Applied Mathematics Letters 25 (2012) 2369–2374

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Increasing the convergence order of an iterative method for nonlinear systems[☆]

Alicia Cordero [a], José L. Hueso [a], Eulalia Martínez [b,*], Juan R. Torregrosa [a]

[a] *Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera, s/n, 46022 València, Spain*
[b] *Instituto de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera, s/n, 46022 València, Spain*

ARTICLE INFO

Article history:
Received 20 April 2012
Accepted 3 July 2012

Keywords:
Nonlinear equation
Iterative function
Convergence order
Efficiency index

ABSTRACT

In this work we introduce a technique for solving nonlinear systems that improves the order of convergence of any given iterative method which uses the Newton iteration as a predictor.

The main idea is to compose a given iterative method of order p with a modification of the Newton method that introduces just one evaluation of the function, obtaining a new method of order $p + 2$.

By applying this procedure to known methods of order three and four, we obtain new methods of order five and six, respectively. The efficiency index and the computational effort of the new methods are checked.

We also perform different numerical tests that confirm the theoretical results and allow us to compare these methods with the ones from which have been derived and with the classical Newton method.

1. Introduction

Let us consider the problem of finding a zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution α of the nonlinear system $F(x) = 0$ with n equations and n unknowns. The best known iterative method is the classical Newton method that converges quadratically under certain conditions.

Recently, for $n = 1$, many robust and efficient methods have been proposed with higher convergence order (see [1–8]), but in most of cases the methods have not been extended to several variables. Few papers for the multidimensional case introduce methods with high order of convergence. In [9], new methods with order of convergence three based on quadrature formulas of order at least one are presented and in [10], variants of the Newton method using fifth-order quadrature formulas get third order convergence. As far as we know, methods with order of convergence four have been developed in [11,12], where the authors present new families of multi-point iterative methods. Order four and five is reached in [13] by means of Adomian decomposition.

In this paper we propose new iterative methods for the multidimensional case with fifth and sixth order of convergence. In order to compare different methods, we use the efficiency index, $p^{1/d}$ (see [14]), where p is the order of convergence and d is the number of functional evaluations per iteration required by the method. This is the most used index. However, in the n-dimensional case, it is also important to take into account the number of operations performed, since for each iteration

☆ This research was supported by Ministerio de Ciencia e Innovación MTM2011-28636-C02-02 and by Vicerrectorado de Investigación. Universitat Politècnica de ValènciaPAID-06-2010-2285.

* Corresponding author.
E-mail addresses: acordero@mat.upv.es (A. Cordero), jlhueso@mat.upv.es (J.L. Hueso), eumarti@mat.upv.es (E. Martínez), jrtorre@mat.upv.es (J.R. Torregrosa).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2012.07.005

a number of linear systems must be solved, for this reason we also use the *computational efficiency index* defined in [15], as $p^{1/(d+op)}$, where op is the number of products/quotients per iteration.

In [15], we remember some known notions and results that we need in order to analyze the convergence of the new methods, in particular, we use the fact that the inverse $[F'(x)]^{-1}$ of the jacobian matrix can be expressed as a power series whose coefficients can be obtained from the formal inverse of the original series.

In Section 2, we introduce a technique for solving nonlinear systems in which the order of convergence p of a given iterative method is improved to $p+2$. The procedure is obtained by composing the method of order p with a modification of the Newton method that introduces only one additional evaluation of function F. We generalize a work presented by Kou et al. in [16], that establishes

Theorem 1. *Let $u_{k+1} = g_3(x_k)$ be a third order method. Performing a new step given by:*

$$x_{k+1} = u_{k+1} - \frac{f(u_{k+1})}{f'(y_k)},$$

where $y_k = x_k - f(x_k)/f'(x_k)$, a new method of order five is obtained.

In [16], Kou et al. consider a procedure for the unidimensional case and improve a third order method to order five. We generalize this result to the multidimensional case and show that any method of order p can be improved to order $p+2$.

In Section 3 we apply the result described in Section 2 to a third-order and a fourth-order method that use the jacobian matrix, $F'(y^{(k)})$, in their iteration function, in order to add just one functional evaluation when composing the methods with a modified Newton method.

Finally, we present some numerical examples to illustrate the efficiency of the studied methods and compare them with the methods from which they have been obtained.

2. Main result

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function. Finding the solutions of the nonlinear system $F(x) = 0$, is a classical problem that appears in many applied mathematical problems. In order to approximate the solution, we use iterative methods.

