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AN OBSTRUCTION TO EMBEDDING GRAPHS IN SURFACES

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It is shown that the genus of an embedding of a graph can be determined by the rank of a certain matrix. Several applications to problems involving the genus of graphs are presented.

1. Introduction

There is much work done in the theory of embeddings of graphs in closed surfaces. Among the most important results is the solution of the Heawood Map Coloring Problem which includes the determination of the genera of complete graphs. (The reader is referred to the Ringel's book [8], and also to monographs by White [16] and Gross and Tucker [6].) Many nice and powerful techniques were developed in order to solve this problem. However, most of the results in the theory of graph embeddings serve to construct embeddings of more or less nice graphs, cf. [12]. To prove that the obtained embeddings are the best possible (which is necessary in computing the genus of a graph), one has to show that a given graph cannot be embedded into a surface of lower genus. Therefore we consider the problem of proving that a graph does not embed in a given surface to be of vital importance. Unfortunately, not many general methods to prove nonembeddability of graphs are known. We mention briefly the most important of these methods:

(a) The Euler's formula, the lengths of shortest cycles in the graph and elementary counting methods give upper bounds on the number of faces and thus lower bounds on the genus.

(b) Forbidden subgraph techniques. It is known that for each surface there is a finite complete set of forbidden minors. However, these sets are very large (except for the plane). This was shown by Robertson and Seymour [10] who also described a global structure of graphs of given genus.

(c) Use of computers and brute force searches.

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(d) For each given genus g there are efficient (i.e. polynomial time) algorithms to check if a graph has genus $\leq g$ [18]. These algorithms are impractical for higher values of g but practically useful for $g = 0$ or 1.

Here we propose a general method, an easy computable algebraic invariant, using the rank of a certain matrix. It provides short algebraic proofs of some well-known theorems concerning embeddings of graphs in surfaces. In the proofs only the elementary properties of the rank of a matrix are used, with no reference to topology. The main theorem also provides a convenient way for obtaining lower bounds on the genus, or the relative genus, of graphs. Usually, this method is not more powerful than the method (a) of above, but in many cases it is much easier to apply. But there are cases where our algebraic method can be used, while (a) fails. See the second example in 4.7.

2. Embedding schemes

It is assumed that the reader is familiar with basic notions of graph theory and embeddings of graphs in closed surfaces. All embeddings will be cellular, surfaces may be orientable or non-orientable. Graphs are finite undirected, multiple edges and loops are allowed. $V(G)$ and $E(G)$ will always denote vertex set and edge set of the graph G, respectively. It will be assumed that all graphs under consideration are connected.

Each edge $e \in E(G)$ gives rise to two oppositely oriented *arcs*, each having the initial vertex at one end of e. For $v \in V(G)$, a *rotation at* v is a cyclic permutation P_v of the set of arcs having initial vertex v. A family $P = (P_v, v \in V(G))$ is a *rotation system* if each P_v is a rotation. P will also be identified with a permutation on the set of all arcs of G having P_v $(v \in V(G))$ as its cycles. Finally, an *embedding scheme* is a pair (P, λ) where P is a rotation system and λ is a mapping $E(G) \to \mathbb{Z}_2$.

It is well known that every orientable embedding of a graph G can be described by an embedding scheme (P, λ) where $\lambda(e) = 0$ for each $e \in E(G)$. By allowing λ to take non-zero values we can describe also all non-orientable embeddings of G. The details can be found in [9, 11, 12]. It is also known that the embedding described by a pair (P, λ) is orientable if and only if each cycle contains even number of edges with $\lambda(e) = 1$. If T is a spanning tree of G and (P, λ) an embedding scheme then there is an embedding scheme (P', λ') such that

(a) (P', λ') yields the same embedding of G as (P, λ), and

(b) for each $e \in E(T)$, $\lambda'(e) = 0$.

