Models with second order properties V: A general principle

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Abstract


We present a general framework for carrying out the constructions in [2–10] and others of the same type. The unifying factor is a combinatorial principle which we present in terms of a game in which the first player challenges the second player to carry out constructions which would be much easier in a generic extension of the universe, and the second player cheats with the aid of ♦. Section 1 contains an axiomatic framework suitable for the description of a number of related constructions, and the statement of the main theorem 1.9 in terms of this framework. In Section 2 we illustrate the use of our combinatorial principle. The proof of the main result is then carried out in Sections 3–5.

Contents

1. Uniform partial orders

We describe a class of partial orderings associated with attempts to manufacture an object of size $\lambda^+$ from approximations of size less than $\lambda$. We also introduce some related notions motivated by the forcing method. The underlying idea is that a sufficiently generic filter on the given partial ordering should give rise to the desired object of size $\lambda^+$.

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We describe a game for two players, in which the first player imposes genericity requirements on a construction, and the second player constructs an object which meets the specified requirements. The main theorem (1.9) is that under certain combinatorial conditions the second player has a winning strategy for this game.

2. Illustrative application

We illustrate the content of our general principle with an example. We show the completeness of the logic \( \mathcal{L}^{<\omega} \), defined by Magidor and Malitz [2] for the \( \lambda^+ \)-interpretation assuming the combinatorial principles Di and \( \diamond_{\lambda^+} \).

3. Commitments

We give a preliminary sketch of the proof of Theorem 1.9. We then introduce the notion of ‘basic data’ which is a collection of combinatorial objects derived from Di and an object called a commitment describing the main features of the second player’s strategy in a given play of the genericity game. We state the main results concerning commitments, and show how Theorem 1.9 follows from these results.

4. Proofs

We prove the propositions stated in Section 3 except we defer the proof of Propositions 3.6 and 3.7 to Section 5. We use Di to show that a suitable collection of ‘basic data’ exists. Then we verify some continuity properties applying to our strategy at limit ordinals.

5. Proof of Proposition 3.7

We prove Proposition 3.7 as well as Proposition 3.6.

Notation

If \((A_\alpha : \alpha < \delta)\) is a increasing sequence of sets we write \(A_{<\delta} = \bigcup_{\alpha < \delta} A_\alpha \).

Throughout the paper, \(\lambda\) is a cardinal such that \(\lambda^{<\lambda} = \lambda\).

\[\mathcal{P}_{<\lambda}(A) = \{B \subseteq A : |B| < \lambda\}\]

\(\text{otp}(u)\) will mean the order type of \(u\). Trees are well-founded, and if \(T\) is a tree, \(\eta \in T\), we write \(\text{len}(\eta)\) for \(\text{otp}\{v \in T : v < \eta\}\) (the level at which \(\eta\) occurs in \(T\)).

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1. Uniform partial orders

We will present an axiomatic framework for the construction of objects of size \( \lambda^+ \) from approximations of size less than \( \lambda \), under suitable set-theoretical hypotheses. The basic idea is that we are constructing objects which can fairly easily be forced to exist in a generic extension, and we replace the forcing construction by the explicit construction of a sufficiently generic object in the ground model.

We begin with the description of the class of partial orderings to which our methods apply. Our idea is that an ‘approximation’ to the desired final object is built from a set of ordinals \( u \in \lambda^+ \) of size less than \( \lambda \). Furthermore, though there will be many such sets \( u \), there will be at most \( \lambda \) constructions applicable to an arbitrary set \( u \). We do not axiomatize the notion of a ‘construction’ in any detail; we merely assume that the approximations can be coded by pairs \( (\alpha, u) \), where \( \alpha < \lambda \) is to be thought of as a code for the particular construction applied to \( u \). An additional feature, suggested by the intuition just described, is captured in the ‘indiscernibility’ condition below, which is a critical feature of the situation—though trivially true in any foreseeable application.

**Definition 1.1.** A standard \( \lambda^+ \)-uniform partial order is a partial order \( \leq \) defined on a subset \( P \) of \( \lambda \times P_{<\lambda}(\lambda^+) \) satisfying the following conditions, where for \( p = (\alpha, u) \) in \( P \) we write \( \text{dom} p = u \), and call \( u \) the domain of \( p \).

1. If \( p \leq q \) then \( \text{dom} p \subseteq \text{dom} q \).
2. For all \( p, q, r \in P \) with \( p \leq q \leq r \) there is \( r' \in P \) so that \( p, q \leq r' \leq r \) and \( \text{dom} r' = \text{dom} p \cup \text{dom} q \).
3. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \), then it has a least upper bound \( q \), with domain \( \bigcup_{i<\delta} \text{dom} p_i \); we will write \( q = \bigcup_{i<\delta} p_i \), or more succinctly: \( q = p_{\delta} \).
4. For all \( p \in P \) and \( \alpha < \lambda^+ \) there exists a \( q \in P \) with \( q \leq p \) and \( \text{dom} q = \text{dom} p \cap \alpha \); furthermore, there is a unique maximal such \( q \), for which we write \( q = p^\uparrow \alpha \).
5. For limit ordinals \( \delta \), \( p \uparrow \delta = \bigcup_{\alpha<\delta} p \uparrow \alpha \).
6. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \), then
   \[ \left( \bigcup_{i<\delta} p_i \right) \uparrow \alpha = \bigcup_{i<\delta} (p_i \uparrow \alpha). \]
7. (Indiscernibility) If \( p = (\alpha, u) \in P \) and \( h : v \rightarrow v' \subseteq \lambda^+ \) is an order-isomorphism then \( (\alpha, v') \in P \). We write \( h[p] = (\alpha, h[v]) \). Moreover, if \( q \leq p \) then \( h[q] \leq h[p] \).
8. (Amalgamation) For every \( p, q \in P \) and \( \alpha < \lambda^+ \), if \( p \uparrow \alpha \leq q \) and \( \text{dom} p \cap \text{dom} q = \text{dom} p \cap \alpha \), then there exists \( r \in P \) so that \( p, q \leq r \).

It should be remarked that a standard \( \lambda^+ \)-uniform partial order comes with the additional structure imposed on it by the domain and restriction functions. We
will call a partial order $\lambda^+-\text{uniform}$ if it is isomorphic to a standard $\lambda^+$-uniform partial ordering. It follows that although a $\lambda^+$-uniform is isomorphic to a standard one as a partial order, there will be an induced notion of domain and restriction. The elements of such a partial order will be called approximations, rather than ‘conditions’, as we are aiming at a construction in the ground model.

Observe that $p \upharpoonright \alpha = p$ iff $\text{dom } p \subseteq \alpha$. Note also that for $p \subseteq q$ in $P$, $p \upharpoonright \alpha \subseteq q \upharpoonright \alpha$. (As $p \upharpoonright \alpha$, $q \upharpoonright \alpha \subseteq q$, there is $r \subseteq q$ in $P$ with $p \upharpoonright \alpha$, $q \upharpoonright \alpha \subseteq r$ and $\text{dom } r = \text{dom } p \upharpoonright \alpha \cup \text{dom } q \upharpoonright \alpha - \text{dom } q \upharpoonright \alpha$; hence $r = q \upharpoonright \alpha$ by maximality of $q \upharpoonright \alpha$, and $p \upharpoonright \alpha \subseteq q \upharpoonright \alpha$.)

It is important to realize that in intended applications there will be $\lambda$-many comparable elements of a $\lambda^+$-uniform partial order which have the same domain (see the first example of the next section).

Typically the only condition that requires attention in concrete cases is the amalgamation condition. It is therefore useful to have a weaker version of the amalgamation property available which is sometimes more conveniently verified, and which is equivalent to the full amalgamation condition in the presence of the other (trivial) hypotheses. Such a version is:

**Weak Amalgamation.** For every $p, q \in P$, and $\alpha < \lambda^+$, if $p \upharpoonright \alpha \subseteq q$, $\text{dom } p \subseteq \alpha + 1$, and $\text{dom } q \subseteq \alpha$, then there exists $r \in P$ with $p, q \subseteq r$.

To prove amalgamation from weak amalgamation, we define a continuous increasing chain of elements $r_\beta \in P$ for $\beta \geq \alpha$ so that

1. $\text{dom}(r_\beta) \subseteq \beta$, and
2. $r_\beta \geq p \upharpoonright \beta$, $q \upharpoonright \beta$.

Let $r_\alpha = q \upharpoonright \alpha$. For limit ordinals, use conditions 3 and 5 of the definition of uniform partial order.

Suppose we have defined $r_\beta$ and $\beta \notin \text{dom}(p) \cup \text{dom}(q)$. Let $r_{\beta + 1} = r_\beta$.

If $\beta \in \text{dom}(q) \setminus \text{dom}(p)$ then $p \upharpoonright \beta + 1 = p \upharpoonright \beta$. Applying weak amalgamation to $r_\beta$ and $q \upharpoonright \beta$. Using condition 2 of the definition now, we can define $r \upharpoonright \beta + 1$.

If $\beta \in \text{dom}(p) \setminus \text{dom}(q)$ then we can apply weak amalgamation to $p \upharpoonright \beta + 1$ and $r_\beta$.

Since these are all the possibilities, let $\gamma = \sup(\text{dom}(p) \cup \text{dom}(q))$ and so $r_\gamma \geq p, q$. This verifies amalgamation.

**Notation.** For $p, q \in P$ we write $p \leq_{sd} q$ to mean $p \subseteq q$ and $\text{dom } p = \text{dom } q$. (Here ‘sd’ stands for ‘same domain’.) If $p, q \in P$ then we write $p \perp q$ if $p$ and $q$ are incompatible, i.e., there is no $r$ so that $p \subseteq r$ and $q \subseteq r$.

We define the collapse $p^{\text{col}}$ of an approximation as $h[p]$ where $h$ is the canonical order isomorphism between $\text{dom } p$ and $\text{otp}(\text{dom } p)$.
**Convention.** For the remainder of this section we fix a standard $\lambda^+$-uniform partial order $\mathbb{P}$, and we let

$$\mathbb{P}_\alpha = \{ p \in \mathbb{P} : \text{dom } p \subseteq \alpha \}$$

for $\alpha < \lambda^+$. Note that $\mathbb{P}_{\lambda^+} = \mathbb{P}$.

Be forewarned that the following definition does not follow the standard set-theoretic use of the term 'ideal'.

**Definition 1.2.**
1. For $\alpha < \lambda^+$, a $\lambda$-generic ideal $G$ in $\mathbb{P}_\alpha$ is a subset of $\mathbb{P}_\alpha$ satisfying:
   (a) $G$ is closed downward;
   (b) if $Q \subseteq G$ and $|Q| < \lambda$ then $Q$ has an upper bound in $G$; and
   (c) for every $p \in \mathbb{P}_\alpha$, if $p \not\in G$ then $p$ is incompatible with some $q \in G$.

Gen$(\mathbb{P}_\alpha)$ is the set of $\lambda$-generic ideals of $\mathbb{P}_\alpha$.

2. If $G \in \text{Gen}(\mathbb{P}_\alpha)$ then
   $$\mathbb{P}/G = \{ p \in \mathbb{P} : p \text{ is compatible with every } r \in G \}.$$

Note that $p \in \mathbb{P}/G$ iff $p \uparrow \alpha \in G$.

3. We say an increasing sequence $(g_i : i < \delta)$ is cofinal in $G \in \text{Gen}(\mathbb{P}_\alpha)$ if $G = \{ r \in \mathbb{P}_\alpha : \text{for some } i, r \in g_i \}$. Every $G \in \text{Gen}(\mathbb{P}_\alpha)$ has a cofinal sequence of length $\lambda$ (possibly constant in degenerate cases). We will often write $(g_\beta)_{\beta < \delta}$ to mean $(g_\beta : \beta < \lambda)$.

4. We will say that $G$ is generic if $G \in \text{Gen}(\mathbb{P}_\alpha)$ for some $\alpha$.

**Lemma 1.3.** Let $G_i \in \text{Gen}(\mathbb{P}_\alpha)$ for $i < \delta$ be an increasing sequence of sets, and $\alpha = \sup_i \alpha_i$. Then there is a unique minimal $\lambda$-generic ideal of $\mathbb{P}_\alpha$ containing $\bigcup_{i < \delta} G_i$. This ideal will be denoted $G_{i<\delta}$.