In this paper, we present a technique that consists in composing an iterative method of order p with a modification of the Newton method, giving a two-step method:

$$\begin{aligned} z^{(k)} &= \phi(x^{(k)}, y^{(k)}) \\ w^{(k)} &= z^{(k)} - F'(y^{(k)})^{-1}F(z^{(k)}), \end{aligned} \tag{1}$$

where $z^{(k)} = \phi(x^{(k)}, y^{(k)})$ is the iteration function of a method of order p, that uses $F'(y^{(k)})$ in its iteration function, and $y^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)})$ is the classical Newton method of convergence order two.

In order to analyze the order of the iterative method defined by (1) we establish the following result:

Theorem 2. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in a neighborhood D of α, that is a solution of the system $F(x) = 0$, whose jacobian matrix is continuous and nonsingular in D. Then, for an initial approximation sufficiently close to α, the method defined by (1) has order of convergence $p+2$.*

Proof. Taylor's expansion of $F(x^{(k)})$ around α gives

$$F(x^{(k)}) = F'(\alpha)\left(e^{(k)} + C_2{e^{(k)}}^2 + C_3{e^{(k)}}^3 + O({e^{(k)}}^4)\right), \tag{2}$$

where $C_k = \frac{1}{k!}F'(\alpha)^{-1}F^{(k)}(\alpha)$, $k \geq 2$, and $e^{(k)} = x^{(k)} - \alpha$

Then

$$F'(x^{(k)}) = F'(\alpha)\left(I + 2C_2e^{(k)} + 3C_3{e^{(k)}}^2 + O({e^{(k)}}^3)\right) \tag{3}$$

and inverting (3), (see [15]), we have

$$F'(x^{(k)})^{-1} = (I - 2C_2e^{(k)} + (4C_2^2 - 3C_3){e^{(k)}}^2)F'(\alpha)^{-1} + O({e^{(k)}}^3). \tag{4}$$

Let us denote $d^{(k)} = y^{(k)} - \alpha = x^{(k)} - \alpha - F'(x^{(k)})^{-1}F(x^{(k)})$. From (2) and (3) one has

$$d^{(k)} = C_2{e^{(k)}}^2 + (-2C_2^2 + 2C_3){e^{(k)}}^3 + O({e^{(k)}}^4). \tag{5}$$

Using (5) in the expansion of $F'(y^{(k)})$, we obtain

$$F'(y^{(k)}) = F'(\alpha)\left(I + 2C_2^2{e^{(k)}}^2 + 2C_2(-2C_2^2 + 2C_3){e^{(k)}}^3 + O({e^{(k)}}^4)\right). \tag{6}$$

Table 1
Efficiency indices for different values of n.

EI	N $2^{\frac{1}{n+n^2}}$	M_3 $3^{\frac{1}{n+2n^2}}$	M_5 $5^{\frac{1}{2n+2n^2}}$	M_4 $4^{\frac{1}{2n+2n^2}}$	M_6 $6^{\frac{1}{3n+2n^2}}$
$n=2$	1.1224621	1.1161232	1.1435298	1.1224620	1.1365335
$n=3$	1.0594631	1.0537075	1.0693595	1.0594631	1.0686129
$n=4$	1.0352649	1.0309874	1.0410564	1.0352649	1.0415623
$n=5$	1.0233739	1.0201756	1.0271870	1.0233739	1.0279490
$n=10$	1.0063212	1.0052452	1.0073425	1.0063212	1.0078207
$n=20$	1.0016517	1.0013407	1.0019178	1.0016517	1.0020856
$n=30$	1.0007456	1.0006005	1.0008657	1.0007456	1.0009485

Then

$$F'(y^{(k)})^{-1} = \left(I - 2C_2^2 {e^{(k)}}^2 + (4C_2^3 - 4C_2C_3){e^{(k)}}^3 + O({e^{(k)}}^4)\right) F'(\alpha)^{-1}. \tag{7}$$