3. Main theorem

Let T be a spanning tree of the graph G, and let (P, λ) be an embedding scheme which is trivial on T, i.e. $\lambda(e) = 0$ for $e \in E(T)$. If e_1, e_2 are edges in the

co-tree $G\backslash E(T)$ we say that these edges *overlap* (w.r.t. P, λ and T) if either $e_1 = e_2$ and $\lambda(e_1) = 1$, or $e_1 \neq e_2$ and the embedding of $T + e_1 + e_2$ induced by $(P, 0)$ is non-planar. Here 0 is the everywhere 0 assignment, regardless of the original λ. The *overlap matrix* A w.r.t. (P, λ) and T is a 01-matrix with rows and columns indexed by edges in $G\backslash E(T)$ and with ef-entry A_{ef} equal to 1 if e and f overlap, and $A_{ef} = 0$ otherwise. Note that the embedding of G described by (P, λ) is orientable if and only if A has zero diagonal.

Theorem 3.1. *Let T be a spanning tree of G. If G is embedded in a surface S, let (P, λ) be an embedding scheme describing this embedding which is trivial on T. If A is the corresponding overlap matrix and* rank A *is the rank of A as a matrix over* GF(2) *then*:

(a) *If S is orientable then the genus $\gamma(S)$ of the embedding is equal to* $\frac{1}{2}$ rank A.

(b) *If S is non-orientable then the non-orientable genus is* $\tilde{\gamma}(S) = \operatorname{rank} A$.

Proof. Contracting T to a point on S preserves the topological type of S and the overlappings. If m is the number of edges in $E' = E(G)\backslash E(T)$ and r is the number of 2-cells of the embedding, then by Euler's formula,

$$\chi(S) = 1 - m + r = \begin{cases} 2 - 2\gamma(S), & \text{if } S \text{ is orientable,} \\ 2 - \tilde{\gamma}(S), & \text{if } S \text{ is non-orientable.} \end{cases}$$

Consequently, it suffices to show that $\operatorname{rank} A = m - (r - 1)$, or equivalently that $\dim(\ker A) = r - 1$.

Let C be a 2-cell of the embedding with T contracted to a point, and let $e_1, \ldots, e_k$, be the consecutive edges on the boundary of C. Given $f \in E'$ and traveling along the boundary of C, count how many times the edges e_i overlap with f. Since we come after k steps (edges $e_1, \ldots, e_k$) back to the initial point on C, the total number of overlaps with f must be even. In other words, if x_C is the characteristic vector of the edges $\{e_1, \ldots, e_k\}$ then $A^{\mathrm{T}} x_C = 0$. The vectors x_C generate the space of boundaries (subsapce of the cycle space) which is known to be $(r-1)$-dimensional on a surface with r cells. Consequently, $\dim(\ker A^{\mathrm{T}}) = \dim(\ker A) \geq r - 1$.

Let now x be a GF(2)-vector such that $A^{\mathrm{T}} x = 0$. Let $e_1, \ldots, e_k$ be those edges of E' for which the corresponding coordinate in x is equal to 1. The condition $A^{\mathrm{T}} x = 0$ means that each $f \in E'$ overlaps with even number of $e_1, \ldots, e_k$. Now remove from the surface S all edges except $e_1, \ldots, e_k$ (and T). Going around the point corresponding to T, mark the regions between any two consecutive arcs of $e_1, \ldots, e_k$ alternately a and b. No edge $f \in E'$ can join a and b since f is overlapping with even many e_i's. Consequently, x is a linear combination of all vectors x_C where C is a 2-cell whose boundary is in a region marked a. Therefore $\dim(\ker A^{\mathrm{T}}) \leq r - 1$. This completes the proof. □

Remark. It was noted by the referee that Theorems A and B of Goldstein and Turner [5] which seem to be only a special case of our result (cf. 4.1) are, in fact,

equivalent to Theorem 3.1 (only orientable case) via a simple construction which assigns with our G, T, P, λ, S a cubic graph H embedded in S such that a hamilton cycle of H bounds a 2-cell and the overlap matrix with respect to this cell is equal to our overlap matrix A.