**Proof.** We may suppose that $\delta$ is a regular cardinal, $\delta \leq \lambda$. If $\delta = \lambda$ then it is clear that $\bigcup_{i < \delta} G_i \in \text{Gen}(\mathbb{P}_\alpha)$. Suppose now that $\delta < \lambda$. For $i < \delta$ fix an increasing continuous sequence $(g'_i)^{\forall \gamma < \lambda}$ cofinal in $G_i$. Fix $i < j < \delta$. There is a club $C_{ij} \subseteq \lambda$ such that for all $\gamma \in C_{ij}$, $g'_i \uparrow \gamma = g'_j \uparrow \gamma$. Let $C = \bigcap_{i < j < \delta} C_{ij}$. If $\beta \in C$ then define $g_\beta = \bigcup_{i < \beta} g'_i \in \mathbb{P}_\alpha$. Then the downward closure of $(g_\beta : \beta < \lambda)$ is the required generic set in $\mathbb{P}_\alpha$. $\square$

The notion of $\lambda$-genericity is of course very weak. In order to get a notion adequate for the applications, we need to formalize the notion of a uniform family of dense sets.

**Definition 1.4.**
1. For $\alpha < \lambda^+$ and $G \in \text{Gen}(\mathbb{P}_\alpha)$ or $G = \emptyset$ (in which case, in what follows, read $\mathbb{P}$ for $\mathbb{P}/G$) we say
   $$D : \{(u, w) : u \subseteq w \in \mathcal{P}_< \lambda^+ \} \to \mathcal{P}(\mathbb{P})$$
is a density system over $G$ if:

(a) for every $(u, w)$, $D(u, w) \subseteq \{ p \in \mathbb{P}/G : \text{dom } p \subseteq w \}$;

(b) for every $p, q \in \mathbb{P}/G$, if $p \in D(u, w)$, $p \leq q$ and $\text{dom } q \subseteq w$ then $q \in D(u, w)$;

(c) (Density) for every $(u, w)$ and every $p \in \mathbb{P}/G$, with $\text{dom } p \subseteq w$, there is $q \geq p$ in $D(u, w)$; and

(d) (Uniformity) for every $(u_1, w_1)$, $(u_2, w_2)$, if $w_1 \cap \alpha = w_2 \cap \alpha$ and there is an order-isomorphism $h : w_1 \rightarrow w_2$ such that $h[u_1] = u_2$, then for every $p \in \mathbb{P}/G$ with $\text{dom } p \subseteq w_1$

$$p \in D(u_1, w_1) \text{ if and only if } h[p] \in D(u_2, w_2).$$

The term 'density system' will refer to density systems over some $G \in \text{Gen}(\mathbb{P}_\alpha)$, for some $\alpha$, and we write '0-density system' for density system over $\emptyset$.

2. For $G \in \text{Gen}(\mathbb{P}_\alpha)$ and $D$ any density system, we say $G$ meets $D$ if for all $u \in \mathcal{P}_{<\lambda}(\gamma)$ there is $v \in \mathcal{P}_{<\lambda}(\gamma)$ so that $u \subseteq v$ and $G \cap D(u, v) \neq \emptyset$.

We give now two examples of density systems which will be important in the proof of Theorem 1.9. Both examples use the following notion. A closed set $X$ of ordinals will be said to be $\lambda$-collapsed if $0 \in X$ and for any $\alpha \leq \sup X$, $[\alpha, \alpha + \lambda] \cap X \neq \emptyset$. An order-isomorphism $h : Y \leftrightarrow X$ between closed sets of ordinals will be called a $\lambda$-isometry if for every pair $\alpha \leq \beta$ in $Y$ and every $\delta \leq \lambda$, $\beta = \alpha + \delta$ if and only if $h(\beta) = h(\alpha) + \delta$. Every closed set of ordinals is $\lambda$-isometric with a unique $\lambda$-collapsed closed set; the corresponding $\lambda$-isometry will be called the $\lambda$-collapse of $Y$, and more generally the $\lambda$-collapse of any set $Y$ of ordinals is defined as the restriction to $Y$ of the $\lambda$-collapse of its closure. Observe that a $\lambda$-collapsed set of fewer than $\lambda$ ordinals is bounded below $\lambda \times \lambda$ (ordinal product).

**Example 1.5.** We shall show that there is a family $\mathcal{D}$ of at most $\lambda^+$ 0-density systems such that for any $\alpha < \lambda^+$, if $G \in \text{Gen}(\mathbb{P}_\alpha)$ meets all $D \in \mathcal{D}$ then $\mathbb{P}/G$ is again $\lambda^+$-uniform. (The amalgamation property must be verified.)

**Construction.** For $p, q \in \mathbb{P}_{\lambda \times \lambda}$ and $\delta < \lambda \times \lambda$ (where $\lambda \times \lambda$ is the ordinal product), we define a density system $D_{p, q, \delta}$ as follows. Let $u = (\text{dom } p \cup \text{dom } q) \cap \delta$. For

$$u' \subseteq w' \in \mathcal{P}_{<\lambda}(\lambda^+),$$

if there is an order-isomorphism $h : w' \rightarrow w \subseteq \delta$ with $h[u'] = u$, then let

$$D_{p, q, \delta}(u', w') = \{ r : \text{dom } r \subseteq w' \text{ and either there does not exist } s \geq p, q, h[r], \text{ or there exists } s \geq p, q \}
\text{ so that } s \uparrow \delta = h[r].$$

This definition is independent of the choice of $h$.

If there is no such $h$ then let $D_{p, q, \delta}(u', w') = \{ r : \text{dom } p \subseteq w' \}$. We claim that $D_{p, q, \delta}$ is a 0-density system. It suffices to check the density condition for $u \subseteq w \subseteq \delta$, and this is immediate.
Application. We will now show that if $G \in \text{Gen}(\mathbb{P}_\alpha)$ meets every density system of the form $D_{p,q,r,h}$ then $\mathbb{P}/G$ is $\lambda^+$-uniform. In order to view $\mathbb{P}/G$ as encoded by elements of $\mathcal{L} \times \mathbb{P}_{\leq \lambda}(\lambda^+)$, we let $h: \lambda^+ \setminus \alpha \leftrightarrow \lambda^+$ be an order-isomorphism, and replace $(\beta, u)$ in $\mathbb{P}/G$ by $(\beta', h[u \setminus \alpha])$ where $\beta'$ is just a code for the pair $(\beta, u \cap \alpha)$. We need only check the amalgamation condition 8 of the definition.

Let $p, q \in \mathbb{P}/G$, $\beta < \lambda^+$ with $p \upharpoonright \beta \leq q$ and $\text{dom} \ q \cap \text{dom} \ p = \text{dom} \ p \cap \beta$. We must find $r \geq p, q$ with $r \in \mathbb{P}/G$. Let $X = \text{dom} \ p \cup \text{dom} \ q \cup \{\alpha\}$ and let $h_0: X \to X'$ be the $\lambda$-collapse of $X$. Let $p' = h_0[p]$, $q' = h_0[q]$, $\alpha' = h_0(\alpha)$, and $u = \text{dom} \ q \cap \alpha$. Now choose $w \subseteq \alpha$ with $|w| < \lambda$ and $r \in G \cap D_{p', q': \alpha}^\lambda(u, w)$. Since $X'$ is $\lambda$-isomorphic with $X$, we can extend $h_0$ to an order-isomorphism

$$h: X \cup w \to X' \cup w'$$

with $h[w] = w' \subseteq \alpha'$.

We claim that there is $s \geq p', q'$ so that $h[s] = h[r]$. If suffices to find some $s \geq p, q, r$. Since $p \upharpoonright \alpha$, $q \upharpoonright \alpha$, $r$ are all in $G$, we may take $r' \geq p \upharpoonright \alpha$, $q \upharpoonright \alpha$, $r$ in $G$. Since $q \in \mathbb{P}/G$ and $r' \in G$ then by amalgamation we can find $\hat{q} \geq q$, $r'$ with $\text{dom} \ \hat{q} = \text{dom} \ q \cup \text{dom} \ r'$. But now $\text{dom} \ p \cap \text{dom} \ \hat{q} = \text{dom} \ p \cap \beta$ and $p \upharpoonright \beta \leq \hat{q}$, so we can find $s \geq p$, $\hat{q}$. This is the desired $s$.

As $r' \in D_{p', q': \alpha}^\lambda(u, w)$, it now follows that there exists $s \geq p', q'$ so that $s \upharpoonright \alpha' = h[r]$, and hence $h^{-1}[s] \geq p, q$ and $(h^{-1}[s]) \upharpoonright \alpha = r$. So $h^{-1}[s] \in \mathbb{P}/G$ and $h^{-1}[s] \geq p, q$, verifying condition 8 for $\mathbb{P}/G$.

Example 1.6. The next example will be useful in the following situation. Suppose we have $G \in \text{Gen}(\mathbb{P}_\lambda)$, $\beta > \alpha$, and we want to build $G' \supseteq G$ with $G' \in \text{Gen}(\mathbb{P}_\beta)$. To ensure the genericity of $G'$ we must arrange that for all $q \in \mathbb{P}_\beta$, either $q \in G'$ or else $q$ is incompatible with some $g \in G'$. We will find another family of at most $\lambda$-density systems $D_{p,q,r,h}$ which make it possible to construct a suitable $G' \supseteq G$ if $G$ meets all $D_{p,q,r,h}$ (from Example 1.5) and $D_{p,q,r,h}$.

Construction. For $p, q, r \in \mathbb{P}_{\lambda \times \lambda}$, $\delta < \lambda \times \lambda$ such that:

$$p \upharpoonright \delta \leq r; \quad \text{dom} \ r \subseteq \delta; \quad \text{and there does not exist} \ s \geq p, q, r,$$

we define $D_{p,q,r,h}$ as follows.

Let $u = (\text{dom} \ p \cup \text{dom} \ q) \cap \delta$. For $u' \subseteq w' \in \mathbb{P}_{\lambda}(\lambda^+)$, if there is an order-isomorphism $h: w' \to w$ where $w \subseteq \delta$ and $h[u'] = u$ then let

$$D_{p,q,r,h}(u', w') = \{s: \text{dom} \ s \subseteq w' \text{ and } h[s] \text{ is incompatible with } r,$$

or $h[s] \geq r \text{ and there is some } t \geq p \text{ so that}$

$$t \upharpoonright \delta \leq h[s] \text{ and } t \text{ is incompatible with } q\}. $$

If there is no such $h$ then let $D_{p,q,r,h}(u', w') = \{s: \text{dom} \ s \subseteq w'\}$.

We claim that $D_{p,q,r,h}$ is a 0-density system. Again we check only the density condition for $u \subseteq w \subseteq \delta$. So we have $s \in \mathbb{P}$, $\text{dom} \ s \subseteq w$, and $s$ is compatible with $r$. We seek $s' \in D_{p,q,r,h}(u, w)$. 


Choose \( s' \supseteq r, s \) with domain \( \text{dom } r \cup \text{dom } s; \) so \( \text{dom } s' \subseteq \delta. \) Then \( s' \supseteq r \supseteq p \uparrow \delta \) and \( \text{dom } s' \cap \text{dom } p = \text{dom } p \cap \delta, \) so we can choose \( t \supseteq s', p \) so that \( \text{dom } (t) = \text{dom } s' \cup \text{dom } p, \) and hence \( t \) is incompatible with \( q \) (since there is no \( t' \supseteq p, q, r). \) Now \( t \uparrow \delta \supseteq s' \supseteq r, s, \) so if \( s'' = t \uparrow \delta \) then \( s'' \supseteq r, s, \) and \( s'' \in D_{p,q,r} (u', w') \) as desired.