Consider now the expansion of $F(z^{(k)})$ about α,

$$F(z^{(k)}) = F'(\alpha)\left(C_2{b^{(k)}}^2 + C_3{b^{(k)}}^3 + O({b^{(k)}}^4)\right), \tag{8}$$

where $b^{(k)} = z^{(k)} - \alpha$. By substituting it in $a^{(k)} = \omega^{(k)} - \alpha = z^{(k)} - \alpha - F'(y^{(k)})^{-1}F(z^{(k)})$, and using (7) one obtains

$$\begin{aligned} a^{(k)} &= \left(-C_2{b^{(k)}}^2 - C_3{b^{(k)}}^3 + O({b^{(k)}}^4)\right) + \left(2C_2^2 b^{(k)} + 2C_2^3{b^{(k)}}^2 + 2C_2^2C_3{b^{(k)}}^3 + O({b^{(k)}}^4)\right){e^{(k)}}^2 \\ &\quad + \left(\left(-4C_2^3 + 4C_2C_3\right)b^{(k)} - C_2\left(4C_2^3 - 4C_2C_3\right){b^{(k)}}^2 \right. \\ &\quad \left. -\ C_3\left(4C_2^3 - 4C_2C_3\right){b^{(k)}}^3 + O({b^{(k)}}^4)\right){e^{(k)}}^3 + O({e^{(k)}}^4). \end{aligned} \tag{9}$$

Due to the fact that the method defined by ϕ is of order p, the error equation is $b^{(k)} = A{e^{(k)}}^p + O({e^{(k)}}^{p+1})$ and so $a^{(k)}$ is at least of order $p+2$. $\square$

3. Extended methods—efficiency index

We apply (1) to a third-order method and to a fourth order method that use $F'(y)$ in their iteration function. Firstly let us consider the third order method, (M_3), introduced by Frontini and Sormani [17]. By using Theorem 1, we obtain the fifth-order method, denoted by M_5:

$$\begin{aligned} z^{(k)} &= x^{(k)} - 2(F'(y^{(k)}) + F'(x^{(k)}))^{-1}F(x^{(k)}), \\ w^{(k)} &= z^{(k)} - F'(y^{(k)})^{-1}F(z^{(k)}). \end{aligned}$$

Now we consider the fourth order method, (M_4), introduced by Cordero et al. in [13]. In order to obtain a sixth-order method, denoted by M_6, we compose it with the modified Newton method:

$$\begin{aligned} z^{(k)} &= y^{(k)} - F'(x^{(k)})^{-1}[2I - F'(y^{(k)})F'(x^{(k)})^{-1}]F(y^{(k)}), \\ w^{(k)} &= z^{(k)} - F'(y^{(k)})^{-1}F(z^{(k)}). \end{aligned}$$

Remember that in both methods $y^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)})$.

Table 1 shows the efficiency indices for different sizes of the nonlinear system. Notice that, for any value of n, the new methods M_5 and M_6 are, respectively, more efficient than M_3 and M_4, the third and fourth order methods from which they are derived, and that for $n > 3$ the new method M_6 is always the most efficient.

It is important to point out that, with this technique, we get efficient methods if they use $F'(y^{(k)})$ in the first step, because in this case we add just one functional evaluation.

Table 2 shows the computational efficiency indices of the described methods for some values of n. The results show that the new method M_5 is always better than M_3, while M_6 is better than M_4 only for $n = 1, 2$. Nevertheless, for big sized problems (see $n = 10, 20, 30$), the new method M_6 has computational efficiency index better than the Newton method.

4. Numerical results

In this section we compare the performance of the numerical methods introduced in our work, M_5 and M_6 with the third and fourth order methods from which they are derived, M_3 and M_4, and with the classical Newton method, in order to check their effectiveness.

Table 2
Computational efficiency indices for different values of n.

	N	M_3	M_5	M_4	M_6
$n = 2$	1.0594631	1.0512048	1.0551131	1.0472941	1.0482809
$n = 3$	1.0241896	1.0201756	1.0216911	1.0205959	1.0205696
$n = 4$	1.0124545	1.0102243	1.0109339	1.0112425	1.0109853
$n = 5$	1.0073230	1.0059561	1.0063315	1.0069556	1.0066582
$n = 10$	1.0012844	1.0010273	1.0010664	1.0014603	1.0012899
$n = 20$	1.0001992	1.0001583	1.0001606	1.0002719	1.0002191
$n = 30$	1.0000641	1.0000508	1.0000510	1.0000959	1.0000735

Table 3
Numerical results for nonlinear systems.