Remark. Theorem 3.1 is, in fact, a homology-type result. It is easily seen that it is equivalent to the following. Each edge e of E' determines a unique cycle in G (with the edges of T), and can be viewed as a (homological) 1-cycle. Denote by γ_e the corresponding element in the homology group $H_1(S;\mathbb{Z}_2)$. Then the mapping ψ: row space of $A \to H_1(S;\mathbb{Z}_2)$, which maps the row corresponding to e to γ_e, is an isomorphism of $\mathbb{Z}_2$-modules.

In the case of orientable embeddings there is also an integer (oriented) version of Theorem 3.1. Orient each edge $e \in E'$ in arbitrary way. With respect to e, the arcs corresponding to the edges in $E' \setminus \{e\}$ fall into two classes: either starting "inside" or "outside" w.r.t. e. The former means that the arc is, according to the local rotation, between the starting and terminal arc of e, and outside means between terminal and initial. If α, β are (oriented) edges of E', put:

(a) $A_{\alpha\beta} := 0$, if α and β do not overlap.

(b) $A_{\alpha\beta} := 1$, if α, β overlap and β starts outside w.r.t. α and terminates inside w.r.t. α.

(c) $A_{\alpha\beta} := -1$, if α, β overlap and β starts inside w.r.t. α and ends outside w.r.t. α.

The obtained matrix $A = (A_{\alpha\beta})$, $\alpha, \beta \in E'$ is the *oriented overlap matrix* and is considered as a matrix over the ring of integers.

Theorem 3.2. *With the notations of Theorem* 3.1, λ *being trivial and* A *the oriented overlap matrix, the genus of* S *is*

$$\gamma(S) = \tfrac{1}{2} \operatorname{rank} A.$$

Proof. Proceed along the lines of the proof of Theorem 3.1. The 2-cells are now oriented, and for a 2-cell C, the vector x_C is a $(0, 1, -1)$-vector, where -1 occurs if the corresponding edge appears on the boundary of C with the orientation not being coherent with the orientation of C. The second part, that x with $A^{\mathrm{T}}x = 0$ corresponds to a bounding set, is trivial by Theorem 3.1 since from its proof it also follows that if $A^{\mathrm{T}}x$ has only even coordinates then x bounds (and thus $A^{\mathrm{T}}x = 0$). □

Remarks. (a) The rank of A does not depend on the choice of orientations of edges in E'.

(b) A is a $(0, 1, -1)$-matrix which is skew symmetric, i.e. $A_{\alpha\beta} = -A_{\beta\alpha}$.

(c) It follows that the rank of A is always even. The same result for the $\mathbb{Z}_2$-rank is obvious since the $\mathbb{Z}_2$-rank of any symmetric 01-matrix with zero diagonal is even.

(d) The rank over integers is equal to the rank over the reals.

Example. If G is the graph of *Fig. 1* with the edges of T drawn thicker, and orientations of the edges in E' as indicated then the oriented overlap matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

with the rows and columns corresponding respectively to a, b, c, d.

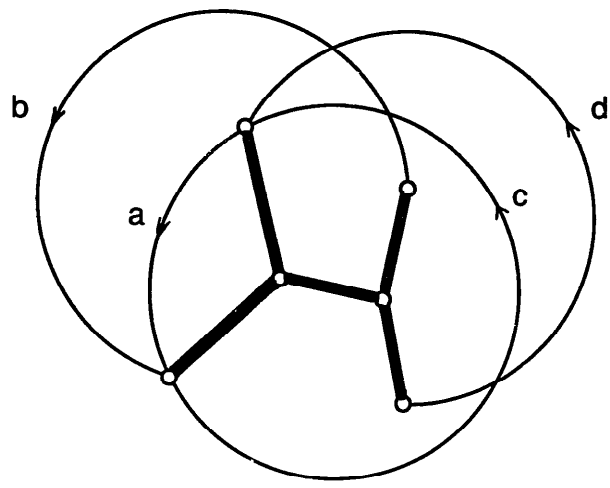

Fig. 1.