**Application.** We return to the situation in which we have \( G \in \text{Gen}(\mathbb{P}_\alpha), \beta > \alpha, \) and we want to build \( G' \supseteq G \) with \( G' \in \text{Gen}(\mathbb{P}_\beta), \) assuming that \( G \) meets all \( D_{p,q,r} \) and \( D_{p,q,r} (u', w''). \) We will naturally take \( G' \) to be the downward closure of a sequence \((g_i)_{\alpha \prec \lambda}\) which is constructed inductively, taking suprema at limit ordinals. At successor stages, suppose that the \( \alpha \)th term of our sequence has just been constructed, and let \( p = g_\alpha. \) Suppose \( q \in \mathbb{P}_\beta \) is fixed. We wish to ‘decide’ \( q: \) that is, we seek \( \hat{p} \supseteq p \) so that either \( \hat{p} \) is incompatible with \( q, \) or else \( \hat{p} \supseteq q. \)

If \( p \) is already incompatible with \( q \) then let \( \hat{p} = p. \) Otherwise, let \( X = \text{dom } p \cup \text{dom } q \cup \{ \alpha \} \) and let \( h: X \rightarrow X' \) be the \( \lambda \)-collapse of \( X. \) Let \( p' = h[p], q' = h[q], \) and \( \alpha' = h(\alpha). \) If \( u = X \cap \alpha, \) choose \( w \supseteq u \) and \( r \in G \cap D_{p,q,r} (u, w). \) Extend \( h \) to an order-preserving function from \( X \cup w \) to \( X' \cup w' \subseteq \alpha', \) and let \( r' = h[r]. \)

Suppose first that there is some \( s \supseteq p', q', r'. \) We may suppose that \( \text{dom } s = \text{dom } p' \cup \text{dom } q'. \) In this case let \( \hat{p} = h^{-}[s]. \) As \( \hat{p} \uparrow \alpha \subseteq r, \) we have \( \hat{p} \in \mathbb{P}/G, \) and \( q \) is decided by \( \hat{p}. \)

Now suppose alternatively that there is no \( s \supseteq p', q', r'. \) We may assume that \( p \uparrow \alpha \subseteq r \) since \( p \uparrow \alpha \in G \) and \( G \) is directed. Let:

\[
Y = \text{dom } p \cup \text{dom } q \cup \text{dom } r \cup \{ \alpha \},
\]

and let \( k: Y \rightarrow Y'' \) be the \( \lambda \)-collapse of \( Y. \) Let \( p'' = k(p), q'' = k(q), r'' = k(r), \) and \( \alpha'' = k(\alpha). \) Then \( p'' \uparrow \alpha'' \subseteq r'', \) and there is no \( s \supseteq p'', q'', r''. \)

Let \( v = (\text{dom } q \cup \text{dom } r) \cap \alpha, \) and choose \( z \supseteq v \) and \( s \in G \cap D_{p,q,r} (u, z). \) We can extend \( k \) to an order-isomorphism from \( Y \cup z \) to \( Y'' \cup z'' \) with \( k[z] = z'' \subseteq \alpha''. \) Let \( s'' = k[s]. \)

Certainly \( r'' \) and \( s'' \) are compatible since \( r, s \in G. \) As \( s \) belongs to \( D_{p'', q'', r''} (u, z), \) we have \( k[s] \supseteq r'', \) and there is some \( t'' \supseteq p'' \) so that \( t'' \uparrow \alpha'' \subseteq s'' \) and \( t'' \) is incompatible with \( q''; \) in other words, \( s \supseteq r, \) and there is some \( \hat{p} > p \) so that \( \hat{p} \uparrow \alpha \subseteq s \) and \( \hat{p} \) is incompatible with \( q. \) Then \( \hat{p} \in \mathbb{P}/G, \) and \( \hat{p} \) decides \( q. \)

We now introduce the genericity game. Our main theorem will state that the second player has a winning strategy in this game, under certain combinatorial conditions.

**Definition 1.7.** Let \( \mathbb{P} \) be a \( \lambda^+ \)-uniform partial order. The **genericity game** for \( \mathbb{P} \) is the two-player game of length \( \lambda^+ \) played according to the following rules:

1. At the \( \alpha \)th move, player II will have chosen an increasing sequence of ordinals \( \xi_\beta < \lambda^+, \) and will have defined an increasing sequence of \( \lambda \)-generic ideals \( G_\beta \) on \( \mathbb{P}_{\xi_\beta} \) for all \( \beta < \alpha. \) Player I will choose an element \( g_\alpha \in \mathbb{P}/G_{\xi_\alpha} \) and will also
choose at most $\lambda$ density systems $D^\alpha_i$ over $G_{<\alpha}$. Note that $G_{<\alpha} \in \text{Gen}(\mathcal{P}_{<\alpha})$ by Lemma 1.3.

2. After player I has played his $i$th move, player II will pick an ordinal $\xi_i$ and a $\lambda$-generic ideal of $\mathcal{P}_{<\alpha}$.

Player II wins the $\mathcal{P}$-game if the sequences $\xi_\alpha$ and $G_\alpha$ are increasing, and for all $\alpha$, all indices $i$ occurring at stage $\alpha$: $g_\alpha \in G_\alpha$, and for all $\beta \geq \alpha$, $G_\beta$ meets $D^\alpha_i$.

Our main theorem uses the following combinatorial principle.

**Definition 1.8.** Suppose $\lambda$ is a regular cardinal. $D_\lambda$ asserts that there are sets $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| < \lambda$ for every $\alpha < \lambda$, such that for all $A \in \mathcal{A}$:

$$\{ \alpha \in \lambda: A \cap \alpha \in \mathcal{A}_\alpha \}$$

is stationary.

Easily, $\Diamond_\lambda$ or $\lambda$ strongly inaccessible (or even $\lambda = \aleph_0$) implies $D_\lambda$. Also, Kunen showed that $D_{\lambda^+}$ implies $\Diamond_\lambda$. Gregory has shown that if GCH holds and $\text{cf}(\kappa) > \aleph_0$ then $\Diamond_{\lambda^+}$ holds. It is useful to note that $D_\lambda$ implies $\lambda^{<\lambda} = \lambda$.

**Theorem 1.9.** $D_\lambda$ implies that player II has a winning strategy for the $\mathcal{P}$-game.

This theorem will be proved in Sections 3–5. We illustrate its use in the next section.

2. **Illustrative application**

In this section we give an example of an application of the combinatorial principle described in Section 1.

In [2], Magidor and Malitz introduce a logic $\mathcal{L}^{<\omega}$ which has a new quantifier $Q^n$ for each $n \in \omega$, in addition to the usual first order connectives and quantifiers. The $\kappa$-interpretation of the formula $Q^n \varphi(x_1, \ldots, x_n, \bar{y})$ is

"there is a set $A$ of cardinality $\kappa$ so that for any $x_1, \ldots, x_n \in A$, $\varphi(x_1, \ldots, x_n, \bar{y})$ holds."

They then give a list of axioms which are sound for the $\kappa$-interpretation when $\kappa$ is regular, and show that these axioms are complete for the $\kappa$-interpretation under the assumption of $\Diamond^{<\kappa}$. They ask whether these axioms are complete for the $\lambda^+$-interpretation. We will show that their axioms are complete when both $D_\lambda$ and $\Diamond_{\lambda^+}(\{ \delta < \lambda^+: \text{cf}(\delta) = \lambda \})$ hold. This will explain a remark at the end of [5]. See Hodges [1] for a treatment in the same vein for the $\kappa$-interpretation.

Fix a complete $\mathcal{L}^{<\omega}$ theory $T$, $|T| \leq \lambda$. Let $Q = Q^1$. We may assume that associated to each formula $\varphi$ with free variables $x_1, \ldots, x_n, y_1, \ldots, y_m$, $\mathcal{L}$ contains an $(m + 1)$-ary function $F_\varphi$, so that $T$ proves $Q\varphi(x = x)$ and, for any
fixed \(y_1, \ldots, y_m, F_q(-, y_1, \ldots, y_m)\) is one-to-one and
\[
Q^n \forall x q(x_1, \ldots, x_n, y_1, \ldots, y_m) \rightarrow \forall z_1 \cdots z_n \left( \bigwedge_{i<j} z_i = z_j \rightarrow \bigwedge_{\alpha \in \text{Sym}(n)} q(F_q(z_{\alpha(1)}, \vec{y}), \ldots, F_q(z_{\alpha(n)}, \vec{y}), \vec{y}) \right).
\]
Strictly, it is not necessary to make this conservative extension to our language and theory but it is convenient when handling the inductive step corresponding to \(Q^n\).

We add new constants \(\{y_\alpha : \alpha < \lambda^+\}\) and \(\{x_\alpha^i : \alpha < \lambda^+, i < \lambda\}\) to \(L_1\) to obtain a language \(L_1\). The set of constants \(\{y_\alpha\} \cup \{x_\alpha^i : i < \lambda\}\) is called the set of \(\alpha\)-constants and \(y_\alpha\) is called the special \(\alpha\)-constant. A constant is said to be a \(\omega\)-constant if it is a \(\beta\)-constant for some \(\beta \in \omega\); in particular a constant is a \((<\alpha)\)-constant if it is a \(\beta\)-constant for some \(\beta < \alpha\).

We define a partial order \(\mathcal{P}^\alpha\) as follows: \(p \in \mathcal{P}\) iff
1. \(p\) is a set of \(L_{<\alpha}^\omega\) sentences consistent with \(T\);
2. \(|p| < \lambda\);
3. \(p\) is closed under conjunction and existential quantification; and
4. if \(\varphi(y_\alpha, \vec{z}) \in p\) and the \(\vec{z}\) are \((<\alpha)\)-constants, then \(Qy \varphi(y, \vec{z}) \in p\).

We now indicate how \(\mathcal{P}\) may be viewed as a standard \(\lambda^+\)-uniform partial order. We order \(\mathcal{P}\) by inclusion. Let \(\mathcal{P}_\alpha\) be
\[
\{p \in \mathcal{P} : \text{all constants occurring in a formula of } p \text{ are } (<\alpha)\text{-constants}\}.
\]
The elements of \(\mathcal{P}_\alpha\) will be called templates. For any template \(p\), there is a least \(\beta\) so that all formulas in \(p\) use only constants from \(\{y_i : i < \beta\} \cup \{x_i^j : i < \beta, j < \lambda\}\). Call this \(\beta_p\).

For any template \(p\) and any \(w \subseteq \lambda^+\) so that \(\text{otp}(w) = \beta_p\), fix an order-isomorphism \(h : \beta_p \rightarrow w\). Define \(p(w)\) as the set of formulas obtained by replacing \(x_i^j\) and \(y_i\) by \(x_i^{h(i)}\) and \(y_i^{h(i)}\) respectively for \(i < \beta_p\). Every element of \(\mathcal{P}\) can be obtained in this way from a template.

Let \(t\) be any bijection between the set of templates and \(\lambda\). Identify \(\mathcal{P}\) with the set \(\{(\{p\}, w) : p \in \mathcal{P}_2, w \in \mathcal{P}_{<\lambda}(\lambda^+)\text{ where otp}(w) = \beta_p\}\) by sending \((\{p\}, w)\) to \(p(w)\). Throughout the rest of this section we will treat \(\mathcal{P}\) as if it were in standard form although in practice we will use its original definition. We claim that \(\mathcal{P}\) is \(\lambda^+\)-uniform; it suffices to check the amalgamation condition 8.

The following notation will be convenient. If \(\varphi(y_\alpha_1, \vec{z}_1, y_\alpha_2, \vec{z}_2, \ldots, y_\alpha_n, \vec{z}_n)\) is a formula with \(\alpha_1 > \alpha_2 > \cdots > \alpha_n\), and \(\vec{z}\) is a collection of \([\alpha_{i+1}, \alpha_i)\)-constants, then the string \(S\) of quantifiers:
\[
\exists \vec{x}_n Qy_n \cdots \exists \vec{x}_1 Qy_1
\]
is called standard for \(\varphi\) where the \(x\)'s quantify over the \(z\)'s. Its dual is denoted \(S^*\):
\[
\forall \vec{x}_n \neg Qy_n \neg \cdots \forall \vec{x}_1 \neg Qy_1 \neg
\]
If \(p\) is a set of fewer than \(\lambda\) formulas of \(L_{<\alpha}^\omega\) which is closed under conjunction,
then the following are equivalent:
1. \( p \leq q \) for some \( q \in \mathbb{P} \);
2. \( S\varphi \in T \) for all \( \varphi \in p \) where \( S \) is standard for \( \varphi \).

For \( p \in \mathbb{P} \) and \( \alpha < \lambda^+ \), we have:

\[
p \upharpoonright \alpha = \{ \varphi \in p : \text{all constants in } \varphi \text{ are } < \alpha \text{-constants} \}.
\]

To show that \( \mathbb{P} \) satisfies amalgamation, we will show that it satisfies weak amalgamation. Suppose \( p \in \mathbb{P}_{\alpha+1} \), \( q \in \mathbb{P}_\alpha \) and \( p \upharpoonright \alpha \equiv q \).

