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Some Congruence Properties of Three Well-Known Sequences: Two Notes

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The three sequences mentioned in the title are Ramanujan's τ-function, the coefficients c_n of Klein, Fricke, and Shimura, and the sequence a_n of Apèry numbers. In the first note, it is shown that $c_n \equiv \tau(n) (\text{mod } 11)$. In the second note it is shown that for a prime p, $a_{p+1} \equiv 25 + 60p (\text{mod } p^2)$.

A CONNECTION BETWEEN $\tau(n)$ AND THE COEFFICIENTS c_n OF KLEIN, FRICKE, AND SHIMURA

Recall that Ramanujan's τ-function is defined by

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \tag{1}$$

and that Klein and Fricke defined the c_n by

$$\sum_{n=1}^{\infty} c_n x^n = x \prod_{n=1}^{\infty} (1 - x^n)^2 (1 - x^{11n})^2. \tag{2}$$

Shimura (*Crelle's J.*, 1965) made a beautiful application of the c_n to reciprocity laws in nonsolvable extensions of the field of rational numbers.

Both c_n and $\tau(n)$ are known to be multiplicative. For a prime p, Hasse has shown that

$$|c_p| < 2p^{1/2} \tag{3}$$

and only recently Deligne has established that

$$|\tau(p)| < 2p^{11/2}.$$

In this note it is observed that $c_n \equiv \tau(n)$ (mod 11) and it is shown that this

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congruence when combined with other known congruences, allows one to easily compute for a prime p, the values of c_p from the values of $\tau(p)$ up to at least $p = 757$.

PROPOSITION. $c_n \equiv \tau(n) \pmod{11}$.

Proof. For power series $f(x)$ and $g(x)$ with integral coefficients, write $f(x) \equiv g(x) \pmod{n}$, if for each nonnegative integer k, the coefficients of x^k in $f(x)$ and $g(x)$ are congruent (mod n). Since $1 - x^{11n} \equiv (1 - x^n)^{11} \pmod{11}$, from (2)

$$\sum c_n x^n \equiv x \prod (1 - x^n)^2 (1 - x^n)^{22} \pmod{11}.$$

But by (1), the right side is $\sum \tau(n) x^n$.

Other known congruences for c_n when the modulus is 2 or 5 are as follows: S. Chowla and M. J. Cowles (*Crelle's J.*, 1977) proved that for a prime $p \neq 11$, (a) if $p \equiv 2, 6, 7, 8$, or $10 \pmod{11}$, then $2 \mid c_p$; and (b) if $p \equiv 1, 3, 4, 5$, or $9 \pmod{11}$, then $2 \mid c_p$ if and only if there are integers u and v such that $p = u^2 + 11v^2$. It is also know that for a prime $p \neq 11$, $c_p \equiv p + 1 \pmod{5}$. Curiously enough it seems hard to get a reference for this last result although it is well known to those acquainted with the theory of modular forms.

As an example of a computation of c_p from $\tau(p)$, take $p = 251$. Since $251 \equiv 9 \pmod{11}$ and $251 \neq u^2 + 11v^2$, from (b) above, c_{251} is an odd integer. Also $c_p \equiv 2 \pmod{5}$ and from (3), $|c_{251}| \leqslant 31$. Hence c_{251} is one of the numbers: -23, -13, -3, 7, 17, and 27. Now from D. H. Lehmer's table (*Duke Math. J.*, 1943) for $\tau(n)$ ($n \leqslant 300$), $\tau(251) = 12983053545252$. Since $(2 + 2 + 4 + 3 + 0 + 8 + 2) - (5 + 5 + 5 + 5 + 3 + 9 + 1) \equiv 10 \pmod{11}$, $c_{251} \equiv \tau(251) \equiv 10 \pmod{11}$. Thus $c_{251} = -23$. In this way one can verify Trotter's table for c_p ($p \leqslant 2000$) in Shimura's paper (*Crelle's J.*, 1965) up to $p = 757$. This method of determining the values of c_p from those of $\tau(p)$ (found by Lehmer for $p \leqslant 10{,}000$, but not published) could be extended if congruences for moduli other than 2, 5, and 11 were known.

A CONGRUENCE ON APÈRY NUMBERS

Apèry introduced the numbers a_n ($n \geqslant 0$) defined by the recurrence relation

$$a_0 = 1, \qquad a_1 = 5,$$

and

$$n^3 a_n - (34n^3 - 51n^2 + 27n - 5)\, a_{n-1} + (n - 1)^3\, a_{n-2} = 0 \tag{4}$$

in his proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} (1/n^3)$. One of the startling facts established by Apery is that all of the a_n's are integers. In [1] it is shown that

$$\text{for all primes } p, \qquad a_p \equiv 5 \pmod{p^2} \tag{5}$$

In this note it is shown that for all primes p,

$$a_{p+1} \equiv 25 + 60p \pmod{p^2}$$

LEMMA. *For $n \geqslant 2$, $a_{n+1} \equiv (5 + 12n)\, a_n \pmod{n^2}$.*

Proof. From (4),

$$(n+1)^3 a_{n+1} - [34(n+1)^3 - 51(n+1)^2 + 27(n+1) - 5]\, a_n + n^3 a_{n-1} = 0.$$

Then

$$(3n+1)\, a_{n+1} - [34(3n+1) - 51(2n+1) + 27(n+1) - 5]\, a_n \equiv 0 \pmod{n^2};$$

and so $(3n+1)\, a_{n+1} \equiv (5 + 27n)\, a_n \pmod{n^2}$. Multiply both sides by $1 - 3n$ and use the fact that $1 - 9n^2 \equiv 1 \pmod{n^2}$, thus obtaining: $a_{n+1} \equiv (5 + 12n)\, a_n \pmod{n^2}$.

PROPOSITION. *For all primes p, $a_{p+1} \equiv 25 + 60p \pmod{p^2}$.*

Proof. By the lemma, $a_{p+1} \equiv (5 + 12p)\, a_p \pmod{p^2}$ and from (5), $a_p \equiv 5 \pmod{p^2}$.

COROLLARY. *For all primes p, $a_{p+1} \equiv 25 \pmod{p}$.*

REFERENCES

1. S. CHOWLA, J. COWLES, AND M. COWLES, Congruence properties of Apèry numbers, to appear.