Its rank is 2. The vectors of the 2-cells are $x_1 = (0, -1, -1, 1)^T$, $x_2 = (-1, 1, 1, -1)^T$, $x_3 = (1, 0, 0, 0)^T$, and they span the 2-dimensional kernel of A.

4. Applications

4.1. *A facial cycle*

Let C be a cycle in G. If we want to consider only those embeddings of G where C appears as a cycle bounding a 2-cell of the embedding (an embedding *relative* to C, as it is called in [14]) then the results of Section 3 can be strengthened. Take a spanning tree T of G which contains all but one edge of C. Then this edge is not overlapping with any other edge, and the overlapping of other edges depends on the consecutivity of their ends on C. (If they have common ends then this depends also on rotation system.) This result, restricted to cubic graphs, C a hamiltonian cycle and S orientable, was proved earlier by Brahana [3], Cohn and Lempel [4], Marx [7], and Goldstein and Turner [5].

Example. Let M_n be the Möbius ladder graph on $2n$ vertices. It consists of a cycle C of length $2n$ and n diagonals, each connecting two vertices opposite on the cycle C. Considering only relative orientable embeddings with C bounding a

2-cell, the overlap matrix is $A = J - I$ where J is the all-1-matrix and I is the identity. Its rank is n if $n \equiv 0 \pmod 2$ and is $n-1$ if n is odd. Therefore the genus is equal to $\lfloor n/2 \rfloor$.

It is interesting to note that the non-orientable relative genus of M_n w.r.t. C is equal to 1. If we take $\lambda = 1$ on all diagonals of M_n then the overlap matrix turns out to be $A = J$ which has rank 1. These graphs were used in [1] to construct examples of graphs with non-orientable genus 1 and genus arbitrarily high.

4.2. *A bounding cycle*

If, similarly as in 4.1, we demand that C be a bounding cycle on the surface, we might proceed as in 4.1. The only distinction is that we split the edges in $E' = E(G) \setminus E(T)$ into two classes X, Y depending on whether (both) ends of an edge lie "outside" or "inside" C. Now the genus will be the sum of (half of) the ranks of overlap matrices of X and Y, respectively.

Example. What is the minimal genus of a surface in which M_n has an embedding with C a bounding cycle? In any case, X and Y have matrices of the form $J - I$ (of appropriate dimensions). So, the orientable genus is $\lfloor p/2 \rfloor + \lfloor r/2 \rfloor$ where $p = |X|$, $r = |Y|$ and $p + r = n$. It follows that the minimum is equal to $\lfloor (n-1)/2 \rfloor$. It is interesting to mention that $\gamma(M_n) = 1$ $(n \geqslant 3)$.

4.3. *Nonplanarity*

Since in the plane every cycle bounds, it follows from 4.2 that for every cycle C in G the edges in E' can be split into classes X, Y such that no two edges in the same class overlap. In other words, the overlap graph is bipartite. By 4.2, this condition is also sufficient for planarity. Note that this is very close to the Tutte's characterization of planarity [16]. See also [15].

4.4. *Additivity of the genus*

A well known result of Battle et al. [2] states that the genus of a graph G is equal to the sum of the genera of its blocks. This follows easily by Theorem 3.1, since any spanning tree of G consists of spanning trees of its 2-connected components, and the edges in different blocks do not overlap in minimal embeddings. Hence any overlap matrix has a block diagonal structure with the rank equal to the sum of the ranks of its blocks. We point out that this also proves the non-orientable genus version proved by Stahl and Beineke [13].

4.5. *Genus and non-orientable genus*

Consider an orientable genus embedding of G. Choose a spanning tree T and put $\lambda(e) = 1$ on one edge $e \in E(G) \setminus E(T)$. The overlap matrix changes in one

entry only, so its rank can be affected by at most 1. Consequently, $\gamma(G) \leq 2 \cdot \tilde{\gamma}(G) + 1$. See [17].