Suppose \( \varphi(\bar{x}, y, \bar{z}) \in p \) where \( \bar{x} \) is all the \( \alpha \)-variables except \( y \), and \( \bar{z} \) is the \( < \alpha \)-variables. Then

\[
\exists \bar{x} \varphi(\bar{x}, y, \bar{z}) \in p \upharpoonright \alpha.
\]

If \( \psi \in q \) then

\[
S\exists \bar{x} (\psi \land \varphi)
\]

where \( S \) is a standard sequence, is equivalent to

\[
S(\psi \land \exists \bar{x} \varphi).
\]

Since both of the conjuncts are in \( q \), this last sentence is in \( T \). This verifies weak amalgamation.

Now the strategy is to build a set \( G \) which is the union of generics so that the constant structure derived from \( G \) will form a model of \( T \) under the \( \lambda^+ \)-interpretation. More precisely, we introduce an equivalence relation \( \sim \) on the set of nonspecial constants \( \lambda = \{ x^\alpha : \alpha < \lambda^+, j < \lambda \} \) by:

\[
a \sim b \iff "a = b" \in G.
\]

Let \( \bar{G} = \{ a/\sim : a \in A \} \) and define the functions and relations on \( \bar{G} \) in the usual manner. We want to ensure that for any formula \( \varphi \) in \( \mathcal{L}_{\lambda^+}^{\omega} \) we will have:

\[
\bar{G} \vDash \varphi(a_1/\sim, \ldots, a_n/\sim) \iff \varphi(a_1, \ldots, a_n) \in G.
\]  

(1)

If (1) is true, its proof naturally proceeds by induction on the complexity of formulas. We now describe a strategy for Player I in the genericity game which can only be defeated by achieving (1). In other words, we will specify density systems and elements \( g \in \mathcal{G} \), to be played by Player I, such that a proper response by Player II ensures that \( G \) allows an inductive argument for (1) to be carried out. Our discussion will be somewhat informal, stopping well short of actually writing down the density systems in many cases.

We begin with the treatment of the ordinary existential quantifier. Whenever \( \exists x \varphi(x, \bar{z}) \in G \) we will want (eventually) to have some \( a \in A \) so that \( \varphi(a, \bar{z}) \in G \). In particular, for every \( \varphi \) there will be some \( a \in A \) so that \( y_\alpha = a \in G \). The density systems which ensure this condition is met will in fact be 0-density systems.

Next we consider the quantifier \( Q \). For each formula \( Qx \varphi(x) \) which is put into \( G \), at cofinally many subsequent stages we wish to add the formula \( \varphi(y) \) for an
unused special constant $y$. The first player will play such formulas as \("g_\alpha\) from time to time. We will also have to deal with the case in which $\neg Qx \varphi(x)$, and we will return to this in a moment.

We now consider the quantifier $Q^n$. Suppose that the formula $Q^n \varphi(x, y)$ is in $G$ at some stage. This is where we use the function $F_q$. If $G$ is a model of $T$ then it has cardinality $\lambda^+$. Moreover, $F_q(-, y)$ is one-to-one. Since $Q^n \varphi(x, y)$ is in $G$, so is

$$\forall z_1 \cdots z_n \left( \bigwedge_{i<j} z_i \neq z_j \to \bigwedge_{\alpha \in \Sym(n)} \varphi(F_q(z_{\alpha(1)}), \ldots, F_q(z_{\alpha(n)}), y, y) \right).$$

It follows that the range of $F_q(-, y)$ is homogeneous for $\varphi$.

We are now left with the cases in which formulas of the form $\neg Q^n x \varphi$ ($n \geq 1$) are placed in $G$. We deal first with the case $n = 1$. For this case, we define a number of density systems depending on the following parameters;

1. $j_0, \ldots, j_{m-1} < \lambda$;
2. a formula $\varphi(x, y_0, \ldots, y_{m-1})$; and
3. a function $f : m \to m$.

We associate with these data a density system $D$. If $\otp(u) \neq m + 1$, we let $D(u, w)$ be degenerate:

$$D(u, w) = \{ p \in \mathbb{P} : \dom(p) \subseteq w \}.$$ 

If $\otp(u) = m + 1$ then let $g : m + 1 \to u$ be an order preserving map, let $h = gf$ and set $\beta = g(m) = \max u$, and:

$$\psi(x) = \varphi(x, x_{j_0}^{h(0)}, \ldots, x_{j_{m-1}}^{h(m-1)}).$$

We will then let $D(u, w)$ consist of those $p \in \mathbb{P}$ for which, setting $\alpha = \min(w)$, we have:

1. $\dom(p) \subseteq w$;
2. If $\neg Qx \psi(x) \in p$, then either $\neg \psi(x_\beta) \in p$ or $x_\beta = x_\alpha \in p$ for some $i < \lambda$.

We shall verify the density condition on $D$. Suppose $q \in \mathbb{P}$ and $\neg Qx \psi(x) \in q$. The extension of $q$ we are about to construct will only involve the adjunction of formulas with $(\leq \beta)$-constants, so we may assume that $q$ itself contains only $(\leq \beta)$-constants.

If we cannot complete $q \cup \{ \neg \psi(x_\beta) \}$ to an element of $\mathbb{P}$, then there is some $\chi \in q$ so that:

$$S\exists x (\chi \land \neg \psi(x)) \notin T$$

where $S\exists x$ is a standard sequence for the formula $\chi \land \neg \psi$. Note that by the assumption that $\beta$ is the maximal element of $\dom(q)$, we may assume that the final quantifier in the standard sequence is an existential quantifier on the constant $x$ in $\psi$.

By the axioms for the $Q$-quantifier, for any $\theta \in q$ such that $T \vdash \theta \to \chi$,

$$S\exists x (\theta \land \psi(x)) \in T.$$
as $\neg Qx \psi(x) \in q$, repeated use of the $Q$-quantifier axiom:

$$Qx \exists y \Delta(x, y) \rightarrow \exists x Qy \Delta(x, y) \lor Qy \exists x \Delta(x, y)$$

shows that $\exists x S(\theta \land \psi(x)) \in T$.

If we now choose a constant $x_i^n$ not occurring in $q$, where $\alpha = \min(w)$, one can conclude that $q \cup \{x_i^n\}$ can be completed to an element of $P$.

It is easy to see that if the foregoing density systems are met, then we can carry out the argument from right to left in condition (1) above for $\varphi = Qx \psi(x)$. We turn now to the treatment of the quantifiers $Q^n$ for $n > 1$.

By applying Fodor’s Lemma to the map sending $\delta$ to $\operatorname{dom}(f(\delta)) \cap \delta$ we obtain:

**Lemma 2.1.** If $S \subseteq \{\delta : \operatorname{cf}(\delta) = \lambda\}$ is stationary and $f : S \rightarrow P$ then there is a stationary $S' \subseteq S$, a template $p$ and $\sigma < \lambda^+$ so that for $\delta \in S'$, $f(\delta) = p(w_\delta)$ where $w_\delta = \operatorname{dom}(f(\delta))$ and $w_\delta \cap \delta \subseteq \sigma$.

It will be convenient to treat conditions as if they were single formulas. Extending our previous notation, for $p \in P$ and $S$ a standard sequence covering some of the variables in $p$, we will write $S(p)$ for the set:

$$\{S, \varphi : \varphi \in p\}$$

where $S_p$ is the standard sequence for $\varphi$ which we think of as a subsequence of the possibly infinite standard sequence $S$.

Let $(A_\delta)_{\operatorname{cf}(\delta) = \lambda}$ be a $\diamondsuit$-sequence. For $u, v$ sets of ordinals, we write $u < v$ if for all $\beta \in u$, $\beta < \min v$.

If $\operatorname{cf}(\delta) = \lambda$ and $G_\delta \in \operatorname{Gen}(P_\delta)$, we will define certain associated density systems over $G_\delta$ which depend on the following additional parameters:

1. an $i < \lambda$;
2. a formula $\varphi(x_1, \ldots, x_n, \bar{y})$ (we will suppress the $\bar{y}$);
3. some $k$ with $0 \leq k \leq n$;
4. templates $p_1, \ldots, p_k$; and
5. ordinals $\gamma_j < \beta_{p_j}$ for $1 \leq j < k$.

The density system $D$ that depends on this particular set of parameters will be taken to have $D(u, w)$ degenerate unless:

1. $u = \{\zeta\} \cup \bigcup_{1 \leq j \leq k} w_j$;
2. $\delta < \zeta < w_k \setminus \delta < \cdots < w_1 \setminus \delta$;
3. $w_j = \beta_{p_j}$;
4. $\bigcup_{1 \leq j \leq k} p_j(w_j)$ can be extended to a member of $P$;

in which case we adopt the following notation. Let $\zeta_j$ be the $\gamma_j$th element of $w_j$, and write $z^e$ for $x_i^n$. Note that since $\gamma_j < \beta_{p_j}$ we will have $\zeta_j > \delta$ and hence $\zeta < \zeta_2 < \cdots < \zeta_1$. Define the set $r(\alpha_1, \ldots, \alpha_{n-k+1})$ for $\alpha_1 < \cdots < \alpha_{n-k+1} \in A_\delta$ to be

$$S\left(\bigwedge_{1 \leq j \leq k} p_j(w_j) \land \neg \varphi(z^{\alpha_1}, \ldots, z^{\alpha_{n-k+1}}, z^{\zeta}, z^{\zeta_2}, \ldots, z^{\zeta_1})\right)$$

where $S$ covers all the $(> \zeta)$-variables.
We now define $D(u, w)$ as the set of $q \in \mathcal{P}/G$ with $\text{dom}(q) \subseteq w$ which satisfy one of the following three conditions:

1. $q \perp \bigcup_{i<j<k} p_i(w_j)$; or
2. $\bigcup_{i<j<k} p_i(w_j) \subseteq q$ and for some $\alpha_1 < \cdots < \alpha_{n-1} \in A_\delta$, $r(\alpha_1, \ldots, \alpha_{n-1}) \subseteq q$; or
3. $\bigcup_{i<j<k} p_i(w_j) \subseteq q$ and for all $\alpha_1 < \cdots < \alpha_{n-1} \in A_\delta$:

$$S^*_0(q \rightarrow S^*_k(p_k(w_k) \rightarrow \cdots \rightarrow S^*_1(p_1(w_1) \rightarrow \varphi(z^{n_0}, \ldots, z^{n_{a+n-1}}, z^{\bar{r}}, z^{\bar{s}}, \ldots) \cdots)) \in G_\delta.$$ 

The third condition means that for every $\alpha_1 < \cdots < \alpha_{n-1} \in A_\delta$, there is a $\chi \in q$ and $\psi_j \in p_j(w_j)$ so that

$$S^*_0(\chi \rightarrow S^*_k(\psi_k \rightarrow \cdots \rightarrow S^*_1(\psi_1 \rightarrow \varphi(z^{n_0}, \ldots, z^{n_{a+n-1}}, z^{\bar{r}}, z^{\bar{s}}, \ldots) \cdots)) \in G_\delta,$$

where $S_j$ covers all the $(\geq \delta)$-variables in $\psi_j$ for $j > 0$, and $S_0$ covers all the $(\geq \delta)$-variables in $\chi$. Notice that the only overlap among the variables occur in the $(< \delta)$-variables.

Now suppose $G \models Q''(\bar{y}, a, -)$. We would like to argue that $Q''(\bar{y}, a, -) \in G$. For convenience we will suppress the parameters $\bar{a}$. We may also assume that $T \models q(\bar{x}) \rightarrow \bigwedge_{i<j} x_i \neq x_j$.

Since $G \models Q''(\bar{y}, a, -)$, there is a $\lambda^*$-homogeneous set $B \subseteq G$ for $q$. We may assume there is an $i < \lambda$ so that every $b \in B$ is of the form $x_i^\gamma/\sim$ for some $\gamma$. Let $A = \{ \gamma: x_i^\gamma/\sim \in B \}$ and $\alpha$ is the least such in a given $\sim$-class. For any $\delta$ so that $\text{cf}(\delta) = \lambda$, let $\xi_\delta = \min(A \setminus A_\delta)$. Note that if $A \cap \delta = A_\delta$ then $\xi_\delta > \delta$.