$F(x)$	$x^{(0)}$	Iterations					ρ				
		N	M_3	M_5	M_4	M_6	N	M_3	M_5	M_4	M_6
(a)	$[1, 1]$	9	6	5	6	5	2.0	3.0	5.0	4.0	6.0
	$[0, -1]$	9	6	4	5	4	2.0	3.0	5.0	4.0	6.0
(b)	$[1, 1]$	10	7	5	6	5	2.0	3.0	3.5	4.0	6.0
	$[-1, 2]$	11	7	6	6	5	2.0	3.0	5.0	4.0	6.0
(c)	$[1, 0.5, 1]$	10	7	5	7	5	2.0	2.83	5.0	4.0	5.1
	$[1, 1, 2]$	10	8	6	8	5	2.0	3.0	5.0	4.0	6.0
(d)	$[2, \ldots, 2]$	9	7	5	6	5	2.0	3.0	5.0	4.0	6.0
	$[-4, \ldots, -4]$	11	7	6	6	5	2.0	3.0	5.0	4.0	6.0
(e)	$[1, \ldots, 1]$	13	9	7	8	6	2.0	3.0	5.2	4.1	6.2
	$[-2, \ldots, -2]$	14	9	7	8	7	2.0	3.0	5.4	4.1	6.4

Consider the nonlinear systems defined by the following functions and their solutions:

(a) $F(x_1, x_2) = (x_1^2 - 4x_2 + x_2^2, 2x_1 - x_2^2 - 2)$, $\alpha \approx (0.3542, 1.1364)$, $\beta \approx (0.3542, -1.1364)$.

(b) $F(x_1, x_2) = (x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 + 0.5)$, $\alpha = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\beta = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

(c) $F(x_1, x_2, x_3) = (\cos(x_2) - \cos(x_1), x_3^{x_1} - 1/x_2, e^{x_1} - x_3^2)$, $\alpha \approx (0.9096, 0.6613, 1.5758)$.

(d) $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$f_i(x) = x_i x_{i+1} - 1, \quad i = 1, 2, \ldots, n-1$$
$$f_n(x) = x_n x_1 - 1.$$

When n is odd, the exact zeros of $F(x)$ are $\alpha = (1, 1, \ldots, 1)$ and $\beta = (-1, -1, \ldots, -1)$. Results appearing in Table 3 are obtained for $n = 31$.

(e) $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, $n > 3$, such that

$$f_k(x) = \sum_{1 \le i < j \le n, i,j \ne k} x_i x_j, \quad k = 1, 2, \ldots, n-1$$
$$f_n(x) = \sum_{1 \le i < j \le n, i,j \ne k} x_i x_j - 1.$$

A zero of $F(x)$ is

$$\alpha_i = \sqrt{\frac{2}{(n-1)(n-2)}}, \quad i < n; \qquad \alpha_n = -\frac{n-3}{\sqrt{2(n-1)(n-2)}},$$

and symmetrically,

$$\beta_i = -\sqrt{\frac{2}{(n-1)(n-2)}}, \quad i < n; \qquad \beta_n = \frac{n-3}{\sqrt{2(n-1)(n-2)}}.$$

The results that appear in Table 3 are obtained for $n = 30$.

Nowadays, high order methods are important because numerical applications use high precision in their computations. For this reason, numerical computations have been carried out using variable precision arithmetic that uses floating point representation of 200 decimal digits of mantissa in MATLAB 7.1. Some of the examples used in our test appear in [13,11].

In Table 3, we calculate the number of iterations and the estimated order of convergence ρ, given by (see [10]):

$$\rho = \frac{\ln\left(\left\|x^{(k+1)} - x^{(k)}\right\| / \left\|x^{(k)} - x^{(k-1)}\right\|\right)}{\ln\left(\left\|x^{(k)} - x^{(k-1)}\right\| / \left\|x^{(k-1)} - x^{(k-2)}\right\|\right)} \tag{10}$$

Table 4
Numerical results stepwise for example (c) with $x^{(0)} = [1, 0.5, 1]$.

	Method	$\|x^{(k)} - x^{(k-1)}\|$	$\|F(x^{(k)})\|$
$k = 1$	N	0.9300	0.8606
	M_3	0.5616	0.2353
	M_5	0.5986	0.0370
	M_4	2.1706	1.1870
	M_6	0.6403	0.0754
$k = 2$	N_2	0.3365	0.0763
	M_3	0.0704	0.0011
	M_5	0.0147	3.3170e−008
	M_4	0.4286	0.0082
	M_6	0.0651	1.4496e−007
$k = 3$	N	0.0687	0.0021
	M_3	0.0015	7.7680e−009
	M_5	4.7902e−008	2.7887e−036
	M_4	0.0067	2.2736e−009
	M_6	2.3791e−007	7.9758e−039
$k = 4$	N_2	0.0038	1.0642e−005
	M_3	1.2151e−008	3.6915e−024
	M_5	4.1532e−036	1.2675e−176
	M_4	4.1168e−009	2.5769e−033
	M_6	1.1971e−038	1.5508e−226
$k = 5$	N	1.7620e−005	2.9328e−010
	M_3	5.7816e−024	3.9896e−070
	M_5	1.8965e−176	0
	M_4	3.9118e−033	2.6800e−129
	M_6	2.3169e−226	0
$k = 6$	N_2	4.4084e−010	1.9684e−019
	M_3	6.2555e−070	5.0462e−208
	M_5	1.8965e−176	0
	M_4	4.0047e−129	0
	M_6	2.3169e−226	0

