, one can compute more precisely:

$$z^3 - wz^2 = \frac{1}{9}\tau(y^2z + y^8z^3)$$

$$\kappa^3 + w^6\kappa = \tau(y + \frac{1}{3} (5\sqrt{-2} - 2)z^4y^5)$$

Here $\sqrt{-2}$ is a 3-adic integer, congruent to 4 mod 9, so that $\frac{1}{3} (5\sqrt{-2} - 2)$ is again a 3-adic integer.

6. ON COMPLEX ORIENTED COHOMOLOGY OF $BO(k)$

In contrast with the previous section, we will focus on $p = 2$. Let $E$ be a complex oriented spectrum. Recall that

$$E^*BU(k) = E^*[[c_1, \ldots, c_k]].$$

Here $c_i$ are the Chern classes, which can be characterized as follows. Let

$$(\mathbb{CP}^\infty)^\times \to BU(k)$$

be the standard inclusion. Then for any complex oriented spectrum $E$, the map in $E^*$-cohomology induced by (57) is injective. We can write

$$E^*(\mathbb{CP}^\infty)^\times \cong E^*[[x_1, \ldots, x_k]],$$

and then $c_i$ is the $i$th elementary symmetric polynomial in the indeterminates $x_1, \ldots, x_k$. Now let $i(x) = -x$ be the inverse function for the formal group law $F$. Define

$$\tilde{c}_j = \sigma_j(i(x_1), \ldots, i(x_k))$$

($\sigma_i$ is the $i$th elementary symmetric polynomial). Now we have the complexification map

$$\phi: BO(k) \to BU(k).$$

Since the complexification of the canonical $k$-bundle on $BO(k)$ (as the complexification of any real bundle) is self-conjugate, in $E^*BO(k)$ we have

$$c_i = \tilde{c}_i.$$

**Definition 6.1.** Call a complex oriented spectrum $E$ 2-generic if $E^*$ is 2-torsion-free and $E^*[[x]]/[2]x$ is flat over $E^*$.

The following result has been known in many special cases (see [11, 5, 7]).
THEOREM 6.2. Let E be a 2-generic complex oriented spectrum. Then
\begin{equation}
E^*BO(k) = E^*[[c_1, \ldots, c_k]]/(c_i - \tilde{c}_i, \ i = 1, \ldots, k) 
\end{equation}
(58)
\begin{equation}
E_{red}^*BO(k)_+ = \Sigma^2k E^*[[c_1, \ldots, c_k]]/\left(c_i - \tilde{c}_i, \ i = 1, \ldots, k - 1, \frac{c_k - \tilde{c}_k}{c_k}\right). 
\end{equation}
(59)
Moreover, both
\[ E^*BO(k), E_{red}^*BO(k)_+ \]
are 2-torsion free.

The proof of Theorem 6.2 will contain several ingredients. First of all, we need an idea what the relations \(c_i = \tilde{c}_i\) look like.

LEMMA 6.3. In \(E^*BO(k)\), we have
\[ c_k - \tilde{c}_k = q_k c_k \]
for a certain power series \(q_k\) in the \(c_i\)'s. Further, we have
\[ q_{k-1} \equiv 2 - q_k \ mod \ c_k. \]
(60)

Proof. In \(E^*BO(k)\), we have
\[ \tilde{c}_k = \prod_{i=1}^{k} i(x_i). \]
(61)
On the right hand side, the \(i\)th factor is a power series in \(x_i\) without a constant term. Thus, the product is divisible by \(c_k = \prod_{i=1}^{k} x_i\). Put
\[ t_k = \frac{\tilde{c}_k}{c_k}. \]
To compare \(t_k\) and \(t_{k-1}\), consider the map fixing \(x_1, \ldots, x_{k-1}\) and sending \(x_k\) to zero. Under this map, \(i(x_i)/x_i\) remains fixed for \(j < k\), while \(i(x_k)/x_k \mapsto -1\). Thus,
\[ t_{k-1} \equiv -t_k \mod x_k \]
as claimed. \(\Box\)

Next, we must develop an inductive model of \(BO(k)\). To this end, recall the standard Gysin cofibration sequence
\[ BO(k - 1)_+ \to BO(k)_+ \to BO(k)_+^\gamma \]
(62)
where \(\gamma_k\) is the canonical real \(k\)-bundle on \(BO(k)\) and \(X^\gamma\), for a vector bundle \(\gamma\) over a space \(X\) denotes the Thom space of the bundle \(\gamma\).