We will now produce the following data. There will be:

1. stationary sets $S_k$ for $0 \leq k \leq n$ with $S_{k+1} \subseteq S_k$ for all $k < n$ and $S_0 = \{ \delta: \text{cf}(\delta) = \lambda, \text{ and } A \cap \delta = A_\delta \}$;
2. templates $p_k$ for $1 \leq k \leq n$, and ordinals $\gamma_k$ so that $\gamma_k < \beta_n$;
3. a domain $w^k_\delta$ of the same order type as $\beta_{p_k}$ for each $\delta \in S_k$; a $\sigma_k < S_k$ so that if $\delta \in S_k$ then $w^k_\delta \cap \delta \subseteq \sigma_k$; let $\xi_\delta^k$ be the $\gamma_k$th element of $w^k_\delta$;
4. For $0 < k < n$, if $\delta < \delta_k < \cdots < \delta_1 \in S_k$ are chosen so that $\delta_1 \subseteq \delta_2 \subseteq \cdots \subseteq \delta_k \subseteq \delta_1$ and $D$ is the density system over $G_\delta$ corresponding to $i$, $q$, $k$, $p_1, \ldots, p_k$ and $\gamma_1, \ldots, \gamma_k$, then

$$p_{k+1}(w^{k+1}_\delta) \in D(\{ \xi_\delta^{k+1} \} \cup w^k_\delta \cup \cdots \cup w^1_\delta, w^{k+1}_\delta) \cap G.$$ 

Using Lemma 2.1 and the fact that $G$ meets all the density systems introduced at stages $\delta \in S_0$, this is straightforward.

Now suppose $\delta_n < \cdots < \delta_1 \in S_n$, so that $w^1_{\delta_1} \setminus \delta_n < \cdots < w^1_{\delta_1} \setminus \delta_1$. Let $q_k = p_k(w^k_{\delta_k})$.

Since $B$ is a homogeneous set for $q$, it follows that $q(z^{n_0}, \ldots, z^{n_{a+n-1}}, z^{\bar{r}}, z^{\bar{s}}, \ldots) \in G$, \ldots
for every $\alpha_0 < \cdots < \alpha_2 < \zeta_\lambda$ in $\mathcal{A}_\lambda$. Since $q_i \in G$, using the density systems defined before, we conclude that

$$S_i^*(q_i \rightarrow \varphi(z^{\geq \alpha_2}, \ldots, z^{\geq \alpha_i})) \in G$$

where $S_i$ covers the $w^{i}_{\beta_i}$-variables. Proceeding by induction and using the definition of the density systems, we conclude that

$$S_n^*(q_n \rightarrow S_{n-1}^*(q_{n-1} \rightarrow \cdots \rightarrow S_1^*(q_1 \rightarrow \varphi(z^{\geq \alpha_2}, \ldots, z^{\geq \alpha_1}))) \in G$$

where $S_i$ covers the $w^{i}_{\beta_i}$-variables.

Of course, $S_n q_n \subseteq G$, so by the Magidor–Malitz axioms, $Q^\nu \varphi(\bar{x}) \in G$ and we finish.

3. Commitments

In this section we begin the proof of Theorem 1.9. Our main goal at present is to formulate a precise notion of a ‘commitment’ (that is, a commitment to enter a dense set — or in model-theoretic terms, to omit a type). We will also formulate the main properties of these commitments, to be proved in Sections 4–5, and we show how to derive Theorem 1.9 from these facts.

Before getting into the details, we give an outline of the proof of Theorem 1.9.

General overview

Suppose that we wish to meet only the following very simple constraints. We are given some 0-density systems $D_i$ over for $i < \lambda$, and some $g_0 \in P_i$, and we seek a $\lambda$-generic ideal $G_0$ containing $g_0$, and meeting each $D_i$. Let $\beta = \lambda \cup \text{sup}(\text{dom}(g_0))$, and enumerate $P_i$ as $(r_i : i < \lambda)$. Then we may construct $G_0$ by generating an increasing sequence $(g_\delta)_{\delta < \lambda}$ beginning with the specified $g_0$, and taking $G_0$ to be the downward closure of $(g_\delta)$. We will run through this in some detail.

Our first obligation is to make $G_0$ $\lambda$-generic in $P_{\beta}$. We will say that $r \in P_{\beta}$ has been decided if we have chosen some $g_\delta \in P_{\beta}$ so that either $r \perp g_\delta$ or else $r \equiv g_\delta$. If the sequence $(g_\delta)_{\delta < \lambda}$ ultimately decides every $r \in P_{\beta}$, then $G_0$ will be $\lambda$-generic in $P_{\beta}$. At stage $\delta + 1$ we will ensure that $r_\delta$ is decided. This takes care of the basic $\lambda$-genericity requirement. At limit stages we can let $g_\delta$ be anything greater than $g_{< \delta}$. We will also take pains at limit stages to meet the specified density systems $D_i$. We enumerate the pairs $(u, D_i)$ with $u \in P_{<\lambda}(\beta)$, using $\lambda^{\lt \lambda} = \lambda$, assigning one such pair to each limit ordinal $\delta < \lambda$. Suppose that $(u, D_i)$ is considered at stage $\delta$. Let $v = \text{dom} g_{< \delta}$. By the density condition on $D_i$, we can find $g_\delta \triangleright g_{< \delta}$ with $g_\delta \in D_i(u, v)$.

Thus it is easy to deal with $\lambda$ constraints of the type arising in one play of our genericity game. Our strategy in that game will rely on this sort of straightforward ‘do what you must when you have the time’ approach, but will encounter
difficulties in ‘keeping up’ at limit stages in the game. We will use $\text{Dl}_\lambda$ to ‘guess’ what additional commitments should be made with regard to various density systems $D_\gamma$, so that any generic set which we construct subsequently which meets these commitments will meet each $D_\gamma$. The commitments themselves retain the feature that each of them can easily be met when necessary; deciding when these commitments should be met requires another use of $\text{Dl}_\lambda$.

At stage 0, Player I selects some density systems, to which we may add all the density systems for Examples 1.5 and 1.6. From these we construct some stage 0 commitments $^0p$, and a $G_0$ meeting $^0p$.

At stage $\delta$ in the play of the game, Player II is attempting to extend $G_{<\delta}$ to a suitable $G_\delta$. (At limit stages we also will need to check that $G_{<\delta}$ continues to meet suitable commitments.) Since $G_{<\delta}$ meets all the previous commitments, in particular it meets all the density systems of Examples 1.5 and 1.6, and therefore $\mathbb{P}/G_{<\delta}$ is $\lambda^+$-uniform. Consequently the construction of $G_\delta$, described at the outset also works in $\mathbb{P}/G_{<\delta}$. Hence we need only construct new commitments $^\delta p$, add them to our previous commitments, and construct $G_\delta$ meeting $^\delta p$ as above. In this way, Player II wins the game.

There is a certain difficulty involved in coping with the freedom enjoyed by Player I (in terms of obligations accumulating at limit stages in the game). There are a priori $\lambda^+$ sets $u \in \mathcal{S}_{<\lambda}(\alpha')$ that may require attention. On the other hand, at a given stage $\delta$ we are only prepared to consider fewer than $\lambda$ such sets. However, by uniformity, it will be sufficient to consider pairs $(u, w) \in \mathcal{P}_{<\lambda}(\beta + \lambda) \times \mathcal{P}_{<\lambda}(\beta + \lambda)$, and hence $\lambda$ such pair suffice. This still leaves Player II at a disadvantage, but with the aid of $\text{Dl}_\lambda$, at limit stages we will guess a relevant set of $u$'s of size less than $\lambda$.

It remains to show that this strategy can be implemented, and works.

We introduce the notion of basic data which will be provided by $\text{Dl}_\lambda$.

**Definition 3.1.** A collection of basic data will contain

1. trees $T_\alpha$, subsets of $\mathcal{P}_\lambda$ (but not suborders), with orders $<_\alpha$, for every $\delta < \lambda$;
2. for every generic set $G \in \text{Gen}(\mathcal{P}_\alpha)$ for some $\alpha < \lambda^+$, two stationary subsets of $\lambda$, $S(G)$ and $S'(G)$ and a club $C$ so that $C \cap S'(G) \subseteq S(G)$; and
3. for every $\delta < \lambda$, a set $U_\delta \subseteq \mathcal{P}_{<\lambda}(\lambda)$

with the following properties

1. $|T_\alpha| < \lambda$, $|U_\delta| < \lambda$ for every $\delta < \lambda$,
2. if $p \in T_\alpha$ then $\text{len}(p) = \text{dom}(p)$,
3. if $p \subset_{<\delta} q$ and $\text{len}(p) = \alpha$ then $p = q \upharpoonright \alpha$,
4. if $p \in T_\alpha$ and $\alpha \subseteq \text{dom}(p)$ then $p \upharpoonright \alpha \in T_\alpha$,
5. if $(g_\delta)_\alpha$ is a cofinal sequence for a generic set $G \in \text{Gen}(\mathcal{P}_\alpha)$ then there is a club $C$ so that for $\delta \in C \cap S(G)$, $(g_{<\delta})^\text{coll} \in T_\alpha$,
6. if $G$ and $G'$ are generic sets so that $G \subseteq G'$ then there is a club $C$ so that $C \cap S(G') \subseteq S(G)$, and
7. (Oracle property) for $\alpha < \lambda^+$ and $G \in \text{Gen}(\mathcal{P}_\alpha)$, $u \in \mathcal{P}_{<\lambda}(\alpha)$ and $\alpha =$
Models with second order properties. V

\[ \bigcup_{\delta < \lambda} w_\delta \text{ a continuous increasing union with } w_\delta \in P_{<\lambda}(\alpha) \text{ and } u \subseteq w_\delta \text{ then there is a club } C \text{ so that for every } \delta \in C \cap S'(G) \text{ there is } u' \in U_\delta \text{ so that } (w_\delta, u) \equiv (\text{otp}(w_\delta), u'). \]

**Remarks.**

1. Although there is the possibility of confusion between the orders \(<_\delta\) and \(<\) on \(P_\lambda\), we will use \(<\) for both and the context should usually make it clear which we mean.

2. The following will be true of the trees that we eventually build although this property will not be needed in the proof: if \(q \in T_\delta\) then there is a generic set \(G\) with a cofinal sequence \((g_\alpha)_\alpha\) so that \(q = (g^{col}_\alpha) \upharpoonright \beta\) for some \(\alpha\) and \(\beta\).

3. If \((g_\alpha)_\alpha\) and \((g'_\alpha)_\alpha\) are cofinal sequences for \(G\) and \(G'\) then there is a club \(C\) so that if \(\delta \in C\) and \(\eta = \text{dom}(g^{col}_\alpha)\) then \(g^{col}_\alpha = (g^{col}_\alpha)^{col} \upharpoonright \eta\). In condition 6, we may assume that for particular cofinal sequences, \(C\) satisfies this property as well as \(C \cap S(G') \subseteq S(G)\). We will often use this version of condition 6.

4. It is important to notice the following about \(p \in T_\delta\) for which \(\text{dom}(p)\) is a limit ordinal. If \(\alpha < \text{dom}(p)\) then \(p \upharpoonright \alpha \in T_\delta\) and \(p \upharpoonright \alpha < p\). Hence, any such \(p\) is the limit of those elements of \(T_\delta\) which are less than it.

**Lemma 3.2** (\(\text{Di}_1\)). *There is a collection of basic data.*

We leave the proof of this until the next section. For the rest of the paper except for the proof of Lemma 3.2, we will fix a particular choice of basic data using the notation of Definition 3.1.

**Definition 3.3.** A weak commitment is a sequence \(p = (p^\delta: \delta < \lambda)\) where \(p^\delta: T_\delta \rightarrow P_\lambda\) with the following properties:

1. \(p^\delta(\eta) \in P_{\text{len}(\eta)}\) (we usually write \(p^\delta_\eta\) for \(p^\delta(\eta)\)),

2. if \(\eta \leq v \in T_\delta\) then \(p^\delta_\eta \supseteq p^\delta_v \upharpoonright \text{len}(\eta)\).