for every method and different initial guesses. The stopping criterion used is $\left\|x^{(k+1)} - x^{(k)}\right\| + \left\|F(x^{(k)})\right\| < 10^{-120}$. Therefore, we check that the iterates converge to a limit and that this limit is a solution of the nonlinear system.

Finally, Table 4 shows the distance between two consecutive approximations $\|x^{(k)} - x^{(k-1)}\|$ and the value of $\|F(x^{(k)})\|$ in each iteration for example (c). It can be observed that from the third iteration on the results of our methods M_5 and M_6 outperform the results of the other methods.

5. Conclusions

In this paper we introduce a technique for accelerating the order of convergence of a given iterative process, that uses the Newton iteration as a predictor, from p to $p+2$, with only one additional evaluation of the function. We apply this technique to two particular methods of third and fourth order obtaining two new methods of fifth and sixth order of convergence. The efficiency indices of these new methods improve the ones corresponding to the methods from which they have been derived.

The theoretical results have been checked with some numerical examples.

References

[1] X. Li, C. Mu, J. Ma, C. Wang, Sixteenth-order method for nonlinear equations, Applied Mathematics and Computation 215 (10) (2010) 3754–3758.
[2] W. Bi, H. Ren, Q. Wu, Three-step iterative methods with eighth-order convergence for solving nonlinear equations, Journal of Computational and Applied Mathematics 255 (2009) 105–112.
[3] J.R. Sharma, R. Sharma, A family of modified Ostrowski's methods with accelerated eighth order convergence, Numerical Algorithms 54 (2010) 445–458.
[4] W. Bi, Q. Wu, H. Ren, A new family of eighth-order iterative methods for solving nonlinear equations, Applied Mathematics and Computation 214 (2009) 236–245.
[5] R. Thukral, M.S. Petkovic, A family of three-point methods of optimal order for solving nonlinear equations, Journal of Computational and Applied Mathematics 233 (2010) 2278–2284.
[6] A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa, New modifications of Potra–Pták's method with optimal fourth and eighth order of convergence, Journal of Computational and Applied Mathematics 234 (2010) 2969–2976.
[7] L. Liu, X. Wang, Eighth-order methods with high efficiency index for solving nonlinear equations, Applied Mathematics and Computation 215 (2010) 3449–3454.
[8] L. Liu, X. Wang, New eighth-order methods for solving nonlinear equations, Journal of Computational and Applied Mathematics 234 (2010) 1611–1620.
[9] M. Frontini, E. Sormani, Third-order methods from quadrature formulae for solving systems of nonlinear equations, Applied Mathematics and Computation 149 (2004) 771–782.

[10] A. Cordero, Juan.R. Torregrosa, Variants of Newton's method using fifth-order quadrature formulas, Applied Mathematics and Computation 190 (2007) 686–698.
[11] G.H. Nedzhibov, A family of multi-point iterative methods for solving systems of nonlinear equations, Journal of Computational and Applied Mathematics 222 (2008) 244–250.
[12] M.T. Darvishi, A. Barati, A fourth-order method from quadrature formulae to solve systems of nonlinear equations, Applied Mathematics and Computation 188 (2007) 257–261.
[13] A. Cordero, E. Martínez, J.R. Torregrosa, Iterative methods of order four and five for systems of nonlinear equations, Journal of Computational and Applied Mathematics 231 (2009) 541–551.
[14] A.M. Ostrowski, Solutions of Equations and Systems of Equations, Academic Press, New York, London, 1966.
[15] A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa, A modified Newton–Jarrat's composition, Numerical Algorithms 55 (1) (2010) 87–99.
[16] J. Kou, Y. Li, X. Wang, Some modification of Newton's method with fifth-order convergence, Journal of Computational and Applied Mathematics 209 (2007) 146–152.
[17] M. Frontini, E. Sormani, Some variants of Newton's method with third-order convergence, Applied Mathematics and Computation 140 (2003) 419–426.