But now, note that we can apply \((?)^\gamma_k\) to the sequence (62). We obtain a cofibration sequence
\[ BO(k - 1)_+^\gamma \to BO(k)_+^\gamma \to BO(k)_+^\gamma \cdot \Sigma BO(k - 1)_+^\gamma. \]
Since the pullback of \(\gamma_k\) to \(BO(k - 1)\) is obviously \(\gamma_{k-1} \oplus 1\), we can rewrite this as
\[ BO(k)_+^\gamma \to BO(k)_+^\gamma \cdot \Sigma BO(k - 1)_+^\gamma. \]
(63)
Moreover, note that \(2\gamma_k\), as the double of any real bundle, is complex oriented, and hence smashing with or taking a function spectrum of (63) into a complex-oriented spectrum will
identify $BO(k)^{2n}$ with $\Sigma^{2k}BO(k)_+$. Our point is that we can assemble (62), (63) into an exact couple (and hence spectral sequence) calculating a complex-oriented cohomology theory on $BO(k)$, from the same cohomology theory on $BO(k - 1)_+, T^{\infty-1}BO(k - 1)_+$. (Of course, there will be an analogous spectral sequence calculating the cohomology of $T^{\infty}BO(k)_+$, but we only need to consider one of these.)

To obtain the spectral sequence, simply apply $F(?, E)$, with $E$ a complex oriented spectrum, to (62), (63). We obtain (co)fibration sequences of spectra:

\[
F(BO(k)_+, E) \rightarrow F(BO(k - 1)_+, E) \rightarrow F(BO(k)_+, \Sigma E)
\]

\[
F(BO(k)_+, \Sigma E) \rightarrow F(BO(k - 1)_+, E) \rightarrow F(\Sigma^{2k - 7}BO(k)_+, E)
\]

\[
F(\Sigma^{2k - 2}BO(k)_+, E) \rightarrow F(\Sigma^{2k - 2}BO(k - 1)_+, E) \rightarrow F(\Sigma^{2k - 2}BO(k)_+, \Sigma E)
\]

\[
F(\Sigma^{2k - 2}BO(k)_+, \Sigma E) \rightarrow F(\Sigma^{2k - 2}BO(k - 1)_+, E) \rightarrow F(\Sigma^{2k - 4}BO(k)_+, E)
\]

(64)

(65)

(The pattern then becomes periodic). An exact couple is extracted from this pattern by setting the left and right terms to be $D$, while the middle terms are $E$. Now we are facing the same issue as in Section 3: since the resolution is 2-periodic, there is one homological and one cohomological exact couple. Similarly as in the Eilenberg–Moore case, the cohomological exact couple is the convergent one: this is because the composition of the obvious maps

\[E^*BO(k) \rightarrow E^*BO(k)_+ \rightarrow E^*BO(k)_+^{\Sigma_2}\]

is (by definition) $c_k$, and the inverse limit of these maps is 0. (This can be established, for example by comparing the Atiyah–Hirzebruch spectral sequences for $E^*BU(k)$ and $E^*BO(k)$: multiplication by $c_k$ increases filtration degree by $\geq 2k$.)