We define an order on weak commitments by \(p \leq q\) if for almost all \(\delta\) (i.e. on a club), \(p^\delta \equiv q^\delta\) pointwise. We say that \(q\) is stronger than \(p\).

We will identify two weak commitments \(p\) and \(q\) if \(p \leq q\) and \(q \leq p\).

**Notation.** From the fixed collection of basic data one can extract a critical weak commitment. Define \(*p = (p^\delta: \delta < \lambda)\) where \(*p^\delta_\eta = \eta\).

**Definition 3.4.** A commitment is a weak commitment which is stronger than \(*p\).

**Definition 3.5.** Suppose \(G\) is generic with a cofinal sequence \((g_\alpha)_\alpha\) and \(p\) is a commitment. We say \(G\) meets \(p\) if there is a club \(C \in \lambda\) so that for every \(\delta \in C \cap S(G)\), \(\eta_\delta = (g^{<\delta}_\alpha)^{col} \in T_\delta\) and there is \(r_\delta \in G\) so that \(\text{dom}(r_\delta) = \text{dom}(g^{<\delta}_\alpha)\) and 

\[ r^{\text{col}}_\delta \equiv p^\delta_\eta \text{ for all } \eta \leq \eta_\delta. \]
Remark. If \( h : \text{len}(\eta_\delta) \to \text{dom}(g_{\prec \lambda}) \) is an order-isomorphism then in the above definition the existence of \( r_\delta \) is equivalent to saying that \( h[p^\alpha_\eta] \in G \) for all \( \eta \leq \eta_\delta \).

**Proposition 3.6.** If \( D_i, i < \lambda, \) are 0-density systems and \( g \in \mathcal{P}_\gamma \) then there is a commitment \( q \) (\( \geq^* p \)) and some \( G \in \text{Gen}(\mathcal{P}_\gamma) \), so that:

1. \( g \in G \),
2. \( G \) meets \( q \), and
3. if \( \gamma \leq \gamma' < \lambda' \) and \( G' \in \text{Gen}(\mathcal{P}_{\gamma'}) \) meets \( q \), then \( G' \) meets each \( D_i \).

**Proposition 3.7.** Suppose \( G \in \text{Gen}(\mathcal{P}_\alpha) \) and \( G \) satisfies

1. for all \( g \in \mathcal{P}/G, h \in \mathcal{P} \) there is \( g' \in \mathcal{P}/G \) with \( g' \gg g \) and either \( g' \gg h \) or \( g' \uparrow h \), and
2. \( \mathcal{P}/G \) is \( \lambda^+ \)-uniform.

For \( i < \lambda, \) let \( D_i \) be a density system over \( G \), and suppose \( g \in \mathcal{P}_\gamma/G \) where \( \alpha \leq \gamma < \lambda^+ \) and \( p \) is some commitment that is met by \( G \). Then there is a commitment \( q \geq p \), and some \( G^* \in \text{Gen}(\mathcal{P}_\gamma) \), so that:

1. \( G \subseteq G^* \), \( g \in G^* \),
2. \( G^* \) meets \( q \),
3. if \( \gamma \leq \gamma' < \lambda^+ \) and \( G' \in \text{Gen}(\mathcal{P}_{\gamma'}) \) contains \( G \) and meets \( q \), then \( G' \) meets each \( D_i \).

**Lemma 3.8.** Let \( (^\alpha p)_{\alpha < \kappa} \) be an increasing sequence of commitments with \( \kappa < \lambda^+ \). Then the sequence has a least upper bound.

**Notation 3.9.** With the notation of the preceding lemma, we write

\[
\bigcup_{\alpha < \delta} ^\alpha p \quad \text{or} \quad ^\delta p
\]

for the least upper bound of the commitments \( ^\alpha p \).

**Proposition 3.10.** Suppose \( G_\alpha \in \text{Gen}(\mathcal{P}_{\xi_\alpha}) \) meets \( ^\alpha p \) for all \( \alpha < \delta \), where \( \delta < \lambda^+ \), and the \( G_\alpha \) and \( ^\alpha p \) are increasing. Then \( G_{\delta < \delta} \) meets \( ^\delta p \).

By combining these results we immediately obtain a proof of Theorem 1.9.

**Proof of Theorem 1.9.** We define a sequence of ordinals \( \xi_\alpha \), a sequence of commitments \( ^\alpha p \), and a sequence of \( \lambda \)-generic ideals \( G_\alpha \in \text{Gen}(\mathcal{P}_{\xi_\alpha}) \), so that:

1. \( \xi_{\alpha < \delta} \leq \xi_\delta ; \quad ^\delta p \leq \delta p ; \quad G_{\delta < \delta} \subseteq G_\delta \) for \( \delta < \lambda^+ \);
2. \( G_\alpha \) meets the commitment \( ^\alpha p \);
3. if \( \xi_\alpha \leq \beta \) and \( G \in \text{Gen}(\mathcal{P}_\beta) \) contains \( G_\alpha \) and meets the commitment \( ^\alpha p \), then \( G \) meets each \( \alpha \)-density system \( D_i \) over \( G_\alpha \) proposed by Player I at stage \( \alpha \) of the genericity game.
At stage 0, Player I provides some \( g_0 \in p \) and at most \( \lambda \) many 0-density systems. To these Player II adds all the 0-density systems mentioned in Examples 1.5 and 1.6. We now apply Proposition 3.6 to all these 0-density systems and \( g_0 \). This will provide us with \( G_0, \zeta_0, \) and \( \alpha^\varphi \).

At stage \( \delta \), we will have \( \zeta_{<\delta}, G_{<\delta}, \varphi^\zeta \) defined, and by Proposition 3.10 \( G_{<\delta} \) meets \( \varphi^\zeta \). Now since \( G_0 \subseteq G_{<\delta}, G_{<\delta} \) meets \( \alpha^\varphi \) and hence meets each of the 0-density systems from Examples 1.5 and 1.6. It follows that \( G_{<\delta} \) satisfies the condition on the generic set in Proposition 3.7. By Proposition 3.7 a suitable choice of \( \zeta_{<\delta}, G_{<\delta}, \zeta^\varphi \) can then be made.

Now we verify that Player II wins the genericity game using this strategy. By construction, \( g_\alpha \in G_\alpha \) for all \( \alpha \). Suppose that \( D \) is a density system over \( G_{<\alpha} \) selected by Player I at stage \( \alpha \) of the genericity game, and \( \beta \geq \alpha \). As \( G_\beta \) meets the commitment \( \beta^\varphi, \beta^\varphi \geq ^\alpha \varphi, \) and \( G_\beta \supseteq G_\alpha \), it follows that \( G_\beta \) meets \( D \). \( \square \)

4. Proofs

In this section we give the proofs of the results stated in the previous section except the proofs of Propositions 3.6 and 3.7 which are deferred to the next section.

Proof of Lemma 3.2. The Lemma states that there is a collection of basic data. Let \( (v_\alpha)_{\alpha < \lambda} \) be an enumeration of \( P_{<\lambda}(\lambda) \) so that \( v_\delta \subseteq \delta \) for all \( \delta \). Let \( V_\delta = \{v_\beta: \beta < \delta \} \) for \( \delta < \lambda \). Using \( D\lambda \) and an encoding of
\[
(\lambda \times \lambda) \cup (\lambda \times P_{<\lambda}(\lambda))
\]
by \( \lambda \), we can find sets \( R_\delta \subseteq P(\delta \times \delta) \) and \( G_\delta \subseteq P(\delta \times V_\delta) \), such that \( |R_\delta|, |G_\delta| < \lambda \) for all \( \delta < \lambda \), and for any \( R \subseteq \lambda \times \lambda, G \subseteq \lambda \times P_{<\lambda}(\lambda) \), the set:
\[
\{\delta: R \cap (\delta \times \delta) \in R_\delta \text{ and } G \cap (\delta \times V_\delta) \in G_\delta\}
\]
is stationary.

Before defining the basic data we establish some notation. For each \( \alpha < \lambda^+ \), we select a bijection \( i_\alpha: \alpha \leftrightarrow |\alpha| \). For simplicity we assume \( |\alpha| = \lambda \) throughout our notation below. For \( \delta < \lambda \), let \( \alpha_\delta \) be the order type of \( i_\alpha^{-1}[\delta] \), and let \( \pi_\alpha: \alpha \cap i_\alpha^{-1}[\delta] = \alpha_\delta, j_\alpha = \pi_\alpha \circ i_\alpha^{-1}: \delta \leftrightarrow \alpha_\delta \).

Let
\[
R_\alpha = \{(i_\alpha(\beta), i_\alpha(\gamma)): \beta < \gamma < \alpha\};
\]
\[
R_{<\delta} = R_\delta \cap (\delta \times \delta) \quad (\delta < \lambda).
\]
Then \( j_{\alpha\delta}(\delta, R_{<\delta}) \approx (\alpha_\delta, <) \). It will be important that \( j_{\alpha\delta} \) is determined by \( R_{<\delta} \).
If $G$ is a $\lambda$-generic ideal in $\mathbb{P}_\alpha$ with $\alpha < \lambda^+$, let:

\[
\begin{align*}
\hat{G} &= \{(\beta, i_\alpha[u]): (\beta, u) \in G\}, \\
\hat{G}_\delta &= \hat{G} \cap (\delta \times V_\delta), \\
G_\delta &= \{(\beta, u) \in G: (\beta, i_\alpha[u]) \in (\delta \times V_\delta)\}, \\
\hat{G}_\delta &= \{(\beta, \pi_{\alpha\delta}[u]): (\beta, u) \in G_\delta\}.
\end{align*}
\]

Again, we can go directly from $\hat{G}_\delta$ to $\hat{G}_\delta$ by applying $i_{\alpha\delta}$. Observe also that $\pi_{\alpha\delta}$ induces an isomorphism $\pi^*_{\alpha\delta}: \hat{G}_\delta \cong \hat{G}_\delta$. We are primarily interested in this collapsing map $\pi^*_{\alpha\delta}$, but $\hat{G}$ provides a better 'encoding' of $G$ because the sets $\hat{G}_\delta$ increase with $\delta$, while the sets $\hat{G}_\delta$ do not.

Let $C(G)$ be the set of $\delta < \lambda$ for which $G_\delta$ contains a cofinal increasing subsequence. Then $C(G)$ is a club in $\lambda$. For $\delta \in C(G)$, $\hat{G}_\delta$ has a least upper bound, which will be denoted $\hat{G}_\delta$.

We are now ready to define the basic data. For $\delta < \lambda$ we define $T_\delta$ as:

\[
\left\{ p \in \mathbb{P}: \exists \alpha < \lambda^+ \exists G \in \text{Gen}(\mathbb{P}_\alpha) \exists \gamma: \delta \in C(G), \begin{array}{c}
\hat{G}_\delta \in \mathcal{G}_\delta, R_{\alpha\delta} \in \mathcal{R}_\delta, \text{ and } p = \left[ \bigcup \hat{G}_\delta \right] \uparrow \gamma.
\end{array}\right\}
\]

Notice that dom$(p)$ is an ordinal for every $p \in T_\delta$. To see that $|T_\delta| < \lambda$, we use the fact that $\hat{G}_\delta, R_{\alpha\delta}$ together determine $\hat{G}_\delta$, and also that any $p$ in $\mathbb{P}$ has fewer than $\lambda$ distinct restrictions. For $p, q \in T_\delta$, define the order by: $p < q$ if $p = q | \text{dom}(p)$.

Now for $G$ a $\lambda$-generic ideal in $\mathbb{P}_\alpha$ with $\alpha < \lambda^+$, fix a cofinal sequence $(g^G_\delta)_{\delta < \lambda}$ in $G$, and set:

\[
\begin{align*}
S(G) &= \{ \delta < \lambda: [g^G_\delta]^{\text{col}} \in T_\delta \}, \\
S'(G) &= \{ \delta < \lambda: \hat{G}_\delta \in \mathcal{G}_\delta \text{ and } R_{\alpha\delta} \in \mathcal{R}_\delta \}, \\
U_\delta &= \{ u: \exists v \in V_\delta \exists R \in \mathcal{R}_\delta \exists \alpha < \lambda^+ (\delta, v, R) = (\alpha, u, <) \}.
\end{align*}
\]

Clearly $U_\delta \subseteq \mathcal{P}_{<\lambda}(\lambda)$ and $|U_\delta| < \lambda$. It is also straightforward to see that $S'(G)$ is stationary.