4.6. *Interpolation theorem and maximum genus for non-orientable embeddings*

Ringel [9] proved that the maximum non-orientable genus of any connected graph G is equal to its Betti number $\beta(G) = |E(G)| - |V(G)| + 1$. It is also well known [11] that a connected graph G has 2-cell embeddings in all non-orientable surfaces with genera between $\tilde{\gamma}(G)$ and $\beta(G)$. There is a purely algebraic proof of these results using overlap matrices. The details are omitted.

4.7. *Obstruction to embeddings*

We left the most important application of the results of Section 3 and 4.1, 4.2 to the very end. An easy computable invariant, rank of the overlap matrix, can be used to obtain lower bounds on the genus of graphs.

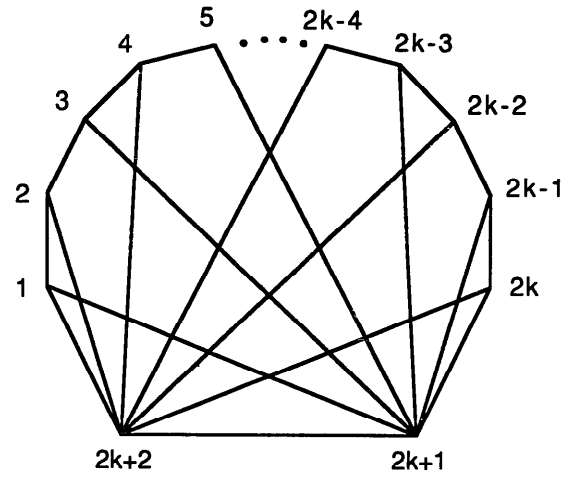

Fig. 2.

Let G_k be the graph represented on *Fig. 2.* It has vertices $1, 2, \ldots, 2k+2$ spanning a cycle C (in cyclic order), and $2k$ additional edges, so that $2k+1$ is joined to $1, 3, 5, \ldots, 2k-1$ and $2k+2$ is adjacent to $2, 4, 6, \ldots, 2k$. Then any overlap matrix w.r.t. C is of the form:

$$A = \begin{bmatrix} X & L \\ L & Y \end{bmatrix}$$

where X, L, and Y are $k \times k$ matrices, X and Y depending on rotation system and L is lower right triangular 01-matrix, i.e. $L_{ij} = 0$ if $j < k + 1 - i$, and $L_{ij} = 1$ otherwise. Consequently, $\operatorname{rank} A \geq k$, and the orientable genus of G_k relative to C is at least $\lceil k/2 \rceil$. Similarly, the non-orientable relative genus (w.r.t. C) of G_k is at least k. (Indeed, it it easily seen that $\tilde{\gamma}(G_k) = k$.)

We mention that the same bounds can also be obtained by the method (a) of the Introduction.

In our last example Euler's formula (method (a)) does not work. We will prove that the graph $K_8 - E(C_4)$ has genus 2. According to Euler's formula, this graph could be embedded in the torus but any such embedding will be triangular. Let v be a vertex in $K_8 - E(C_4)$ of degree 7 and assume we have an embedding of $K_8 - E(C_4)$ into the torus. This is, of course, equivalent to having an embedding of the graph $(K_8 - E(C_4)) - v \approx K_7 - E(C_4)$ into the torus relative to a hamilton cycle C of this graph. Denote the vertices on C respectively by 1, 2, 3, 4, 5, 6, 7. It is easy to see that we may assume w.l.o.g. that the edge 13 is not in our graph. After that, we have only two non-isomorphic possibilities for the cycle C_4: 1375 or 1374. In each of the cases, the overlap matrix A contains the submatrix S with rows corresponding to edges 16, 27, 35 and columns 16, 27, and 24. It is equal to:

$$S = \begin{array}{r} \\ 16 \\ 27 \\ 35 \end{array} \begin{array}{c} \begin{array}{ccc} 16 & 27 & 24 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & x \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

where the value of x depends on the rotation. S has rank equal to 3, indpendent of x being 0 or 1. Thus rank $A \geqslant 3$ which shows that the genus cannot be one.

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