Thus, we have proved the following result:

**Lemma 6.4.** For a complex-oriented spectrum $E$, there is a conditionally convergent spectral sequence

\[E^p_1 = E_{p,q}BO(k),\]

where

\[E^p_1 = \begin{cases} E^{-m(2k - 7) + r}(BO(k - 1))_+ & \text{for } p = 2m \\ E^{-m(2k - 2) + r}(BO(k - 1))_+^{\Sigma_2} & \text{for } p = 2m + 1. \end{cases}\]

By applying the Thom diagonal, we see that the spectral sequence of Lemma 6.4 is a spectral sequence of $E^*BO(k)$-modules, and hence of $E^*BU(k)$-modules.

**Proof of Theorem 6.2.** An induction on $k$. First of all, when $k = 1$, $O(1) = \mathbb{Z}/2$, and the statement is obvious. Next, assume the statements are true for $k - 1$. To prove the statement for $BO(k)$, we will use the spectral sequence of Lemma 6.4. We claim that this spectral sequence collapses to $E_2$. We know the $E_1$-term, need to calculate $d_1$. In effect, by the module spectral sequence structure, it suffices to compute the image under $d_1$ of the element $1 \in E_1^{p,q}BO(k) = E^{p,q}BO(k - 1)$ to $E_2^{p,q}BO(k) = E^{p,q}_2BO(k)$. It is easy to see that $1 \in E_1^{p,q}$ is represented by $c_k \in E^*BO(k)$, hence all elements of $E_1^{p,q}$ are permanent cycles. Thus, to prove that the spectral sequence collapses to $E_2$, it suffices to prove

\[d_1 : E^*\text{red}BO(k - 1)^{\Sigma_2} \rightarrow E^*BO(k - 1) \text{ is injective.} \]

(65)
To this end, consider the diagram

\[
\begin{array}{cccc}
\Sigma BO(k-1)^{2^{m-1}} & \longrightarrow & BO(k) \\
\downarrow & & \downarrow \\
\Sigma^2 BO(k-1)^{2^{m-1}} & \longrightarrow & BO(k) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Sigma^2 BO(k-1)^{2^{m-1}}.
\end{array}
\]

(66)

Here the columns are cofibration sequences, and the unlabelled maps are the obvious ones. \(u\) is the inclusion,

\[ d_1 = (wv)^* \]  

Moreover, the middle column is obtained by pulling the usual Gysin cofibration sequence associated with the fibration

\[ BO(k-1) \rightarrow BO(k) \]

back to \( BO(k-1) \subset BO(k) \), and applying \((?)^{2^m}\). Thus, \( X \) is the homotopy pushout of the diagram

\[
\begin{array}{cccc}
\Sigma^2 BO(k-2)^{2^{m-1}} & \longrightarrow & \Sigma^2 BO(k-1)^{2^{m-1}} \\
\downarrow & & \downarrow \\
\Sigma^2 BO(k-1)^{2^{m-1}} & \longrightarrow & X.
\end{array}
\]

(67)

Now by inspection, \( su \) is the sum of the two canonical maps \( i, j: \Sigma^2 BO(k-1)^{2^{m-1}} \rightarrow X \) (see (67)), while \( tt = tj = Id \). We conclude that

\[ wvu = tsu = 2. \]

Since \( \overline{K}(\eta)^*_{red} BO(k-1)^{2^{m-1}} \) is 2-torsion free, 2 is injective, and hence so is \( d_1 \).

Thus, the spectral sequence of Lemma 6.4 collapses to \( E_2 \). Next, we need to show that

\[ d_1(1) = \eta \cdot \frac{c_k - \overline{c}_k}{c_k} \quad \text{for a unit} \ \eta. \]

(68)

(This implies the inductive statement about \( E^* BO(k) \), and also about \( E^*_{red} BO(k)^{2^m} \) by factoring out \( E^* BO(k-1) \).