Let $C_1$ be

\[
\left\{ \delta \in C(G): g^G_{\lambda}\delta = \bigcup G_\delta \right\}.
\]

Then $C_1$ is a club in $\lambda$, and if $\delta \in S'(G) \cap C_1$ then $(g^G_{\lambda}\delta)^{\text{col}} \in T_\delta$ so $S(G)$ is stationary. If $(g^G_\delta)_{\delta}$ is any other cofinal sequence for $G$ then there is a club

\[
C = \{ \delta: g'_{\lambda}\delta = g^G_{\lambda}\delta \}
\]

and for every $\delta \in C \cap S(G)$, $(g^G_{\lambda}\delta)^{\text{col}} \in T_\delta$.

Let $G \subseteq G^*$ be two $\lambda$-generic ideals in $\mathbb{P}_\alpha, \mathbb{P}_\alpha^*$ with $S(G), S(G^*)$ determined
by cofinal sequences \((g_{\delta}^{G})_{\delta}, (g_{\delta}^{G_{*}})_{\delta}\) respectively. If one considers \(C = \{\delta: g_{\delta}^{G} \upharpoonright \alpha = g_{\delta}^{G_{*}}\}\), it is easy to see that \(C \cap S(G^{*}) \subseteq S(G)\).

It remains to verify the oracle property 7 of Definition 3.1. We fix \(\alpha < \lambda^{+}, G\) \(\lambda\)-generic in \(P_{\alpha}, u \in P_{\lambda}(\alpha)\), and we let \(\alpha = \bigcup_{\beta < \lambda} w_{\beta}\) be a continuous increasing union with each \(|w_{\beta}| < \lambda\) and \(u \subseteq w_{\beta}\). One some club \(C\), otp \(w_{\delta} = \alpha_{\delta}\) and if \(v_{\alpha} \subseteq \delta\) then \(\alpha < \delta\). So \((w_{\delta}, u) = (\alpha_{\delta}, \pi_{\alpha\delta}[u])\). For \(\delta \in C \cap S'(G)\) we have

\[(\delta, i_{\alpha}[u], R_{\alpha\delta}) = (\alpha_{\delta}, \pi_{\alpha\delta}[u], <) = (w_{\delta}, u, <).\]

Hence, \(\pi_{\alpha\delta}[u] \in U_{\delta}\). □

**Notation 4.1.** In the next few results we make systematic use of the diagonal intersection of clubs. If \((C_{\alpha})_{\alpha < \lambda}\) is a sequence of clubs in \(\lambda\), the diagonal intersection is defined corresponding as:

\[\Delta_{\alpha}C_{\alpha} = \left\{ \delta < \lambda: \delta \in \bigcap_{\alpha < \delta} C_{\alpha} \right\}.\]

The diagonal intersection of such a sequence of clubs is again a club.

**Proof of Lemma 3.8.** Let \((^p)_{\alpha < \kappa}\) be an increasing sequence of commitments with \(\kappa < \lambda^{+}\). We claim that the sequence has a least upper bound. We may take \(\kappa\) to be a regular cardinal, with \(\kappa < \lambda^{+}\). We deal with the case \(\kappa = \lambda\); for \(\kappa < \lambda\) our use of a diagonal intersection below would reduce to an ordinary intersection.

For \(\beta < \lambda\) let \(C_{\beta}\) be a club such that for all \(\alpha < \beta\):

\[^{\alpha}p^{\delta} \triangleq p^{\delta}\] pointwise for \(\delta \in C_{\beta}\).

Let \(C = \Delta_{\beta}C_{\beta}\). For \(\delta \in C\) and \(\eta \in T_{\delta}\), let \(p_{\eta}^{\delta} = \bigcup_{\alpha < \delta} ^{\alpha}p_{\eta}^{\delta}\). Then \(p\) is a commitment. We have \(^{\eta}p \triangleq p\) since \(^{\alpha}p^{\delta} \triangleq p^{\delta}\) pointwise for \(\delta \in C \setminus \alpha\).

Now we will check that \(p\) is the least upper bound of the sequence as a commitment. Let \(q\) be a second upper bound. Let

\[C_{\eta}^{*} = \{ \delta < \lambda: q^{\delta} \geq ^{\alpha}p^{\delta}\] pointwise\},

and let \(C^{*} = \Delta_{\alpha}C_{\alpha}^{*}\). For \(\delta \in C \cap C^{*}\), and \(\eta \in T_{\delta}\), we have:

\[^{\alpha}p_{\eta}^{\delta} \geq \bigcup_{\alpha < \delta} ^{\alpha}p_{\eta}^{\delta} = p_{\eta}^{\delta}\]

It follows that \(p\) is the least upper bound. □

We divide Proposition 3.10 into two parts.

**Proposition 4.2.** Suppose that \((G_{i})_{\iota < \kappa}\) is an increasing sequence with \(G_{i} \in \text{Gen}(P_{\alpha})\), \(\kappa < \lambda^{+}\), and that each \(G_{i}\) meets a fixed commitment \(p\). Then \(\bigcup_{i} G_{i}\) also meets the commitment \(p\).
Proof. Let $G = \bigcup_{i<n} G_i$. By consulting the proof of Lemma 1.3, we can see that there is a cofinal sequence $(g_j)_{j<\lambda}$ for $G$ so that if $g_j^i = g_j \upharpoonright \alpha_i$ then $(g_j^i)_{i<\alpha}$ is a cofinal sequence in $G_i$. For each $i<\lambda$, let $C_i$ be a club demonstrating that $G_i$ meets $p$. In other words, for $\delta \in C_i \cap S(G_i)$ we have $r_\delta^i \in G$ with:

1. $\text{dom}(r_\delta^i) = \text{dom}(g_{<\delta}^i)$, $\eta_\delta = (g_{<\delta}^i)^{\text{col}} \in T_\delta$; and
2. $[r_\delta^i]^{\text{col}} \supseteq p_\eta^\delta$ for all $\eta \leq \eta_\delta$.

By Definition 3.1.5, we may also suppose that $C_i \cap S(G) \subseteq S(G_i)$.

We consider the case $\kappa = \lambda$ (use ordinary intersection instead of diagonal intersection when $\kappa < \lambda$).

Let

$$C = \Delta_i C_i \cap \left\{ \delta < \lambda; \sup \text{dom} g_{<\delta} = \sup \alpha_i \right\}.$$

If $\delta \in C \cap S(G)$ then we can find $r_\delta \in G$ so that $r_\delta \supseteq r_\delta^i$ for $i < \delta$ and

$$\text{dom}(r_\delta) = \bigcup_{i<\delta} \text{dom}(g_{<\delta}^i) = \bigcup_{i<\delta} \text{dom}(g_{<\delta} \upharpoonright \alpha_i) = \text{dom}(g_{<\delta}).$$

Clearly, $r_\delta^{\text{col}} \supseteq p_\eta^\delta$ for all $\eta < (g_{<\delta})^{\text{col}} = \eta_\delta \in T_\delta$. However, since $p_\eta^\delta \upharpoonright \text{len}(\eta) \leq p_\eta^\delta$ for all $\eta \leq \eta_\delta$ and $p_\eta^{\delta} = \bigcup_{\eta < \eta_\delta} p_\eta^\delta \upharpoonright \text{len}(\eta)$, $r_\delta^{\text{col}} \supseteq p_\eta^{\delta}$.

\textbf{Proposition 4.3.} Suppose $G \in \text{Gen}(\mathcal{P}_\alpha)$ meets an increasing sequence of commitments $(\gamma^\mathcal{P}; \gamma < \kappa)$ where $\kappa < \lambda$. Then $G$ meets $\bigcup_{\gamma < \kappa} \gamma^\mathcal{P}$.

\textbf{Proof.} Again, we treat only the case when $\kappa = \lambda$. Let $(g_i)_{i<\lambda}$ be a cofinal sequence in $G$. For each $\gamma$, let $C_\gamma$ witness the fact that $G$ meets $\gamma^\mathcal{P}$. That is, for all $\delta \in C_\gamma \cap S(G)$ there is $r_\delta \in G$ so that:

1. $\text{dom}(r_\delta) = \text{dom} g_{<\delta}$, $\eta_\delta = (g_{<\delta})^{\text{col}} \in T_\delta$; and
2. $(r_\delta)^{\text{col}} \supseteq p_\eta^\delta$ for all $\eta \leq \eta_\delta$.

We may also suppose that $C_\gamma$ witnesses the relation $\gamma^\mathcal{P} \leq \gamma^\mathcal{P}$ for $i < \gamma$. Hence we may assume $r_\delta^i \prec r_\delta^\gamma$ for $\delta \in C_\gamma \cap S(G)$. Let $C = \Delta_\gamma C_\gamma$.

For $\delta \in C \cap S(G)$, let $r_\delta = \bigcup_{i<\delta} r_\delta^i$. Then on $C \cap S(G)$, dom $r_\delta = \text{dom} g_{<\delta}$ and $r_\delta^{\text{col}} - \bigcup_{i<\delta} [r_\delta^i]^{\text{col}} \supseteq \bigcup_{i<\delta} p_\eta^\delta = <^\delta p_\eta^\delta$ for all $\eta \leq \eta_\delta$. The last equality follows from the proof of Lemma 3.8. \qed

Proposition 3.10 is an immediate consequence of the preceding two propositions.

5. Proof of Proposition 3.7

We recall the statement of Proposition 3.7.
Proposition 3.7. Suppose $G \in \text{Gen}(\mathbb{P}_\alpha)$ and $G$ satisfies

1. for all $g \in \mathbb{P}/G$, $h \in \mathbb{P}$ there is $g' \in \mathbb{P}/G$ with $g' \geq g$ and either $g' \geq h$ or $g' \perp h$, and
2. $\mathbb{P}/G$ is $\lambda^+$-uniform.

For $i < \lambda$, let $D_i$ be a density system over $G$, and suppose $g \in \mathbb{P}_\gamma/G$ where $\alpha \leq \gamma < \lambda^+$ and $p$ is some commitment that is met by $G$. Then there is a commitment $q \equiv p$, and some $G^* \in \text{Gen}(\mathbb{P}_\gamma)$, so that:

1. $G \subseteq G^*$, $g \in G^*$;
2. $G^*$ meets $q$;
3. if $\gamma \leq \gamma' < \lambda^+$ and $G' \in \text{Gen}(\mathbb{P}_\gamma)$ contains $G$ and meets $q$, then $G'$ meets each $D_i$.

We are also obliged to prove Proposition 3.6 as well. The proof is very similar to the proof of Proposition 3.7 and so we will only highlight the formal differences at the end of the proof.

Proof of Proposition 3.7. Let $\gamma = \bigcup_{\delta < \lambda} \omega_\delta$ be a continuous increasing union with $|\omega_\delta| < \lambda$. Set $\gamma_\delta = \text{opt}(\omega_\delta)$ and choose $\zeta_\delta$ so that $\gamma_\delta + \zeta_\delta \geq \text{ht}(T_\delta)$. Let

$$h_\delta : \gamma_\delta + \zeta_\delta \rightarrow \omega_\delta \cup \{\gamma, \gamma + \zeta_\delta\}$$

be an order-isomorphism.

Fix a cofinal sequence $(g_\delta)_\delta$ for $G$. Since $G$ meets $p$, there is a club $C$ so that for all $\delta \in C \cap S(G)$ we have $g_\delta^{\text{col}} := \eta_\delta \in T_\delta$, and there is $r_\delta \in G$ so that $\text{dom}(r_\delta) = \omega_\delta \cap \alpha$ and $r_\delta^{\text{col}} \equiv p_\delta^\eta$ for all $\eta < \eta_\delta$. We may also assume that $\text{dom}(g_\delta^{<\lambda}) = \omega_\delta \cap \alpha$ for all $\delta \in C$.