To see (68), let \( I \) (resp. \( J \)) be the ideal in \( E^* BO(k-1) \) generated by \( (c_k - \overline{c}_k)/c_k \) (resp. \( d_1(1) \)). Since the relation \( c_k - \overline{c}_k \) holds in \( E^* BO(k) \), we have

\[ I \subseteq J. \]

(69)

Next, observe that

\[ u^* I = u^* J = 2E^*_{red} BO(k-1)^{2^{m-1}} \]

(by Lemma 6.3). But since \( u^* \) is injective on \( J \), equality must hold in (69), and hence (68) follows.
The only remaining statement to be proved is that
\[ A_k = E^*[[c_1, \ldots, c_k]]/(c_i - \tilde{c}_i) \]
and
\[ T\!A_k = E^*[[c_1, \ldots, c_k]]/(c_i - \tilde{c}_i, i = 1, \ldots, k, (c_k - \tilde{c}_k)/c_k) \text{ are 2 torsion free.} \]

First of all, we already proved that there is a short exact sequence
\[ 0 \to T\!A_k \to A_k \to A_{k-1} \to 0 \]
so, by the induction hypothesis, it suffices to consider \( T\!A_k \).

Now consider the projection
\[ \phi: T\!A_k \to T\!A_k/(c_k - 1). \]
We claim that \( T\!A_k/(c_k - 1) \) is torsion free. In effect, consider the filtration of \( T\!A_k/(c_k - 1) \) by powers of \((c_k)\). The associated graded object is
\[ T\!A_{k-2}[[c_k]] \]
by Lemma 6.3, which is 2-torsion free by the induction hypothesis.

Thus, it suffices to prove
\[ \text{Ker } \phi \text{ is 2-torsion free.} \] (71)

Now let \( t = c_k - 1 \). We need to prove
\[ 2t \neq 0 \in T\!A_k. \] (72)
Without loss of generality (by periodicity),
\[ \deg(t) = 0. \]
Thus,
\[ c_k - 1 \neq 0 \in A_{k-1}/2. \] (73)

Now consider the map
\[ \psi: T\!A_k/\text{Ker}(c_k - 1) \to E^*[[c_1, \ldots, c_k - 1, x_k]]/(c_1 - c_1, \ldots, c_k - 1, \tilde{c}_k, x_k) \]
\[ 1 \to \frac{1}{c_k - 1}. \]
Here by \( \tilde{c}_i \) we mean \( \sigma_i(i(x_1), \ldots, i(x_k)) \), written as a polynomial in \( c_1, \ldots, c_k, x_k \). Note that by (73),
\[ \psi(z) \neq 0. \] (74)

But now note that the right hand side of \( \psi \) is isomorphic to
\[ A_{k-1} \otimes E^* T\!A_1. \]
Here the generators of \( A_{k-1} \) are \( q_i = \sigma_i(x_1, \ldots, x_{k-1}) \). Put also
\[ \tilde{q}_i = \sigma_i(i(x_1), \ldots, i(x_{k-1})). \]
Now to verify the compatibility of the relations, compute:
\[ c_i - \tilde{c}_i = q_i - \tilde{q}_i + \sum_{1 \leq a_1 < \ldots < a_k \leq k - 1} (x_{a_1} \ldots x_{a_k} - i(x_{a_1}) \ldots i(x_{a_k}) i(x_k)) \]
\[ = q_i - \tilde{q}_i + x_k(q_{i-1} + \tilde{q}_{i-1}). \]
Now by the induction hypothesis, $2\psi(z) \neq 0$, hence $z \neq 0 \in TA_k/Ker(c_{k-1})$. Thus, $2t = 2c_{k-1}z \neq 0 \in TA_k$, as claimed, (70) is proved.

Acknowledgements—I am very indebted to Neil Strickland for valuable comments, for his substantial help in verifying the calculational aspects of this paper and finally for detecting a fatal error in a previous version of this paper with the aid of a supercomputer at MIT. I am very grateful to Geoffrey Falk, Kevin Lee, Doug Ravenel, Brooke Shipley and Steve Wilson for helpful discussions.

REFERENCES


Department of Mathematics
University of Michigan
East Hall
525 East University Avenue
MI 48109-1109
U.S.A.