Now we build the commitment $q$. If $\delta \notin C \cap S(G)$ then let $q^\delta = p^\delta$. Fix then $\delta \in C \cap S(G)$. For $i < \delta$, $\xi \leq \xi_\delta$ and $u \in U_\delta$, $u \subseteq \gamma_\delta + \xi$, let

$$D_\delta(u) = \{r \in \mathbb{P}_{\gamma_\delta + \xi} : h_\delta[r] \in D_i(h_\delta[u], \omega_\delta \cup \{\gamma, \gamma + \xi\})\}.$$

Let $\mathbb{P}_C[T_\delta]$ be the set of functions $\bar{p} : I_\delta \rightarrow \mathbb{P}_\lambda$ so that

1. $\bar{p}(\eta) \in \mathbb{P}_{\text{len}(\eta)}$ for all $\eta \in T_\delta$,
2. if $\eta \leq \nu$ then $\bar{p}(\eta) \geq \bar{p}(\nu) \upharpoonright \text{len}(\eta)$, and
3. if $\eta$ is comparable with $\eta_\delta$ then $h_\delta[\bar{p}(\eta)] \in \mathbb{P}/G$.

We will write $\bar{p}_\delta$ for $\bar{p}(\eta)$.

Remark. Since $G$ meets $p$, if $\delta \in C \cap S(G)$ then $p^\delta \in \mathbb{P}_C[T_\delta]$. To see this, we must show that if $\eta$ is comparable to $\eta_\delta$ then $h_\delta[p_\delta^\eta] \in \mathbb{P}/G$. If $\eta \leq \eta_\delta$ then since $\delta \in C \cap S(G)$, $h_\delta[p_\delta^\eta] \in \mathbb{P}/G$. Suppose $\eta \geq \eta_\delta$. Now $p_\delta^\eta \upharpoonright \text{len}(\eta_\delta) \equiv p_\delta^{\eta_\delta}$ and since $\omega_\delta \cap \alpha = \text{dom}(g^{<\lambda})$ we have $h_\delta[p_\delta^{\eta_\delta}] \upharpoonright \alpha \equiv h_\delta[p_\delta^\eta]$ so $h_\delta[p_\delta^\eta] \in \mathbb{P}/G$.

Proposition 5.1. There is a $q^\delta \in \mathbb{P}_C[T_\delta]$ with $q^\delta \equiv p^\delta$ pointwise and so that for every $u \in U_\delta$, $i < \delta$ if $\eta' \in T_\delta$, $\eta' \geq \eta_\delta$ with $\text{len}(\eta') = \gamma_\delta + \xi$ and $u \subseteq \gamma_\delta + \xi$ then $q^\delta_{\eta'} \in D_\delta(u)$. 
To obtain this \( q^\varnothing \) we use a claim whose proof we postpone.

**Claim 5.2.** If \( \bar{q} \in \mathcal{P}_G[T_\delta], \ u \in U_\delta, \ i < \delta, \ \zeta \leq \xi_\delta \) and \( \eta^* \in T_\delta, \ \eta^* \geq \eta_\delta \) with \( \text{len}(\eta^*) = \gamma_\delta + \zeta \) and \( u \subseteq \gamma_\delta + \zeta \) then there is \( \bar{r} \in \mathcal{P}_G[T_\delta] \) so that \( \bar{r} \geq \bar{q} \) pointwise and \( \bar{r}_\eta \in D_\eta(u) \).

**Proof of Proposition 5.1.** To get the required \( q^\varnothing \), one starts with \( p^\varnothing \), at limit stages take unions and at successor stages use the claim applied to some particular \( i < \delta, \ \zeta \leq \xi_\delta, \ u \in U_\delta \) and \( \eta^* \in T_\delta \). After at most \( |U_\delta| \cdot |\delta| \cdot |\xi_\delta| \cdot |T_\delta| \) stages we will have produced \( q^\varnothing \). \( \square \)

Now we turn to the construction of \( G^* \), a \( \lambda \)-generic ideal in \( \mathcal{P}_{\gamma} \) meeting \( q \) with \( G \subseteq G^* \) and \( g \in G^* \), hence completing the proof of Proposition 3.7.

Fix an enumeration \( (s_\delta)_\delta \) of \( \mathcal{P}_\gamma \). \( G^* \) will be the downward closure of an increasing sequence \( (g^*_\delta)_\delta \) which is constructed inductively starting with \( g^*_\varnothing = g \). We shall guarantee that \( g^*_\delta \in \mathcal{P}_{\gamma}/G \) for each \( \delta \).

At stage \( \delta \), if \( \delta \in C \cap S(G) \), \( \text{dom}(g^*_\delta) = w_\delta \), \( (g^*_\delta)^{\text{col}} = \eta^* \in T_\delta \) then let \( h : \text{dom}(\eta^*) \to \text{dom}(g^*_\delta) \) be an order-isomorphism and let \( \bar{g}_\delta \in \mathcal{P}/G \) be chosen so that \( \text{dom}(\bar{g}_\delta) = \text{dom}(g^*_\delta) \), \( \bar{g}_\delta \geq g^*_\delta \) and \( \bar{g}_\delta \geq h_\delta[g^*_\delta] \) for every \( \eta < \eta^* \). This can be accomplished because \( \mathcal{P}/G \) is \( \lambda^\varnothing \)-uniform by assumption and we guaranteed that \( h_\delta[g^*_\delta] \in \mathcal{P}/G \) when we built the commitment \( q^\varnothing \).

If any of the above conditions fail, let \( g^*_\delta = g^*_\varnothing \). In either case, use the assumption on \( G \) to find \( g^*_\delta \) so that \( g^*_\delta \in \mathcal{P}_{\gamma}/G \) with \( \bar{g}_\delta \geq g^*_\varnothing \) and either \( g^*_\delta \geq s_\delta \) or \( g^*_\delta \perp s_\delta \).

It follows easily now that \( G^* \in \text{Gen}(\mathcal{P}_{\gamma}) \), \( G \subseteq G^* \) and \( g \in G^* \). We now show that \( G^* \) meets \( q \). Choose \( C_1 \) so that \( C_1 \cap S(G^*) \subseteq C \cap S(G) \) and for all \( \delta \in C_1 \), \( \text{dom}(g^*_\varnothing) = w_\delta \) and \( (g^*_\varnothing)^{\text{col}} = \eta^* \in T_\delta \) then let \( h : \text{dom}(\eta^*) \to \text{dom}(g^*_\delta) \) be an order-isomorphism and let \( \bar{g}_\delta \in \mathcal{P}/G \) be chosen so that \( \text{dom}(\bar{g}_\delta) = \text{dom}(g^*_\delta) \), \( \bar{g}_\delta \geq g^*_\delta \) and \( \bar{g}_\delta \geq h_\delta[g^*_\delta] \) for every \( \eta < \eta^* \). This follows from considerations at stage \( \delta \), \( g^*_\delta \in G^* \) and \( (g_\delta)^{\text{col}} \geq q^*_\eta \) for all \( \eta \leq \eta^* \). It follows that \( G^* \) meets \( q \).

Now suppose \( G^* \) meets \( q \), \( G^* \in \text{Gen}(\mathcal{P}_{\gamma}) \) contains \( G \) with \( \gamma \equiv \gamma^* < \lambda^+ \). Fix a cofinal sequence \( (g^*_\delta)_\delta \) for \( G^* \), a density system \( D_\delta \) and \( u \in \mathcal{P}_{<\lambda}(\gamma^*) \). We want to find \( w \) so that \( u \subseteq w \) and \( D_\delta(u, w) \cap G^* \neq \emptyset \). Write \( \gamma^* \setminus \gamma \) as a continuous increasing union \( \bigcup_{\delta \in \lambda} w^*_\delta \) with \( w^*_\delta \in \mathcal{P}_{<\lambda}(\gamma^* \setminus \gamma) \).

There is a club \( C_2 \) with the following properties:
1. if \( \delta \in C_2, \ (g^*_\varnothing)^{\text{col}} \mid \alpha = g^*_\varnothing \) and \( \text{dom}(g^*_\varnothing) = w_\delta \cup w^*_\delta \);
2. \( C_2 \cap S'(G^*) \subseteq C \cap S(G) \subseteq C_2 \cap S(G) \);
3. for \( \delta \in C_2 \cap S(G^*) \) there is \( r_\delta \in G^* \) with \( \text{dom}(g^*_\varnothing) = \text{dom}(r_\delta) \), \( g^*_\varnothing \leq r_\delta \), \( (g^*_\varnothing)^{\text{col}} = \eta^*_\delta \in T_\delta \) and \( r_\delta^{\text{col}} \geq q^*_\eta \) for all \( \eta \leq \eta^*_\delta \); and
4. for \( \delta \in C_2 \cap S'(G^*) \) there is \( u' \in U_\delta \) so that
   \[ f_\delta : (\text{otp}(w_\delta \cup w^*_\delta), u') = (w_\delta \cup w^*_\delta, u) \).

This can be obtained by referring to the definition of basic data, Definition 3.1 and Lemma 3.2. In particular, condition 4 follows from the oracle property.
Now choose \( \delta \in C_i \cap S'(G') \) with \( i < \delta \). Let \( \zeta = \text{otp}(w^\delta) \). Since \( \text{dom}(g^\delta) = w^\delta \cup w^\delta \) and \( \eta^\delta = (g^\delta)^{\text{col}} \in T^\delta \), we have that \( \gamma^\delta + \zeta \leq \text{ht}(T^\delta) \). Moreover, \( \eta^\delta = g^\delta \in T^\delta \) and \( \eta^\delta \leq \eta^\delta \). Hence,

\[ h^\delta(q^\delta_{\eta^\delta}) \in D_i(h^\delta[u'], w^\delta \cup [\gamma, \gamma + \zeta]) \]

and \( r^\delta_{\text{col}} \equiv q^\delta_{\eta^\delta} \) with \( r^\delta \in G' \).

By the indiscernibility of the density systems, we have

\[ f_\delta[q^\delta_{\eta^\delta}] \in D_i(u, w^\delta \cup w^\delta) \]

since

\[ f_\delta h^{-1} : (w^\delta \cup [\gamma, \gamma + \zeta], h^\delta[u']) = (w^\delta \cup w^\delta, u). \]

By the indiscernibility of \( P \), we have \( r^\delta \cong f_\delta[q^\delta_{\eta^\delta}] \). Since \( r^\delta \in G' \),

\[ f_\delta[q^\delta_{\eta^\delta}] \in G' \cap D_i(u, w^\delta \cup w^\delta) \]

so \( G \) meets \( D_i \). \( \square \)

It remains to prove Claim 5.2.

**Proof of Claim 5.2.** Consider the set

\[ S = \{ h^\delta[q^\delta_{\eta^\delta}] : \eta \leq \eta^* \} \]

which is a subset of \( \mathbb{P}/G \). By the compatibility condition in the definition of \( \mathbb{P}_G[T^\delta] \), \( S \) is also a compatible set so we can choose \( r^*_{\eta^\delta} \in \mathbb{P}_{\text{len}(\eta^*)} \) so that \( r^*_{\eta^\delta} \equiv q^\delta_{\eta^\delta} \) for all \( \eta \leq \eta^* \) and \( h^\delta[r^*_{\eta^\delta}] \in \mathbb{P}/G \) since \( \mathbb{P}/G \) is \( \lambda^+ \)-uniform.

Now choose \( r^*_{\eta^\delta} \in D^*_i(u) \) so that \( r^*_{\eta^\delta} \equiv r^\delta_{\eta^\delta} \). This is possible since \( D_i \) is a density system over \( G \). Define

\[ \tilde{r}_\eta = \begin{cases} r^*_{\eta^\delta}, & \text{if } \eta \leq \eta^*, \\ q^\delta_{\eta^\delta}, & \text{otherwise}. \end{cases} \]

It is easy to check that \( \tilde{r} \in \mathbb{P}_G[T^\delta] \). \( \square \)

To obtain a proof of 3.6, make the following changes in the above proof. In the statement of 3.6, there is no \( G \) or \( p \) so that the start of the proof, one must consider all \( \delta < \lambda \). The definition of \( D^*_i(u) \) is the same. We replace \( \mathbb{P}_G[T^\delta] \) with \( \mathbb{P}[T^\delta] \) which is the same as \( \mathbb{P}_G[T^\delta] \) but there is no third condition. With few formal changes, Claim 5.2 can be proved which allows one to build the required \( q^\delta \cong p^\delta \).

The rest of the proof is almost identical except that instead of referring to the two conditions on \( G \) in the statement of Proposition 3.7, one uses the fact that \( \mathbb{P} \) already possesses these qualities by virtue of being \( \lambda^+ \)-uniform.
References


